On the Spectrum of an Oseen–Type Operator Arising from Flow past a Rotating Body

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Abstract

We present the description of the spectrum of a linear perturbed Oseen–type operator which arises from equations of motion of a viscous incompressible fluid in the exterior of a rotating compact body. Considering the operator in the function space $L^2_\sigma(\Omega)$ we prove that the essential spectrum consists of an infinite set of overlapping parabolic regions in the left half–plane of the complex plane. Our approach is based on a reduction to invariant closed subspaces of $L^2_\sigma(\Omega)$ and on a Fourier series expansion with respect to an angular variable in a cylindrical coordinate system attached to the axis of rotation in the case of an axially symmetric body. In the general case we exploit stability properties of the essential spectrum.

AMS Subject Classification: Primary: 35 Q 35; secondary: 35 P 99, 76 D 07
Keywords: eigenvalues, essential spectrum, modified Oseen problem, rotating obstacle

1 Motivation and introduction

Suppose that $\mathcal{B}$ is a compact body in $\mathbb{R}^3$ which is rotating about the x_1–axis with a constant angular velocity $\omega > 0$. Denote by $\Omega(t)$ the exterior of $\mathcal{B}$ at time t and assume that $\Omega(t)$ is a domain with boundary of class $C^{1,1}$.

The flow of a viscous incompressible fluid in the exterior of the body $\mathcal{B}$ can be described by the Navier–Stokes equation and the equation of continuity in the space–time region $\{(\boldsymbol{x},t) \in \mathbb{R}^3 \times \mathcal{I};\ t \in \mathcal{I},\ \boldsymbol{x} \in \Omega(t)\}$ where $\mathcal{I}$ is an interval on the time axis. The disadvantage of this description is the variability of the spatial domain $\Omega(t)$. Therefore, many authors use a time–dependent transformation of spatial coordinates which in fact also represents the rotation about the x_1 axis such that the body $\mathcal{B}$ is fixed and its exterior is just $\Omega(0)$ in the new coordinate system. The system of equations after the transformation reads

$$\partial_t \boldsymbol{u} - \nu\Delta\boldsymbol{u} - \omega(\boldsymbol{e}_1 \times \boldsymbol{x}) \cdot \nabla\boldsymbol{u} + \omega\boldsymbol{e}_1 \times \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p \;=\; \boldsymbol{f} \qquad (1.1)$$

$$\nabla\cdot\boldsymbol{u} \;=\; 0 \qquad (1.2)$$

in $\Omega(0) \times I$, where $\boldsymbol{e}_1$ is the unit vector oriented in the direction of the x_1–axis. The condition of the adherence of the fluid to the body on the boundary, after the transformation, has the form

$$\boldsymbol{u}(\boldsymbol{x},t) \;=\; \omega\boldsymbol{e}_1 \times \boldsymbol{x}, \qquad \boldsymbol{x} \in \partial\Omega(0). \qquad (1.3)$$

In order to simplify the notation, we shall write only Ω instead of $\Omega(0)$.

Among a series of results on qualitative properties of the system (1.1)–(1.3) and related linear problems, let us mention T. Hishida [15], [16], [17], G. P. Galdi [9], [10], R. Farwig, T. Hishida, D. Müller [5], R. Farwig [3], [4], Š. Nečasová [23], M. Geissert, H. Heck, M. Hieber [11], S. Kračmar, Š. Nečasová, P. Penel [19], R. Farwig, J. Neustupa [6] and R. Farwig, Š. Nečasová, J. Neustupa [7].

We shall use the usual function spaces and notation:

- $\boldsymbol{n}$ is the outer normal vector on $\partial\Omega$.
- $(\,.\,,\,.\,)_{0,2}$ and $\|\,.\,\|_{0,2}$ are the scalar product and the norm in $L^2(\Omega)^3$, respectively.
- $W_0^{1,2}(\Omega)$ is the subspace of the Sobolev space $W^{1,2}(\Omega)$ consisting of functions vanishing on $\partial\Omega$ in the sense of traces. As is well-known, $W_0^{1,2}(\Omega)$ equals the closure of $C_0^\infty(\Omega)$ in the norm of $W^{1,2}(\Omega)$.
- $\|\,.\,\|_{k,2}$ denotes the norm in $W^{k,2}(\Omega)^3$, $k \in \mathbb{N}$.
- $C_{0,\sigma}^\infty(\Omega)$ denotes the space of all divergence–free functions from $C_0^\infty(\Omega)^3$.
- $L^2_\sigma(\Omega)$ is the closure of $C_{0,\sigma}^\infty(\Omega)$ in $L^2(\Omega)^3$. The space $L^2_\sigma(\Omega)$ can be characterized as the space of all divergence–free (in the sense of distributions) vector functions $\boldsymbol{u}$ from $L^2(\Omega)^3$ such that $\boldsymbol{u}\cdot\boldsymbol{n} = 0$ on $\partial\Omega$ in the sense of traces ([8], pp. 111–115).
- Π_σ denotes the orthogonal projection of $L^2(\Omega)^3$ onto $L^2_\sigma(\Omega)$.

Suppose that $\boldsymbol{U}^*$ is a steady strong solution of the problem (1.1)–(1.3) such that

$$|\nabla \boldsymbol{U}^*| \in L^{3/2}(\Omega) \cap L^3(\Omega), \tag{1.4}$$

$$\lim_{R\to+\infty} \operatorname*{ess\,sup}_{|x|>R} |\boldsymbol{U}^* - \boldsymbol{U}^*_\infty| \,=\, 0 \tag{1.5}$$

where $\boldsymbol{U}^*_\infty = (\gamma, 0, 0)$, $\gamma \in \mathbb{R}$. The function $\boldsymbol{U} := \boldsymbol{U}^* - \boldsymbol{U}^*_\infty$ equals $\omega \boldsymbol{e}_1 \times \boldsymbol{x} - (\gamma, 0, 0)$ on $\partial\Omega$. Combining this information with the Sobolev inequality, see e.g. [8], p. 31, we can deduce that $\boldsymbol{U}$ satisfies $\boldsymbol{U} \in L^s(\Omega)^3$ for all $3 \le s < +\infty$. In order to study the behavior of solutions near the steady solution $\boldsymbol{U}^*$, we put $\boldsymbol{u} = \boldsymbol{U}^* + \boldsymbol{v} = (\gamma, 0, 0) + \boldsymbol{U} + \boldsymbol{v}$. Then the perturbation $\boldsymbol{v}$ is a solution of the problem given by the equations

$$\begin{aligned} &\partial_t \boldsymbol{v} - \nu\Delta\boldsymbol{v} - \omega(\boldsymbol{e}_1 \times \boldsymbol{x})\cdot\nabla\boldsymbol{v} + \omega\boldsymbol{e}_1 \times \boldsymbol{v} + \gamma\partial_1\boldsymbol{v} + (\boldsymbol{U}\cdot\nabla)\boldsymbol{v} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{U} \\ &\qquad + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} + \nabla p \,=\, \boldsymbol{0}, \\ &\nabla\cdot\boldsymbol{v} \,=\, 0 \end{aligned}$$

in $\Omega \times \mathcal{I}$ (where $\mathcal{I}$ is a time interval) and by the boundary condition

$$\boldsymbol{v}(\boldsymbol{x}, t) \,=\, 0 \qquad \text{for } \boldsymbol{x} \in \partial\Omega.$$

This problem can be written in the form of the operator equation

$$\partial_t \boldsymbol{v} \,=\, L^\omega_\gamma \boldsymbol{v} + N\boldsymbol{v} \tag{1.6}$$

in $L^2_\sigma(\Omega)$ where

$$L^\omega_\gamma \boldsymbol{v} \;=\; A^\omega_\gamma \boldsymbol{v} + B\boldsymbol{v}, \tag{1.7}$$

$$A^\omega_\gamma \boldsymbol{v} \;=\; \Pi_\sigma \nu\Delta\boldsymbol{v} + \Pi_\sigma[\omega(\boldsymbol{e}_1 \times \boldsymbol{x})\cdot\nabla\boldsymbol{v} - \omega\boldsymbol{e}_1 \times \boldsymbol{v} - \gamma\partial_1\boldsymbol{v}], \tag{1.8}$$

$$B\boldsymbol{v} \;=\; -\Pi_\sigma[(\boldsymbol{U}\cdot\nabla)\boldsymbol{v} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{U}], \tag{1.9}$$

$$N\boldsymbol{v} \;=\; -\Pi_\sigma(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}. \tag{1.10}$$

The operators A^{ω}_{γ} and L^{ω}_{γ} are defined in the same domains

$$D(A^{\omega}_{\gamma}) \;=\; D(L^{\omega}_{\gamma}) \;=\; \Big\{\boldsymbol{v} \in W^{2,2}(\Omega)^3 \cap W^{1,2}_0(\Omega)^3 \cap L^2_{\sigma}(\Omega);\; \omega(\boldsymbol{e}_1 \times \boldsymbol{x}) \cdot \nabla \boldsymbol{v} \in L^2(\Omega)^3\Big\}$$

which are dense subsets of $L^2_{\sigma}(\Omega)$. The information on spectra of the linear operators A^{ω}_{γ} and L^{ω}_{γ} plays a fundamental role in studies of the evolution equation (1.6). Whereas the case $\gamma = 0$ was treated in detail in our paper [6], here we consider the **important case** $\gamma \neq 0$. Our main theorem now reads as follows:

Theorem 1.1 *(i) The essential spectrum $\sigma_{ess}(A^{\omega}_{\gamma})$ has the form*

$$\sigma_{ess}(A^{\omega}_{\gamma}) \;=\; \Lambda^{\omega}_{\gamma} \;:=\; \{\lambda = \alpha + \mathrm{i}\beta + \mathrm{i}k\omega \in \mathbb{C};\; \alpha, \beta \in \mathbb{R},\; k \in \mathbb{Z},\; \alpha \leq -\nu\beta^2/\gamma^2\}, \tag{1.11}$$

i.e., it consists of an infinite union of equally shifted filled parabolas in the left half–plane of $\mathbb{C}$, see Fig. 1.

(ii) The operator A^{ω}_{γ} is not normal.

(iii) If λ is an eigenvalue of A^{ω}_{γ} then $\mathrm{Re}\,\lambda < 0$.

(iv) If the body $\mathcal{B}$ (and therefore also the domain Ω) is axially symmetric about the x_1–axis, then $\sigma(A^{\omega}_{\gamma}) = \sigma_{ess}(A^{\omega}_{\gamma}) = \Lambda^{\omega}_{\gamma}$.

(v) The operator L^{ω}_{γ} has the same essential spectrum as A^{ω}_{γ}.

(vi) $\sigma(L^{\omega}_{\gamma}) = \sigma_{ess}(L^{\omega}_{\gamma}) \cup \Gamma$ where Γ consists of an at most countable set of isolated eigenvalues of L^{ω}_{γ} which can possibly cluster only at points of $\sigma_{ess}(L^{\omega}_{\gamma})$; each of them has finite algebraic multiplicity.

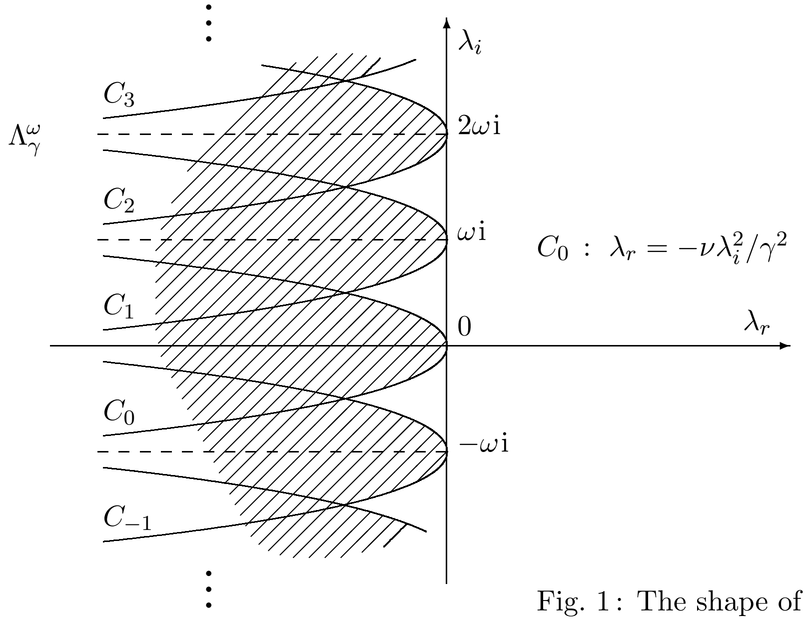

Fig. 1: The shape of set $\Lambda^{\omega}_{\gamma}$

The proof of statements (i)–(iii) and (v), (vi) is given in Section 6. Statement (iv) is proved in Section 5.

