Flow of an electrorheological fluids between eccentric rotating cylinders

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FLOW OF AN ELECTRORHEOLOGICAL FLUID BETWEEN ECCENTRIC ROTATING CYLINDERS

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Dedicated to professor R. R. Huilgol on the occasion of his seventieth birthday.

Abstract. Electrorheological fluids have numerous potential applications in vibration dampers, brakes, valves, clutches, exercise equipment, etc. The flows in such applications are complex three dimensional flows. Most models that have been developed to describe the flows of electrorheological fluids are one dimensional models. Here, we discuss the behaviour of two fully three dimensional models for electrorheological fluids. The models are such that they reduce, in the case of simple shear flows with the intensity of the electric field perpendicular to the streamlines, to the same constitutive relation, but they would not be identical in more complicated three dimensional settings. In order to show the difference between the two models we study the flow of these fluids between eccentrically placed rotating cylinders kept at different potentials, in the setting that corresponds to technologically relevant problem of flow of electrorheological fluid in journal bearing. Even though the two models have quite a different constitutive structure, due to the assumed forms for the velocity and pressure fields, the models lead to the same velocity field but to different pressure fields. This finding illustrates the need for considering the flows of fluids described by three dimensional constitutive models in complex geometries, and not restricting ourselves to flows of fluids described by one dimensional models or simple shear flows of fluids characterized by three dimensional models.

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1. Introduction

Electrorheological fluids are fluids that change their material properties on the application of an external electric field. The fact, that the material properties (particularly the viscosity) can be controlled by the external electric field leads, as it was shown in the pioneering work by Winslow [1949], to many potential applications in industrial devices such as clutches, brakes and dampers, to name a few. The potential applications of the electrorheological fluids has led to intensive research in this field both with regard to the manufacture of such fluids and the development of mathematical models to describe their response.

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Since electrorheological fluids are usually suspensions of electrically active particles (typically zeolite or silica gel particles) in a non-conducting fluid (typically silicone or mineral based oils), it is possible to try to model the fluids using mixture theory, see for example Rajagopal et al. (1994). This approach is however usually not preferred as there are problems with specifying boundary conditions and the system of equations is also considerably complicated, and thus electrorheological fluids are modelled as a homogenized single continuum. In such an approach, if one wants to fully describe a given fluid, it is sufficient to know a single tensorial constitutive relation of type (in the most simple cases)

\[ f(T, \nabla U, E) = 0. \]  \hspace{1cm} (1.1)

that specifies the relation between the Cauchy stress \( T \), the symmetric part of the velocity gradient \( \nabla U \) and the intensity of the electric field \( E \). (The boundary condition is also in general a constitutive specification that needs to be provided; it is however usually automatically assumed that the velocity field satisfies the no-slip boundary condition, and the “constitutive” nature of the boundary condition is ignored.)

Concerning constitutive equations for electrorheological fluids have been suggested by the experimentalists themselves, see for example Shulman et al. (1989) or Choi et al. (2005) and references therein. In these works, however, the constitutive relation takes the form of a scalar relation between the shear stress, shear rate and the intensity of the electric field—the reason is that the experiments are usually designed in such way that the fluid behaviour is investigated in a simple shear flow and the intensity of the electric field is perpendicular to the streamlines, and, in such a simple setting, only a scalar relation between the above mentioned quantities is necessary. Consequently, the constitutive theory is essentially one-dimensional and a generalization to a fully three-dimensional setting is not obvious in the sense that there exist many three-dimensional constitutive relations that can—in simple shear flows—lead to the same relation between the Cauchy stress, the shear rate and the intensity of the electric field—a particular example is given in the present work.

