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TWO DIMENSIONAL FLOW OF INCOMPRESSIBLE HEAT CONDUCTING NON-NEWTONIAN FLUID - EXISTENCE ANALYSIS

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Abstract. We study a flow of an incompressible homogeneous fluid whose material coefficients can depend on the temperature and the shear-rate. For large class of models we establish the existence of a suitable weak solution for two-dimensional flows of fluid in a bounded domain. The proof relies on the reconstruction of the pressure, that is possible since we consider Navier's slip boundary conditions, and on the so-called L^∞-truncation method, that is used to obtain the strong convergence of the velocity gradient also in the case when the energy equality does not hold and the standard monotone operator theory cannot be used.

Keywords: Heat conducting fluid, non-Newtonian fluid, existence, weak solution, suitable weak solution, L^∞-truncation method

MSC 2000: 35Q30, 35Q72, 76D03, 76A05

1. Introduction

This paper focuses on the existence analysis for an unsteady flow of an incompressible homogeneous heat conductive non-Newtonian fluid in a bounded two-dimensional domain $\Omega \subset \mathbb{R}^2$. Such flow is governed by the following system of partial differential equations (PDE's):

$$\operatorname{div} \boldsymbol{v} = 0\,, \tag{1.1}$$

$$\boldsymbol{v}_{,t} + \operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{v}) - \operatorname{div} \mathbf{S} = -\nabla p + \boldsymbol{f}\,, \tag{1.2}$$

$$E_{,t} + \operatorname{div}(\boldsymbol{v}(E+p)) - \operatorname{div}(\mathbf{S}\boldsymbol{v} - \boldsymbol{q}) = \boldsymbol{f} \cdot \boldsymbol{v}\,, \tag{1.3}$$

that is supposed to be satisfied in $Q := (0,T) \times \Omega$, where $T > 0$ denotes the length of time interest. In (1.1)–(1.3) $\boldsymbol{v} : Q \to \mathbb{R}^2$ denotes the velocity of the fluid; $\mathbf{S} : Q \to \mathbb{R}^{2\times 2}_{\text{sym}}$ stands for the constitutively determined part of the Cauchy stress; $p : Q \to \mathbb{R}$ is the pressure; $E : Q \to \mathbb{R}$ denotes the density of the global energy; $\boldsymbol{q} : Q \to \mathbb{R}^2$ is the heat flux and finally $\boldsymbol{f} : Q \to \mathbb{R}^2$ is the density of the external body forces. For simplicity (but without any essential changes in the proof) we assume that $\boldsymbol{f} \equiv \mathbf{0}$

in Q in what follows. Note that (1.1) represents the incompressibility constrain that is, for homogeneous fluid, equivalent to the balance of mass, (1.2) is the balance of linear momentum and (1.3) stands for the balance of the global energy.

To close the system of PDE's (1.1)–(1.3), it remains to prescribe initial and boundary data. For the initial data we assume that

$$\boldsymbol{v}(0,x) = \boldsymbol{v}_0(x) \quad \text{and} \quad E(0,x) = E_0(x) \qquad \text{in } \Omega. \tag{1.4}$$

For simplicity, we also denote $\Gamma := (0,T) \times \partial\Omega$ and we assume that the boundary data for the velocity field are given by the so-called Navier's slip boundary conditions, i.e., we assume that for some $\gamma \in [0,1)$

$$\boldsymbol{v} \cdot \boldsymbol{n} = 0 \quad \text{and} \quad \gamma \boldsymbol{v}_\tau + (1-\gamma)(\mathbf{S}\boldsymbol{n})_\tau = \mathbf{0} \qquad \text{on } \Gamma\,. \tag{1.5}$$

Finally, for the heat flux $\boldsymbol{q}$ we prescribe the homogeneous Neumann boundary condition, i.e.,

$$\boldsymbol{q} \cdot \boldsymbol{n} = 0 \qquad \text{on } \Gamma\,. \tag{1.6}$$

In (1.5)–(1.6) we denoted by $\boldsymbol{n}$ the unit outward normal vector on $\partial\Omega$, and for any vector $\boldsymbol{w}$, we defined $\boldsymbol{w}_\tau := \boldsymbol{w} - (\boldsymbol{w}\cdot\boldsymbol{n})\boldsymbol{n}$ its projection to the tangent plane. The boundary conditions (1.5) describe an internal flow (no outflow/inflow is allowed) that can slip on the boundary. In addition, $(1.5)_2$ proposes that the velocity of the fluid is proportional to the viscous forces on the boundary $\partial\Omega$. Also note that setting $\gamma = 0$ in $(1.5)_2$ we get the so-called no-stick boundary conditions and setting $\gamma = 1$ we obtain the no-slip boundary conditions for the velocity. It should be mentioned at the very beginning that while the case $\gamma \in [0,1)$ is included in our analysis, the case $\gamma = 1$ is not covered since we are not able to construct the pressure p that is needed in the global energy balance (1.3). To simplify further notation, we also define a slip parameter $\alpha \geq 0$ as

$$\alpha := \frac{\gamma}{1-\gamma}.$$

On the other hand the Neumann boundary condition (1.6) for $\boldsymbol{q}$ can be easily relaxed and we are also able to treat the Dirichlet boundary conditions for the temperature and mixed boundary conditions as well. Since such different setting does not make any essential changes in the proof, we do not include them into our analysis.

Next, we describe the constitutive relations for $\mathbf{S}$, E, $\boldsymbol{q}$ we are interested in. First, the density of the total energy E is given as the sum of the kinetic and the internal energy

$$E := \frac{1}{2}|\boldsymbol{v}|^2 + e\,, \tag{1.7}$$

where $e : Q \to \mathbb{R}_+$ is the density of the internal energy. For the heat flux $\boldsymbol{q}$ we consider the generalized Fourier-law

$$\boldsymbol{q} = \hat{\boldsymbol{q}}(\theta, \mathbf{D}(\boldsymbol{v}), \nabla\theta) = -\hat{\kappa}(\theta, |\mathbf{D}(\boldsymbol{v})|)\nabla\theta\,, \tag{1.8}$$

here $\hat{\kappa} : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ denotes the heat conductivity of the fluid and $\theta : Q \to \mathbb{R}_+$ is the temperature. We also employ the notation for the symmetric part of the velocity gradient $\mathbf{D}(\boldsymbol{v}) := \frac{1}{2}(\nabla\boldsymbol{v} + (\nabla\boldsymbol{v})^T)$. Note that the flux $\boldsymbol{q}$ is allowed to be dependent on the shear rate $|\mathbf{D}(\boldsymbol{v})|$, we refer to [5] for more details concerning physical importance of such a setting.

In order to cover general class of fluids without making any other assumptions on the heat capacity of the fluid, we rather reformulate (1.8) such that $\boldsymbol{q}$ is given by the internal energy and by its gradient rather than by the temperature and the temperature gradient. To do so, we assume that the internal energy e is a smooth function of θ with a smooth inverse[1], i.e.

$$e = e(\theta) \quad \text{and} \quad \theta = \theta(e)\,.$$

Using this assumption, we can consider e as a primitive quantity. Therefore (1.8) can be rewritten as

$$\boldsymbol{q} = \boldsymbol{q}^*(e, \mathbf{D}(\boldsymbol{v}), \nabla e) = -\kappa^*(e, |\mathbf{D}(\boldsymbol{v})|)\nabla e\,, \tag{1.9}$$

where we set

$$\kappa^*(e, |\mathbf{D}(\boldsymbol{v})|) := \hat{\kappa}(\theta(e), |\mathbf{D}(\boldsymbol{v})|)\theta'(e)\,. \tag{1.10}$$

In this paper, we will assume that the constitutively determined part of the Cauchy stress $\mathbf{S}$ depends on the internal energy (to be precise on the temperature that is however function of the energy) and on the symmetric part of the velocity gradient

$$\mathbf{S} = \mathbf{S}^*(e, \mathbf{D}(\boldsymbol{v}))\,. \tag{1.11}$$

Next, we shortly discuss the second law of thermodynamics (the entropy inequality) and its equivalence to the energy inequality, or/and to the notion of the suitable weak solution to generalized Navier-Stokes-Fourier system (see [5], [6] for details). First, as it is usual in many textbooks of the continuum mechanics, we take the scalar product of (1.2) with $\boldsymbol{v}$ and subtract the result from (1.3) to obtain the balance of internal energy

$$e_t + \operatorname{div}(e\boldsymbol{v}) + \operatorname{div}\boldsymbol{q} = \mathbf{S}\cdot\mathbf{D}(\boldsymbol{v})\,. \tag{1.12}$$

Such procedure is however not rigorous as we only deal with weak solutions in general[2] and one is forced to replace the equality (1.12) by the inequality

$$e_t + \operatorname{div}(e\boldsymbol{v}) + \operatorname{div}\boldsymbol{q} \geq \mathbf{S}\cdot\mathbf{D}(\boldsymbol{v})\,. \tag{1.13}$$

To justify possible inequality sign in (1.13), we show that (1.13) is equivalent to the entropy inequality provided that the temperature is strictly positive and therefore any (weak) solution to (1.1)–(1.3) and (1.13) can be considered as the physical one. First, we recall that the entropy S, the internal energy e and the temperature θ are related through

$$\frac{1}{\theta} := \frac{\partial S}{\partial e}\,.$$

[1]The assumption that e is an invertible function of θ is valid for most fluids, since the energy e is usually strictly increasing function of the temperature.

[2]It would be rigorous if we would know that $\boldsymbol{v}$ is a possible test function in a weak formulation of (1.2).

Hence, multiplying (1.13) by positive quantity $\frac{1}{\theta}$, and using (1.1) leads to

$$S_{,t} + \operatorname{div}(S\boldsymbol{v}) + \operatorname{div}\frac{\boldsymbol{q}}{\theta} \geq \frac{1}{\theta}\left(\mathbf{S}\cdot\mathbf{D}(\boldsymbol{v}) - \frac{\boldsymbol{q}\cdot\nabla\theta}{\theta}\right), \tag{1.14}$$

which is nothing else than the entropy inequality. Although this step was again formal, it can be deduced rigorously since $\frac{1}{\theta}$ will be always a possible test function in (1.13) for the class of fluids we are interested in.

On the other hand subtracting (1.13) from (1.3) we get the kinetic energy inequality

$$|\boldsymbol{v}|_{,t} + \operatorname{div}(\boldsymbol{v}(|\boldsymbol{v}|^2 + 2p)) - 2\operatorname{div}(\mathbf{S}\boldsymbol{v}) + 2\mathbf{S}\cdot\mathbf{D}(\boldsymbol{v}) \leq 2\boldsymbol{f}\cdot\boldsymbol{v}\,. \tag{1.15}$$

Note that weak solutions to (1.1)–(1.2) satisfying also (1.15) are called suitable weak solutions introduced in [8], where the inequality (1.15) plays the important role for proving partial regularity results for three-dimensional Navier-Stokes equations. Thus, having equivalence of (1.15) and (1.13), it is natural to call a weak solution the *suitable* one if it solves (1.1)–(1.3) in a weak sense and if it in addition satisfies in a weak sense the energy inequality (1.13) (that is equivalent to the entropy inequality (1.14)). Therefore our goal in this paper is to find $(\boldsymbol{v}, p, e, \mathbf{S}, \boldsymbol{q}, E)$ that solve in a weak sense (1.1)–(1.6) and (1.13), and solve (1.7), (1.9) and (1.11) point-wisely in Q. Such $(\boldsymbol{v}, p, e, \mathbf{S}, \boldsymbol{q}, E)$ we call the suitable weak solution.

The structure of the paper is following. In Section 2 we introduce the structural assumptions on $\mathbf{S}^*$ and κ^* that are considered in the paper. Then we explain used notation and formulate the main existence theorem of the paper. Next, in Section 3 we recall several important tools and auxiliary results used in the proof of the main theorem. Then in Section 4 we formally derive *a priori* estimates on $(\boldsymbol{v}, p, e, \boldsymbol{q}, \mathbf{S})$ and show that the weak formulation introduced in the main theorem is really meaningful. Finally, in Section 5 we prove the main theorem. We mainly follow the paper [5], where three-dimensional case is treated but with more restrictive assumptions on $\mathbf{S}^*$ and $\boldsymbol{q}^*$.

2. Assumptions on $\mathbf{S}^*$ and κ^*

In the paper we assume that $\mathbf{S}^* : \mathbb{R}_+ \times \mathbb{R}^{2\times 2}_{\mathrm{sym}} \to \mathbb{R}^{2\times 2}_{\mathrm{sym}}$ and $\kappa^* : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ are continuous mappings. Moreover, we assume that there are $r \in (1,\infty)$ and $\beta \in \mathbb{R}$ such that for all $e \in \mathbb{R}_+$ and all $\mathbf{B}, \mathbf{D} \in \mathbb{R}^{2\times 2}_{\mathrm{sym}}$, $\mathbf{B} \neq \mathbf{D}$

$$C_1(|\mathbf{D}|^r - 1) \leq \mathbf{S}^*(e,\mathbf{D})\cdot\mathbf{D}\,, \qquad |\mathbf{S}^*(e,\mathbf{D})| \leq C_2(|\mathbf{D}|^{r-1} + 1)\,, \tag{2.1}$$

$$[\mathbf{S}^*(e,\mathbf{D}) - \mathbf{S}^*(e,\mathbf{B})]\cdot(\mathbf{D} - \mathbf{B}) > 0\,, \tag{2.2}$$

$$C_3 e^\beta \leq \kappa^*(e, |\mathbf{D}|) \leq C_4 e^\beta\,, \tag{2.3}$$

i.e., we assume that $\mathbf{S}^*$ is r-coercive, has $(r-1)$-growth and is strictly monotone w.r.t. $\mathbf{D}$. Since we will always have that $e \geq e_{\min} > 0$, where $e_{\min}$ is the minimum (infimum) of the initial internal energy e_0, we can relax the assumptions (2.1)–(2.3) and consider them to be valid only for $e \geq e_{\min} > 0$. We refer to [5], [6], [11] for physical examples of $\mathbf{S}^*$ and $\boldsymbol{q}^*$ satisfying (2.1)–(2.3).