2 Preliminaries

All function spaces needed in the following are considered to be spaces of complex–valued functions.

Lemma 2.1 *There exists $c_1 > 0$ such that if $\boldsymbol{v} \in D(A^\omega_\gamma)$ and $A^\omega_\gamma \boldsymbol{v} = \boldsymbol{f}$, then*

$$\|\boldsymbol{v}\|_{2,2} + \|(\omega \boldsymbol{e}_1 \times \boldsymbol{x}) \cdot \nabla \boldsymbol{v}\|_{0,2} \;\leq\; c_1(\gamma)\, \big(\|\boldsymbol{f}\|_{0,2} + \|\boldsymbol{v}\|_{0,2}\big)\,. \tag{2.1}$$

Proof. The equation $A^\omega_\gamma \boldsymbol{v} = \boldsymbol{f}$ means that $A^\omega_0 \boldsymbol{v} = \boldsymbol{f} + \Pi_\sigma \gamma\, \partial_1 \boldsymbol{v}$. Applying the results from [5] ($\Omega = \mathbb{R}^3$) or from [16] (Ω being an exterior domain in $\mathbb{R}^3$) to the solution of the equation $A^\omega_0 \boldsymbol{v} = \boldsymbol{g}$ (with $\boldsymbol{g} = \boldsymbol{f} + \Pi_\sigma \gamma\, \partial_1 \boldsymbol{v}$), we obtain

$$\begin{aligned} \|\boldsymbol{v}\|_{2,2} + \|(\omega \boldsymbol{e}_1 \times \boldsymbol{x}) \cdot \nabla \boldsymbol{v}\|_{0,2} \;&\leq\; c_1(0)\, \big(\|A^\omega_0 \boldsymbol{v}\|_{0,2} + \|\boldsymbol{v}\|_{0,2}\big) \\ &\leq\; c_1(0)\, \big(\|\boldsymbol{f}\|_{0,2} + |\gamma|\, \|\Pi_\sigma \partial_1 \boldsymbol{v}\|_{0,2} + \|\boldsymbol{v}\|_{0,2}\big)\,. \end{aligned}$$

Interpolating suitably the norm $\|\Pi_\sigma \partial_1 \boldsymbol{v}\|_{0,2}$ between the norms $\|\boldsymbol{v}\|_{0,2}$ and $\|\boldsymbol{v}\|_{2,2}$, we arrive at (2.1). □

Lemma 2.2 *A^ω_γ is a closed operator in $L^2_\sigma(\Omega)$ and its adjoint has the form*

$$(A^\omega_\gamma)^* \boldsymbol{v} \;=\; \Pi_\sigma \nu \Delta \boldsymbol{v} - \Pi_\sigma[\omega(\boldsymbol{e}_1 \times \boldsymbol{x}) \cdot \nabla \boldsymbol{v} - \omega \boldsymbol{e}_1 \times \boldsymbol{v} - \gamma \partial_1 \boldsymbol{v}] \;=\; A^{-\omega}_{-\gamma} \boldsymbol{v} \tag{2.2}$$

with $D\big((A^\omega_\gamma)^*\big) = D(A^\omega_\gamma)$.

Proof. The operator A^ω_γ generates a C_0–semigroup, see [11]. Hence it is closed and there exists $\xi_0 \in \mathbb{R}$ such that $\xi \in \rho(A^\omega_\gamma)$ and $R(\zeta I - A^\omega_\gamma)$, the range of $\zeta I - A^\omega_\gamma$, equals $L^2_\sigma(\Omega)$ for $\xi > \xi_0$.

Let us denote by T^ω_γ the operator on the right hand side of (2.2) with $D(T^\omega_\gamma) = D(A^\omega_\gamma)$, i.e. $T^\omega_\gamma = A^{-\omega}_{-\gamma}$. Then T^ω_γ is closed and $R(\zeta I - T^\omega_\gamma) = L^2_\sigma(\Omega)$ if $\zeta > 0$ is sufficiently large. Using integration by parts, we can verify that

$$\big(\boldsymbol{u},\, A^\omega_\gamma \boldsymbol{v}\big)_{0,2} \;=\; \big(T^\omega_\gamma \boldsymbol{u},\, \boldsymbol{v}\big)_{0,2}$$

for all $\boldsymbol{u} \in D(T^\omega_\gamma)$ and $\boldsymbol{v} \in D(A^\omega_\gamma)$. It means that the operators A^ω_γ and T^ω_γ are adjoint to each other and $T^\omega_\gamma \subset (A^\omega_\gamma)^*$, see T. Kato [18], p. 167.

Suppose that $\boldsymbol{u} \in D((A^\omega_\gamma)^*)$. Then there exists $\boldsymbol{w} \in D(T^\omega_\gamma)$ such that $[\zeta I - (A^\omega_\gamma)^*]\boldsymbol{u} = (\zeta I - T^\omega_\gamma)\boldsymbol{w}$. Multiplying both sides of this identity by $\boldsymbol{v} \in D(A^\omega_\gamma)$, we arrive at

$$\big(\boldsymbol{u},\, (\zeta I - A^\omega_\gamma)\boldsymbol{v}\big)_{0,2} \;=\; \big(\boldsymbol{w},\, (\zeta I - A^\omega_\gamma)\boldsymbol{v}\big)_{0,2}\,.$$

As this holds for all $\boldsymbol{v} \in D(A^\omega_\gamma)$, we get $\boldsymbol{u} = \boldsymbol{w} \in D(T^\omega_\gamma)$ and consequently, $D((A^\omega_\gamma)^*) \subset D(T^\omega_\gamma)$. Thus, $(A^\omega_\gamma)^* = T^\omega_\gamma$. □

Lemma 2.3 *If $\boldsymbol{v} \in D(A^\omega_\gamma)$, then both the terms $\omega(\boldsymbol{e}_1 \times \boldsymbol{x}) \cdot \nabla \boldsymbol{v} - \omega \boldsymbol{e}_1 \times \boldsymbol{v}$ and $\gamma \partial_1 \boldsymbol{v}$ belong to $L^2_\sigma(\Omega)$.*

Proof. It was already shown in [6] that $\omega(\boldsymbol{e}_1 \times \boldsymbol{x}) \cdot \nabla \boldsymbol{v} - \omega \boldsymbol{e}_1 \times \boldsymbol{v} \in L^2_\sigma(\Omega)$.

The space $C^\infty_{0,\sigma}(\Omega)$ is dense in $D(A^\omega_\gamma)$ in the topology of $W^{1,2}(\Omega)^3$; hence given $\boldsymbol{v} \in D(A^\omega_\gamma)$, there exists a sequence $\boldsymbol{v}^n \in C^\infty_{0,\sigma}(\Omega)$ such that $\boldsymbol{v}^n \to \boldsymbol{v}$ in $W^{1,2}(\Omega)^3$. Let ψ be a function from $W^{1,2}_{loc}(\Omega)$ such that $\nabla\psi \in L^2(\Omega)^3$. Then we have

$$\int_\Omega \gamma \partial_1 \boldsymbol{v} \cdot \nabla\psi \, \mathrm{d}\boldsymbol{x} \;=\; \lim_{n\to+\infty} \int_\Omega \gamma\partial_1 \boldsymbol{v}^n \cdot \nabla\psi \, \mathrm{d}\boldsymbol{x} \;=\; -\lim_{n\to+\infty} \int_\Omega \mathrm{div}\,(\gamma\partial_1\boldsymbol{v}^n)\,\psi\,\mathrm{d}\boldsymbol{x} \;=\; 0.$$

Thus the function $\gamma\partial_1\boldsymbol{v}$ is orthogonal to the subspace of all gradients in $L^2(\Omega)^3$, which implies that it belongs to $L^2_\sigma(\Omega)$, see e.g. G. P. Galdi [8], p. 103. □

Lemma 2.3 enables us to omit the projection Π_σ in front of the terms in the brackets on the right hand side of (1.8) and (2.2). The operator A^ω_γ can therefore be simplified to

$$\begin{aligned} A^\omega_\gamma \boldsymbol{v} &= A^0_0\boldsymbol{v} + \omega(\boldsymbol{e}_1 \times \boldsymbol{x}) \cdot \nabla\boldsymbol{v} - \omega\boldsymbol{e}_1 \times \boldsymbol{v} - \gamma\partial_1\boldsymbol{v} \\ &= A^0_\gamma\boldsymbol{v} + \omega(\boldsymbol{e}_1 \times \boldsymbol{x}) \cdot \nabla\boldsymbol{v} - \omega\boldsymbol{e}_1 \times \boldsymbol{v} \qquad (2.3) \end{aligned}$$

where $A^0_0 \equiv \nu\Pi_\sigma\Delta$ is the Stokes operator in $L^2_\sigma(\Omega)$ with domain $D(A^0_0) = W^{2,2}(\Omega)^3 \cap W^{1,2}_0(\Omega)^3 \cap L^2_\sigma(\Omega)$. The Stokes operator A^0_0 is selfadjoint in $L^2_\sigma(\Omega)$, see e.g. Y. Giga, H. Sohr [12]. Moreover, A^0_γ is the usual Oseen operator with the same domain as A^0_0. By analogy, the adjoint operator to A^ω_γ can be simplified to

$$\begin{aligned} (A^\omega_\gamma)^*\boldsymbol{v} &= A^0_0\boldsymbol{v} - \omega(\boldsymbol{e}_1 \times \boldsymbol{x}) \cdot \nabla\boldsymbol{v} + \omega\boldsymbol{e}_1 \times \boldsymbol{v} + \gamma\partial_1\boldsymbol{v} \\ &= A^0_{-\gamma}\boldsymbol{v} - \omega(\boldsymbol{e}_1 \times \boldsymbol{x}) \cdot \nabla\boldsymbol{v} + \omega\boldsymbol{e}_1 \times \boldsymbol{v} \qquad (2.4) \end{aligned}$$

Lemma 2.4 *The operator B, defined by (1.9), is A^ω_γ–compact.*

Proof. We have proved in [6] that the operator B is A^ω_0–compact. The proof in the case of $\gamma \neq 0$ can be done in the same way. The crucial step is an appropriate application of Lemma 2.1, which enables us to deduce that the boundedness of two sequences $\{\boldsymbol{\phi}_n\}$ and $\{A^\omega_\gamma\boldsymbol{\phi}_n\}$ in $L^2_\sigma(\Omega)$ implies the boundedness of $\{\boldsymbol{\phi}_n\}$ in $W^{2,2}(\Omega)^3$. □

Lemmas 2.2 and 2.4 imply that the operator L^ω_γ is closed in $L^2_\sigma(\Omega)$, see [18], p. 194.