On the other hand, it is possible to apply the classical tools that have been developed to generate constitutive relations (see for example Truesdell and Noll (1965) and Spencer (1971)) to derive fully three dimensional constitutive relations for electrorheological fluids, see for example Wineman and Rajagopal (1993) and Rajagopal and Růžička (1999, 2001). Having a class of three dimensional constitutive relations will allow one to assess the validity and usefulness of such models, and we do this by considering special flows that correspond to flows in domains that are relevant to experiments. For instance, one can use these constitutive relations to determine fluid behaviour in simple shear flows, viscometric flows and nearly viscometric flows.\(^1\) We need to be in a position to differentiate between competing models and thus if different constitutive relations are indistinguishable in simple shear flows, then it is necessary to find a different experimental setting that would allow us to differentiate between the constitutive relations that are indistinguishable in shear flows, and consequently would allow the experimenter to decide which constitutive relation is the right one for description of the fluid in complicated flows that go beyond simple shear flows.

In the present work we study the flow of electrorheological fluids between two eccentrically placed cylinders, the flow being driven by steady rotation of the inner cylinder. The aim of our study is threefold. First, we want to investigate the flow of electrorheological fluids in a geometry that goes beyond a simple “shear flow geometry” considered in the literature (see for example Atkin et al. (1991), Rajagopal and Wineman (1992), Gavin et al. (1996) and Choi et al. (2005)), but is still simple enough to allow us to determine the influence of the various parameters on the flow. Second, we want to illustrate the issues related to the constitutive modelling that we have pointed out in the previous paragraphs, namely we want to examine the possibility of an experimental design that would allow us to differentiate between three dimensional models that are indistinguishable in simple shear flows. Third, the geometry corresponds to the journal bearing geometry and it is a technologically relevant geometry.\(^2\) Note that although the electrorheological fluids are used in hydrodynamic lubrication, most results in this field are, however, based on crude approximations—see for example Bouzidane and Thomas (2008)—and not on detailed modelling of the flow in relevant geometries.

The paper is organized as follows. In Section 2 we introduce the governing equations for electrorheological fluids and discuss the assumptions that lead to the system of governing equations. In Section 3 we introduce two particular fully three dimensional models for the Cauchy stress tensor that are clearly different for general three dimensional flows and, in the next Section, we discuss their behaviour in simple shear flows and we show that the two models coincide in simple shear flows. Finally, in Section 4 we propose the problem of flow of electrorheological fluids between eccentric rotating cylinders and we show that even in this more complicated geometry the two models lead to the same velocity field though the pressure fields differ. This points to the need for a fully three dimensional flow field to study the efficacy of the two models. The problem of flow in a journal bearing is solved numerically and in Section 5 comment on the influence of various parameters on the solution.

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1See for example Coleman et al. (1966), Truesdell (1974) and Pipkin and Owen (1967) for discussion on viscometric and nearly viscometric flows.

2One of the biggest problems with large rotating machinery in power plants is the losses due to the decrease in the load carrying capacity of the bearing due to the decrease in the viscosity as a consequence of the increase in the temperature in the bearing, which is estimated in the billions of dollars. One could compensate for this decrease in viscosity by applying an electric field, thereby increasing the viscosity.
2. Governing equations

If we assume that the electromagnetic field is weak and not changing rapidly and that the velocity of the fluid is small (compared to the speed of light) and the size of the domain is small, then the flow of an isotropic, homogeneous, incompressible, non-conducting dielectric fluid is governed by the following set of equations (see for example Rajagopal and Růžička (1996) or monographs Penfield and Haus (1967), Hutter and van de Ven (1978) and Paci (1978) for details). For the electromagnetic field we can use a quasistatic approximation of Maxwell equations

\[
\text{div}(\varepsilon_0 \varepsilon_r \mathbf{E}) = 0, \quad (2.1a)
\]

\[
\text{rot} \mathbf{E} = 0, \quad (2.1b)
\]

where \( \mathbf{E} \) denotes the intensity of the electric field, \( \varepsilon_r \) is the relative permittivity (a constant) and \( \varepsilon_0 \) denotes the vacuum permittivity (a constant). For the fluid (no external forces) we get the equations

\[
\rho \frac{d\mathbf{v}}{dt} = \mathbf{T} + \varepsilon_0 \varepsilon_r [\nabla \mathbf{E}] \mathbf{E}, \quad (2.2a)
\]

\[
\mathbf{T} = \mathbf{T}^\top, \quad (2.2b)
\]

\[
\text{div} \mathbf{v} = 0, \quad (2.2c)
\]

where \( \rho \) is the density of the fluid, \( \mathbf{v} \) is the velocity field, \( \mathbf{T} \) is the Cauchy stress tensor and \( \mathcal{D} \) is the symmetric part of the velocity gradient \( \mathcal{D} = \frac{1}{2} (\nabla \mathbf{v} + \nabla \mathbf{v}^\top) \).