Next, we introduce a notation of function spaces that are suitable for describing the problem (1.1)–(1.3) and (1.13). We use the standard notation for Sobolev, Lebesgue and Bochner spaces. For the vector-valued functions having zero normal part on the boundary we employ the following notation[3]

$$W^{1,q}_{\boldsymbol{n}} := \{\boldsymbol{v} \in W^{1,q}(\Omega)^2;\ \boldsymbol{v}\cdot\boldsymbol{n} = 0 \text{ on } \partial\Omega\}\,,\ W^{-1,q'}_{\boldsymbol{n}} := (W^{1,q}_{\boldsymbol{n}})^*\,,$$

$$W^{1,q}_{\boldsymbol{n},\mathrm{div}} := \{\boldsymbol{v} \in W^{1,q}_{\boldsymbol{n}};\ \mathrm{div}\,\boldsymbol{v} = 0\}\,,\ W^{-1,q'}_{\boldsymbol{n},\mathrm{div}} := (W^{1,q}_{\boldsymbol{n},\mathrm{div}})^*\,,\ L^q_{\boldsymbol{n},\mathrm{div}} := \overline{\left\{\boldsymbol{v} \in W^{1,q}_{\boldsymbol{n},\mathrm{div}}\right\}}^{\|\cdot\|_q}.$$

Note that above spaces are separable for $q \in [1,\infty)$ and reflexive for $q \in (1,\infty)$. In order to simplify the presentation we also denote for any $r, q \in [1,+\infty]$

$$\begin{aligned} X^{r,q} &:= \{\boldsymbol{u} \in L^r(0,T;W^{1,r}_{\boldsymbol{n}}) \cap L^q(0,T;L^q(\Omega)^2);\ \boldsymbol{u} \in L^2(0,T;L^2(\partial\Omega)^2)\}\,,\\ X^{r,q}_{\mathrm{div}} &:= \{\boldsymbol{u} \in X^{r,q};\ \mathrm{div}\,\boldsymbol{u} = 0\}\,,\end{aligned}$$

and also recall that above spaces are separable for $r, q \in [1,\infty)$ and reflexive for $r, q \in (1,\infty)$. The Lebesgue spaces of functions having zero mean value are denoted by L^q_0, i.e., we define $L^q_0 := \{p \in L^q(\Omega);\ \int_\Omega p \ \mathrm{d}x = 0\}$. Finally, for f and g being scalar-, vector- or tensor-valued functions we define $(f,g)_O := \int_O fg$ whenever $fg \in L^1(O)$. Since the most frequent setting in the paper is $O = \Omega$, we shorten the notation for Ω as $(f,g) := (f,g)_\Omega$. Moreover, for any $g \in X$ and $f \in X^*$ we set $\langle f,g\rangle := \langle f,g\rangle_{X^*,X}$ whenever it is clear from the context which duality pairing is taken into account.

The constant C in the whole paper depends only on the data, i.e., on $\boldsymbol{v}_0$, e_0, Ω, T, r, β. If there is any dependence on other quantities it is clearly denoted in the text.

Having above definitions of function spaces, we can formulate the main theorem of the paper.

Theorem 2.1. *Let $\Omega \in \mathcal{C}^{1,1}$ be a bounded domain in $\mathbb{R}^2$, $r > \frac{3}{2}$ and $\beta > \max(-1, \frac{r}{2(r-1)} - 2)$ be arbitrary. Assume that the initial data satisfy*

$$\boldsymbol{v}_0 \in L^2_{\boldsymbol{n},\mathrm{div}} \quad \text{and} \quad e_0 \in L^1(\Omega);\ e_0(x) \geq e_{min} > 0 \text{ in } \Omega, \tag{2.4}$$

$\mathbf{S}^*$ and $\boldsymbol{q}^*$ satisfy (2.1)–(2.3) with r and β and let $m_E > 1$ be defined as

$$m_E := \begin{cases} \dfrac{2(\beta+2)}{3} & \text{for } \beta \geq r-2 \text{ and } \beta \in (-\frac{1}{2}, 1),\\[2mm] \dfrac{2r(\beta+1)}{2r+\beta-1} & \text{for } \beta \geq r-2 \text{ and } \beta \geq 1,\\[2mm] \dfrac{2(\beta+2)(r-1)}{2r-1+\beta} & \text{for } \beta < r-2 \text{ and } r > 2,\\[2mm] \dfrac{2r(\beta+2)}{2\beta+r+4} & \text{for } \beta < r-2 \text{ and } \frac{3}{2} < r \leq 2. \end{cases} \tag{2.5}$$

[3]For simplicity we write $\boldsymbol{v}$ instead of $\mathrm{tr}\,\boldsymbol{v}$ whenever it is clear from the context that we are restricted on $\partial\Omega$.

Then there exists $(\boldsymbol{v}, p, e, \mathbf{S}, \boldsymbol{q}, E)$ *such that*

$$\boldsymbol{v} \in \mathcal{C}_{weak}(0,T; L^2_{\boldsymbol{n},\mathrm{div}}) \cap X^{r,2r}_{\mathrm{div}}, \tag{2.6}$$

$$\boldsymbol{v}_{,t} \in \left(X^{r,\frac{2r}{2r-3}}_{\mathrm{div}}\right)^* \cap L^{\min(r,r')}(0,T; W^{-1,\min(r,r')}_{\boldsymbol{n}}), \tag{2.7}$$

$$\mathbf{S} \in L^{r'}(0,T; L^{r'}(\Omega)^{2\times 2}), \tag{2.8}$$

$$p \in L^{\min(r,r')}(0,T; L^{\min(r,r')}_0), \tag{2.9}$$

$$e \in L^\infty(0,T; L^1(\Omega)), \qquad e \geq e_{min} \text{ in } Q, \tag{2.10}$$

$$e^{\frac{\beta+\lambda+1}{2}} \in L^2(0,T; W^{1,2}(\Omega)) \quad \text{for all } \lambda < 0, \tag{2.11}$$

$$E \in L^q(0,T; L^q(\Omega)) \cap W^{1,s'}(0,T; (W^{1,s})^*) \tag{2.12}$$

for all $q < \min(2+\beta, r)$ *and all* $s' < \min(\frac{2r}{3}, \frac{2r}{2r-1}, m_E)$,

$$\boldsymbol{q} \in L^m(0,T; L^m(\Omega)^2) \quad \text{for all} \quad 1 \leq m < \frac{4+2\beta}{3+2\beta}, \tag{2.13}$$

such that they satisfy (1.7), (1.9) *and* (1.11) *a.e. in* Q*, and such that they satisfy* (1.2)–(1.3) *and* (1.13) *in the following sense*

$$\begin{aligned}\langle \boldsymbol{v}_{,t}, \boldsymbol{\varphi}\rangle - (\boldsymbol{v}\otimes\boldsymbol{v}, \nabla\boldsymbol{\varphi})_Q + \alpha(\boldsymbol{v}, \boldsymbol{\varphi})_\Gamma + (\mathbf{S}, \mathbf{D}(\boldsymbol{\varphi}))_Q &= (p, \operatorname{div}\boldsymbol{\varphi})_Q \\ \text{for all } \boldsymbol{\varphi} \in L^{\max(r,r')}(0,T; W^{1,\max(r,r')}_{\boldsymbol{n}}&)),\end{aligned} \tag{2.14}$$

$$\begin{aligned}\langle E_{,t}, \varphi\rangle - ((E+p)\boldsymbol{v}, \nabla\varphi)_Q + \alpha(|\boldsymbol{v}|^2, \varphi)_\Gamma + (\mathbf{S}\boldsymbol{v} - \boldsymbol{q}, \nabla\varphi)_Q &= 0 \\ \text{for all } \varphi \in L^\infty(0,T; W^{1,\infty}(\Omega)&)),\end{aligned} \tag{2.15}$$

$$\begin{aligned}-(e, \psi_{,t})_Q - (\boldsymbol{v}e, \nabla\psi)_Q - (\boldsymbol{q}, \nabla\psi)_Q &\geq (\mathbf{S}, \mathbf{D}(\boldsymbol{v})\psi)_Q \\ \text{for all nonnegative } \psi \in \mathcal{D}(0,T; W^{1,\infty}&(\Omega)),\end{aligned} \tag{2.16}$$

and the initial conditions (1.4) *is attained in the following sense*

$$\lim_{t\to 0+} \|\boldsymbol{v}(t) - \boldsymbol{v}_0\|_2^2 + \|e(t) - e_0\|_1 = 0. \tag{2.17}$$

In addition, we see that for $r \geq 2$ we can set in (2.14) $\boldsymbol{\varphi} := \boldsymbol{v}$ and therefore following the argumentation in Section 1 and the procedure described in [5] we can strengthen the statement of Theorem 2.1 in the following way.

Corollary 2.1. *Let all assumptions of Theorem 2.1 be satisfied. Moreover, assume that* $r \geq 2$*. Then there exist* $(\boldsymbol{v}, p, e, \mathbf{S}, \boldsymbol{q}, E)$ *satisfying* (1.7), (1.9), (1.11) *and* (2.6)–(2.17) *and in addition there hold*

$$e \in \mathcal{C}(0,T; L^1(\Omega)) \cap W^{1,1}(0,T; (W^{1,q})^*) \text{ for sufficiently large } q \gg 1 \tag{2.18}$$

and

$$\begin{aligned}\langle e_{,t}, \psi\rangle - (\boldsymbol{v}e, \nabla\psi)_Q - (\boldsymbol{q}, \nabla\psi)_Q &= (\mathbf{S}, \mathbf{D}(\boldsymbol{v})\psi)_Q \\ \text{for all } \psi \in L^\infty(0,T; W^{1,\infty}&(\Omega)).\end{aligned} \tag{2.19}$$

The Navier-Stokes or Navier-Stokes-Fourier like systems are ones of the most studied systems of PDEs coming from mathematical physics. These systems of equations can be split onto two cases. First one, called subcritical, takes place for $r \geq 2$ in two dimensions and $r \geq \frac{11}{5}$ in dimension three, and in this case the velocity can be used as a test function. The second one, called supercritical, when the velocity cannot be used as a test function. While the subcritical cases were studied by many authors, we refer to [10], [9], and the existence theory can be established with help of the Minty method, the supercritical cases were solved very recently. First such result was for Newtonian fluid in spatially periodic setting established in [11] and then extended for Navier's boundary conditions in [6]. The existence analysis for fully nonlinear model satisfying (2.1)–(2.3) was developed for $r \in (\frac{9}{5}, 2]$ and for three dimensions in [5], where in addition the dependence of $\mathbf{S}^*$ on the pressure p is allowed. On the other hand, also some uniform monotonicity is required there. This paper uses the method developed in [5] to extend the existence theory for sub- and supercritical cases in dimension two provided that (2.1)–(2.3) hold. The main novelty consists in introducing the optimal function spaces, that are natural for considered problem and follows from *a priori* estimates, and in establishing the existence of a weak solution whenever *a priori* estimates provides that (2.14)–(2.16) are meaningful.

We would also like to emphasize that our result is nontrivial. It is well known that for non-Newtonian fluids (without coupling with the energy) the regularity of the solution can be established for any $r > 1$ at least in spatially periodic setting (see [15]) and therefore one could think that the same method works also for the full Navier-Stokes-Fourier system. However, once the equations are coupled, i.e., once $\mathbf{S}^*$ depends on e, the resulting system falls into the so-called class of PDE's with critical growth on the right hand side, for which the regularity theory does not hold in general and therefore the method developed in [15] cannot be used and one is forced to use a different procedure.

3. Auxiliary tools

In this section we recall several auxiliary results and important tools that will be used in the proof of Theorem 2.1. First, we recall the theory for Laplace equation.

Lemma 3.1. *Let $\Omega \in \mathcal{C}^{1,1}$ be an open bounded domain and $1 < q < \infty$. There exists a continuous linear operator $\mathcal{N}^{-1} : L^q_0 \to W^{2,q}(\Omega)$ such that*

$$\Delta \mathcal{N}^{-1}(z) = z \text{ in } \Omega, \qquad \nabla \mathcal{N}^{-1}(z) \cdot \boldsymbol{n} = 0 \text{ on } \partial\Omega, \qquad \int_\Omega \mathcal{N}^{-1}(z)\, dx = 0. \tag{3.1}$$

Moreover, for any $\boldsymbol{v} \in W^{1,q}_{\boldsymbol{n}} \cap L^s(\Omega)^2$ we define $\boldsymbol{v}_{\mathrm{div}} := \boldsymbol{v} - \nabla\mathcal{N}^{-1}(\operatorname{div} \boldsymbol{v})$ and the following hold

$$\|\mathcal{N}^{-1}(\operatorname{div}\boldsymbol{v})\|_{2,q} \leq C\|\operatorname{div}\boldsymbol{v}\|_q\,, \qquad \|\boldsymbol{v}_{\mathrm{div}}\|_{1,q} \leq C\|\boldsymbol{v}\|_{1,q}\,, \tag{3.2}$$

$$\|\mathcal{N}^{-1}(\operatorname{div}\boldsymbol{v})\|_{1,s} \leq C\|\boldsymbol{v}\|_s\,, \qquad \|\boldsymbol{v}_{\mathrm{div}}\|_{s} \leq C\|\boldsymbol{v}\|_{s}\,. \tag{3.3}$$

Note that due to the definition of $\boldsymbol{v}_{\mathrm{div}}$ we have $\operatorname{div}\boldsymbol{v}_{\mathrm{div}} = \operatorname{div}\boldsymbol{v} - \Delta\mathcal{N}^{-1}(\operatorname{div}\boldsymbol{v}) = 0$ and $\boldsymbol{v}_{\mathrm{div}} \in W^{1,q}_{\boldsymbol{n},\mathrm{div}}$.

Proof. The proof can be found in [14, Proposition 2.5.2.3, page 131]. □

Next, since $\mathbf{S}^*$ depends only on the symmetric part of the velocity gradient we need to recall the Korn inequality.