It will be further advantageous to work in cylindrical coordinates. We shall denote by x_1, r and φ the cylindrical coordinate system whose axis is the x_1–axis such that the angle φ is measured from the positive part of the x_2–axis towards the positive part of the x_3–axis. The corresponding cylindrical components of vector functions will be denoted by the indices 1, r and φ, e.g. u_1, u_r and u_φ. In order to distinguish between the Cartesian and the cylindrical components of vectors, we shall write the Cartesian components in parentheses and the cylindrical components in brackets. Thus, we have $(u_1, u_2, u_3) \triangleq [u_1, u_r, u_\varphi]$. Using the transformations

$$\begin{aligned} u_r &= u_2\cos\varphi + u_3\sin\varphi, & u_2 &= u_r\cos\varphi - u_\varphi\sin\varphi, \\ u_\varphi &= -u_2\sin\varphi + u_3\cos\varphi, & u_3 &= u_r\sin\varphi + u_\varphi\cos\varphi, \end{aligned}$$

we can calculate that

$$(\boldsymbol{\omega} \times \boldsymbol{x}) \cdot \nabla\boldsymbol{u} - \boldsymbol{\omega} \times \boldsymbol{u} \;=\; \omega\,\partial_\varphi\boldsymbol{u} - (\boldsymbol{\omega} \times \boldsymbol{u}) \;=\; \omega\,\partial_\varphi(u_1, u_2, u_3) - \omega\,(0, -u_3, u_2)$$

$$= \omega\,\partial_\varphi \begin{pmatrix} u_1 \\ u_r\,\cos\varphi - u_\varphi\,\sin\varphi \\ u_r\,\sin\varphi + u_\varphi\,\cos\varphi \end{pmatrix}^T - \omega \begin{pmatrix} 0 \\ -u_r\,\sin\varphi - u_\varphi\,\cos\varphi \\ u_r\,\cos\varphi - u_\varphi\,\sin\varphi \end{pmatrix}^T$$

$$= \omega \begin{pmatrix} \partial_\varphi u_1 \\ (\partial_\varphi u_r)\,\cos\varphi - (\partial_\varphi u_\varphi)\,\sin\varphi \\ (\partial_\varphi u_r)\,\sin\varphi + (\partial_\varphi u_\varphi)\,\cos\varphi \end{pmatrix}^T \triangleq \omega \begin{bmatrix} \partial_\varphi u_1 \\ \partial_\varphi u_r \\ \partial_\varphi u_\varphi \end{bmatrix}^T = \omega\,\partial_\varphi\,[u_1, u_r, u_\varphi].$$

In the following, the vector function $\boldsymbol{u}$ will be identified with $[u_1, u_r, u_\varphi]$; the same holds for other vectors or vector functions. Thus, the relation (2.3) between the operator A^ω_γ and the Stokes operator A^0_0 can be written in the form

$$A^\omega_\gamma \boldsymbol{u} \;=\; A^0_0 \boldsymbol{u} + \omega\,\partial_\varphi \boldsymbol{u} - \gamma\,\partial_1 \boldsymbol{u} \tag{2.5}$$

where A^0_0 now stands for the Stokes operator in cylindrical coordinates.

If T is a closed linear operator in a Hilbert space H, then we shall use the following notions and notation:

- $N(T)$ is the null space of T, $R(T)$ is the range and T^* is the adjoint operator to T.
- $\mathrm{nul}(T)$ is the nullity and $\mathrm{def}(T)$ is the deficiency of T.
- $\mathrm{ind}(T) = \mathrm{nul}(T) - \mathrm{def}(T)$ denotes the index of T.
- $\mathrm{nul}'(T)$ is the approximate nullity and $\mathrm{def}'(T)$ is the approximate deficiency of T.
- $\rho(T)$ denotes the resolvent set of T.
- $\sigma_p(T)$ is the point spectrum of T, $\sigma_c(T)$ its continuous spectrum and $\sigma_r(T)$ its residual spectrum.
- $\sigma(T)$ is the whole spectrum of T $(= \sigma_p(T) \cup \sigma_c(T) \cup \sigma_r(T))$.
- $\sigma_{ess}(T)$ denotes the essential spectrum of T, i.e. the set of $\lambda \in \mathbb{C}$ such that $\mathrm{nul}'(T - \lambda I) = \mathrm{def}'(T - \lambda I) = +\infty$.
- $\widetilde{\sigma}_c(T)$ denotes the set of those $\lambda \in \mathbb{C}$ for which there exists a non–compact sequence $\{\boldsymbol{u}_n\}$ in the unit sphere in H such that $(T - \lambda I)\boldsymbol{u}_n \to \boldsymbol{0}$ for $n \to +\infty$. It is equivalent to the equality $\mathrm{nul}'(T - \lambda I) = +\infty$.
- T is said to be normal if $T^*T = TT^*$.

The definitions of these notions can be found in [18] or in [13], see [6] for the survey of their main properties. For the purposes of this paper, let us recall that $\sigma_p(T)$, $\sigma_r(T)$ and $\sigma_c(T)$ are mutually disjoint, $\sigma(T)$, $\sigma_{ess}(T)$ and $\widetilde{\sigma}_c(T)$ are closed sets in $\mathbb{C}$ and $\sigma_c(T) \subset \sigma_{ess}(T) \subset \widetilde{\sigma}_c(T) \subset \sigma(T)$.

3 The Oseen operator A^0_γ

It is known that the spectra of the Stokes operator A^0_0 satisfy the identities

$$\sigma_p(A^0_0) \;=\; \sigma_r(A^0_0) \;=\; \emptyset, \tag{3.1}$$

$$\sigma(A^0_0) \;=\; \sigma_{ess}(A^0_0) \;=\; \sigma_c(A^0_0) \;=\; (-\infty, 0]. \tag{3.2}$$

(The residual spectrum of A_0^0 is empty because A_0^0 is selfadjoint. The reasons why the point spectrum is also empty are explained in [6]. The identities $\sigma(A_0^0) = \sigma_c(A_0^0) = (-\infty, 0]$ follow from I. M. Glazman [13] and O. A. Ladyzhenskaya [20].)

The spectrum of the Oseen operator A_γ^0 was studied by K. I. Babenko in [1]. Considering the case $\Omega = \mathbb{R}^3$ and assuming that $\lambda \in \Lambda_\gamma^0$ where

$$\Lambda_\gamma^0 \;=\; \big\{\lambda = \alpha + \mathrm{i}\beta \in \mathbb{C};\ \alpha, \beta \in \mathbb{R},\ \alpha \le -\nu\beta^2/\gamma^2\big\}, \tag{3.3}$$

K. I. Babenko mentions a construction (based on the Fourier transform) of a non–compact sequence $\{\boldsymbol{v}^n\}$ in the unit sphere in $L^2_\sigma(\Omega)$ such that $\|(A_\gamma^0 - \lambda I)\boldsymbol{v}^n\|_{0,2} \to 0$ as $n \to +\infty$. Then $\lambda \in \sigma_{ess}(A_\gamma^0)$ and consequently $\Lambda_\gamma^0 \subset \sigma_{ess}(A_\gamma^0)$. On the other hand, the author states that the equation

$$(A_\gamma^0 - \lambda I)\boldsymbol{v} \;=\; \boldsymbol{f}, \tag{3.4}$$

for Re $\lambda > 0$ and $\boldsymbol{f} \in L^2_\sigma(\Omega)$, can be solved by means of a Green's function of the Dirichlet problem with a reference to F. Odqvist [24] for more details concerning the construction of Green's function and its estimates. Furthermore, K. I. Babenko emphasizes that it is not difficult to treat the other cases of $\lambda \in \mathbb{C} - \Lambda_\gamma^0$. Thus, he arrives at Propositions 4 and 5 which imply that $\mathbb{C} - \Lambda_\gamma^0 \subset \rho(A_\gamma^0)$.

Since the information on the spectrum of the operator A_γ^0 is of fundamental importance, in the following theorem we present a complete proof based on a totally different approach.

Theorem 3.1 $\sigma(A_\gamma^0) = \sigma_{ess}(A_\gamma^0) = \Lambda_\gamma^0$.

Proof. I. Let us begin with the inclusion $\mathbb{C} - \Lambda_\gamma^0 \subset \rho(A_\gamma^0)$ to be proved by contradiction. Suppose that $\lambda = \alpha + \mathrm{i}\beta \in (\mathbb{C} - \Lambda_\gamma^0) \cap \sigma(A_\gamma^0)$. Assume that $\lambda \in \sigma_p(A_\gamma^0) \cup \sigma_c(A_\gamma^0)$ at first. Then there exists a sequence $\boldsymbol{v}^n$ in the unit sphere in $L^2_\sigma(\Omega)$ such that

$$(A_\gamma^0 - \lambda I)\boldsymbol{v}^n \;=\; \boldsymbol{\epsilon}^n \;\longrightarrow\; \boldsymbol{0} \qquad \text{in } L^2_\sigma(\Omega) \quad \text{as } n \to +\infty. \tag{3.5}$$

This sequence $\{\boldsymbol{v}^n\}$ can be constant if $\lambda \in \sigma_p(A_\gamma^0)$. We test (3.5) with $\boldsymbol{v}^n$ (in the L^2–sense for complex–valued functions) and get that

$$-\nu\,\|\nabla\boldsymbol{v}^n\|_{0,2}^2 - \lambda\,\|\boldsymbol{v}^n\|_{0,2}^2 - \gamma\,(\partial_1\boldsymbol{v}^n, \boldsymbol{v}^n)_{0,2} \;=\; (\boldsymbol{\epsilon}^n, \boldsymbol{v}^n)_{0,2}\,. \tag{3.6}$$

Note that Re $(\partial_1\boldsymbol{v}^n, \boldsymbol{v}^n)_{0,2} = 0$. Next we consider the real and imaginary part of (3.6) and see that

$$\nu\,\|\nabla\boldsymbol{v}^n\|_{0,2}^2 \;=\; -\alpha\,\|\boldsymbol{v}^n\|_{0,2}^2 - \mathrm{Re}\,(\boldsymbol{\epsilon}^n, \boldsymbol{v}^n)_{0,2} \;\le\; -\alpha + \|\boldsymbol{\epsilon}^n\|_{0,2}\,. \tag{3.7}$$

Using (3.5), we observe that $\alpha \le 0$. From (3.6) and (3.7) we obtain

$$\|\nabla\boldsymbol{v}^n\|_{0,2} \;\le\; \Big(-\frac{\alpha}{\nu} + \frac{1}{\nu}\,\|\boldsymbol{\epsilon}^n\|_{0,2}\Big)^{1/2}, \tag{3.8}$$

as well as

$$\beta \;=\; \beta\,\|\boldsymbol{v}^n\|_{0,2}^2 \;=\; -\gamma\,\mathrm{Im}\,(\partial_1\boldsymbol{v}^n, \boldsymbol{v}^n)_{0,2} - \mathrm{Im}\,(\boldsymbol{\epsilon}^n, \boldsymbol{v}^n)_{0,2}$$

so that

$$|\beta| \;\leq\; \gamma\, \|\nabla \boldsymbol{v}^n\|_{0,2} + \|\boldsymbol{\epsilon}^n\|_{0,2}\,. \tag{3.9}$$

Inserting the estimate (3.8) into (3.9) we are led to the inequality

$$|\beta| \;\leq\; \gamma \left(-\frac{\alpha}{\nu} + \frac{1}{\nu}\, \|\boldsymbol{\epsilon}^n\|_{0,2} \right)^{1/2} + \|\boldsymbol{\epsilon}^n\|_{0,2}\,.$$

As $n \to +\infty$, (3.5) implies that $|\beta| \leq \gamma\sqrt{-\alpha/\nu}$, i.e. $\alpha \leq \nu\beta^2/\gamma^2$. Obviously, this inequality is in contradiction with the assumption $\lambda \in \mathbb{C} - \Lambda^0_\gamma$. Hence $\lambda \in \mathbb{C} - \Lambda^0_\gamma$ cannot belong to $\sigma_p(A^0_\gamma) \cup \sigma_c(A^0_\gamma)$. Now assume that $\lambda \in \sigma_r(A^0_\gamma)$. Then $\overline{\lambda}$ belongs to the point spectrum of the adjoint operator $(A^0_\gamma)^*$; this leads to the same contradiction as if $\lambda \in \sigma_p(A^0_\gamma)$. Thus, $\lambda \in \rho(A^0_\gamma)$ which implies that $\mathbb{C} - \Lambda^0_\gamma \subset \rho(A^0_\gamma)$.

II. Now we will prove that $\Lambda^0_\gamma \subset \widetilde{\sigma}_c(A^0_\gamma)$. Let $\lambda = \alpha + \mathrm{i}\beta \in (\Lambda^0_\gamma)^\circ$ be given; here $(\Lambda^0_\gamma)^\circ$ denotes the interior of Λ^0_γ, i.e. the set of $\alpha + \mathrm{i}\beta \in \mathbb{C}$ such that $\alpha < -\nu\beta^2/\gamma^2$. The number α can be written in the form $\alpha = \alpha_1 + \alpha_2$ where $\alpha_1 = -\nu\beta^2/\gamma^2$ and $\alpha_2 < 0$.