In (2.1) and (2.2) we have already made a constitutive assumption that the polarization \( \mathbf{P} \) is proportional to the intensity of the electric field, thus \( \mathbf{P} = \varepsilon_0 \chi_r \mathbf{E} \), where \( \chi_r \) denotes the electric susceptibility, \( \varepsilon_r = \frac{\varepsilon_0}{\varepsilon_0} + 1 \). Another constitutive assumption that is in fact tacitly introduced in (2.2a) is the absence of internal body couples that leads to the symmetry (2.2b) of the Cauchy stress tensor.

Note that the electric field \( \mathbf{E} \) enters the balance of linear momentum (2.2a) in two ways, “implicitly” in the constitutive equation for the Cauchy stress \( \mathbf{T} \), and “explicitly” as a volume force, \( [\nabla \mathbf{E}] \mathbf{E} \). Obviously, the latter term vanishes if one considers a homogeneous electric field, for example an electric field between two infinite plates, but this is not the case in more complicated geometries. The term is nevertheless completely ignored in many theoretical and experimental works, see for example Nikolakopoulos and Papadopoulos (1997) or Peng and Zhai (2001).

System (2.1) can be reduced to a single equation for the scalar potential \( \phi \),

\[
\Delta \phi = 0, \quad (2.3)
\]

where the intensity of the electric field \( \mathbf{E} \) is related to the potential \( \phi \) through the formula \( \mathbf{E} = -\nabla \phi \). Obviously, the equation for the potential can be solved without knowledge of the pressure and velocity field—the potential (intensity of the electric field) therefore enters system (2.2) as a known “parameter”.

From the perspective of the full Maxwell equations, the study of flow of electrorheological fluids can be seen as a “counterpart” to magnetohydrodynamics. In the first case the magnetic field plays a secondary role and is in fact ignored, and the converse is true in the latter case. In magnetohydrodynamics, see for example Chandrasekhar (1961), the situation is however more complicated since the equations for the magnetic induction \( \mathbf{B} \) can not be solved a priori without any reference to the velocity and pressure fields.

3. Constitutive relations for the Cauchy stress tensor

The experimental investigations show (see for example Shulman et al. (1983), Halsey et al. (1992), Kolias and Dimarogonas (1993), Martin et al. (1994), Abu-Jdayil and Brunn (1997), Krztoń-Maziopta et al. (2005), Belza et al. (2008) and reviews by Gast and Zukoski (1989) and Zukoski (1993)) that electrorheological fluids can exhibit many non-newtonian characteristics: yield stress, shear thinning/thickening and normal stress differences.

In what follows, we focus on electrorheological fluids that do not exhibit yield stress. Although yield stress is observed in many electrorheological fluids which do not exhibit yield stress, see for example Abu-Jdayil and Brunn (1996). Note that a fluid by definition cannot support a shear stress and hence from a philosophical standpoint “yield stress” is not a viable concept. However, it all depends on the time scale, length scale and force scale. A good example is the well

3Recall that the Lorentz transformation for the electromagnetic field reads

\[
\mathbf{E}' = \gamma (\mathbf{E}_\perp + \mathbf{w} \times \mathbf{B}) + \mathbf{E}_\parallel, \quad \mathbf{B}' = \gamma \left( \mathbf{B}_\perp - \frac{1}{c^2} \mathbf{w} \times \mathbf{E} \right) + \mathbf{B}_\parallel,
\]