Lemma 3.2 (Korn inequality)**.** *Let $\Omega \in \mathcal{C}^{0,1}$ be an open bounded domain and $q \in (1,\infty)$. Then for all $\boldsymbol{v} \in W^{1,q}(\Omega)^2 \cap L^2(\Omega)^2$ the following inequality holds*

$$\|\boldsymbol{v}\|_{1,q} \le C\left(\|\mathbf{D}(\boldsymbol{v})\|_q + \|\boldsymbol{v}\|_2\right). \tag{3.4}$$

Proof. We refer to [15] for detailed proof. □

Since we deal with Navier's boundary conditions, we need to control compactness of the trace operator in a proper space. Therefore, we recall the following lemma.

Lemma 3.3. *Let $\Omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain. For $1 < q_1, q_2 < \infty$ and $r > \frac{3}{2}$ we define the set*

$$\mathcal{S} := \left\{\boldsymbol{v}; \boldsymbol{v} \in L^\infty(0,T;L^2(\Omega)^2) \cap L^r(0,T;W^{1,r}_{\boldsymbol{n}}), \boldsymbol{v}_{,t} \in L^{q_1}(0,T;W^{-1,q_2}_{\boldsymbol{n},\mathrm{div}})\right\}.$$

Assume that $\{\boldsymbol{v}^i\}_{i=1}^\infty$ is bounded in $\mathcal{S}$. Then the sequence $\{\boldsymbol{v}_i|_{\partial\Omega}\}_{i=1}^\infty$ is precompact in $L^2(0,T;L^2(\partial\Omega)^2)$.

Proof. We refer to [5, Lemma 1.4] for detailed proof. □

Finally, we introduce two interpolation inequalities that are frequently used in the proof of the main theorem.

Lemma 3.4. *Let $\Omega \in \mathcal{C}^{0,1}$ be a an open bounded domain in $\mathbb{R}^2$. Assume that $1 < r < \infty$ and $\gamma > 0$ are fixed. Then for any $s \in [2, \frac{2r}{2-r}]$, if $r < 2$, for any $s \in [2,\infty)$, if $r \ge 2$, and for any q such that*

$$\frac{1}{q} + \frac{1}{s(r-1)} = \frac{1}{2(r-1)}$$

the following interpolation holds

$$\|f\|_{L^q(0,T;L^s(\Omega))} \le C\|f\|^\beta_{L^\infty(0,T;L^2(\Omega))}\|f\|^{1-\beta}_{L^r(0,T;W^{1,r}(\Omega))} \tag{3.5}$$

with

$$\beta := \frac{r}{s(r-1)} + \frac{r-2}{2(r-1)}.$$

Moreover, for any $1 < s, q < \infty$ such that

$$\frac{2}{q} + \frac{1}{s\gamma} = \frac{1}{\gamma}$$

the following holds

$$\|f\|_{L^q(0,T;L^s(\Omega))} \le C\|f\|^{\frac{1}{s}}_{L^\infty(0,T;L^1(\Omega))}\|f^\gamma\|^{\frac{s-1}{s\gamma}}_{L^2(0,T;W^{1,2}(\Omega))}, \tag{3.6}$$

provided that $f \ge 0$.

Proof. The proof easily follows from the following interpolation inequalities (see [1])

$$\|f\|_s \le C\|f\|_2^\beta \|f\|_{1,r}^{1-\beta},$$
$$\|f\|_s \le C\|f\|_1^{\frac{1}{s}} \|f^\gamma\|_{1,2}^{\frac{s-1}{s\gamma}},$$

and from the Hölder inequality. □

As a direct consequence of this lemma we obtain the following.

Corollary 3.1. *Let all assumptions of Lemma 3.4 be satisfied. Then*

$$\int_0^T \|\boldsymbol{v}\|_{2r}^{2r}\, dt \le C\|\boldsymbol{v}\|_{L^\infty(0,T;L^2(\Omega)^2)}^r \int_0^T \|\boldsymbol{v}\|_{1,r}^r\, dt. \tag{3.7}$$

$$\int_0^T \|f\|_{2\gamma+1}^{2\gamma+1}\, dt \le C\|f\|_{L^\infty(0,T;L^1(\Omega))} \int_0^T \|f^\gamma\|_{1,2}^2\, dt. \tag{3.8}$$

4. A priori estimates

This section is devoted to obtaining formal *a priori* estimates for solutions to (2.14)–(2.16). Thus, assuming that the velocity field is smooth enough, we can set $\boldsymbol{\varphi} := \boldsymbol{v}$ in (2.14) to obtain (note that the convective term vanishes as well as the term with the pressure, we refer to [5], [15] for details)

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{v}\|_2^2 + \alpha\|\boldsymbol{v}\|_{L^2(\partial\Omega)}^2 + (\mathbf{S}, \mathbf{D}(\boldsymbol{v})) = 0. \tag{4.1}$$

Consequently, using (1.11), (2.1) and (3.4), we find that

$$\sup_{t\in(0,T)} \|\boldsymbol{v}\|_2^2 + \int_0^T \|\boldsymbol{v}\|_{1,r}^r + \|\boldsymbol{v}\|_{L^2(\partial\Omega)}^2\, \mathrm{d}t \le C\|\boldsymbol{v}_0\|_2^2, \tag{4.2}$$

that gives the space introduced for the velocity field in Theorem 2.1, see (2.6). Note that (2.8) is then a direct consequence of (2.6) and the assumption (2.1). Next, we derive corresponding estimates also for the internal energy e. Thus, setting $\varphi \equiv 1$ in (2.15), we get (using the definition of E (1.7) and assuming that $e \ge e_{\min}$) that

$$\sup_{t\in(0,T)} \|e\|_1 \le C(\|\boldsymbol{v}_0\|_2^2 + \|e_0\|_1), \tag{4.3}$$

that formally gives (2.10). Finally, setting $\psi := e^\lambda$ in (2.16) with arbitrary $-1 < \lambda < 0$, we get after using the fact that $\operatorname{div} \boldsymbol{v} = 0$ and nonnegativity of the right hand side of (2.16) that

$$\int_0^T \lambda(e^{\lambda-1}\boldsymbol{q}, \nabla e)\, \mathrm{d}t \le C(e_{\min}, \|e\|_{L^\infty(L^1)}, \|\boldsymbol{v}_0\|_2^2). \tag{4.4}$$

Therefore, using (1.9) and (2.3), we find that

$$\begin{aligned} \int_0^T \|\nabla e^{\frac{\lambda+1+\beta}{2}}\|_2^2\, \mathrm{d}t &\le \frac{(\lambda+1+\beta)^2}{4} \int_0^T (e^{\lambda-1+\beta}\nabla e, \nabla e)\, \mathrm{d}t \\ &\le C(\lambda^{-1})\lambda \int_0^T (e^{\lambda-1}\boldsymbol{q}, \nabla e)\, \mathrm{d}t \le C(\lambda^{-1}). \end{aligned} \tag{4.5}$$

Thus, we obtained the corresponding spaces for e as those introduced in Theorem 2.1, see (2.10)–(2.11).

To obtain the desired estimate on the pressure, we first notice that using (2.6) and (4.2) we get

$$\int_0^T \|\boldsymbol{v}\|_{2r}^{2r}\, \mathrm{d}t \le C. \tag{4.6}$$

Thus, setting $\boldsymbol{\varphi} := \nabla\mathcal{N}^{-1}(|p|^{\min(r,r')-2}p - \fint_\Omega |p|^{\min(r,r')-2}p)$ in (2.14) (note that $\boldsymbol{\varphi}\cdot\boldsymbol{n} = 0$ on $\partial\Omega$), we get

$$\int_0^T \|p\|_{\min(r,r')}^{\min(r,r')}\,\mathrm{d}t = \int_0^T (\boldsymbol{v}_{,t},\boldsymbol{\varphi}) + \alpha(\boldsymbol{v},\boldsymbol{\varphi})_{\partial\Omega} + (\mathbf{S},\nabla\boldsymbol{\varphi}) - (\boldsymbol{v}\otimes\boldsymbol{v},\nabla\boldsymbol{\varphi})\,\mathrm{d}t. \tag{4.7}$$

Since $\operatorname{div}\boldsymbol{v} = 0$, the first term vanishes. Next, using Lemma 3.1 we have that $\|\boldsymbol{\varphi}\|_{1,\max(r,r')}^{\max(r,r')} \le C\|p\|_{\min(r,r')}^{\min(r,r')}$ and using the Young inequality and the fact that the trace operator is continuous from $W^{1,2}(\Omega)$ to $L^2(\partial\Omega)$ we get that

$$\int_0^T \|p\|_{\min(r,r')}^{\min(r,r')}\,\mathrm{d}t \le C\left(1 + \int_0^T \|\boldsymbol{v}\|_{L^2(\partial\Omega)}^2 + \|\mathbf{S}\|_{r'}^{r'} + \|\boldsymbol{v}\|_{2r}^{2r}\,\mathrm{d}t\right) \le C, \tag{4.8}$$

where the last inequality follows from (2.6) and (2.8).

In the remaining part of this section we show that the relations (2.14)-(2.16) are well defined for $(\boldsymbol{v}, e, p)$ belonging to the spaces introduced in (2.6), (2.10) and (2.11). The fact that (2.14) is meaningful follows from (2.6), (2.7), (4.6) and (4.8). The corresponding space for time derivative introduced in (2.7) can be obtained by standard interpolations (see [7], [6], [5] for details). Next, using (3.8), (4.3) and (4.5), it is easy to observe for all $\lambda < 0$ that

$$\int_0^T \|e\|_{\lambda+2+\beta}^{\lambda+2+\beta}\,\mathrm{d}t \le C(\lambda^{-1}). \tag{4.9}$$

Next, using (1.9) and (2.3) we get that after using the Hölder inequality, (4.5) and (4.9) that

$$\begin{aligned}\int_0^T \|\boldsymbol{q}\|_{1+\frac{2\lambda+1}{3+2\beta}}^{1+\frac{2\lambda+1}{3+2\beta}}\,\mathrm{d}t &\le C\int_Q |\nabla e^{\frac{\beta+1+\lambda}{2}}|^{1+\frac{2\lambda+1}{3+2\beta}}|e|^{\frac{1-\lambda+\beta}{2}(1+\frac{2\lambda+1}{3+2\beta})}\,\mathrm{d}x\,\mathrm{d}t\\ &\le C(\lambda^{-1})\end{aligned} \tag{4.10}$$

and (2.13) follows. To obtain the first part of (2.12), it is enough to combine (4.6) and (4.9). To obtain also the second part of (2.12), we first deduce by using the Hölder inequality, (4.6), (4.8) and (2.8) that

$$\int_Q \left(|\boldsymbol{v}|^3 + |p||\boldsymbol{v}| + |\mathbf{S}||\boldsymbol{v}|\right)^{\min(\frac{2r}{3},\frac{2r}{2r-1})}\,dx\,\mathrm{d}t \le C. \tag{4.11}$$

Finally, we need to find $m_E > 1$ such that $\int_Q |\boldsymbol{v}e|^q\,\mathrm{d}x\,\mathrm{d}t \le C$ for all $q < m_E$. Once having such m_E we can use (4.11) to obtain the second part of (2.12).

First natural choice how to find such m_E is to use homogeneous (homogeneous means the same integrability w.r.t. x and t) estimates (4.6) and (4.9). Then m_E is given by the relation

$$\frac{1}{m_E} = \frac{1}{2r} + \frac{1}{\beta+2}.$$

Consequently, to guarantee that $m_E > 1$ we need to restrict ourselves to the case

$$\beta > -2\frac{r-1}{2r-1}.$$

However, in the rest of this subsection, we show that the use of homogeneous estimates is not optimal and one can get the better relation for m_E that is introduced in (2.5) and consequently not so restrictive condition for β.

Thus, to find such optimal m_E, we use (2.6), (2.10)–(2.11) and Lemma 3.4. For simplicity, since λ in (2.11) can be arbitrarily closed to zero, we do all estimates with $\lambda = 0$ and finally replace possible equality by the strict inequality as it is done in (2.12). Hence, using the Hölder inequality, we get that

$$\|e\boldsymbol{v}\|_{L^{m_E}(Q)} \le \|\boldsymbol{v}\|_{L^{a_1}(0,T;L^{b_1}(\Omega)^2)}\|e\|_{L^{a_2}(0,T;L^{b_2}(\Omega))} \le C,$$

provided that

$$\frac{1}{m_E} = \frac{1}{a_1} + \frac{1}{a_2} = \frac{1}{b_1} + \frac{1}{b_2}\,, \tag{4.12}$$

$$\frac{1}{a_1} + \frac{1}{b_1(r-1)} = \frac{1}{2(r-1)}, \qquad \frac{1}{a_2} + \frac{1}{b_2(\beta+1)} = \frac{1}{\beta+1} \tag{4.13}$$

and that $1 < a_1, a_2, b_1, b_2 < \infty$. These identities are consequences of (3.5) and (3.6) and the properties (2.6), (2.10)–(2.11). For the last constraint we need to guarantee that

$$a_2 \in (\max(1, \beta+1), \infty)\,,\; a_1 \in (r^{\#}, \infty),\; b_1 \in [2, r^*),\; b_2 \in (1, \infty), \tag{4.14}$$

where, $r^* = \frac{2r}{r-2}$ and $r^{\#} = r$ if $r < 2$ and $r^* = \infty$ and $r^{\#} = 2(r-1)$ for $r \ge 2$. Thus, solving the system of algebraic equations (4.12)–(4.13) we find that

$$\begin{aligned} &\frac{1}{a_1} = \frac{3}{2r} - \frac{\beta+2}{a_2 r}, \qquad \frac{1}{b_1} = \frac{1}{2} + \frac{3(r-1)}{2r} + \frac{(r-1)(\beta+1)}{a_2 r}, \\ &\frac{1}{b_2} = 1 - \frac{\beta+1}{a_2} \end{aligned} \tag{4.15}$$

and therefore for m_E we get

$$\frac{1}{m_E} = \frac{3}{2r} + \frac{r-\beta-2}{a_2 r}. \tag{4.16}$$

Before we choose a_2 such that m_E is maximal, let us recall the possible ranges for a_1, b_1, a_2, b_2 that follows from (4.15). First using the restriction for a_1 and (4.15) we find for $r < 2$ that

$$\frac{2(\beta+2)}{3} \le a_2 \le 2(\beta+2),$$

that implies

$$\max\left(1, 1+\beta, \frac{2(\beta+2)}{3}\right) \le a_2 \le 2(\beta+2)\,.$$

Similarly for $r \ge 2$ we get the restriction

$$\max\left(1, 1+\beta, \frac{2(\beta+2)}{3}\right) \le a_2 < \frac{2(r-1)(\beta+2)}{2r-3},$$

Having these ranges for a_2, we now choose it such that m_E is the largest possible. Thus, going back to (4.16), we see that for $r-2>\beta$ we need to choose a_2 maximal. So for $r<2$ we set $a_2 := 2(\beta+2)$ and therefore (4.16) implies that

$$\frac{1}{m_E}=\frac{1}{r}+\frac{1}{2(\beta+2)} \Leftrightarrow m_E=\frac{2r(\beta+2)}{2\beta+4+r} \tag{4.17}$$

and we see that to get that $m_E>1$ we need to restrict β as

$$\beta>\frac{4-3r}{2(r-1)}. \tag{4.18}$$

Similarly for $r\geq 2$, setting the maximal $a_2=\frac{2(r-1)(\beta+2)}{2r-3}$ we get

$$m_E=\frac{2(r-1)(\beta+2)}{2r+\beta-1} \tag{4.19}$$

that is always strictly greater than 1 since $\beta>-1$.