We shall explicitly define functions $\boldsymbol{v}^n \in L^2_\sigma(\Omega)$ such that $\|\boldsymbol{v}^n\|_{0,2} = 1$, $(A^0_\gamma - \lambda I)\boldsymbol{v}^n \to \boldsymbol{0}$ in $L^2_\sigma(\Omega)$ as $n \to +\infty$ and such that the sequence $\{\boldsymbol{v}^n\}$ does not contain any subsequence, convergent in $L^2_\sigma(\Omega)$. Let us denote by v^n_1, v^n_r and v^n_φ the cylindrical components of $\boldsymbol{v}^n$. Put

$$\begin{aligned}
v^n_1(x_1, r, \varphi) \;&:=\; 0,\\
v^n_r(x_1, r, \varphi) \;&:=\; \kappa_n\, U^n(x_1)\, V^n(r)\, \mathrm{e}^{\mathrm{i}k\varphi},\\
v^n_\varphi(x_1, r, \varphi) \;&:=\; -\frac{1}{\mathrm{i}k}\, \partial_r \left[r\, v^n_r(x_1, r, \varphi) \right] \;=\; -\frac{1}{\mathrm{i}k}\, \kappa_n\, U^n(x_1) \left[V^n(r) + r\, \frac{\mathrm{d}V^n(r)}{\mathrm{d}r} \right] \mathrm{e}^{\mathrm{i}k\varphi}
\end{aligned}$$

where k is an arbitrary, but fixed chosen non–zero integer. Then, obviously, $\boldsymbol{v}^n$ satisfies the condition

$$\nabla \cdot \boldsymbol{v}^n \;\equiv\; \partial_1 v^n_1 + \frac{1}{r}\, \partial_r (r v^n_r) + \frac{1}{r}\, \partial_\varphi v^n_\varphi \;=\; 0.$$

Here the function U^n has the form

$$U^n(x_1) \;:=\; \eta^n_1(x_1)\, Y(x_1) \tag{3.10}$$

where η^n_1 is an infinitely differentiable function on $(-\infty, +\infty)$ such that $0 \leq \eta^n_1 \leq 1$,

$$\eta^n_1(x_1) \;=\; \begin{cases} 0 & \text{for} \quad x_1 \leq -n - n^2 \quad \text{and} \quad n + n^2 \leq x_1, \\ 1 & \text{for} \quad -n^2 \leq x_1 \leq n^2, \end{cases}$$

and $Y(x_1) = \mathrm{e}^{\mathrm{i}ax_1}$. The identity $\alpha_1 = -\nu\beta^2/\gamma^2$ guarantees that the characteristic equation $\nu\zeta^2 - \gamma\zeta - (\alpha_1 + \mathrm{i}\beta) = 0$, corresponding to the equation (3.11) below, has the root $\zeta_1 = \mathrm{i}a$ where $a = -\beta/\gamma$. Thus, the function Y is a bounded non–trivial solution of the ordinary differential equation

$$\nu\, Y''(x_1) - \gamma\, Y'(x_1) - (\alpha_1 + \mathrm{i}\beta)\; Y(x_1) \;=\; 0 \tag{3.11}$$

in the interval $(-\infty, +\infty)$. The function V^n has the form

$$V^n(r) \;:=\; \eta_2^n(r)\, \mathrm{e}^{\mathrm{i}br}; \qquad b = \sqrt{-\frac{\alpha_2}{\nu}} \tag{3.12}$$

where η_2^n is an infinitely differentiable function on $[0, +\infty)$ such that $0 \le \eta_2^n \le 1$ and

$$\eta_2^n(r) \;=\; \begin{cases} 0 & \text{for} \quad 0 \le r \le n \;\; \text{and} \;\; 3n + n^2 \le r, \\ 1 & \text{for} \quad 2n \le r \le 2n + n^2. \end{cases}$$

Both the functions η_1 and η_2 can be chosen so that their derivatives are of the order $1/n$. The definition of V^n guarantees that it satisfies

$$\nu\, \frac{\mathrm{d}^2}{\mathrm{d}r^2}\, V^n(r) - \alpha_2\, V^n(r) \;=\; 0 \tag{3.13}$$

for $2n < r < 2n + n^2$. Finally, the constant κ_n is chosen so that $\|\boldsymbol{v}^n\|_{0,2} = 1$. Thus, the support of $\boldsymbol{v}^n$ is a subset of

$$S^n \;:=\; \big\{ \boldsymbol{x} = [x_1, r, \varphi] \in \mathbb{R}^3;\; -n - n^2 \le x_1 \le n + n^2,\; n \le r \le 3n + n^2,\; 0 \le \varphi < 2\pi \big\}. \tag{3.14}$$

Considering the norm of $\boldsymbol{v}^n$, we can observe that for large n the decisive contribution comes from the integral of $|v_\varphi^n|^2$, namely of its part $\big|(-1/\mathrm{i}k)\, \kappa_n\, U^n\, r\, (\mathrm{d}V^n/\mathrm{d}r)\, \mathrm{e}^{\mathrm{i}k\varphi}\big|^2$, on the region

$$D^n \;:=\; \big\{ \boldsymbol{x} = [x_1, r, \varphi] \in \mathbb{R}^3;\; -n^2 < x_1 < n^2,\; 2n < r < 2n + n^2,\; 0 < \varphi < 2\pi \big\}. \tag{3.15}$$

The integrals of all other parts on other regions are of a lower order in n. Calculating the integral of $\big|(-1/\mathrm{i}k)\, \kappa_n\, U^n\, r\, (\mathrm{d}V^n/\mathrm{d}r)\, \mathrm{e}^{\mathrm{i}k\varphi}\big|^2$ on the domain D^n, we obtain

$$\begin{aligned} &\int_{-n^2}^{n^2} \int_{2n}^{2n+n^2} \int_0^{2\pi} \left| \frac{\kappa_n}{\mathrm{i}k}\, U^n(x_1)\, r\, \frac{\mathrm{d}V^n(r)}{\mathrm{d}r} \right|^2 r\, \mathrm{d}\varphi\, \mathrm{d}r\, \mathrm{d}x_1 \\ &= \;\; 2\pi\, \frac{\kappa_n^2}{k^2} \int_{-n^2}^{n^2} |U^n(x_1)|^2\, \mathrm{d}x_1 \int_{2n}^{2n+n^2} r^3 \left| \frac{\mathrm{d}V^n(r)}{\mathrm{d}r} \right|^2 \mathrm{d}r \\ &= \;\; 2\pi\, \frac{\kappa_n^2}{k^2}\, 2n^2\, \frac{b^2}{4} \Big((2n + n^2)^4 - (2n)^4 \Big). \end{aligned}$$

Here we have used the equalities $\eta_1^n(x_1) = \eta_2^n(r) = 1$, hence $|U^n(x_1)| = |V^n(r)| = 1$ for $(x_1, r, \varphi) \in D^n$. Thus, there exist $n_0 \in \mathbb{N}$ and positive constants c_2 and c_3 (independent of n) such that

$$\forall\, n \in \mathbb{N},\; n \ge n_0 \;:\; \quad \frac{c_2}{n^5} \;\le\; \kappa_n \;\le\; \frac{c_3}{n^5}\, . \tag{3.16}$$

Now looking at $(A_\gamma^0 - \lambda I)\boldsymbol{v}^n$, we can omit the projection Π_σ in front of the Laplace operator in $A_\gamma^0 \boldsymbol{v}^n$ because $\Delta \boldsymbol{v}^n$ is divergence–free and has a compact support in Ω. Thus, $(A_\gamma^0 - \lambda I)\boldsymbol{v}^n = \nu \Delta \boldsymbol{v}^n - \gamma \partial_1 \boldsymbol{v}^n - \lambda \boldsymbol{v}^n$. Calculating the norm of this expression in $L_\sigma^2(\Omega)$, we observe that the contributions coming from $\Omega - D^n$ tend to zero as $n \to +\infty$ because they represent square roots of integrals of functions bounded by $C\kappa_n^2 r^2$ on $S^n - D^n$. Due to (3.16), this contribution is of the order $n^{-1/2}$. Concerning the integral on D^n, the decisive part again comes from $(\nu\Delta - \gamma\partial_1 - \lambda I)v_\varphi^n$, namely from $(\nu\Delta - \gamma\partial_1 - \lambda I)$ applied to the term $(-1/\mathrm{i}k)\,\kappa_n\, U^n\, r\, (\mathrm{d}V^n/\mathrm{d}r)\, \mathrm{e}^{\mathrm{i}k\varphi}$ because of the factor r inside this term. Note that due to (3.11) and (3.13)

$$
\begin{aligned}
&(\nu\Delta - \gamma\partial_1 - \lambda I)\Big(\frac{\kappa_n}{\mathrm{i}k}\, U^n(x_1)\, r\, \frac{\mathrm{d}V^n(r)}{\mathrm{d}r}\, \mathrm{e}^{\mathrm{i}k\varphi}\Big) \\
&= \Big(\nu\partial_1^2 + \nu\partial_r^2 + \frac{\nu}{r}\,\partial_r + \frac{\nu}{r^2}\,\partial_\varphi^2 - \gamma\,\partial_1 - \lambda I\Big)\Big(\frac{\kappa_n}{\mathrm{i}k}\, U^n(x_1)\, r\, \frac{\mathrm{d}V^n(r)}{\mathrm{d}r}\, \mathrm{e}^{\mathrm{i}k\varphi}\Big) \\
&= \frac{\kappa_n}{\mathrm{i}k}\, U^n(x_1)\,\Big(\nu\,\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{\nu}{r}\,\frac{\mathrm{d}}{\mathrm{d}r} - \alpha_2 I\Big)\Big[r\,\frac{\mathrm{d}V^n(r)}{\mathrm{d}r}\Big]\mathrm{e}^{\mathrm{i}k\varphi} \\
&\quad + \frac{\kappa_n}{\mathrm{i}k}\,\Big(\nu Y''(x_1) - \gamma Y'(x_1) - [\alpha_1 + \mathrm{i}\beta]\, Y(x_1)\Big)\Big[r\,\frac{\mathrm{d}V^n(r)}{\mathrm{d}r}\Big]\mathrm{e}^{\mathrm{i}k\varphi} \\
&\quad - \frac{\kappa_n}{\mathrm{i}k}\,\frac{\nu k^2}{r^2}\, U^n(x_1)\; r\; \frac{\mathrm{d}V^n(r)}{\mathrm{d}r}\, \mathrm{e}^{\mathrm{i}k\varphi} \\
&= \Big\{\frac{\kappa_n}{\mathrm{i}k}\; U^n(x_1)\; r\; \frac{\mathrm{d}}{\mathrm{d}r}\Big[\nu\,\frac{\mathrm{d}^2V^n(r)}{\mathrm{d}r^2} - \alpha_2\, V^n(r)\Big] + \frac{\kappa_n}{\mathrm{i}k}\; U^n(x_1)\; 2\nu\; \frac{\mathrm{d}^2V^n(r)}{\mathrm{d}r^2} \\
&\quad + \frac{\kappa_n}{\mathrm{i}k}\; U^n(x_1)\; \frac{\nu}{r}\,\frac{\mathrm{d}}{\mathrm{d}r}\Big[r\,\frac{\mathrm{d}V^n(r)}{\mathrm{d}r}\Big] - \frac{\kappa_n}{\mathrm{i}k}\; U^n(x_1)\; \frac{\nu k^2}{r}\, \frac{\mathrm{d}V^n(r)}{\mathrm{d}r}\Big\}\,\mathrm{e}^{\mathrm{i}k\varphi} \\
&= \frac{\nu\kappa_n}{\mathrm{i}k}\,\Big(-3b^2 + \frac{\mathrm{i}b}{r} - \frac{k^2\,\mathrm{i}b}{r}\Big)\,\mathrm{e}^{\mathrm{i}(ax_1+br)}\,\mathrm{e}^{\mathrm{i}k\varphi}
\end{aligned}
$$

where in the last step we used the simple forms of the functions U^n and V^n on D^n, i.e. $U^n(x_1) = \mathrm{e}^{\mathrm{i}ax_1}$ and $V^n(r) = \mathrm{e}^{\mathrm{i}br}$. Hence

$$
\begin{aligned}
&\Big[\int_{-n^2}^{n^2}\int_{2n}^{2n+n^2}\int_0^{2\pi}\Big|\Big(\nu\Delta - \gamma\partial_1 - \lambda I\Big)\Big(\frac{\kappa_n}{\mathrm{i}k}\, U^n(x_1)\, r\, \frac{\mathrm{d}V^n(r)}{\mathrm{d}r}\, \mathrm{e}^{\mathrm{i}k\varphi}\Big)\Big|^2\, r\;\mathrm{d}\varphi\,\mathrm{d}r\,\mathrm{d}x_1\Big]^{1/2} \\
&\leq\; C(\nu,k,b)\,\kappa_n\,\Big[\int_{-n^2}^{n^2}\int_{2n}^{2n+n^2} r\;\mathrm{d}r\,\mathrm{d}x_1\Big]^{1/2} \;=\; C(\nu,k,b)\,\kappa_n\, n\,\Big[(2n+n^2)^2 - (2n)^2\Big]^{1/2}.
\end{aligned}
$$

The last term tends to zero as $n \to +\infty$ due to (3.16). In this way, we prove that $\|(A_\gamma^0 - \lambda I)\boldsymbol{v}^n\|_{0,2} \to 0$ as $n \to +\infty$.