where \( \mathbf{w} \) is the velocity of the reference frame \( S' \) with respect to the reference frame \( S \) (it is assumed that coordinate axes in \( S' \) and \( S \) remain parallel), \( c \) is the speed of light, \( \gamma = \frac{1}{\sqrt{1 - \frac{|\mathbf{w}|^2}{c^2}}} \), \( \mathbf{E}_\parallel \) denotes the projection of the intensity of the electric field \( \mathbf{E} \) to the direction of the velocity \( \mathbf{w}, \mathbf{E}_\perp = \frac{1}{\gamma} \left( \mathbf{w} \cdot \mathbf{E} \right) \mathbf{w} \), and \( \mathbf{E}_\perp \) denotes the projection of the intensity of the electric field \( \mathbf{E} \) to the direction perpendicular to the direction of the velocity \( \mathbf{w}, \mathbf{E}_\parallel = \frac{1}{\gamma} \left( \mathbf{w} \cdot \mathbf{E} \right) \mathbf{w} \). (Similarly for \( \mathbf{B} \).) Requirement on the smallness of the velocity and the smallness of the domain ensures that the Lorentz transformation can be approximated by a Galilean transformation. Note that the Lorentz transformation does not imply that if \( \mathbf{B} = 0 \) then \( \mathbf{B}' = 0 \), and similarly for \( \mathbf{E} \). If, however, \( \mathbf{E} \) is not very strong and \( |\mathbf{w}| \) is small compared to the speed of light \( c \), then the implication holds. In what follows, this allows us to consider the electric and magnetic field as different physical phenomena and not to consider them as related aspects of the electromagnetic field.
known experiment on asphalt—see for example Edgeworth et al. (1984)—that documents that the material that is apparently solid-like (if the observation time is in minutes or hours) in fact flows on a time scale of years. When yield stress is not considered, it can be shown (see for example Rajagopal and Wineman (1992), Wineman and Rajagopal (1995) and Rajagopal and Růžička (2001)) that a general constitutive relation for a homogeneous, isotropic, incompressible and non-conducting electrorheological fluid reads

\[ T = -pI + S, \]  
\[ \text{(3.1a)} \]

where \( T \) is the Cauchy stress, \( p \) is the Lagrange multiplier associated to the incompressibility constraint, \( I \) denotes the identity tensor and \( S \) is the extra stress given by the formula

\[ S = \alpha_2 (E, D) E \otimes E + \alpha_3 (E, D) D + \alpha_4 (E, D) D^2 + \alpha_5 (E, D) (DE \otimes E + E \otimes DE) \]
\[ + \alpha_6 (E, D) (D^2E \otimes E + E \otimes D^2E), \]
\[ \text{(3.1b)} \]

where \( \alpha_i, i = 2, \ldots, 6 \) are functions of combined invariants of the intensity of the electric field and symmetric part of the velocity gradient \( D = \frac{1}{2} (\nabla v + \nabla v^\top) \). There are way too many material functions that appear in (3.1b) for us to outline a sensible experimental program through which all of them can be characterized. In particular we will focus on two models that belong to class (3.1), namely the models

\[ S_1 = 2\mu_0 D + \mu_1 (DE \otimes E + E \otimes DE), \]
\[ S_2 = 2\mu_0 \left( 1 + \beta |E|^2 \right) D, \]
\[ \text{(3.2a)} \]
\[ \text{(3.2b)} \]

where \( \mu_0, \mu_1 \) and \( \beta \) are constants. Note that in virtue of the constraint of incompressibility \( \text{div} \, v = \text{Tr} \, D = 0 \) we have \( \text{Tr} \, S_2 = 0 \), but, in general, we cannot expect \( \text{Tr} \, S_1 = 0 \)—we discuss the implications of this observation in Section 5.6.