For $\beta\geq r-2$ we choose a_2 minimal, i.e., we choose $a_2 := \max(\beta+1, \frac{2(\beta+2)}{3})$. This leads to

$$\begin{aligned} -\frac{1}{2}\leq \beta\leq 1 \qquad & \Longrightarrow m_E=\frac{2(\beta+2)}{3},\\ 1\leq\beta \qquad & \Longrightarrow m_E=\frac{2r(\beta+1)}{2r+\beta-1}. \end{aligned}$$

Note that in all cases we obtain $m_E>1$.

5. Proof of the main theorem

In order to prove the main theorem, we need several approximations in what follows. First, since we do not know that $e\geq e_{\min}$ *a priori*, we truncate $\mathbf{S}^*$ and $\boldsymbol{q}^*$ as

$$\tilde{\mathbf{S}}^*(e,\mathbf{D}) := \mathbf{S}^*(\max(e,e_{\min}),\mathbf{D}), \qquad \tilde{\boldsymbol{q}}^*(e,|\mathbf{D}|^2,\nabla e) := \boldsymbol{q}^*(\max(e,e_{\min}),|\mathbf{D}|^2,\nabla e).$$

Note that once having the minimum principle for e, the above truncation is not needed.

Further, to be able to use monotone operator theory and to converge from Galerkin approximative scheme to "continuous" problem we need to guarantee that $\boldsymbol{v}$ is a possible test function in (2.14). To have so, we mollify the convective term[4]. Thus, let r be a standard regularization kernel. For $\eta>0$ we define $r_\eta(x) := r(x/\eta)/\eta^2$. Then we find some $\omega_\eta\in\mathcal{D}(\Omega)$ such that $\operatorname{dist}(\operatorname{supp}\omega_\eta,\partial\Omega)\geq\eta$ and $\omega_\eta(x)=1$ for all $x\in\Omega$, $\operatorname{dist}(x,\partial\Omega)\geq 2\eta$. Finally for arbitrary $\boldsymbol{v}\in W^{1,r}_{\boldsymbol{n}}$ we define $\boldsymbol{v}_\eta := ((\boldsymbol{v}\omega_\eta)*r_\eta)_{\mathrm{div}}$. Note that a direct consequence of this definition is that $\operatorname{div}\boldsymbol{v}_\eta=0$ in Ω, $\boldsymbol{v}_\eta\cdot\boldsymbol{n}=0$ on $\partial\Omega$. Moreover, using Lemma 3.1 it is not difficult to observe for all $\boldsymbol{v}\in L^r(0,T;W^{1,r}_{\boldsymbol{n}})$

$$\boldsymbol{v}_\eta \overset{\eta\to 0}{\to} \boldsymbol{v} \text{ strongly in } L^q(Q).$$

[4]This mollification is not needed in case $r\geq 2$ since in this case one can use the velocity field as a test function in (2.14).

Finally, we replace the balance of global energy (1.3) by the balance of internal energy (1.12) for which the standard theory for parabolic equation can be easily used. Thus, our final approximative system takes the following form:

$$\operatorname{div} \boldsymbol{v} = 0\,, \tag{5.1}$$

$$\boldsymbol{v}_{,t} + \operatorname{div}(\boldsymbol{v}_\eta \otimes \boldsymbol{v}) - \operatorname{div} \tilde{\mathbf{S}}^*(e, \mathbf{D}(\boldsymbol{v})) = -\nabla p\,, \tag{5.2}$$

$$e_{,t} + \operatorname{div}(\boldsymbol{v}_\eta e) + \operatorname{div} \tilde{\boldsymbol{q}}^*(e, \mathbf{D}(v), \nabla e) = \tilde{\mathbf{S}}^*(\theta, \mathbf{D}(\boldsymbol{v})) \cdot \mathbf{D}(\boldsymbol{v})\,, \tag{5.3}$$

with the boundary conditions (1.5)–(1.6) and the initial conditions (1.4) where $E_0 := \frac{1}{2}|\boldsymbol{v}_0| + e_0$. The existence of a solution to (5.1)–(5.3) is done by using the "double" Galerkin approximation scheme and the monotone operator theory. Once having a solution to approximative system (5.1)–(5.3), we introduce the equation for the global energy and let $\eta \to 0$. In order to be able to identify limits of nonlinear term we use Aubin-Lions lemma, Div-Curl lemma and compactness of the velocity gradient that is achieved by using the so-called L^∞ truncation method. For more details we refer interested reader to [5] where three-dimensional case is treated.

5.1. **Galerkin approximation.** Let $\{\boldsymbol{w}_j\}_{j=1}^\infty$ be a basis of $W^{1,r}_{\boldsymbol{n},\mathrm{div}}$ that is orthonormal in $L^2(\Omega)^2$ such that $\boldsymbol{w}_j \in W^{1,2r}_{\boldsymbol{n},\mathrm{div}}$ for all $j \in \mathbb{N}$. Assume that $\{w_j\}_{j=1}^\infty$ is a basis of $W^{1,2}(\Omega)$ again orthonormal in $L^2(\Omega)$. Then the Galerkin approximation to (5.1)–(5.3) has the form:
For $k, \ell \in \mathbb{N}$ we look for the functions

$$\boldsymbol{v}^{k,\ell} := \sum_{i=1}^{k} c_i^{k,\ell}(t)\boldsymbol{w}_i \quad \text{and} \quad e^{k,\ell} := \sum_{i=1}^{\ell} d_i^{k,\ell}(t) w_i$$

that satisfy the following system of ordinary differential equations

$$\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{v}^{k,\ell}, \boldsymbol{w}_j) - (\boldsymbol{v}_\eta^{k,\ell} \otimes \boldsymbol{v}^{k,\ell}, \nabla \boldsymbol{w}_j) + (\mathbf{S}^{k,\ell}, \nabla \boldsymbol{w}_j) + \alpha(\boldsymbol{v}^{k,\ell}, \boldsymbol{w}_j)_{\partial\Omega} = 0 \tag{5.4}$$

for $j = 1, \dots, k$ and

$$\frac{\mathrm{d}}{\mathrm{d}t}(e^{k,\ell}, w_j) - (\boldsymbol{v}_\eta^{k,\ell} e^{k,\ell}, \nabla w_j) - (\boldsymbol{q}^{k,\ell}, \nabla w_j) = (\mathbf{S}^{k,\ell} \cdot \mathbf{D}^{k,\ell}, w_j) \tag{5.5}$$

for $j = 1, \dots, \ell$. Here, we set for simplicity

$$\mathbf{D}^{k,\ell} := \mathbf{D}(\boldsymbol{v}^{k,\ell}), \quad \mathbf{S}^{k,\ell} := \tilde{\mathbf{S}}^*(e^{k,\ell}, \mathbf{D}^{k,\ell}), \quad \boldsymbol{q}^{k,\ell} := \tilde{\boldsymbol{q}}^*(e^{k,\ell}, \mathbf{D}^{k,\ell}, \nabla e^{k,\ell}).$$

The system (5.4)–(5.5) is completed by the following initial conditions

$$\boldsymbol{v}^{k,\ell}(0) = \boldsymbol{v}_0^{k,\ell} := \sum_{j=1}^{k} c_{0j}^{k,\ell} \boldsymbol{w}_j = P^k \boldsymbol{v}_0,$$

$$e^{k,\ell}(0) = e_0^{k,\ell} := \sum_{j=1}^{\ell} d_{0j}^{k,\ell} \boldsymbol{w}_j = P^\ell(r_{1/k} * e_0),$$

where P^k and P^ℓ denote the projections of spaces $L^2_{\boldsymbol{n},\mathrm{div}}$, $L^2(\Omega)$, respectively onto the $\operatorname{span}\{\boldsymbol{w}_1, \dots, \boldsymbol{w}_k\}$, $\operatorname{span}\{w_1, \dots, w_\ell\}$, respectively. Note that here we also mollify the initial condition e_0 since we want to use the standard L^2 theory for parabolic equations for which the smooth initial data are needed, which is not our case since we assume only that $e_0 \in L^1(\Omega)$.

The assumptions on $\mathbf{S}^*$ and $\boldsymbol{q}^*$ and the Caratheodory theory for ODE enable us to establish the existence of a solution $c_i^{k,\ell}(t)$ $(i = 1, \dots, k)$ and $d_j^{k,\ell}(t)$ $(j = 1, \dots, \ell)$ to (5.4)–(5.5) defined on $[0, T_0)$ for some $T_0 > 0$ and by using the estimates established in the next subsection we can extend the solution on the whole time interval $(0, T)$.such that

5.1.1. *Estimates uniform w.r.t. ℓ.* First, we derive the uniform estimates (that can however depend on k). Multiplying the j-th equation in (5.4) by $c_j^{k,\ell}$, taking the sum over $j = 1, \dots, k$ and integrating the result over $(0, t)$ for $t \in (0, T)$ (that is nothing else than testing by $\boldsymbol{v}^{k,\ell}$) we get

$$\|\boldsymbol{v}^{k,\ell}(t)\|_2^2 + 2\int_0^t (\mathbf{S}^{k,\ell}, \mathbf{D}^{k,\ell}) + \alpha\|\boldsymbol{v}^{k,\ell}\|^2_{L^2(\partial\Omega)^2}\,\mathrm{d}\tau = \|\boldsymbol{v}_0^{k,\ell}\|_2^2\,. \tag{5.6}$$

Here, we used the fact that $(\boldsymbol{v}_\eta^{k,\ell}\otimes\boldsymbol{v}^{k,\ell}, \nabla\boldsymbol{v}^{k,\ell}) = 0$ (see [5], [15] for details). Therefore, using (2.1) and the Korn inequality we observe that

$$\sup_{t\in(0,T)} \|\boldsymbol{v}^{k,\ell}(t)\|_2^2 + \int_0^T \|\boldsymbol{v}^{k,\ell}\|_{1,r}^r + \|\boldsymbol{v}^{k,\ell}\|^2_{L^2(\partial\Omega)^2}\;\mathrm{d}t \le C, \tag{5.7}$$

where the constant C does not depend on k, ℓ, η. Moreover, the same procedure as in Section 4 gives that

$$\int_0^T \|\boldsymbol{v}^{k,\ell}\|_{2r}^{2r}\,\mathrm{d}t \le C\,. \tag{5.8}$$

It is also an easy consequence of (5.7) and the orthogonality of $\boldsymbol{w}_j$ that

$$\|c_i^{k,\ell}\|_\infty \le C \qquad\qquad \text{for all } i = 1, \dots, k. \tag{5.9}$$

Moreover, using (5.4)–(5.9) we immediately get that

$$\|(c_i^{k,\ell})_{,t}\|_\infty \le C(k) \qquad\qquad \text{for all } i = 1, \dots, k. \tag{5.10}$$

Next, since $\boldsymbol{w}_j \in W^{1,2r}_{\boldsymbol{n},\mathrm{div}}$ we observe by using (2.1) and (5.9) that

$$\int_Q |\mathbf{S}^{k,\ell}|^2|\mathbf{D}^{k,\ell}|^2\,\mathrm{d}x\,\mathrm{d}t \le C(k). \tag{5.11}$$

Thus, multiplying the j-th equation in (5.5) by $d_j^{k,\ell}$, taking the sum over $j = 1, \dots, \ell$ and integrating the result over $(0, t)$ where $t \in (0, T)$ (here we again use the fact that convective term vanishes) we deduce that

$$\|e^{k,\ell}(t)\|_2^2 - 2\int_0^t (\boldsymbol{q}^{k,\ell}, \nabla e^{k,\ell})\,\mathrm{d}\tau = \|e^{k,\ell}(0)\|_2^2 + 2\int_0^t (\mathbf{S}^{k,\ell}, \mathbf{D}^{k,\ell}e^{k,\ell})\,\mathrm{d}\tau.$$

Therefore, using the Gronwall lemma, the definition of $e^{k,\ell}(0)$, the estimate (5.11) and the Hölder inequality we obtain

$$\|e^{k,\ell}\|^2_{L^\infty(0,T;L^2(\Omega))} - \int_0^T (\boldsymbol{q}^{k,\ell}, \nabla e^{k,\ell})\,\mathrm{d}t \le C(k).$$

Finally, to obtain an estimate also for $\nabla e^{k,\ell}$ we define $Q_{\min} := \{(t,x) \in Q; e^{k,\ell}(t,x) \le e_{\min}\}$ and we can estimate the second integral as

$$\begin{aligned} &-(\boldsymbol{q}^{k,\ell}, \nabla e^{k,\ell})_Q = -(\boldsymbol{q}^{k,\ell}, \nabla e^{k,\ell})_{Q\setminus Q_{\min}} - (\boldsymbol{q}^{k,\ell}, \nabla e^{k,\ell})_{Q_{\min}} \\ &\ge C_3 \int_{Q\setminus Q_{\min}} (e^{k,\ell})^\beta |\nabla e^{k,\ell}|^2 \,\mathrm{d}x\,\mathrm{d}t + C_3 \int_{Q_{\min}} e^\beta_{\min} |\nabla e^{k,\ell}|^2 \,\mathrm{d}x\,\mathrm{d}t\,. \end{aligned}$$