The sequence $\{\boldsymbol{v}^n\}$ does not contain any convergent subsequence because the intersection of supports of any infinite family of functions, chosen from $\{\boldsymbol{v}^n\}$, is empty.

Since λ was an arbitrarily chosen number from $(\Lambda_\gamma^0)^\circ$, we have obtained the inclusion $(\Lambda_\gamma^0)^\circ \subset \widetilde{\sigma}_c(A_\gamma^0)$. It means that $\mathrm{nul}'(A_\gamma^0 - \lambda I) = +\infty$. Since the operators A_γ^0 and $(A_\gamma^0)^*$ differ only in the sign in front of $\gamma\partial_1$, we can prove in the same way that $\mathrm{nul}'((A_\gamma^0)^* - \overline{\lambda} I) = +\infty$. It means that $\mathrm{def}'(A_\gamma^0 - \lambda I) = +\infty$ and consequently, $\lambda \in \sigma_{ess}(A_\gamma^0)$. The essential spectrum is a closed set, hence $\Lambda_\gamma^0 \subset \sigma_{ess}(A_\gamma^0)$. □

Theorem 3.1 provides an information on the shape of the whole spectrum $\sigma(A_\gamma^0)$, but it does not specify which numbers λ from $\sigma(A_\gamma^0)$ belong to $\sigma_p(A_\gamma^0)$, $\sigma_c(A_\gamma^0)$ or to $\sigma_r(A_\gamma^0)$. We do answer this question in this paper neither for the operator A_γ^0 nor for the more general operator A_γ^ω.

The Oseen operator A^0_γ generates an analytic semigroup, see T. Miyakawa [22]. Therefore the operator $(-A^0_\gamma)$ is sectorial, see D. Henry [14], p. 20–21. The next theorem states the non–normality of the Oseen operator, which stresses the difference between the Stokes and Oseen operators.

Theorem 3.2 *The Oseen operator A^0_γ is not normal.*

Proof. Constructing a function $\boldsymbol{z} \in D((A^0_\gamma)^* A^0_\gamma)$ which is not in $D(A^0_\gamma (A^0_\gamma)^*)$, we show that the domains $D((A^0_\gamma)^* A^0_\gamma)$ and $D(A^0_\gamma (A^0_\gamma)^*)$ do not coincide.

Let $R > 0$ be so large that the body $\mathcal{B}$ is contained in the interior of the cube $[-R, R]^3$. Recall that $\Omega = \mathbb{R}^3 - \mathcal{B}$. Define the set

$$\begin{aligned} \Omega_{\rm per} \ := \ & \big\{ \boldsymbol{x} = (x_1, x_2, x_3) \in \mathbb{R}^3;\ \exists\, k, l, m \in \mathbb{Z} : \\ & \big(x_1 - kR,\ x_2 - lR,\ x_3 - mR \big) \in [-R, R]^3 - \mathcal{B} \big\}. \end{aligned}$$

Note that $\Omega_{\rm per}$ is a domain in $\mathbb{R}^3$ which consists of infinitely many copies of the set $[-R, R]^3 - \mathcal{B}$, periodically repeated in directions parallel with the x_1–, x_2– and x_3–axis. We shall use the following function spaces:

- $(C_0^\infty)^3_{\rm per}$ is the space of infinitely differentiable vector functions $\boldsymbol{\phi}$ in $\Omega_{\rm per}$, R–periodic in the directions of all Cartesian axes and such that $\text{dist}\,(\text{supp}\,\boldsymbol{\phi};\, \mathcal{B}) > 0$.
- $(L^2)^3_{\rm per}$ is the completion of $(C_0^\infty)^3_{\rm per}$ in the norm identical with the L^2–norm on $(-R, R)^3 - \mathcal{B}$. The spaces $(W_0^{1,2})^3_{\rm per}$ and $(W^{2,2})_{\rm per}$ are defined analogously.
- $(L^2_\sigma)_{\rm per}$ is the closure of the space of divergence–free functions from $(C_0^\infty)^3_{\rm per}$ in $(L^2)^3_{\rm per}$.

Let $(A^0_0)_{\rm per}$ denote the Stokes operator in $(L^2_\sigma)_{\rm per}$ with the dense domain $D((A^0_0)_{\rm per}) = (W^{2,2})^3_{\rm per} \cap (W_0^{1,2})^3_{\rm per} \cap (L^2_\sigma)_{\rm per}$. Then $(A^0_0)_{\rm per}$ has a compact resolvent and its spectrum, as well as the spectrum of $(A^0_0)_{\rm per} - \gamma\partial_1$ (with the same domain), consists of a countable number of isolated eigenvalues with finite multiplicities and negative real parts. Choose an eigenvalue ζ of $(A^0_0)_{\rm per} - \gamma\partial_1$ and denote by $\boldsymbol{u}$ an associated eigenfunction so that the equation

$$(A^0_0)_{\rm per}\boldsymbol{u} - \gamma\partial_1\boldsymbol{u} - \zeta\boldsymbol{u} \ = \ \boldsymbol{0} \tag{3.17}$$

is satisfied in $\Omega_{\rm per}$.

Let us show, by contradiction, that the eigenfunction $\boldsymbol{u}$ can be chosen so that $\partial_1\boldsymbol{u} \not\equiv \boldsymbol{0}$ on $\partial\Omega_{\rm per}$. Assume the opposite, i.e. that all eigenfunctions $\boldsymbol{v}$ of the operator $(A^0_0)_{\rm per} - \gamma\partial_1$, corresponding to the eigenvalue ζ, satisfy $\partial_1\boldsymbol{v} \equiv \boldsymbol{0}$ on $\partial\Omega_{\rm per}$. Then, for each of them, there are two possibilities: either $\partial_1\boldsymbol{v} \equiv \boldsymbol{0}$ in $\Omega_{\rm per}$ (which can be easily excluded) or $\partial_1\boldsymbol{v}$ is also an eigenfunction of $(A^0_0)_{\rm per} - \gamma\partial_1$ corresponding to the same eigenvalue ζ. Since the eigenspace of $(A^0_0)_{\rm per} - \gamma\partial_1$, generated by all such eigenfunctions, is finite–dimensional, we can choose an eigenfunction $\boldsymbol{u}$ so that

$$\gamma\,\partial_1\boldsymbol{u} \ = \ \mu\,\boldsymbol{u} \tag{3.18}$$

with an appropriate constant μ. Since $\boldsymbol{u} = \boldsymbol{0}$ on $\partial\Omega_{\rm per}$ and equation (3.18) is satisfied in $\Omega_{\rm per}$, the integration of (3.18) on line segments parallel with the x_1–axis and starting from the boundary of $\Omega_{\rm per}$ yields that $\boldsymbol{u}=\boldsymbol{0}$ on all such line segments. Thus, $\boldsymbol{u}$ vanishes identically in an open subset of $\Omega_{\rm per}$. Now the unique continuation principle, see e.g. R. Leiss [21], applied to $\boldsymbol{w}=\mathbf{curl}\,\boldsymbol{u}$, shows that $\boldsymbol{w}\equiv\boldsymbol{0}$ in $\Omega_{\rm per}$. Consequently, $\boldsymbol{u}\equiv\boldsymbol{0}$ in $\Omega_{\rm per}$ which is impossible because $\boldsymbol{u}$ is an eigenfunction. Since the assumption that $\partial_1\boldsymbol{u}\equiv\boldsymbol{0}$ on $\partial\Omega_{\rm per}$ leads to contradiction, we have $\partial_1\boldsymbol{u}\not\equiv\boldsymbol{0}$ on $\partial\Omega_{\rm per}$.

Note that $\partial\Omega\subset\partial\Omega_{\rm per}$ and $\partial\Omega_{\rm per}$ consists of infinitely many copies of $\partial\Omega$ repeated periodically with the period R in the direction of each Cartesian coordinate. Now we multiply function $\boldsymbol{u}$ by an infinitely–differentiable cut–off function η_R which equals one in the neighborhood of $\partial\Omega$ and whose support is contained in $(-R,R)^3-\mathcal{B}$, and correct the product $\eta_R\boldsymbol{u}$ by an appropriate function $\boldsymbol{U}_R$ which guarantees that $\mathrm{div}\,(\eta_R\boldsymbol{u}-\boldsymbol{U}_R)=0$. By these means we can obtain a function $\boldsymbol{z}$ in $D((A^0_\gamma)^*A^0_\gamma)$ which coincides with the function $\boldsymbol{u}$ constructed above in the neighborhood of Ω and equals zero outside $(-R,R)^3$. The function $\boldsymbol{z}$ satisfies $\partial_1\boldsymbol{z}\not\equiv\boldsymbol{0}$ on $\partial\Omega$. Then $\boldsymbol{z}$ cannot belong to $D(A^0_\gamma(A^0_\gamma)^*)$ because all functions from $D((A^0_\gamma)^*A^0_\gamma)\cap D(A^0_\gamma(A^0_\gamma)^*)$ satisfy on $\partial\Omega$ the conditions $\boldsymbol{z}=A^0_0\boldsymbol{z}+\gamma\partial_1\boldsymbol{z}=A^0_0\boldsymbol{z}-\gamma\partial_1\boldsymbol{z}=\boldsymbol{0}$, which implies that $\partial_1\boldsymbol{z}=\boldsymbol{0}$ on $\partial\Omega$. □

4 Axially symmetric domains – decomposition of $L^2_\sigma(\Omega)$ and of A^0_γ

We shall assume that the domain $\Omega\subset\mathbb{R}^3$ is axially symmetric with respect to the x_1–axis in this section.

Let k be an integer. We introduce the following spaces and notation:

- $L^2(\Omega)^3_k=\{\boldsymbol{v}\in L^2(\Omega)^3;\ \boldsymbol{v}=\boldsymbol{V}(x_1,r)\,\mathrm{e}^{\mathrm{i}k\varphi}\}$
- $C^\infty_0(\Omega)^3_k=C^\infty_0(\Omega)^3\cap L^2(\Omega)^3_k$
- $C^\infty_{0,\sigma}(\Omega)_k=C^\infty_0(\Omega)^3_k\cap C^\infty_{0,\sigma}(\Omega)$
- $L^2_\sigma(\Omega)_k=$ the closure of $C^\infty_{0,\sigma}(\Omega)_k$ in $L^2(\Omega)^3_k$
- P_k – the orthogonal projection of $L^2(\Omega)^3$ onto $L^2(\Omega)^3_k$
- $(A^0_\gamma)_k$ – the restriction of the operator A^0_γ to the space $L^2_\sigma(\Omega)_k$

Obviously, $L^2(\Omega)^3_k$, $k\in\mathbb{Z}$, is a closed subspace of $L^2(\Omega)^3$, and $L^2_\sigma(\Omega)_k$ is a closed subspace of $L^2_\sigma(\Omega)$. The domain of $(A^0_\gamma)_k$ equals $D(A^0_\gamma)\cap L^2_\sigma(\Omega)_k$.

Each function $\boldsymbol{v}$ from $L^2(\Omega)^3$ can uniquely be written in the form of a convergent Fourier series – with respect to the variable φ – of terms from $L^2(\Omega)^3_k$, $k\in\mathbb{Z}$:

$$\boldsymbol{v}(x_1,r,\varphi)=\sum_{k=-\infty}^{+\infty}\boldsymbol{V}^k(x_1,r)\,\mathrm{e}^{\mathrm{i}k\varphi};\quad \boldsymbol{V}^k(x_1,r)=\frac{1}{2\pi}\int_0^{2\pi}\boldsymbol{v}(x_1,r,\varphi)\,\mathrm{e}^{-\mathrm{i}k\varphi}\,\mathrm{d}\varphi. \tag{4.1}$$

Thus, we have $L^2(\Omega)^3=\ldots\oplus L^2(\Omega)^3_{-2}\oplus L^2(\Omega)^3_{-1}\oplus L^2(\Omega)^3_0\oplus L^2(\Omega)^3_1\oplus L^2(\Omega)^3_2\oplus\ldots$. We have proved in [6] that

$$\Pi_\sigma\,L^2(\Omega)^3_k\ =\ L^2_\sigma(\Omega)\cap L^2(\Omega)^3_k\ =\ L^2_\sigma(\Omega)_k\ =\ P_kL^2_\sigma(\Omega). \tag{4.2}$$

The next lemma generalizes some results from [6].

