Anisotropic Diffusion in Mathematical Visualization

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Preprint no. 2009-017

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Abstract

We propose a method of vector field visualization based on noisy texture smearing. The smearing process is carried out by solving a problem for the Allen-Cahn diffusion PDE with advection. We introduce a parallel algorithm for the numerical solution of the given problem and we present several results of the efficiency benchmark. Finally, an extension to diffusion tensor field visualization is considered together with its application to diffusion tensor magnetic resonance imaging.

1 Introduction

Suppose we have a static vector field \( \mathbf{v} \) defined in a rectangular domain \( \Omega = (0, L^1) \times (0, L^2) \). Our goal is to make its stream lines emerge as smudges by means of smearing a noisy texture on \( \Omega \) in the direction of the field. In addition to smearing, it is sometimes reasonable to make the texture move (advection) in the direction of the vector field.

For the above stated purposes, it is possible to use a diffusion parabolic PDE with an incorporated diffusion anisotropy and with the advection term. (see [5, 1]).

2 Formulation

Let \( p : \mathcal{J} \times \Omega \mapsto \mathbb{R}, p = p(t, \mathbf{x}) \) be the function of texture intensity at each point \( \mathbf{x} \in \Omega \) and at the time \( t \in \mathcal{J} \) where \( \mathcal{J} = (0, T) \) is the time interval. The initial boundary-value problem for the Allen-Cahn equation with advection (see [4]) reads as

\[
\xi^2 \frac{\partial p}{\partial t} + \xi^2 \mathbf{v} \cdot \nabla p = \xi^2 \nabla \cdot \mathbf{T}^0(\nabla p) + f_0(p) + c_0 F \xi
\quad \text{in } \mathcal{J} \times \Omega, \quad (1)
\]

\[
p|_{\partial \Omega} = 0 \quad \text{on } \mathcal{J} \times \partial \Omega, \quad (2)
\]

\[
p|_{t=0} = I \quad \text{in } \Omega \quad (3)
\]

where

\[
f_0(p) = p(1-p) \left( p - \frac{1}{2} \right).
\]
In (1), the term $\nabla \cdot T^0(\nabla p)$ is responsible for anisotropic diffusion of $p$ focused into the direction of the vector field. Consider a vector $\eta = (\eta^1, \eta^2)^T \in \mathbb{R}^2$ and denote the coordinates of $\eta$ in the orthonormal basis $(\frac{\eta}{\|\eta\|}, \frac{\eta}{\|\eta\|}^\perp)$ by $\tilde{\eta}^1, \tilde{\eta}^2$. The anisotropic operator $T^0$ is defined as

$$T^0(\eta) = \Phi^0(\eta) \Phi^0_\eta(\eta)$$

where

$$\Phi^0(\eta) = \sqrt{\alpha \cdot (\tilde{\eta}^1)^2 + \beta \cdot (\tilde{\eta}^2)^2}, \quad \Phi^0_\eta(\eta) = \left( \frac{\partial_{\eta^1} \Phi^0(\eta)}{\partial_{\eta^2} \Phi^0(\eta)} \right).$$

The coefficients $\alpha, \beta$ depend on the vector field and should be chosen such that the absolute value of $T^0$ is largest in the case when the directions of $v$ and $\nabla p$ coincide. Our choice is

$$\alpha = \kappa (1 + \sigma |v|), \quad \beta = \kappa, \quad \kappa, \sigma > 0.$$

The term $v \cdot \nabla p$ in (1) causes texture advection. (see [1, 5]). The polynomial $f_0$ makes nucleation occur during the time. In this context, nucleation is a formation of areas where the value of $p$ is near 0 or 1. As described for example in [5], the parameter $\xi$ is proportional to the diffuse interface layer between such areas. $\xi$ is chosen such that it is small in comparison with the dimensions of $\Omega$. The sense of the parameter $F$ is related to the problem of mean curvature flow and is explained e.g. in [1, 5].

In the context of visualization, if $I: \Omega \rightarrow \mathbb{R}$ represents the intensity of a noisy texture at each point, the solution $p$ will reflect the gradual diffusion of the initial image $I$ with increasing time. Both the state of $p$ at some final time $T$ and the entire solution evolution can be regarded as the result.

**3 Numerical solution**

For numerical solution, we use the method of lines which converts the problem (1-3) to the solution of the system of ODEs in the form

$$\frac{dp}{dt} = f(t, p).$$  \hspace{1cm} (4)

The spatial discretization is carried out by the finite difference method; for the temporal discretization, we employ the 4th-order Runge-Kutta-Merson solver with adaptive time stepping. First, let us introduce the notations

$$h = (h^1, h^2), \quad h^k := \frac{L^k}{m^k}, \quad k \in \{1, 2\},$$

$$x_{i,j} = (x^1_i, x^2_j) = (i \cdot h^1, j \cdot h^2), \quad u_{i,j} = u(x_{i,j}),$$

$$2$$
\[ \omega_h = \{(ih^1, jh^2) \mid i = 1, \ldots, m^1 - 1, j = 1, \ldots, m^2 - 1\}, \]
\[ \bar{\omega}_h = \{(ih^1, jh^2) \mid i = 0, \ldots, m^1, j = 0, \ldots, m^2\}, \quad \gamma_h = \bar{\omega}_h - \omega_h, \]
\[ \mathcal{H}_h = \{w \mid \bar{w}_h \to \mathbb{R}\}, \quad \mathcal{P}_h u = u|_{\bar{\omega}_h}. \]