Thus, combining everything together, we get

$$\sup_{t\in[0,T]} \|e^{k,\ell}(t)\|_2^2 + \int_{Q\setminus Q_{\min}} (e^{k,\ell})^\beta |\nabla e^{k,\ell}|^2 \,\mathrm{d}x\,\mathrm{d}t + \int_{Q_{\min}} e^\beta_{\min} |\nabla e^{k,\ell}|^2 \,\mathrm{d}x\,\mathrm{d}t \le C(k)\,. \tag{5.12}$$

Next, we deduce further estimates on $e^{k,\ell}$ based on (5.12). First, we define

$$\overline{\kappa}(e) := \begin{cases} e^\beta & \text{for } e_{\min} \le e, \\ e^\beta_{\min} & \text{for } e < e_{\min}, \end{cases} \qquad \overline{K}(e) := \begin{cases} \frac{2}{\beta+2} e^{(\beta+2)/2} + \frac{\beta}{\beta+2} e^{(\beta+2)/2}_{\min} & \text{for } e_{\min} \le e, \\ e^{\beta/2}_{\min} e & \text{for } e < e_{\min}. \end{cases}$$

It is easy to check that $\overline{K}'(e) = \sqrt{\overline{\kappa}(e)}$ and that

$$\begin{aligned} \int_Q |\nabla \overline{K}(e^{k,\ell})|^2 \,\mathrm{d}x\,\mathrm{d}t &= e^\beta_{\min} \int_{Q_{\min}} |\nabla e^{k,\ell}|^2 \,\mathrm{d}x\,\mathrm{d}t + \int_{Q\setminus Q_{\min}} (e^{k,\ell})^\beta |\nabla e^{k,\ell}|^2 \,\mathrm{d}x\,\mathrm{d}t \\ &\overset{(5.12)}{\le} C(k). \end{aligned}$$

Consequently, using the first part of (5.12), we have (here we refer to [5] for details)

$$\int_0^T \|\overline{K}(e^{k,\ell})\|^2_{1,2} \le C(k) \tag{5.13}$$

which leads with help of the first part of the estimate (5.12) and the standard interpolation inequality to

$$\int_{Q_{\min}} (e^{k,\ell})^4 \,\mathrm{d}x\,\mathrm{d}t + \int_{Q\setminus Q_{\min}} (e^{k,\ell})^{\beta+4} \,\mathrm{d}x\,\mathrm{d}t \le C(k). \tag{5.14}$$

Consequently, we observe that $\overline{\kappa}(e^{k,\ell}) \in L^\infty(Q)$ for $\beta \le 0$ and that

$$\int_0^T \|\sqrt{\overline{\kappa}(e^{k,\ell})}\|^{\frac{2(\beta+4)}{\beta}}_{\frac{2(\beta+4)}{\beta}} \,\mathrm{d}t \le C(k), \tag{5.15}$$

for $\beta > 0$. Having this, we deduce an estimate on $\boldsymbol{q}^{k,\ell}$. Since

$$|\boldsymbol{q}^{k,\ell}| \le C|\nabla \overline{K}(e^{k,\ell})| \sqrt{\overline{\kappa}(e^{k,\ell})}\,, \tag{5.16}$$

we obtain for $\beta \le 0$ by using (5.15), (5.13) and the Hölder inequality that

$$\int_0^T \|\boldsymbol{q}^{k,\ell}\|_2^2 \,\mathrm{d}t \le C \int_Q |\nabla \overline{K}(e^{k,\ell})|^2 |\overline{\kappa}(e^{k,\ell})| \,\mathrm{d}x\,\mathrm{d}t \le C(k)$$

and similarly for $\beta > 0$ we have

$$\int_Q |\boldsymbol{q}^{k,\ell}|^{\frac{\beta+4}{\beta+2}} \,\mathrm{d}x\,\mathrm{d}t \le C \int_Q |\nabla \overline{K}(e^{k,\ell})|^{\frac{\beta+4}{\beta+2}} (\overline{\kappa}(e^{k,\ell}))^{\frac{\beta+4}{2(\beta+2)}} \,\mathrm{d}x\,\mathrm{d}t$$
$$\le C \left(\int_Q |\nabla \overline{K}(e^{k,\ell})|^2 + |\overline{\kappa}(e^{k,\ell})|^{\frac{\beta+4}{\beta}} \,\mathrm{d}x\,\mathrm{d}t \right) \le C(k)\,.$$

Thus, to summarize, we deduced that

$$\int_Q |\boldsymbol{q}^{k,\ell}|^{m_{\boldsymbol{q}}} \,\mathrm{d}t \le C(k) \qquad \text{where } m_{\boldsymbol{q}} = \min(2, \frac{\beta+4}{\beta+2})\,. \tag{5.17}$$

Similarly, for $\beta \ge 0$ we have from (5.12)

$$\int_Q |\nabla e^{k,\ell}|^2 \,\mathrm{d}x\,\mathrm{d}t \le C(k)$$

and for $-1 < \beta < 0$ we have

$$\begin{aligned} \int_Q |\nabla e^{k,\ell}|^{\frac{\beta+4}{2}} \,\mathrm{d}x\,\mathrm{d}t &\le C \int_Q |\nabla \overline{K}(e^{k,\ell})|^{\frac{\beta+4}{2}} |e^{k,\ell}|^{-\frac{\beta(\beta+4)}{4}} \,\mathrm{d}x\,\mathrm{d}t \\ &\le C \left(\int_Q |\nabla \overline{K}(e^{k,\ell})|^2 + |e^{k,\ell}|^{\beta+4} \,\mathrm{d}x\,\mathrm{d}t \right) \overset{(5.14)}{\le} C(k)\,. \end{aligned} \tag{5.18}$$

Thus, we obtained

$$\int_0^T \|e^{k,\ell}\|_{1,m_e}^{m_e} \,\mathrm{d}t \le C(k) \quad \text{where } m_e := \min(2, \frac{\beta+4}{2}). \tag{5.19}$$

Finally, to deduce also uniform (k-independent) estimate on $e^{k,\ell}_{,t}$, we use (5.12), (5.11) and (5.17) to obtain that

$$\|e^{k,\ell}_{,t}\|_{L^{m_{\boldsymbol{q}}}(0,T,W^{-1,m_{\boldsymbol{q}}}(\Omega))} \le C(k) \tag{5.20}$$

with $m_{\boldsymbol{q}}$ defined in (5.17).

5.1.2. *Limit* $\ell \to \infty$. Using (5.9), (5.10), (5.12), (5.17), (5.19) and (5.20) we can find a not relabeled subsequence such that

$$c^{k,\ell} \rightharpoonup^* c^k \qquad \text{weakly}^* \text{ in } W^{1,\infty}(0,T), \tag{5.21}$$
$$e^{k,\ell} \rightharpoonup e^k \qquad \text{weakly in } L^{m_e}(0,T;W^{1,m_e}(\Omega)), \tag{5.22}$$
$$e^{k,\ell}_{,t} \rightharpoonup e^k_{,t} \qquad \text{weakly in } L^{m_{\boldsymbol{q}}}(0,T;W^{-1,m_{\boldsymbol{q}}}(\Omega)), \tag{5.23}$$
$$\boldsymbol{q}^{k,\ell} \rightharpoonup \overline{\boldsymbol{q}^k} \qquad \text{weakly in } L^{m_{\boldsymbol{q}}}(0,T;L^{m_{\boldsymbol{q}}}(\Omega)). \tag{5.24}$$

Moreover, using Aubin-Lions lemma, Arsela-Ascoli theorem and (5.13) we get again for not relabeled subsequence that

$$e^{k,\ell} \to e^k \qquad \text{strongly in } L^q(0,T;L^q(\Omega)) \quad \text{where } q < \min(4,\beta+4), \tag{5.25}$$

$$c^{k,\ell} \to c^k \qquad \text{strongly in } \mathcal{C}(0,T), \tag{5.26}$$

$$\overline{K}(e^{k,\ell}) \rightharpoonup \overline{K}(e^k) \qquad \text{weakly in } L^2(0,T;W^{1,2}(\Omega)), \tag{5.27}$$

$$\boldsymbol{v}^{k,\ell} \to \boldsymbol{v}^k \qquad \text{strongly in } L^{2r}(0,T;W^{1,2r}_{\boldsymbol{n}})\,. \tag{5.28}$$

Having (5.21)–(5.26), it is easy to let $\ell \to \infty$ in (5.4)–(5.5) and to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{v}^k, \boldsymbol{w}_j) - (\boldsymbol{v}^k_\eta \otimes \boldsymbol{v}^k, \nabla \boldsymbol{w}_j) + (\mathbf{S}^k, \nabla \boldsymbol{w}_j) + \alpha(\boldsymbol{v}^k, \boldsymbol{w}_j)_{\partial\Omega} = 0 \tag{5.29}$$

for all $j = 1, \dots, k$ and

$$\langle e^k_{,t}, \varphi\rangle - (\boldsymbol{v}^k_\eta e^k, \nabla\varphi)_Q - (\overline{\boldsymbol{q}^k}, \nabla\varphi)_Q = (\mathbf{S}^k, \nabla \boldsymbol{v}^k \varphi)_Q \tag{5.30}$$

for all $\varphi \in L^{(m_{\boldsymbol{q}})'}(0,T;W^{1,(m_{\boldsymbol{q}})'}(\Omega))$. Here we denoted $\mathbf{S}^k := \mathbf{S}^*(\max(e_{\min}, e^k), \mathbf{D}(\boldsymbol{v}^k))$. The attainment of initial condition, i.e, that fact that $\boldsymbol{v}^k(0) = P^k(\boldsymbol{v}_0)$ and $e^k(0) = r_{1/k} * e_0$ is standard and we refer to [5]. We also postpone the identification of $\overline{\boldsymbol{q}^k} = \boldsymbol{q}^*(\max(e_{\min}, e), \mathbf{D}(\boldsymbol{v}^k), \nabla e^k)$ to the next subsection, where more difficult case is treated. By using the minimum principle argument, it is also easy to observe that since $e_0 \geq e_{\min}$ and therefore $r_{1/k} * e_0 \geq e_{\min}$ as well, we have that $e^k \geq e_{\min}$ a.e. in Q, we again refer to [5] for details.

5.1.3. *Estimates independent of k.* Similarly as before, multiplying the j-th equation in (5.29) by c^k_j, taking the sum over $j = 1, \dots, k$ and integrating the result over $(0,t)$ for $t \in (0,T)$ we get

$$\|\boldsymbol{v}^k(t)\|_2^2 + 2\int_0^t (\mathbf{S}^k, \mathbf{D}(\boldsymbol{v}^k)) + \alpha\|\boldsymbol{v}^k\|^2_{L^2(\partial\Omega)^2}\,\mathrm{d}\tau = \|\boldsymbol{v}^k_0\|_2^2\,. \tag{5.31}$$

Hence, using (2.1) and the Korn inequality we observe that

$$\sup_{t\in(0,T)} \|\boldsymbol{v}^k(t)\|_2^2 + \int_0^T \|\boldsymbol{v}^k\|^r_{1,r} + \|\mathbf{S}^k\|^{r'}_{r'} + \|\boldsymbol{v}^k\|^2_{L^2(\partial\Omega)^2}\ \mathrm{d}t \leq C \tag{5.32}$$

and

$$\int_0^T \|\boldsymbol{v}^k\|^{2r}_{2r}\,\mathrm{d}t \leq C\,. \tag{5.33}$$

Next, we follow the procedure described in Section 4. First, setting $\varphi \equiv 1$ in (5.30) we get (using that $e \geq e_{\min} > 0$)

$$\sup_{t\in(0,T)} \|e^k(t)\|_1 \leq C\,. \tag{5.34}$$

Next, taking $\varphi = (e^k)^\lambda$ with $-1 < \lambda < 0$ in (5.30) (note that φ is bounded since $e^k \geq e_{\min} > 0$ and it can be deduced that it is a possible test function, see [5] for detailed explanation) and using (5.32) we get similarly as in (4.5) that

$$\int_0^T \|(e^k)^{\frac{\beta+\lambda+1}{2}}\|_{1,2}^2 \,\mathrm{d}t \le C(\lambda^{-1}). \tag{5.35}$$

Similarly, following (4.9), we obtain a uniform bound

$$\int_0^T \|e^k\|_{\beta+2+\lambda}^{\beta+2+\lambda} + \|\boldsymbol{q}_k\|_{1+\frac{2\lambda+1}{3+2\beta}}^{1+\frac{2\lambda+1}{3+2\beta}} \,\mathrm{d}t \le C(\lambda^{-1}) \quad \text{for all} \quad -\frac{1}{2} < \lambda < 0\,. \tag{5.36}$$

It remains to deduce estimates on time derivatives. Thus, using (5.32) and the fact that we mollified the convective term, we can deduce that

$$\|\boldsymbol{v}^k_{,t}\|_{(X^{r,2r}_{\mathrm{div}})^*} \le C(\eta^{-1})\,. \tag{5.37}$$

Similarly, using (5.32) and (5.36) we can deduce that for sufficiently large m we have

$$\|e^k_{,t}\|_{L^1(0,T;W^{-1,m'}(\Omega))} \le C. \tag{5.38}$$

5.1.4. *Limit $k \to \infty$.* Using (5.32) and (5.37), we see that we can find a not relabeled subsequence such that

$$\boldsymbol{v}^k_{,t} \rightharpoonup \boldsymbol{v}_{,t} \qquad \text{weakly in } (X^{r,2r}_{\mathrm{div}})^*, \tag{5.39}$$

$$\boldsymbol{v}^k \rightharpoonup^* \boldsymbol{v} \qquad \text{weakly}^* \text{ in } L^\infty(0,T;L^2(\Omega)^2), \tag{5.40}$$

$$\boldsymbol{v}^k \rightharpoonup \boldsymbol{v} \qquad \text{weakly in } X^{r,2r}_{\mathrm{div}}, \tag{5.41}$$

$$\mathbf{S}^k \rightharpoonup \mathbf{S} \qquad \text{weakly in } L^{r'}(0,T;L^{r'}(\Omega)^{2x2}), \tag{5.42}$$

and therefore after using Aubin-Lions lemma and Lemma 3.3

$$\boldsymbol{v}^k \to \boldsymbol{v} \qquad \text{strongly in } L^q(0,T;L^q(\Omega)^2) \text{ for all } q \in [1,2r), \tag{5.43}$$

$$\boldsymbol{v}^k \to \boldsymbol{v} \qquad \text{strongly in } L^2(0,T;L^2(\partial\Omega)^2). \tag{5.44}$$