Lemma 4.1 *Let $k \in \mathbb{Z}$. Then $(A^0_\gamma)_k$ is a closed operator in $L^2_\sigma(\Omega)_k$ with the dense domain $D((A^0_\gamma)_k)$; moreover $D((A^0_\gamma)_k) = P_k[D(A^0_\gamma)]$, $R((A^0_\gamma)_k) \subset L^2_\sigma(\Omega)_k$.*

Proof. The operator $(A^0_\gamma)_k$ is closed because it is the restriction of the closed operator A^0_γ onto a closed subspace of $L^2_\sigma(\Omega)$. The domain of $(A^0_\gamma)_k$ is the set of functions from $L^2_\sigma(\Omega)_k$, that belong to $W^{2,2}(\Omega)^3 \cap W^{1,2}_0(\Omega)^3$. This set contains $C^\infty_{0,\sigma}(\Omega)$, hence it is dense in $L^2_\sigma(\Omega)_k$.

Let $\boldsymbol{v} \in D(A^0_\gamma) \equiv W^{2,2}(\Omega)^3 \cap W^{1,2}_0(\Omega)^3 \cap L^2_\sigma(\Omega)$ and let (4.1) be its Fourier expansion in the variable φ. Then $\boldsymbol{V}^k(x_1, r)\, \mathrm{e}^{\mathrm{i}k\varphi} \equiv P_k\boldsymbol{v} \in W^{2,2}(\Omega)^3$, and, due to the axial symmetry of Ω and the boundary condition satisfied by $\boldsymbol{v}$ on $\partial\Omega$, $\boldsymbol{V}^k(x_1, r)\, \mathrm{e}^{\mathrm{i}k\varphi}$ also belongs to $W^{1,2}_0(\Omega)^3$. Using the equation $\operatorname{div} \boldsymbol{v} = 0$ and the orthogonality of the functions $\operatorname{div}\big[\boldsymbol{V}^k(x_1, r)\, \mathrm{e}^{\mathrm{i}k\varphi}\big]$ in $L^2(\Omega)^3$ (for different k), we can prove that $\operatorname{div}\big[\boldsymbol{V}^k(x_1, r)\, \mathrm{e}^{\mathrm{i}k\varphi}\big] = 0$. Hence $\boldsymbol{V}^k(x_1, r)\, \mathrm{e}^{\mathrm{i}k\varphi} \in L^2_\sigma(\Omega)_k$ and consequently, $P_k[D(A^0_\gamma)] \subset D((A^0_\gamma)_k)$.

On the other hand, if $\boldsymbol{v} \in D((A^0_\gamma)_k)$, then it belongs to $D(A^0_\gamma)$, and since $P_k\boldsymbol{v} = \boldsymbol{v}$, it also belongs to $L^2(\Omega)^3_k$. Hence $\boldsymbol{v} \in D(A^0_\gamma) \cap L^2(\Omega)^3_k = D(A^0_\gamma) \cap L^2_\sigma(\Omega)_k = P_k[D(A^0_\gamma)]$.

If $\boldsymbol{v} \in D((A^0_\gamma)_k)$, then $\Delta\boldsymbol{v}$, $\partial_1\boldsymbol{v} \in L^2(\Omega)^3_k$, and due to (4.2), $A^0_\gamma\boldsymbol{v} = \nu\Pi_\sigma\Delta\boldsymbol{v} - \gamma\partial_1\boldsymbol{v} \in L^2_\sigma(\Omega)_k$. Hence A^0_γ is reduced onto $L^2_\sigma(\Omega)_k$. □

Lemma 4.2 *Let $k \in \mathbb{Z}$. Then $\sigma((A^0_\gamma)_k) = \sigma_{ess}((A^0_\gamma)_k) = \Lambda^0_\gamma$ where Λ^0_γ is the parabolic region in $\mathbb{C}$ defined by (3.3): $\Lambda^0_\gamma = \big\{\lambda = \alpha + \mathrm{i}\beta \in \mathbb{C};\ \alpha,\, \beta \in \mathbb{R},\ \alpha \le -\nu\beta^2/\gamma^2\big\}$.*

Proof. The operator $(A^0_\gamma)_k$ is a part of A^0_γ, hence $\sigma((A^0_\gamma)_k) \subset \sigma(A^0_\gamma) = \Lambda^0_\gamma$.

On the other hand, for $\lambda \in (\Lambda^0_\gamma)^\circ$, we have shown the existence of a non–compact sequence $\boldsymbol{v}^n \in L^2_\sigma(\Omega)$ such that $\|\boldsymbol{v}^n\|_{0,2} = 1$ and $(A^0_\gamma - \lambda I)\boldsymbol{v}^n \to \boldsymbol{0}$ in $L^2_\sigma(\Omega)$ as $n \to +\infty$ in the proof of Theorem 3.1, part II. The construction of $\boldsymbol{v}^n$ involved the choice of an arbitrary non–zero integer k. An easy examination shows that the functions $\boldsymbol{v}^n$ actually belong not only to $L^2_\sigma(\Omega)$, but to $L^2_\sigma(\Omega)_k$. Thus, we obtain that $(\Lambda^0_\gamma)^\circ \subset \widetilde{\sigma}_c((A^0_\gamma)_k)$ for $k \neq 0$. Using the same arguments as at the end of the proof of Theorem 3.1, we deduce that $\Lambda^0_\gamma \subset \sigma_{ess}((A^0_\gamma)_k)$ for $k \neq 0$. It completes the proof in the case when $k \neq 0$.

The case $k = 0$ must be treated separately. Suppose that $\lambda = \alpha + \mathrm{i}\beta \in (\Lambda^0_\gamma)^\circ$. Let us construct a non–compact sequence $\{\boldsymbol{v}^n\}$ in the unit sphere in $L^2_\sigma(\Omega)_0$ such that $(A^0_\gamma - \lambda I)\boldsymbol{v}^n \to \boldsymbol{0}$ as $n \to +\infty$. The requirement that $\boldsymbol{v}^n \in L^2_\sigma(\Omega)_0$ means that $\boldsymbol{v}^n \equiv [v^n_1, v^n_r, v^n_\varphi]$ does not depend on φ. Then the condition $\operatorname{div} \boldsymbol{v}^n = 0$ says that $\partial_1(rv^n_1) + \partial_r(rv^n_r) = 0$. This equation is automatically satisfied if $\boldsymbol{v}^n$ has the cylindrical components

$$\boldsymbol{v}^n(x_1, r) \,=\, \Big[\,\frac{1}{r}\,\partial_r\psi^n(x_1, r),\ -\frac{1}{r}\,\partial_1\psi^n(x_1, r),\ 0\,\Big]. \tag{4.3}$$

Put $\psi^n(x_1, r) = \delta_n\, U^n(x_1)\, V^n(r)$ where U^n and V^n are the same functions as in the proof of Theorem 3.1, i.e. the functions given by (3.10) and (3.12), and where the factor δ_n must be chosen so that $\|\boldsymbol{v}^n\|_{0,2} = 1$. Calculating the norm of $\|\boldsymbol{v}^n\|_{0,2}$, we can observe that for large n the decisive contribution comes from the integral on D^n, see (3.15). The contribution coming from $\Omega - D^n$ is of a lower order in powers of n. The cut–off functions η^n_1 and η^n_2 are both equal to 1 on D^n. Hence $U^n(x_1) = \mathrm{e}^{\mathrm{i}ax_1}$, where $a = -\beta/\gamma$, and $V^n(r) = \mathrm{e}^{\mathrm{i}br}$, where $b = \sqrt{-\alpha_2/\nu}$, see the proof of Theorem 3.1. Thus, $\psi(x_1, r) = \delta_n\, \mathrm{e}^{\mathrm{i}(ax_1+br)}$ on D^n

and

$$
\begin{aligned}
\int_{D^n} |\boldsymbol{v}^n|^2 \, \mathrm{d}\boldsymbol{x} &= \int_{-n^2}^{n^2} \int_{2n}^{2n+n^2} \int_0^{2\pi} \big(|v_1^n|^2 + |v_r^n|^2 \big) \, \mathrm{d}\varphi \, r \, \mathrm{d}r \, \mathrm{d}x_1 \\
&= 2\pi \int_{-n^2}^{n^2} \int_{2n}^{2n+n^2} \frac{1}{r} \big(|\partial_r \psi^n|^2 + |\partial_1 \psi^n|^2 \big) \, \mathrm{d}r \, \mathrm{d}x_1 \\
&= 2\pi \, \delta_n^2 \int_{-n^2}^{n^2} \int_{2n}^{2n+n^2} \frac{1}{r} \, (a^2+b^2) \, \mathrm{d}r \, \mathrm{d}x_1 \\
&= 2\pi \, \delta_n^2 \, (a^2+b^2) \, 2n^2 \Big(\ln(2n+n^2) - \ln(2n) \Big)
\end{aligned}
$$

The condition that this tends to 1 as $n \to +\infty$ leads to the existence of constants c_4, $c_5 > 0$ and $n_0 \in \mathbb{N}$ such that

$$
\forall\, n \in \mathbb{N},\ n \geq n_0 \ : \quad \frac{c_4}{n\sqrt{\ln n}} \ \leq\ \delta_n \ \leq\ \frac{c_5}{n\sqrt{\ln n}} \, . \tag{4.4}
$$

Suppose that n is so large that the support of $\boldsymbol{v}^n$ is a subset of Ω. Then $(A_\gamma^0 - \lambda I)\boldsymbol{v}^n = \nu\Delta\boldsymbol{v}^n - \gamma\partial_1\boldsymbol{v}^n - \lambda\boldsymbol{v}^n$ and so $\|(A_\gamma^0 - \lambda I)\boldsymbol{v}^n\|_{0,2}^2$ equals

$$
\delta_n^2 \int_{S^n - D^n} \big|\nu\Delta\boldsymbol{v}^n - \gamma\partial_1\boldsymbol{v}^n - \lambda\boldsymbol{v}^n\big|^2 \, \mathrm{d}\boldsymbol{x} + \delta_n^2 \int_{D^n} \big|\nu\Delta\boldsymbol{v}^n - \gamma\partial_1\boldsymbol{v}^n - \lambda\boldsymbol{v}^n\big|^2 \, \mathrm{d}\boldsymbol{x} \tag{4.5}
$$

where S^n is defined by (3.14). The integrands are less than or equal to C/r^2. Thus, the first term in (4.5) can be estimated from above by

$$
\begin{aligned}
& C\delta_n^2 \int_{-n-n^2}^{n+n^2} \mathrm{d}x_1 \left(\int_n^{2n} + \int_{2n+n^2}^{3n+n^2} \right) \frac{r\,\mathrm{d}r}{r^2} + C\delta_n^2 \left(\int_{-n-n^2}^{-n^2} + \int_{n^2}^{n+n^2} \right) \mathrm{d}x_1 \int_n^{3n+n^2} \frac{r\,\mathrm{d}r}{r^2} \\
\leq\ & C\delta_n^2 \, (2n+2n^2) \left(\ln\frac{2n}{n} + \ln\frac{3n+n^2}{2n+n^2} \right) + C\delta_n^2 \left(2n \, \ln\frac{3n+n^2}{n} \right).
\end{aligned}
$$

Using (4.4), we verify that the right hand side tends to zero as $n \to +\infty$. In the second term in (4.5), we use the identity $\psi^n(x_1, r) = \delta_n \, \mathrm{e}^{\mathrm{i}(ax_1+br)}$ which holds on D^n. Note that the function $Y(x_1) = \mathrm{e}^{\mathrm{i}ax_1}$ satisfies the differential equation (3.11) and that the function e^{ibr} satisfies the differential equation (3.13). Then calculating the expression $\nu\Delta\boldsymbol{v}^n - \gamma\partial_1\boldsymbol{v}^n - \lambda\boldsymbol{v}^n$ we find that $|\nu\Delta\boldsymbol{v}^n - \gamma\partial_1\boldsymbol{v}^n - \lambda\boldsymbol{v}^n| \leq C\delta_n/r^2$. Consequently the second term in (4.5) can be estimated by

$$
C\delta_n^2 \int_{-n^2}^{n^2} \int_{2n}^{2n+n^2} \int_0^{2\pi} \frac{1}{r^4} \, \mathrm{d}\varphi \, r \, \mathrm{d}r \, \mathrm{d}x_1 \ =\ C\delta_n^2 \, n^2 \int_{2n}^{2n+n^2} \frac{\mathrm{d}r}{r^3} \ \leq\ C\delta_n^2 \, .
$$

Due to (4.4), the right hand side tends to zero as $n \to +\infty$. Hence we have shown that $\|(A_\gamma^0 - \lambda I)\boldsymbol{v}^n\|_{0,2} \to 0$ as $n \to +\infty$. The sequence $\{\boldsymbol{v}^n\}$ is non–compact because the intersection of the supports of any infinite family of functions chosen from $\{\boldsymbol{v}^n\}$ is empty. Since all the functions $\boldsymbol{v}^n$ belong to $L^2_\sigma(\Omega)_0$, we have proved that $\lambda \in \widetilde{\sigma}_c((A_\gamma^0)_0)$. Applying once again the same arguments as at the end of the proof of Theorem 3.1, we observe that $\lambda \in \sigma_{ess}((A_\gamma^0)_0)$ and due to the closedness of $\sigma_{ess}((A_\gamma^0)_0)$, we obtain the inclusion $\Lambda_\gamma^0 \subset \sigma_{ess}((A_\gamma^0)_0)$. This completes the proof in the case $k = 0$. □

Since $-(A_\gamma^0)_k$ is, by definition, the reduction of the sectorial operator $-A_\gamma^0$ on the space $L^2_\sigma(\Omega)_k$, $-(A_\gamma^0)_k$ is a sectorial operator in $L^2_\sigma(\Omega)_k$.