The reason for the choice of these two models is the following. If we consider models of type \( \text{(3.1a)} \), and we are interested in fluids where the generalized viscosity depends only on the intensity of the electric field, that do not exhibit normal stress differences effect, we see that \( S = \alpha_3 (E) D + \alpha_5 (E) (DE \otimes E + E \otimes DE) \) is—for these fluids—a natural choice for constitutive relation for the extra stress. The model \( S = \alpha_3 (E) D + \alpha_5 (E) (DE \otimes E + E \otimes DE) \) however coincides—in simple shear flows with the intensity of the electric field perpendicular to the streamlines—with a simpler model \( S = \alpha_3 (E) D \), see Section 4. Clearly, the models cannot be expected to be equivalent in more complicated settings, and one can ask what the effects of the tensorial term \( DE \otimes E + E \otimes DE \) are in such settings. Comparison of the models \( \text{(3.2a)} \) gives us an opportunity to address this question that has implications with respect to the constitutive modelling of electrorheological fluids, see Section 1. Let us now briefly comment on the behaviour of models \( \text{(3.2a)} \) and \( \text{(3.2b)} \) in simple shear flows.

4. Simple shear flows

4.1. Plane Poiseuille flow. Let us consider the flow of electrorheological fluids \( \text{(3.2)} \) between infinite parallel plates kept at different potentials \( \text{v} \) (see Figure 1a). Obviously, the intensity of the electric field is perpendicular to the plates—\( E = E^\phi e_\phi \), where \( E^\phi \) is a constant. If we assume that the velocity field has the form \( v(x, y) = \hat{v}^\phi (y) e_x \), then the Cauchy stress is, according to models \( \text{(3.2a)} \) and \( \text{(3.2b)} \), given by the formula

\[ T = -pI + \left( \mu_0 + \frac{\mu_1}{2} \left( E^\phi \right)^2 \right) \left[ \begin{array}{c} 0 \\ d\hat{v}^\phi \\ 0 \end{array} \right], \quad \text{and} \quad T = -pI + \mu_0 \left( 1 + \beta \left( E^\phi \right)^2 \right) \left[ \begin{array}{c} 0 \\ d\hat{v}^\phi \\ 0 \end{array} \right], \]
\[ \text{(4.1)} \]

respectively. Clearly, if we set \( \beta = \frac{\mu_1}{2\mu_0} \) in \( \text{(3.2b)} \), then both models lead, in Poiseuille flow, to the same formula for the Cauchy stress. In this sense, any experiment that uses measurements in Poiseuille flow can not be used to distinguish model \( \text{(3.2a)} \) from \( \text{(3.2b)} \).

4.2. Cylindrical Couette flow. Similarly, if we consider the flow between coaxial rotating cylinders kept at different potentials \( \text{v} \) (see Figure 1a), we see that the intensity of the electric field is given by the formula \( E = E^\phi e_\phi \), where \( E^\phi \) is a known function of the radial variable \( r \). If we assume that the velocity field has the form \( v(r, \phi) = \omega (r) e_x \), where \( \omega (r) \) is the angular frequency, then the Cauchy stress is, according to models \( \text{(3.2a)} \) and \( \text{(3.2b)} \), given by the formula

\[ T = -pI + \left( \mu_0 + \frac{\mu_1}{2} \left| E^\phi (r) \right|^2 \right) \left[ \begin{array}{c} 0 \\ r \frac{d\omega}{dr} \\ 0 \end{array} \right], \quad \text{and} \quad T = -pI + \mu_0 \left( 1 + \beta \left| E^\phi (r) \right|^2 \right) \left[ \begin{array}{c} 0 \\ r \frac{d\omega}{dr} \\ 0 \end{array} \right], \]
\[ \text{(4.2)} \]

respectively. Again, if we set \( \beta = \frac{\mu_1}{2\mu_0} \) in \( \text{(3.2b)} \), then both models lead to the same formula for the Cauchy stress, and the models are indistinguishable in this type of flow.

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4. We deliberately use the term “the Lagrange multiplier associated with the constraint of incompressibility” instead of “the pressure”. Later we show that these notions are not equivalent and that we must carefully differentiate between these two notions. For further discussion on this issue see also Rajagopal and Srinivasan (2004), Rajagopal (2005) and Huilgol (2009).

5. The invariants are \( |E|^2 \), \( Tr \, D \), \( Tr \, D^2 \), \( Tr \, D^3 \), \( Tr \, (DE \otimes E) \) and \( Tr \, (D^2E \otimes E) \).