In the same way, using (5.35) and (5.36) we can find a subsequence that is again not relabeled such that

$$e^k \rightharpoonup e \qquad \text{weakly in } L^q(0,T;L^q(\Omega)) \text{ for all } q < 2+\beta, \tag{5.45}$$

$$\boldsymbol{q}^k \rightharpoonup \boldsymbol{q} \qquad \text{weakly in } L^q(0,T;L^q(\Omega)^2) \text{ for } q < \frac{4+2\beta}{3+2\beta}\,, \tag{5.46}$$

$$(e^k)^{\frac{\beta+\lambda+1}{2}} \rightharpoonup \overline{e^{\frac{\beta+\lambda+1}{2}}} \qquad \text{weakly in } L^2(0,T;W^{1,2}(\Omega)). \tag{5.47}$$

In order to show the strong convergence of e^k, inspired by [12], we use the Div-Curl Lemma[5]. Indeed, defining $\boldsymbol{w}^k := (e^k, e^k v_1^k + q_1^k, e^k v_2^k + q_2^k)$ and $\boldsymbol{u}^k := ((e^k)^m, 0, 0)$ for some $0 < m < \min(1, (\beta+2)/2)$, we see with help of (5.32) that $\mathrm{Div}_{t,x}\boldsymbol{w}^k$ is bounded in $L^1(Q)$ and with help of (5.35) we have that $\mathrm{Curl}_{t,x}\boldsymbol{u}^k$ is bounded in $L^2(Q)$. Therefore, applying Div-Curl Lemma we observe, after using the bound (5.36) and the strong convergence result (5.43), that

$$\boldsymbol{w}^k \cdot \boldsymbol{u}^k \rightharpoonup \boldsymbol{w} \cdot \boldsymbol{u},$$

[5]In fact if β is sufficiently large one could simply use the Aubin-Lions lemma.

with $\boldsymbol{w} := (e, ev_1 + \overline{q_1}, e^k v_2) + \overline{q_2}$ and $\boldsymbol{u} := (\overline{e^m}, 0, 0)$. Consequently, we get

$$(e^k)^{m+1} \rightharpoonup e\overline{e^m}.$$

Thus, using the convexity of the $(m+1)$-th power, we have that $e^{m+1} \le e\overline{e^m}$ and consequently, since e is positive a.e. in Q we obtain $e^m \le \overline{e^m}$. On the other hand, we have from the concavity (as $m \in (0,1)$) that $e^m \ge \overline{e^m}$ and therefore $e^m = \overline{e^m}$. Since the m-the power is strictly concave, this relation implies (for a subsequence) that

$$e^k \to e \qquad \text{a.e. in } Q. \tag{5.48}$$

Thus using (5.43), (5.45), the definition of m_E (2.5), (5.48) and the estimates done in Section 4 we observe that

$$e^k \to e \qquad \text{strongly in } L^q(0,T;L^q(\Omega)) \text{ for all } q < 2+\beta, \tag{5.49}$$

$$e^k \boldsymbol{v}^k \to e\boldsymbol{v} \qquad \text{strongly in } L^m(0,T;L^m(\Omega)^2) \text{ for all } m < m_E. \tag{5.50}$$

First, we identify the limit of (5.29). Having (5.39)–(5.44), it is easy to obtain (by using the fact that $\boldsymbol{w}_j$ is a basis of $W^{1,r}_{\boldsymbol{n}}$)

$$\begin{aligned} &\langle \boldsymbol{v}_{,t}, \boldsymbol{w}\rangle - (\boldsymbol{v}_\eta \otimes \boldsymbol{v}, \nabla \boldsymbol{w})_Q + (\mathbf{S}, \nabla \boldsymbol{w})_Q + \alpha(\boldsymbol{v}, \boldsymbol{w})_\Gamma = 0 \\ &\qquad\qquad\qquad\qquad \text{for all} \quad \boldsymbol{w} \in X^{r,2r}_{\mathrm{div}}. \end{aligned} \tag{5.51}$$

Moreover, it is standard to show that the initial condition is attained in the following sense (we refer to [15] for details)

$$\lim_{t\to 0_+} \|\boldsymbol{v}(t) - \boldsymbol{v}_0\|_2^2 = 0 \tag{5.52}$$

and since $\boldsymbol{v}$ is a possible test function in (5.51), one can also obtain (by using (5.40)) that

$$\boldsymbol{v} \in \mathcal{C}(0,T;L^2(\Omega)^2).$$

Hence, to finish convergence result for (5.29), it remains to show that

$$\mathbf{S} = \mathbf{S}^*(e, \mathbf{D}(\boldsymbol{v})) \quad \text{a.a. in} \quad Q. \tag{5.53}$$

To show this, we use the Minty method. First, we set $\boldsymbol{w} := \boldsymbol{v}\chi_{[0,t]}$ in (5.51) (note that this choice is possible since $\boldsymbol{v} \in X^{r,2r}_{\mathrm{div}}$) to observe

$$\|\boldsymbol{v}(t)\|_2^2 + 2\int_0^t (\mathbf{S}, \mathbf{D}(\boldsymbol{v})) + \alpha\|\boldsymbol{v}\|^2_{L^2(\partial\Omega)^2}\, \mathrm{d}\tau = \|\boldsymbol{v}_0\|_2^2, \tag{5.54}$$

where we used the fact that the convective term vanishes. Thus, letting $k \to \infty$ in (5.31) and using weak lower semicontinuity of norm combined with (5.40), strong convergence (5.44) and also the strong convergence $P^k(\boldsymbol{v}_0) \to \boldsymbol{v}_0$ in L^2 we find that[6]

$$\limsup_{k\to\infty} (\mathbf{S}^k, \mathbf{D}(\boldsymbol{v}^k))_Q \le (\mathbf{S}, \mathbf{D}(\boldsymbol{v}))_Q. \tag{5.55}$$

[6]This relation holds for a.a. $t \in (0,T)$ and therefore one can deduce its validity also for T.

Next, using the Lebesgue dominated convergence theorem, the assumption (2.1) and (5.49), we can find that for all $\mathbf{B} \in L^r(Q)^{2\times 2}$

$$\mathbf{S}^*(e^k, \mathbf{B}) \to \mathbf{S}^*(e, \mathbf{B}) \quad \text{strongly in} \quad L^{r'}(Q) \tag{5.56}$$

as $k \to \infty$. Therefore, using the monotonicity of $\mathbf{S}^*$, i.e., (2.2), we can finally deduce that for all $\mathbf{B} \in L^r(Q)^{2\times 2}$

$$\begin{aligned}
0 &\le \limsup_{k\to\infty} (\mathbf{S}^k - \mathbf{S}^*(e^k, \mathbf{B}), \mathbf{D}(\boldsymbol{v}^k) - \mathbf{B})_Q \\
&= \limsup_{k\to\infty} (\mathbf{S}^k, \mathbf{D}(\boldsymbol{v}^k))_Q - \lim_{k\to\infty} (\mathbf{S}^k, \mathbf{B})_Q - \lim_{k\to\infty} (\mathbf{S}^*(e^k, \mathbf{B}), \mathbf{D}(\boldsymbol{v}^k) - \mathbf{B})_Q \\
&\underset{(5.56),(5.41)}{\overset{(5.55),(5.42)}{\le}} (\mathbf{S}, \mathbf{D}(\boldsymbol{v}))_Q - (\mathbf{S}, \mathbf{B})_Q - (\mathbf{S}^*(e, \mathbf{B}), \mathbf{D}(\boldsymbol{v}) - \mathbf{B})_Q \\
&= (\mathbf{S} - \mathbf{S}^*(e, \mathbf{B}), \mathbf{D}(\boldsymbol{v}) - \mathbf{B})_Q \,.
\end{aligned}$$

Consequently, setting $\mathbf{B} := \mathbf{D}(\boldsymbol{v}) \pm \varepsilon \tilde{\mathbf{B}}$ dividing by ε and letting $\varepsilon \to 0_+$ leads to the identity

$$0 = (\mathbf{S} - \mathbf{S}^*(e, \mathbf{D}(\boldsymbol{v})), \tilde{\mathbf{B}})_Q \quad \text{for arbitrary} \quad \tilde{\mathbf{B}} \in L^r(Q)^{2\times 2} \tag{5.57}$$

and we see that (5.53) holds. Moreover, since $\mathbf{S}^*$ is strictly monotone, see (2.2), we immediately deduce from (5.57) that (for subsequence)

$$\mathbf{D}(\boldsymbol{v}^k) \to \mathbf{D}(\boldsymbol{v}) \quad \text{a.e. in} \quad Q. \tag{5.58}$$

In addition, repeating the above procedure with the setting $\mathbf{B} := \mathbf{D}(\boldsymbol{v})$ we directly obtain that

$$[\mathbf{S}^k - \mathbf{S}^*(e^k, \mathbf{D}(\boldsymbol{v}))] \cdot \mathbf{D}(\boldsymbol{v}^k - \boldsymbol{v}) \to 0 \quad \text{strongly in} \quad L^1(Q).$$

Consequently, we have for any $\varphi \in L^\infty(0, T; L^\infty(\Omega))$

$$\begin{aligned}
0 &= \lim_{k\to\infty} (\mathbf{S}^k - \mathbf{S}^*(e^k, \mathbf{D}(\boldsymbol{v})), \mathbf{D}(\boldsymbol{v}^k - \boldsymbol{v})\varphi)_Q \\
&= \lim_{k\to\infty} (\mathbf{S}^k, \mathbf{D}(\boldsymbol{v}^k)\varphi)_Q - \lim_{k\to\infty} (\mathbf{S}^k, \mathbf{D}(\boldsymbol{v})\varphi)_Q - \lim_{k\to\infty} (\mathbf{S}^*(e^k, \mathbf{D}(\boldsymbol{v})), \mathbf{D}(\boldsymbol{v}^k - \boldsymbol{v})\varphi)_Q \\
&= \lim_{k\to\infty} (\mathbf{S}^k, \mathbf{D}(\boldsymbol{v}^k)\varphi)_Q - (\mathbf{S}, \mathbf{D}(\boldsymbol{v})\varphi)_Q,
\end{aligned}$$

where we used (5.42) to identify the second limit and (5.41) and (5.56) to identify the third limit. Hence, we obtained

$$\mathbf{S}^k \cdot \mathbf{D}(\boldsymbol{v}^k) \rightharpoonup \mathbf{S} \cdot \mathbf{D}(\boldsymbol{v}) \quad \text{weakly in} \quad L^1(0, T; L^1(\Omega)). \tag{5.59}$$

Consequently, using (5.30) and (5.59), one can strengthen the information (5.38) and deduce that for a subsequence

$$e^k_{,t} \rightharpoonup e_{,t} \quad \text{weakly in} \quad L^1(0, T; W^{-1,m'}(\Omega)) \tag{5.60}$$

for some sufficiently large m. Thus, using (5.45)–(5.50) and (5.59)–(5.60), we can let $k \to \infty$ in (5.30) to observe

$$\begin{aligned}
&\langle e_{,t}, \varphi\rangle - (\boldsymbol{v}_\eta e, \nabla\varphi)_Q - (\boldsymbol{q}, \nabla\varphi)_Q = (\mathbf{S}, \mathbf{D}(\boldsymbol{v})\varphi)_Q \\
&\qquad \text{for all } \varphi \in L^\infty(0, T; L^\infty(\Omega)) \cap L^m(0, T; W^{1,m}(\Omega)) \text{ for some } m \gg 1\,.
\end{aligned} \tag{5.61}$$

Hence to finish the convergence $k \to \infty$ it remains to show that

$$\boldsymbol{q} = \boldsymbol{q}^*(e, \mathbf{D}(\boldsymbol{v}), \nabla e) \quad \text{a.e. in} \quad Q. \tag{5.62}$$

Using (1.9) we see that it is enough to show that

$$\kappa^*(e^k, |\mathbf{D}(\boldsymbol{v}^k)|)\nabla e^k \rightharpoonup \kappa^*(e, |\mathbf{D}(\boldsymbol{v})|)\nabla e \quad \text{weakly in} \quad L^1(0,T; L^1(\Omega)^2). \tag{5.63}$$

Since we can rewrite this term as

$$\kappa^*(e^k, |\mathbf{D}(\boldsymbol{v}^k)|)\nabla e^k = \frac{2}{\beta+\lambda+1}\nabla(e^k)^{\frac{\beta+\lambda+1}{2}}\kappa^*(e^k, |\mathbf{D}(\boldsymbol{v}^k)|)(e^k)^{\frac{1-\beta-\lambda}{2}},$$

we can use (5.47) and therefore to prove (5.63) it is enough to show that

$$\kappa^*(e^k, |\mathbf{D}(\boldsymbol{v}^k)|)(e^k)^{\frac{1-\beta-\lambda}{2}} \to \kappa^*(e, |\mathbf{D}(\boldsymbol{v})|)e^{\frac{1-\beta-\lambda}{2}} \quad \text{strongly in} \quad L^2(Q). \tag{5.64}$$

Since we already established point-wise convergence (at least for a subsequence) of $\mathbf{D}(\boldsymbol{v}^k)$ and e^k, see (5.49) and (5.58), it is enough to show that

$$\int_Q \left|\kappa^*(e^k, |\mathbf{D}(\boldsymbol{v}^k)|)(e^k)^{\frac{1-\beta-\lambda}{2}}\right|^{2+\varepsilon} \mathrm{d}x\,\mathrm{d}t \le C \tag{5.65}$$

for some $\varepsilon > 0$. Indeed, having (5.65) and point-wise convergence, one can use the Vitali theorem to deduce (5.64). Thus, using (2.3), assuming that $\max(-\frac{1}{2}, -1-\beta) < \lambda < 0$ and setting $\varepsilon := \frac{2(2\lambda+1)}{1+\beta-\lambda} > 0$ we get

$$\begin{aligned}\int_Q \left|\kappa^*(e^k, |\mathbf{D}(\boldsymbol{v}^k)|)(e^k)^{\frac{1-\beta-\lambda}{2}}\right|^{2+\varepsilon} \mathrm{d}x\,\mathrm{d}t &\le C\int_Q (e^k)^{\frac{(1+\beta-\lambda)(2+\varepsilon)}{2}}\,\mathrm{d}x\,\mathrm{d}t \\ &= C\int_Q (e^k)^{\beta+2+\lambda}\,\mathrm{d}x\,\mathrm{d}t \overset{(5.36)}{\le} C(\lambda^{-1}).\end{aligned}$$

Thus, the proof of (5.62) is complete. The proof of attainment of e_0 is postponed to the next subsection, where a more difficult case is treated.