5 Axially symmetric domains Ω – the operator A_γ^ω and its decomposition

Let $k \in \mathbb{Z}$. We shall denote by $(A_\gamma^\omega)_k$ the restriction of A_γ^ω to $L^2_\sigma(\Omega)_k$. The domain of $(A_\gamma^\omega)_k$ is the same as the domain of $(A_\gamma^0)_k$, i.e.,

$$D((A_\gamma^\omega)_k) = D((A_\gamma^0)_k) \equiv W^{2,2}(\Omega)^3 \cap W_0^{1,2}(\Omega)^3 \cap L^2_\sigma(\Omega)_k.$$

If $\boldsymbol{u} \in L^2_\sigma(\Omega)_k$, then it has the form $\boldsymbol{u}(x_1, r, \varphi) = \boldsymbol{U}(x_1, r)\, \mathrm{e}^{\mathrm{i}k\varphi}$ and $\partial_\varphi \boldsymbol{u} = \mathrm{i}\, k\, \boldsymbol{U}\, \mathrm{e}^{\mathrm{i}k\varphi} = \mathrm{i}\, k\, \boldsymbol{u}$. Therefore, $(A_\gamma^\omega)_k$ can be rewritten as

$$(A_\gamma^\omega)_k \boldsymbol{u} \;=\; (A_\gamma^0)_k \boldsymbol{u} + \omega\, \partial_\varphi \boldsymbol{u} \;=\; (A_\gamma^0)_k \boldsymbol{u} + \mathrm{i}k\omega\, \boldsymbol{u}. \tag{5.1}$$

Thus, $(A_\gamma^\omega)_k$ is a closed and densely defined operator in $L^2_\sigma(\Omega)_k$. The representation (5.1) of the operator $(A_\gamma^\omega)_k$ and Lemma 4.2 imply that

$$\sigma((A_\gamma^\omega)_k) \;=\; \sigma_{ess}((A_\gamma^\omega)_k) \;=\; \{\lambda \in \mathbb{C};\; \lambda - \mathrm{i}k\omega \in \Lambda_\gamma^0\}. \tag{5.2}$$

The next lemma provides the information on the spectrum of the full operator A_γ^ω. It confirms that statement (iv) of Theorem 1.1 is true.

Lemma 5.1 *$\sigma(A_\gamma^\omega) = \sigma_{ess}(A_\gamma^\omega) = \Lambda_\gamma^\omega$ where set Λ_γ^ω is defined by (1.11):*
$\Lambda_\gamma^\omega = \{\lambda = \alpha + \mathrm{i}\beta + \mathrm{i}k\omega \in \mathbb{C};\; \alpha, \beta \in \mathbb{R},\; k \in \mathbb{Z},\; \alpha \le -\nu\beta^2/\gamma^2\}$.

Proof. Each operator $(A_\gamma^\omega)_k$, $k \in \mathbb{Z}$, is a part of the operator A_γ^ω, hence $\sigma_{ess}((A_\gamma^\omega)_k) \subset \sigma_{ess}(A_\gamma^\omega)$. Thus, $\cup_{k\in\mathbb{Z}}\, \sigma_{ess}((A_\gamma^\omega)_k) = \Lambda_\gamma^\omega \subset \sigma_{ess}(A_\gamma^\omega) \subset \sigma(A_\gamma^\omega)$.

It remains to prove the opposite inclusion, i.e. that $\sigma(A_\gamma^\omega) \subset \Lambda_\gamma^\omega$ or equivalently that $(\mathbb{C} - \Lambda_\gamma^\omega) \subset \rho(A_\gamma^\omega)$. Suppose that $\lambda \equiv \alpha + \mathrm{i}\beta \in \mathbb{C} - \Lambda_\gamma^\omega$. We will show that the operator $A_\gamma^\omega - \lambda I$ has a bounded inverse in $L^2_\sigma(\Omega)$. Let $\boldsymbol{f} \in L^2_\sigma(\Omega)$ with Fourier expansion

$$\boldsymbol{f}(x_1, r, \varphi) \;=\; \sum_{k=-\infty}^{+\infty} \boldsymbol{f}_k(x_1, r)\; \mathrm{e}^{\mathrm{i}k\varphi},$$

where $\boldsymbol{f}_k\, \mathrm{e}^{\mathrm{i}k\varphi} \in L^2_\sigma(\Omega)_k$, be given. Let us at first solve the equation $((A_\gamma^\omega)_k - \lambda I)\boldsymbol{w}_k = \boldsymbol{f}_k\, \mathrm{e}^{\mathrm{i}k\varphi}$ in $L^2_\sigma(\Omega)_k$. Putting $\boldsymbol{w}_k = \boldsymbol{u}_k\, \mathrm{e}^{\mathrm{i}k\varphi}$ and using (5.1), we observe that this equation is equivalent with

$$(A_\gamma^0)_k(\boldsymbol{u}_k\, \mathrm{e}^{\mathrm{i}k\varphi}) - (\alpha + \mathrm{i}\beta - \mathrm{i}k\omega)(\boldsymbol{u}_k\, \mathrm{e}^{\mathrm{i}k\varphi}) \;=\; \boldsymbol{f}_k\, \mathrm{e}^{\mathrm{i}k\varphi}. \tag{5.3}$$

Due to Lemma 4.2 we have $\alpha + \mathrm{i}\beta - \mathrm{i}k\omega \in \rho((A_\gamma^0)_k)$ for all $k \in \mathbb{Z}$. Since the operator $-(A_\gamma^0)_k$ is sectorial, we deduce from resolvent estimates for sectorial operators, see e.g. D. Henry [14], p. 23, that there exists a constant $M > 0$, independent of k, such that

$$\|\boldsymbol{u}_k\|_{0,2} \;=\; \|\boldsymbol{u}_k\, \mathrm{e}^{\mathrm{i}k\varphi}\|_{0,2} \;\le\; \frac{M}{1 + |k|}\, \|\boldsymbol{f}_k\, \mathrm{e}^{\mathrm{i}k\varphi}\|_{0,2} \;=\; \frac{M}{1 + |k|}\, \|\boldsymbol{f}_k\|_{0,2}\,. \tag{5.4}$$

Then the series $\sum_{k=-\infty}^{+\infty} \boldsymbol{u}_k\, \mathrm{e}^{\mathrm{i}k\varphi}$ converges in $L^2_\sigma(\Omega)$ and $\boldsymbol{u} = \sum_{k=-\infty}^{+\infty} \boldsymbol{u}_k\, \mathrm{e}^{\mathrm{i}k\varphi}$ satisfies the estimate

$$\|\boldsymbol{u}\|_{0,2}^2 \;=\; \sum_{k=-\infty}^{+\infty} \|\boldsymbol{u}_k\|_{0,2}^2 \;\leq\; \sum_{k=-\infty}^{+\infty} \frac{M^2}{(1+|k|)^2}\, \|\boldsymbol{f}_k\|_{0,2}^2 \;\leq\; M^2\, \|\boldsymbol{f}\|_{0,2}^2\,. \tag{5.5}$$

From the equation (5.3) and the estimate (5.4), we have

$$\big\|(A_\gamma^\omega)_k(\boldsymbol{u}_k\, \mathrm{e}^{\mathrm{i}k\varphi})\big\|_{0,2} \;\leq\; \|\boldsymbol{f}_k\, \mathrm{e}^{\mathrm{i}k\varphi}\|_{0,2} + |\alpha+\mathrm{i}\beta|\, \|\boldsymbol{u}_k\, \mathrm{e}^{\mathrm{i}k\varphi}\|_{0,2} \;\leq\; C\, \|\boldsymbol{f}_k\, \mathrm{e}^{\mathrm{i}k\varphi}\|_{0,2}$$

where C is independent of k. Using these inequalities and the closedness of the operator A_γ^ω, we deduce that $\boldsymbol{u} \in D(A_\gamma^\omega)$ and $(A_\gamma^\omega - \lambda I)\boldsymbol{u} = \boldsymbol{f}$. This information, together with (5.5), completes the proof. □

6 General exterior domains – the operators A_γ^ω and L_γ^ω

Using the same procedure as in the proof of Theorem 3.2, we can show that the operator A_γ^ω is not normal, i.e., that the statement (ii) of Theorem 1.1 is true.

If $\lambda = \alpha + \mathrm{i}\beta$ is an eigenvalue of A_γ^ω and $\boldsymbol{v}$ is a corresponding eigenfunction, then, multiplying the equation $A_\gamma^\omega \boldsymbol{v} = \lambda\, \boldsymbol{v}$ by $\overline{\boldsymbol{v}}$ and integrating on Ω, we obtain the identity $-\nu\, \|\nabla \boldsymbol{v}\|_{0,2}^2 \;=\; \alpha\, \|\boldsymbol{v}\|_{0,2}^2$; compare with that part of the proof of Theorem 3.1 which lead to (3.8). This verifies Theorem 1.1 (iii).

Let $R_0 = \max\{|\boldsymbol{x}|;\; \boldsymbol{x} \in \mathcal{B}\}$ and $\Omega_R = \Omega \cap B_R(\boldsymbol{0})$.

Lemma 6.1 *Let $\lambda \in \widetilde{\sigma}_{\mathrm{c}}(A_\gamma^\omega)$. Then there exists $R > R_0$ and a non–compact sequence $\{\boldsymbol{u}^n\}$ in $D(A_\gamma^\omega)$ such that $\|\boldsymbol{u}^n\|_{0,2} = 1$, $\boldsymbol{u}^n = \boldsymbol{0}$ in Ω_R and*

$$(A_\gamma^\omega - \lambda I)\, \boldsymbol{u}^n \;\longrightarrow\; 0 \quad \textit{in } L^2_\sigma(\Omega) \quad \textit{for } n \to +\infty. \tag{6.1}$$

Proof. The condition $\lambda \in \widetilde{\sigma}_{\mathrm{c}}(A_\gamma^\omega)$ means that $\mathrm{nul}'(A_\gamma^\omega - \lambda I) = +\infty$. Then there exists an orthonormal sequence $\{\boldsymbol{v}^n\}$ in $L^2_\sigma(\Omega)$ such that

$$(A_\gamma^\omega - \lambda I)\, \boldsymbol{v}^n \;=\; \boldsymbol{\epsilon}^n \;\longrightarrow\; 0 \quad \text{in } L^2_\sigma(\Omega) \quad \text{for } n \to +\infty; \tag{6.2}$$

the construction of the sequence $\{\boldsymbol{v}^n\}$ is based on Lemma IV.2.3 in [18] and is explained in [6]. Obviously $\{\boldsymbol{v}^n\}$ converges to the zero function weakly in $L^2_\sigma(\Omega)$. Using (6.2) and the estimate (2.1), we get that the sequence $\{\boldsymbol{v}^n\}$ is bounded in $W_0^{1,2}(\Omega)^3 \cap W^{2,2}(\Omega)^3$. Then there exists a subsequence, again denoted by $\{\boldsymbol{v}^n\}$, which is weakly convergent to $\boldsymbol{0}$ in $W_0^{1,2}(\Omega)^3 \cap W^{2,2}(\Omega)^3$. Moreover, $\{\omega(\boldsymbol{e}_1 \times \boldsymbol{x}) \cdot \nabla \boldsymbol{v}^n\}$ converges weakly to $\boldsymbol{0}$ in $L^2_\sigma(\Omega)$. Suppose that $R \geq R_0 + 3$ is a fixed number. The compact imbedding $W^{2,2}(\Omega_R)^3 \hookrightarrow$ $\hookrightarrow W^{1,2}(\Omega_R)^3$ yields

$$\boldsymbol{v}^n \;\longrightarrow\; \boldsymbol{0} \qquad \text{strongly in } W^{1,2}(\Omega_R)^3. \tag{6.3}$$

The first part of (6.2) can be written in the form

$$\nu\Delta \boldsymbol{v}^n + \omega(\boldsymbol{e}_1 \times \boldsymbol{x}) \cdot \nabla \boldsymbol{v}^n - \omega \boldsymbol{e}_1 \times \boldsymbol{v}^n - \gamma\partial_1 \boldsymbol{v}^n - \lambda \boldsymbol{v}^n + \nabla q^n \;=\; \boldsymbol{\epsilon}^n \tag{6.4}$$

where q^n is an appropriate scalar function. It follows from (6.4) and (2.1) that $\nabla q^n \to \mathbf{0}$ weakly in $L^2(\Omega)^3$. Thus, the functions q^n, which are given uniquely up to an additive constant by (6.4), can be chosen so that $q^n \to q \equiv const.$ strongly in $L^2(\Omega_R)$. We may even assume that $q = 0$.