In the sense of (5), we introduce the following discrete substitutes for derivatives, gradient and divergence:

\[ u_{x^1, i,j} = \frac{u_{i,j} - u_{i-1,j}}{h^1}, \quad u_{x^2, i,j} = \frac{u_{i+1,j} - u_{i,j}}{h^1}, \]
\[ u_{x^1, i,j} = \frac{u_{i,j} - u_{i,j-1}}{h^2}, \quad u_{x^2, i,j} = \frac{u_{i,j+1} - u_{i,j}}{h^2}, \]
\[ \bar{\nabla}_h u = (u_{x^1}, u_{x^2}), \quad \nabla_h u = (u_{x^1}, u_{x^2}), \]
\[ \nabla_h \cdot V = V_{x^1}^1 + V_{x^2}^2, \quad \bar{\nabla}_h \cdot V = V_{x^1}^1 + V_{x^2}^2, \quad V = (V^1, V^2)^T. \]

Using the above definitions, we assemble the semi-discrete scheme of the problem (1-3) for the unknown grid function \( p^h : J \to \mathcal{H}_h \) which represents the vector of functions of time \( p \) in (4):

\[ \xi^2 \frac{dp^h}{dt} + \xi^2 \mathcal{P}_h(v) \cdot \bar{\nabla}_h p^h = \xi^2 \nabla_h \cdot \left(T^h(\bar{\nabla}_h p^h)\right) + f_0(p^h) + c_0 F \xi \quad \text{in } J \times \omega_h, \]
\[ p^h|_{\gamma_h} = 0 \quad \text{on } J \times \gamma_h, \]
\[ p^h(0) = \mathcal{P}_h I \quad \text{in } \omega_h. \]

The work [4] contains a detailed convergence analysis, proving that the above numerical scheme converges to the unique weak solution of the problem (1-3). However, the homogeneous Dirichlet boundary condition can be altered in the computational studies.

The results of the numerical algorithm based on some sample vector fields are displayed in Figure 2. Color visualization has been achieved by separately solving the above problem for the R, G, B components of the image.

### 4 Parallelization

The parallel implementation of the numerical algorithm has been developed by means of the MPI library (see [3]) in order to allow reasonably fast calculations on large grids. Very fine grids are necessary e.g. for the convergence verification of several numerical scheme modifications. The idea of parallelization of the finite difference algorithm is to divide (decompose) the grid \( \omega_h \) into blocks, each of those being handled by a different process. Our choice was to compose a block of several rows of the grid. The processes belonging to the adjacent blocks need to interchange (synchronize) data in order to complete each step of the Runge-Kutta method.
Since the method of lines is extremely demanding on the amount of synchronization, much attention has been paid to benchmarking and scalability improvement of the code. Using the nonblocking communication operations, we are able to optimize the flow of the calculation by requesting the operations as soon as possible and completing them as late as possible. Since the synchronized data is used for calculation of the border nodes of the blocks only, we can calculate the value of the right hand side of (4) in the interior of the block before the communication is complete.

Extensive efficiency benchmarks have been performed on the CLX Linux cluster at CINECA, Italy. Some efficiency results are shown in Figure 1.

5 Application in MR Tractography

Diffusion tensor magnetic resonance imaging (DT-MRI, see [6]) represents a framework for water molecule diffusion measurement in human brain tissue. The result of the measurement can be interpreted as a tensor field $D$ indicating the main directions of water diffusion in each volume element. Using the choice

$$T^0(\eta) = D\eta$$

in (1), we are able to employ our visualization approach to reveal the streamlines of the tensor field. Since the strongest water diffusion occurs along neural fibers, the described technique is capable of effectively displaying the main neural tracts in any planar cut through the volume of interest. A sample result is displayed in Figure 3.
Figure 2: Sample vector field visualizations.

Figure 3: MR tractography using anisotropic diffusion. Colorized by fractional anisotropy (see [6]).
6 Conclusions

We have developed and thoroughly tested an optimized parallel algorithm for the numerical solution of the anisotropic diffusion problem (1-3). The results show that it is possible to create a well scalable parallel implementation of the method of lines despite the huge amount of necessary communication. Thanks to the computing power of the CLX cluster at CINECA, several successful convergence tests could be completed for different numerical schemes.

Acknowledgements: This work was carried out under the HPC-EUROPA project (RII3-CT-2003-506079), with the support of the European Community - Research Infrastructure Action under the FP6 Structuring the European Research Area Programme.

References


Publications