5.1.5. *Reconstruction of the pressure p.* In this subsection we reconstruct the pressure p corresponding to the equation (5.51). We recall the theory developed in [7], [5] where the existence of the pressure $p \in L^{r'}(Q)$ is established and where the following identity is shown.

$$\begin{aligned}\langle \boldsymbol{v}_{,t}, \boldsymbol{w}\rangle - (\boldsymbol{v}_\eta \otimes \boldsymbol{v}, \nabla \boldsymbol{w})_Q + (\mathbf{S}, \nabla \boldsymbol{w})_Q + \alpha(\boldsymbol{v}, \boldsymbol{w})_\Gamma = (p, \operatorname{div}\boldsymbol{w})_Q \\ \text{for all} \quad \boldsymbol{w} \in X^{r,2r}.\end{aligned} \tag{5.66}$$

Moreover, the pressure p is given as $p := p_1 + p_2 + p_3$ with $⨍_\Omega p_i = 0$ for $i = 1,2,3$ and the particular pressures solve for a.a. $t \in (0,T)$ and all $\varphi \in W^{2,r}(\Omega)$ such that $\nabla\varphi \in W^{1,r}_{\boldsymbol{n}}$ the following identities

$$(p_1, \triangle\varphi) = (\mathbf{S}, \nabla^2\varphi), \tag{5.67}$$

$$(p_2, \triangle\varphi) = -(\boldsymbol{v}_\eta \otimes \boldsymbol{v}, \nabla^2\varphi), \tag{5.68}$$

$$(p_3, \triangle\varphi) = (\boldsymbol{v}, \nabla\varphi)_{\partial\Omega}. \tag{5.69}$$

5.2. **Limit $\eta \to 0$.** In this subsection we complete the proof of Theorem 2.1. To do so, we first denote $(\boldsymbol{v}^\eta, e^\eta, p^\eta, \mathbf{S}^\eta, \boldsymbol{q}^\eta)$ a solution to (5.66) and (5.61).

5.2.1. *Uniform estimates and weak convergence result.* Using weak lower semicontinuity of norms, the Fatou lemma and the uniform estimates (5.32)–(5.36), we have

$$
\begin{aligned}
\sup_{t\in(0,T)} (\|\boldsymbol{v}^\eta(t)\|_2^2 + \|e^\eta\|_1) + \int_0^T \|\boldsymbol{v}^\eta\|_{1,r}^r + \|\boldsymbol{v}^\eta\|_{2r}^{2r} + \|\mathbf{S}^\eta\|_{r'}^{r'} + \|\boldsymbol{v}^\eta\|_{L^2(\partial\Omega)^2}^2 \,\mathrm{d}t \\
+ \int_0^T \|(e^\eta)^{\frac{\beta+\lambda+1}{2}}\|_{1,2}^2 + \|\boldsymbol{q}^\eta\|_{1+\frac{2\lambda+1}{3+2\beta}}^{1+\frac{2\lambda+1}{3+2\beta}} + \|e^\eta\|_{\beta+2+\lambda}^{\beta+2+\lambda} \,\mathrm{d}t \le C(\lambda^{-1}).
\end{aligned} \tag{5.70}
$$

Next, we deduce a uniform bound also for the pressure. First, setting $\varphi := \mathcal{N}^{-1}(p_3^\eta)$ in (5.69) we get with help of Lemma 3.1

$$
\|p_3^\eta\|_2^2 = (\boldsymbol{v}^\eta, \nabla\varphi)_{\partial\Omega} \le C\|\boldsymbol{v}^\eta\|_{L^2(\partial\Omega)^2}\|\varphi\|_{2,2} \le C\|\boldsymbol{v}^\eta\|_{L^2(\partial\Omega)^2}\|p_3^\eta\|_2
$$

Consequently, using (5.70) we have that

$$
\int_0^T \|p_3^\eta\|_2^2 \,\mathrm{d}t \le C. \tag{5.71}
$$

Similarly, setting $\varphi := \mathcal{N}^{-1}(|p_1^\eta|^{r'-2}p_1^\eta - \fint_\Omega |p_1^\eta|^{r'-2}p_1^\eta \,\mathrm{d}x)$, we get by using Lemma 3.1

$$
\|p_1^\eta\|_{r'}^{r'} = (\mathbf{S}^\eta, \nabla^2\varphi) \le \|\mathbf{S}^\eta\|_{r'}\|\nabla^2\varphi\|_r \le C\|\mathbf{S}^\eta\|_{r'}\|p_1^\eta\|_{r'}^{r'-1}
$$

and by using (5.70) we deduce that

$$
\int_0^T \|p_1^\eta\|_{r'}^{r'} \,\mathrm{d}t \le C. \tag{5.72}
$$

Finally, setting $\varphi := \mathcal{N}^{-1}(|p_2^\eta|^{r-2}p_2^\eta - \fint_\Omega |p_2^\eta|^{r-2}p_2^\eta \,\mathrm{d}x)$, we again obtain with help of Lemma 3.1

$$
\|p_2^\eta\|_r^r = -(\boldsymbol{v}_\eta^\eta \otimes \boldsymbol{v}^\eta, \nabla^2\varphi) \le \|\boldsymbol{v}_\eta^\eta \otimes \boldsymbol{v}^\eta\|_r\|\nabla^2\varphi\|_{r'} \le C\|\boldsymbol{v}^\eta\|_{2r}^2\|p_2^\eta\|_r^{r-1}
$$

and consequently (5.70) implies

$$
\int_0^T \|p_2^\eta\|_r^r \,\mathrm{d}t \le C. \tag{5.73}
$$

Thus, combining (5.71)–(5.73) we see that the following uniform bound holds

$$
\int_0^T \|p^\eta\|_{\min(r,r')}^{\min(r,r')} \,\mathrm{d}t \le C. \tag{5.74}
$$

Thus, using (5.70), (5.74) and the identity (5.66) we obtain

$$
\int_0^T \|\boldsymbol{v}_{,t}^\eta\|_{W_{\boldsymbol{n}}^{-1,\min(r,r')}}^{\min(r,r')} \,\mathrm{d}t \le C, \tag{5.75}
$$

and also using (5.70) we have that, see [7], [5] for details

$$
\|\boldsymbol{v}^\eta\|_{\left(X_{\mathrm{div}}^{r,\frac{2r}{2r-3}}\right)^*} \le C. \tag{5.76}
$$

Moreover, following the procedure in the previous subsection one can observe that

$$
\|e_{,t}^\eta\|_{L^1(0,T;W^{-1,\sigma'}(\Omega))} \le C \quad \text{for sufficiently large} \quad \sigma. \tag{5.77}
$$

Therefore, using (5.70), (5.74)–(5.76), Lemma 3.3, the Aubin-Lions lemma and the Div-Curl lemma, we can extract a not relabeled subsequence and find $(\boldsymbol{v}, p, e, \boldsymbol{q}, \mathbf{S})$ such that[7]

$$
\begin{aligned}
&(5.78) && \boldsymbol{v}^\eta_{,t} \rightharpoonup \boldsymbol{v}_{,t} && \text{weakly in } L^{\min(r,r')}(0,T; W_{\boldsymbol{n}}^{-1,\min(r,r')}) \cap (X_{\mathrm{div}}^{r,\frac{2r}{2r-3}})^*\,,\\
&(5.79) && \boldsymbol{v}^\eta \rightharpoonup^* \boldsymbol{v} && \text{weakly}^* \text{ in } L^r(0,T; W_{\boldsymbol{n}}^{1,r}) \cap L^\infty(0,T; L^2(\Omega)^2)\,,\\
&(5.80) && \boldsymbol{v}^\eta \to \boldsymbol{v} && \text{strongly in } L^q(0,T; L^q(\Omega)^2) \text{ for all } q < 2r\,,\\
&(5.81) && \boldsymbol{v}^\eta \to \boldsymbol{v} && \text{strongly in } L^2(0,T; L^2(\Omega)^2)\,,\\
&(5.82) && e^\eta \to e && \text{strongly in } L^n(0,T; L^n(\Omega)) \text{ for all } n < 2+\beta\,,\\
&(5.83) && (e^\eta)^{\frac{\beta+\lambda+1}{2}} \rightharpoonup e^{\frac{\beta+\lambda+1}{2}} && \text{weakly in } L^2(0,T; W^{1,2}(\Omega)) \text{ for all } -1 < \lambda < 0\,,\\
&(5.84) && p^\eta \rightharpoonup p && \text{weakly in } L^{\min(r,r')}(0,T; L^{\min(r,r')}(\Omega))\,,\\
&(5.85) && \mathbf{S}^\eta \rightharpoonup \mathbf{S} && \text{weakly in } L^{r'}(0,T; L^{r'}(\Omega)^{2\times 2})\,,\\
&(5.86) && \boldsymbol{q}^\eta \rightharpoonup \boldsymbol{q} && \text{weakly in } L^m(0,T; L^m(\Omega)^2) \text{ for all } m < \frac{4+2\beta}{3+2\beta}\,.
\end{aligned}
$$

Having these convergence results, one can easily let $\eta \to 0$ in (5.66) to obtain (2.14). It is also standard to deduce the first part of (2.17). Moreover, defining $E^\eta := \frac{1}{2}|\boldsymbol{v}^\eta|^2 + e^\eta$, setting $\boldsymbol{w} := \boldsymbol{v}^\eta \varphi$ in (5.51), adding the result to (5.61) and letting $\eta \to 0$ it is not difficult to obtain (2.15), we refer to [5] for details. Hence, to finish the proof, it is enough to show the second part of (2.17), i.e., the attainment of e_0, and to show the point-wise convergence of $\mathbf{D}(\boldsymbol{v}^\eta)$. Indeed, once having this convergence result, we can easily show that $\mathbf{S} = \mathbf{S}^*(e, \mathbf{D}(\boldsymbol{v}))$ and similarly as in preceding subsection that $\boldsymbol{q} = \boldsymbol{q}^*(e, \mathbf{D}(\boldsymbol{v}), \nabla e)$ a.e. in Q. Moreover, one can also use the Fatou lemma and to let $\eta \to 0$ in (5.61) to obtain (2.10) and (2.16).

5.3. **Point-wise convergence of $\mathbf{D}(\boldsymbol{v}^\eta)$.** In this subsection we use the so-called L^∞ truncation method (applied first for scalar parabolic equation in [4], see also [3], [2], and generalized for Navier-Stokes-like system in [13]) to obtain the point-wise convergence of the velocity gradient. In fact, we slightly improve the method such that no diagonal procedure for extracting subsequence is needed. First, we define

$$
g^\eta := |\nabla \boldsymbol{v}^\eta|^r + |\nabla \boldsymbol{v}|^r + (|\mathbf{S}^\eta| + |\mathbf{S}^*(e^\eta, \mathbf{D}(\boldsymbol{v}))|)(|\mathbf{D}(\boldsymbol{v}^\eta)| + |\mathbf{D}(\boldsymbol{v})|) \tag{5.87}
$$

and from (2.1) and (5.70) it follows that

$$
\int_Q g^\eta \,\mathrm{d}x\,\mathrm{d}t \le C \tag{5.88}
$$

independently of η. Next, we prove the essential observation that will be used in what follows.