Denote by η an infinitely differentiable cut–off function in Ω such that

$$\eta(\boldsymbol{x}) \;=\; \begin{cases} 0 & \text{if} \quad |\boldsymbol{x}| < R-2, \\ 1 & \text{if} \quad |\boldsymbol{x}| > R-1, \end{cases}$$

and $0 \le \eta(\boldsymbol{x}) \le 1$ if $R-2 \le |\boldsymbol{x}| \le R-1$. Put $\boldsymbol{u}^n = \eta \boldsymbol{v}^n - \boldsymbol{V}^n$ where $\operatorname{div} \boldsymbol{V}^n = \nabla\eta \cdot \boldsymbol{v}^n$. Although $\boldsymbol{V}^n$ is not given uniquely, the results on solutions of the equation $\operatorname{div} \boldsymbol{V} = \boldsymbol{f}$, see e.g. [2], show that the function $\boldsymbol{V}^n$ can be chosen such that $\operatorname{supp} \boldsymbol{V}^n \subset \{\boldsymbol{x} \in \Omega;\; R-3 < |\boldsymbol{x}| < R\}$ and there exist $c_6 > 0$ such that

$$\|\boldsymbol{V}^n\|_{2,2} \;=\; \|\boldsymbol{V}^n\|_{2,2;\,\Omega_R} \;\le\; c_6\, \|\nabla\eta \cdot \boldsymbol{v}^n\|_{1,2;\,\Omega_R} \;\longrightarrow\; 0 \tag{6.5}$$

as $n \to +\infty$. (Here $\|\,.\,\|_{2,2;\,\Omega_R}$ and $\|\,.\,\|_{1,2;\,\Omega_R}$ denote the norm in $W^{2,2}(\Omega_R)^3$ and in $W^{1,2}(\Omega_R)^3$, respectively.) The function $\boldsymbol{u}^n$ is divergence–free, equals $\mathbf{0}$ in Ω_{R-3}, equals $\boldsymbol{v}^n$ in $\Omega - \Omega_R$ and belongs to $L^2(\Omega)^3$. Due to the properties of the functions η and $\boldsymbol{V}^n$ we get $\boldsymbol{u}^n \in D(A^{\omega}_{\gamma})$. Obviously $\boldsymbol{u}^n$ satisfies

$$\begin{aligned}
&\nu\Delta\boldsymbol{u}^n + \omega(\boldsymbol{e}_1 \times \boldsymbol{x}) \cdot \nabla\boldsymbol{u}^n - \omega\boldsymbol{e}_1 \times \boldsymbol{u}^n - \gamma\partial_1\boldsymbol{u}^n - \lambda\boldsymbol{u}^n + \nabla(\eta q^n) \\
&\quad = \; \eta\left[\nu\Delta\boldsymbol{v}^n + \omega(\boldsymbol{e}_1 \times \boldsymbol{x}) \cdot \nabla\boldsymbol{v}^n - \omega\boldsymbol{e}_1 \times \boldsymbol{v}^n - \gamma\partial_1\boldsymbol{u}_v - \lambda\boldsymbol{v}^n\right] + 2\nu\nabla\eta \cdot \nabla\boldsymbol{v}^n \\
&\qquad + \nu(\Delta\eta)\boldsymbol{v}^n - \nu\Delta\boldsymbol{V}^n + \left[\omega(\boldsymbol{e}_1 \times \boldsymbol{x}) \cdot \nabla\eta\right] \boldsymbol{v}^n - \gamma\,(\partial_1\eta)\,\boldsymbol{v}^n - \omega(\boldsymbol{e}_1 \times \boldsymbol{x}) \cdot \nabla\boldsymbol{V}^n \\
&\qquad + \omega\boldsymbol{e}_1 \times \boldsymbol{V}^n + \gamma\partial_1\boldsymbol{V}^n + \lambda\boldsymbol{V}^n + \nabla(\eta q^n) \\
&\quad = \; \eta\boldsymbol{\epsilon}^n + 2\nu\nabla\eta \cdot \nabla\boldsymbol{v}^n + \nu(\Delta\eta)\boldsymbol{v}^n - \nu\Delta\boldsymbol{V}^n + \left[\omega(\boldsymbol{e}_1 \times \boldsymbol{x}) \cdot \nabla\eta\right] \boldsymbol{v}^n - \gamma\,(\partial_1\eta)\,\boldsymbol{v}^n \\
&\qquad - \omega(\boldsymbol{e}_1 \times \boldsymbol{x}) \cdot \nabla\boldsymbol{V}^n + \omega\boldsymbol{e}_1 \times \boldsymbol{V}^n + \gamma\partial_1\boldsymbol{V}^n + \lambda\boldsymbol{V}^n + (\nabla\eta)q^n
\end{aligned} \tag{6.6}$$

where $\eta\boldsymbol{\epsilon}^n \to \mathbf{0}$ in $L^2(\Omega)^3$ due to (6.2), and $\nu[2\nabla\eta \cdot \nabla\boldsymbol{v}^n + (\Delta\eta)\boldsymbol{v}^n] \to \mathbf{0}$ in $L^2(\Omega)^3$ because $\nabla\eta$ and $\Delta\eta$ are supported in Ω_R and due to (6.3). The terms $\left[\omega(\boldsymbol{e}_1 \times \boldsymbol{x}) \cdot \nabla\eta\right] \boldsymbol{v}^n$ and $\gamma\,(\partial_1\eta)\,\boldsymbol{v}^n$ also tend to zero in in $L^2(\Omega)^3$ for the same reasons. Furthermore, all terms involving $\boldsymbol{V}^n$ tend to $\mathbf{0}$ in $L^2(\Omega)^3$ due to (6.5). Finally, $(\nabla\eta)q^n \to \mathbf{0}$ in $L^2(\Omega)^3$ because $q^n \to 0$ in $L^2(\Omega_R)$ and $\nabla\eta$ is supported in Ω_R. Thus,

$$\nu\Delta\boldsymbol{u}^n + \omega(\boldsymbol{e}_1 \times \boldsymbol{x}) \cdot \nabla\boldsymbol{u}^n - \omega\boldsymbol{e}_1 \times \boldsymbol{u}^n - \gamma\partial_1\boldsymbol{u}^n - \lambda\boldsymbol{u}^n + \nabla(\eta q^n) \;\longrightarrow\; \mathbf{0} \quad \text{in } L^2(\Omega)^3$$

for $n \to +\infty$, and therefore $\{\boldsymbol{u}^n\}$ satisfies (6.1). Moreover, we have

$$\|\boldsymbol{u}^n\|_{0,2}^2 \;\ge\; \int_{|x|>R} |\boldsymbol{u}^n(\boldsymbol{x})|^2 \,\mathrm{d}\boldsymbol{x} \;=\; \int_{|x|>R} |\boldsymbol{v}^n(\boldsymbol{x})|^2 \,\mathrm{d}\boldsymbol{x} \;\longrightarrow\; 1 \quad \text{for } n \to +\infty$$

because $\|\boldsymbol{v}^n\|_{0,2} = 1$ and due to (6.3). If we divide each of the functions $\boldsymbol{u}^n$ by its norm in $L^2_{\sigma}(\Omega)$ and denote the new function again by $\boldsymbol{u}^n$, we obtain the sequence $\{\boldsymbol{u}^n\}$ with all the properties stated in Lemma 6.1. Finally, the orthonormality of $\{\boldsymbol{v}^n\}$ and (6.3) imply the non–compactness of the sequence $\{\boldsymbol{u}^n\}$. □

We denote by $\widehat{A}^{\omega}_{\gamma}$ the operator which is defined in the same way as A^{ω}_{γ}, however on the whole space $\mathbb{R}^3$ rather than on the exterior domain $\Omega \subset \mathbb{R}^3$. Obviously, the operator $\widehat{A}^{\omega}_{\gamma}$ has all the properties derived in Sections 4 and 5.

Lemma 6.2 $\widetilde{\sigma}_c(A^\omega_\gamma) = \widetilde{\sigma}_c(\widehat{A}^\omega_\gamma) = \Lambda^\omega_\gamma$.

Proof. Suppose that $\lambda \in \widetilde{\sigma}_c(A^\omega)$. Let $R > 0$ and $\{\boldsymbol{u}^n\}$ be a number and a sequence, respectively, with the properties named in Lemma 6.1. All functions $\boldsymbol{u}^n$, extended by zero from Ω to the whole space $\mathbb{R}^3$, belong to the domain of operator $\widehat{A}^\omega_\gamma$. Thus, (6.1) shows that $\lambda \in \widetilde{\sigma}_c(\widehat{A}^\omega_\gamma)$.

On the other hand, if $\lambda \in \widetilde{\sigma}_c(\widehat{A}^\omega_\gamma)$ then we can use analogous arguments and prove that λ also belongs to $\widetilde{\sigma}_c(A^\omega_\gamma)$. □

Let us show that $\widetilde{\sigma}_c(A^\omega_\gamma) \subset \sigma_{ess}(A^\omega_\gamma)$; the opposite inclusion is trivial. For $\lambda \in \widetilde{\sigma}_c(A^\omega_\gamma)$ we have that $\mathrm{nul}'(A^\omega_\gamma - \lambda I) = +\infty$. Moreover, $\overline{\lambda} \in \widetilde{\sigma}_c((A^\omega_\gamma)^*) = \widetilde{\sigma}_c(A^{-\omega}_{-\gamma})$, so that $\mathrm{nul}'((A^\omega_\gamma)^* - \overline{\lambda} I) = +\infty$. Hence $\mathrm{def}'(A^\omega_\gamma - \lambda I) = +\infty$ which shows that $\lambda \in \sigma_{ess}(A^\omega_\gamma)$, see [18], p. 234. We have thus proved Theorem 1.1 (i). Theorem IV.5.35 in [18] and Lemma 2.4 imply that the essential spectrum of the operator L^ω_γ is the same as $\sigma_{ess}(A^\omega_\gamma)$; therefore it is also given by (1.11). Moreover, since $\mathrm{ind}(L^\omega_\gamma - \lambda I) = 0$ in $\mathbb{C} - \sigma_{ess}(L^\omega_\gamma)$ and due to Theorem IV.5.31 in [18], $\mathbb{C} - \sigma_{ess}(L^\omega_\gamma)$ can contain at most countably many eigenvalues λ of L^ω_ω, which can cluster only on the boundary of $\mathbb{C} - \sigma_{ess}(L^\omega_\gamma)$ and $0 < \mathrm{nul}(L^\omega_\gamma - \lambda I) = \mathrm{def}(L^\omega_\gamma - \lambda I) < +\infty$ at each of them. This implies Theorem 1.1 (vi).

Acknowledgement. The work was supported by the Nečas Center for Mathematical Modelling (project No. LC06052), by the Academy of Sciences of the Czech Republic (Institutional Research Plan No. AV0Z10190503) and by the Grant Agency of the Czech Academy of Sciences (grant No. IAA100190612). It was also performed as a part of the cooperation between TU Darmstadt and CTU Prague.

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