[7]Here we use the same procedure as in the previous subsection to show point-wise convergence of e^η

Lemma 5.1. *Let g^η be defined through (5.87). Then for all $j \in \mathbb{N}$ and all $\eta > 0$ there exists a constant $L_{j,\eta} \in (2^{-2^{2^{j+1}}}, 2^{-2^{2^j}})$ and the set $E_{j,\eta} \subset Q$ defined as*

$$E_{j,\eta} := \{(t,x) \in Q;\ L_{j,\eta}^2 \le |\boldsymbol{v}^\eta(t,x) - \boldsymbol{v}(t,x)| < L_{j,\eta}\}$$

such that

$$\int_{E_{j,\eta}} g^\eta \, dx \, dt \le 2^{-j}. \tag{5.89}$$

Proof. First, for fixed j, η we define recurrently for $k = 0, \ldots, 2^j$

$$\begin{aligned} L_0 &:= 2^{-2^{2^j}}, \\ L_k &:= L_{k-1}^2 \qquad \Longrightarrow\ L_k = 2^{-2^{2^j} 2^k} \end{aligned}$$

and introduce the sets

$$E_k := \{(t,x) \in Q;\ L_k^2 \le |\boldsymbol{v}^\eta - \boldsymbol{v}| < L_k\}.$$

Since E_k are disjoint, we find by using (5.88) that

$$\sum_{k=1}^{2^j} \int_{E_k} g^\eta \, \mathrm{d}x \, \mathrm{d}t = \int_{\cup_k E_k} g^\eta \, \mathrm{d}x \, \mathrm{d}t \le \int_Q g^\eta \, \mathrm{d}x \, \mathrm{d}t \le C.$$

Therefore there exists $k^* \in \{1, \ldots, 2^j\}$ such that

$$\int_{E_{k^*}} g^\eta \, \mathrm{d}x \, \mathrm{d}t \le 2^{-j}. \tag{5.90}$$

Thus, we can finally define $L_{j,\eta} := L_{k^*}$ and we see that (5.89) is satisfied, so the proof is complete. □

We use Lemma 5.1 in the following way. First, we fix arbitrary $j \in \mathbb{N}$. For such fixed j we find $\{L_{j,\eta}\}_{\eta>0}$ such that (5.89) holds. Further, we define $\boldsymbol{u}^{j,\eta}$ and the sets $Q_{j,\eta}$ as

$$\begin{aligned} \boldsymbol{u}^{j,\eta} &:= (\boldsymbol{v}^\eta - \boldsymbol{v}) \left(1 - \min\{\frac{|\boldsymbol{v} - \boldsymbol{v}^\eta|}{L_{j,\eta}}, 1\}\right), \\ Q_{j,\eta} &:= \{(t,x) \in Q;\ |\boldsymbol{v} - \boldsymbol{v}^\eta| < L_{j,\eta}\}. \end{aligned} \tag{5.91}$$

Since $|\boldsymbol{u}^{j,\eta}| \le 1$ in Q, we deduce from (5.79) and (5.80) that

$$\boldsymbol{u}^{j,\eta} \rightharpoonup \boldsymbol{0} \qquad \text{weakly in } L^r(0,T; W^{1,r}_{\boldsymbol{n}}), \tag{5.92}$$

$$\boldsymbol{u}^{j,\eta} \to \boldsymbol{0} \qquad \text{strongly in } L^s(0,T; L^s(\Omega)^2) \text{ for all } s < \infty, \tag{5.93}$$

$$\boldsymbol{u}^{j,\eta} \to \boldsymbol{0} \qquad \text{strongly in } L^2(0,T; L^2(\partial\Omega)^2) \tag{5.94}$$

as $\eta \to 0_+$. Next, we can observe that

$$\begin{aligned} \int_Q |\operatorname{div} \boldsymbol{u}^{j,\eta}|^r \, \mathrm{d}x \, \mathrm{d}t &= \int_{Q_{j,\eta}} \left| (\boldsymbol{v}^\eta - \boldsymbol{v}) \cdot \nabla \frac{|\boldsymbol{v}^\eta - \boldsymbol{v}|}{L_{j,\eta}} \right|^r \mathrm{d}x \, \mathrm{d}t \\ &\le C L_{j,\eta}^r \int_{Q_{j,\eta} \setminus E_{j,\eta}} g^\eta \, \mathrm{d}x \, \mathrm{d}t + C \int_{E_{j,\eta}} g^\eta \, \mathrm{d}x \, \mathrm{d}t \overset{(5.89)}{\le} C 2^{-j}. \end{aligned} \tag{5.95}$$

Consequently, using Lemma 3.1 we deduce

$$\int_0^T \|\boldsymbol{u}^{j,\eta} - \boldsymbol{u}^{j,\eta}_{\mathrm{div}}\|^r_{1,r}\,\mathrm{d}t = \int_0^T \|\nabla\mathcal{N}^{-1}(\operatorname{div}\boldsymbol{u}^{j,\eta})\|^r_{1,r}\,\mathrm{d}t \le C2^{-j}. \tag{5.96}$$

Moreover, using (5.93) and (3.3) we get

$$\boldsymbol{u}^{j,\eta}_{\mathrm{div}} \to \mathbf{0} \qquad \text{strongly in } L^s(0,T;L^s(\Omega)^2) \quad \text{for all } s \le \infty\,. \tag{5.97}$$

Finally, we set $\boldsymbol{w} := \boldsymbol{u}^{j,\eta}_{\mathrm{div}}$ in (5.66) and let $\eta \to 0_+$. First, we discuss the term with the time derivative. Since $\operatorname{div}\boldsymbol{v}^\eta = \operatorname{div}\boldsymbol{v} = 0$, we have

$$\langle \boldsymbol{v}^\eta_{,t}, \boldsymbol{u}^{j,\eta}_{\mathrm{div}}\rangle = \langle \boldsymbol{v}^\eta_{,t} - \boldsymbol{v}_{,t}, \boldsymbol{u}^{j,\eta}\rangle + \langle \boldsymbol{v}_{,t}, \boldsymbol{u}^{j,\eta}_{\mathrm{div}}\rangle\,.$$

First, having (5.92)–(5.94) and (5.97) we obtain that

$$\boldsymbol{u}^{j,\eta} \rightharpoonup \mathbf{0} \quad \text{weakly in} \quad X^{r,\frac{2r}{2r-3}}_{\mathrm{div}} \tag{5.98}$$

as $\eta \to 0$. Consequently, (5.78) implies that

$$\lim_{\eta\to 0_+} \langle \boldsymbol{v}_{,t}, \boldsymbol{u}^{j,\eta}_{\mathrm{div}}\rangle = 0. \tag{5.99}$$

For the remaining term we can observe (defining $\boldsymbol{w}^\eta := \boldsymbol{v}^\eta - \boldsymbol{v}$)

$$\langle \boldsymbol{v}^\eta_{,t} - \boldsymbol{v}_{,t}, \boldsymbol{u}^{j,\eta}\rangle = \langle \boldsymbol{w}^\eta_{,t}, \boldsymbol{w}^\eta(1 - \min(\frac{|\boldsymbol{w}^\eta|}{L_{j,\eta}}, 1))\rangle = G(|\boldsymbol{w}^\eta(T)|) - G(|\boldsymbol{w}^\eta(0)|) \ge 0,$$

where the function G is defined as

$$G(x) := \begin{cases} x^2(\frac{1}{2} - \frac{x}{3L_{j,\eta}}) & \text{for } x \le L_{j,\eta}, \\ \frac{L^2_{j,\eta}}{6} & \text{for } x > L_{j,\eta}, \end{cases}$$

and where we also used the fact that $|\boldsymbol{w}^\eta(0)| = 0$. Thus, we observed that

$$\liminf_{\eta\to 0_+} \langle \boldsymbol{v}^\eta_{,t}, \boldsymbol{u}^{j,\eta}_{\mathrm{div}}\rangle \ge 0. \tag{5.100}$$

Similarly, using (5.79), (5.97) and integration by parts we find that

$$\lim_{\eta\to 0_+} -(\boldsymbol{v}^\eta_\eta \otimes \boldsymbol{v}, \nabla\boldsymbol{u}^{j,\eta}_{\mathrm{div}})_Q = \lim_{\eta\to 0_+} (\boldsymbol{v}^\eta_\eta \otimes \boldsymbol{u}^{j,\eta}_{\mathrm{div}}, \nabla\boldsymbol{v})_Q = 0. \tag{5.101}$$

Hence, (5.100)–(5.101) imply that

$$\limsup_{\eta\to 0_+} (\mathbf{S}^\eta, \mathbf{D}(\boldsymbol{u}^{j,\eta}_{\mathrm{div}}))_Q \le 0 \tag{5.102}$$

and by using (5.96), (5.85) and the same procedure as in the preceding subsection we derive

$$\limsup_{\eta\to 0_+} (\mathbf{S}^*(e^\eta, \mathbf{D}(\boldsymbol{v}^\eta)) - \mathbf{S}^*(e^\eta, \mathbf{D}(\boldsymbol{v})), \mathbf{D}(\boldsymbol{u}^{j,\eta}))_Q \le 2^{-j}. \tag{5.103}$$

Therefore, using the definition of $\boldsymbol{u}^{j,\eta}$ we can rewrite the last inequality as

$$
\begin{aligned}
0 &\le \limsup_{\eta\to 0_+} (\mathbf{S}^*(e^\eta, \mathbf{D}(\boldsymbol{v}^\eta)) - \mathbf{S}^*(e^\eta, \mathbf{D}(\boldsymbol{v})), \mathbf{D}(\boldsymbol{v}^\eta - \boldsymbol{v}))_{Q_{j,\eta}} \\
&\le 2^{-j} + C\int_{Q_{j,\eta}} (|\mathbf{S}^\eta| + |\mathbf{S}^*(e^\eta, \mathbf{D}(\boldsymbol{v}))|)(|\nabla \boldsymbol{v}^\eta| + |\nabla \boldsymbol{v}|)\frac{|\boldsymbol{v}^\eta - \boldsymbol{v}|}{L_{j,\eta}} \,\mathrm{d}x\,\mathrm{d}t \\
&\le 2^{-j} + CL_{j,\eta}\int_{Q_{j,\eta}\setminus E_{j,\eta}} g^\eta \,\mathrm{d}x\,\mathrm{d}t + C\int_{E_{j,\eta}} g^\eta \,\mathrm{d}x\,\mathrm{d}t \overset{(5.89)}{\le} C2^{-j}.
\end{aligned} \tag{5.104}
$$

Finally, using (5.80) we deduce that

$$|Q \setminus Q_{j,\eta}| \to 0 \quad \text{as} \quad \eta \to 0,$$

and therefore by using the Hölder inequality and (5.104) we have

$$
\begin{aligned}
&\limsup_{\eta\to 0_+} \int_Q |(\mathbf{S}^*(e^\eta, \mathbf{D}(\boldsymbol{v}^\eta)) - \mathbf{S}^*(e^\eta, \mathbf{D}(\boldsymbol{v}))) \cdot \mathbf{D}(\boldsymbol{v}^\eta - \boldsymbol{v})|^{\frac{1}{2}} \,\mathrm{d}x\,\mathrm{d}t \\
&\quad = \limsup_{\eta\to 0_+} \int_{Q\setminus Q_{j,\eta}} \ldots + \int_{Q_{j,\eta}} \ldots \\
&\quad \overset{(5.88)}{\le} C\left(\limsup_{\eta\to 0_+} |Q\setminus Q_{j,\eta}| + (\mathbf{S}^*(e^\eta, \mathbf{D}(\boldsymbol{v}^\eta)) - \mathbf{S}^*(e^\eta, \mathbf{D}(\boldsymbol{v})), \mathbf{D}(\boldsymbol{v}^\eta - \boldsymbol{v}))_{Q_{j,\eta}}\right)^{\frac{1}{2}} \\
&\quad \overset{(5.104)}{\le} C2^{-\frac{j}{2}}.
\end{aligned}
$$

Since j can be chosen arbitrarily, we see that the last limit is zero and by using strict monotonicity of $\mathbf{S}^*$, i.e., the assumption (2.2) and the strong convergence of e^η (5.82), one can conclude that

$$\mathbf{D}(\boldsymbol{v}^\eta) \to \mathbf{D}(\boldsymbol{v}) \quad \text{a.e. in} \quad Q.$$

5.4. **Initial conditions.** This subsection is devoted to the proof of the second part of (2.17). Here we assume that the first part of (2.17) holds (see [15] for detailed proof). First, we set $\varphi := \chi_{(0,t)}$ in (5.30) and $\boldsymbol{w} := \boldsymbol{v}^\eta \chi_{(0,t)}$ and sum the resulting identities to obtain (we neglect the boundary term since it has a correct sign)

$$\frac{1}{2}\|\boldsymbol{v}^\eta(t)\|_2^2 + \|e^\eta(t)\|_1 \le \frac{1}{2}\|\boldsymbol{v}_0\|_2^2 + \|e_0\|_1. \tag{5.105}$$

Thus, letting $\eta \to 0_+$ and using (5.80), (5.82) and the Fatou lemma we obtain for a.a. $t \in (0,T)$

$$\frac{1}{2}\|\boldsymbol{v}(t)\|_2^2 + \|e(t)\|_1 \le \frac{1}{2}\|\boldsymbol{v}_0\|_2^2 + \|e_0\|_1. \tag{5.106}$$

Moreover, one can redefine $\boldsymbol{v}$ and e on the zero measure subset of $(0,T)$ so that (5.106) holds for all $t \in (0,T)$. Therefore, using the first part of (2.17) end letting $t \to 0_+$ in (5.106) we deduce that

$$\limsup_{t\to 0_+} \|e(t)\|_1 \le \|e_0\|_1. \tag{5.107}$$

Next, setting $\varphi := \chi_{(0,t)} \frac{\psi}{\sqrt{e^\eta}}$ in (5.61) with arbitrary $\psi \ge 0$ such that $\psi \in W^{1,\infty}(\Omega)$ (such setting is possible since all terms are meaningful) we deduce that (using the nonnegativity of ψ)

$$\frac{1}{2}(\sqrt{e^\eta(t)}, \psi) - \int_0^t (\boldsymbol{q}^\eta, (e^\eta)^{-\frac{1}{2}} \nabla\psi)\, \mathrm{d}\tau \ge \frac{1}{2}(\sqrt{e_0}, \psi). \tag{5.108}$$

Hence, using (5.70), (5.82) and (5.86) we can let $\eta \to 0_+$ in (5.108) to deduce for a.a. $t \in (0,T)$

$$\frac{1}{2}(\sqrt{e(t)}, \psi) - \int_0^t (\boldsymbol{q}, e^{-\frac{1}{2}} \nabla\psi)\, \mathrm{d}\tau \ge \frac{1}{2}(\sqrt{e_0}, \psi), \tag{5.109}$$

which can be again extended onto the whole time interval $(0,T)$. Thus letting $t \to 0_+$, we have

$$\liminf_{t\to 0_+} (\sqrt{e(t)}, \psi) \ge (\sqrt{e_0}, \psi) \tag{5.110}$$

for all smooth nonnegative ψ and by using the density argument and (2.10) for all nonnegative $\psi \in L^2(\Omega)$. Therefore, we have

$$\begin{aligned}\limsup_{t\to 0_+} \|\sqrt{e(t)} - \sqrt{e_0}\|_2^2 &= \limsup_{t\to 0_+} \|e(t)\|_1 + \|e_0\|_1 - 2(\sqrt{e(t)}, \sqrt{e_0}) \\ &\overset{(5.107),(5.110)}{\le} 2\|e_0\|_1 - 2\|e_0\|_1 = 0\end{aligned}$$

and (2.17) follows. Thus the proof of Theorem 2.1 is complete. The proof of Corollary 2.1 is then the consequence of the standard theory for the heat equation with L^1-right-hand side and of the fact that for $r \ge 2$ the velocity is a possible test function in (2.14).

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