6. We refer the reader to Rajagopal and Wineman (1992) for details.

7. For a careful discussion of this type of problem see for example Rajagopal and Wineman (1992).
5. Flow between eccentric cylinders

We have seen that models (3.2) lead to the same results in (idealized) Poiseuille and Couette type experiments. In both experiments, the intensity of the electric field is perpendicular to the streamlines. One needs to guess the flow field that could be used to differentiate between the two models. A problem that can be expected to help us to distinguish between the two models is for example the problem of flow between two eccentrically placed parallel cylinders, see Figure 1b. The cylinders are kept at different potentials, $\phi_1$ (inner cylinder) and $\phi_2$ (outer cylinder), and the cylinders are rotating with constant angular velocities $\Omega_1$ and $\Omega_2$ about their axes. The radius of the inner cylinder is $R_1$ and the radius of the outer cylinder is $R_2$, the distance between the axes of the cylinders (the eccentricity) is $e$. In the idealized case, we can assume that the cylinders are infinite, and consequently we can essentially treat the problem as a two dimensional problem—there is no flow in the direction of axes of the cylinders and the velocity and pressure fields are the same on each cross section perpendicular to the axes of the cylinders. Furthermore, it is reasonable to assume that the flow is steady.

This type of problem has been already extensively studied within the context of electrically inactive fluids because it provides a simple model for the flow in a journal bearing (see for example Dai et al. (1992) and Gwynlyw et al. (1996)). We can in fact use much of the available “classical” apparatus to solve the problem for electrorheological fluids, this is especially true concerning the numerical treatment of the problem. Since it is possible to find an analytical solution to the Laplace equation (2.3), we can explicitly calculate all terms that include $E$, and we get almost the same problem as for the flow of the classical Navier–Stokes fluid in a journal bearing. The difference is that the viscosity is in our case a known function of $E$ and hence a known function of position.

5.1. Dimensionless form of the governing equations. Let us now convert (2.3) and (2.2) to a dimensionless form. Let $L$ be a characteristic length scale, $E_{\text{ref}}$ the characteristic intensity of the electric field and $V$ the characteristic velocity. Using these characteristic quantities a dimensionless version of the governing equations reads

\[
\frac{dv^*}{dt^*} = \text{div}^* T^* + \bar{\varepsilon}_0 \bar{\varepsilon}_r [\nabla^* E^*] E^*, \tag{5.1a}
\]
\[\text{div}^* v^* = 0, \tag{5.1b}\]
\[\Delta^* \phi^* = 0, \tag{5.1c}\]
where the star denotes dimensionless quantities \( v^* = \frac{v}{V} \), \( t^* = \frac{t}{L} \), \( x^* = \frac{x}{L} \), \( E^* = \frac{E}{E_{rel}} \), \( \phi^* = \frac{\phi}{E_{rel} L} \). Dimensionless form of constitutive equations (5.2) is
\[
T^* = -p^* I + 2\mu_0 \mathbb{D}^* + \mu_1 \left( \mathbb{D}^* \mathbb{E}^* \otimes \mathbb{E}^* + \mathbb{E}^* \otimes \mathbb{D}^* \mathbb{E}^* \right),
\]
(5.2a)
\[
T^* = -p^* I + 2\mu_0 \left( 1 + \beta \left| \mathbb{E}^* \right|^2 \right) \mathbb{D}^*,
\]
(5.2b)
where \( \varepsilon_0 = \frac{\varepsilon_0 \varepsilon_0 \rho V^2}{p^* V^*} \), \( \mu_0 = \frac{\mu_0 \rho V^2}{V^*} \), \( \mu_1 = \frac{\mu_1 \rho V^2}{V^*} \), \( \beta = \beta \mathcal{E}_{rel} \). Obviously some of these parameters can be used to determine the usual dimensionless numbers, \( \mathcal{R} = \frac{1}{\mu_0} \) is the Reynolds number (the ratio of inertial to the viscous forces), \( \mathcal{M} = \frac{1}{\varepsilon_0} \) is the Mason number (the ratio of inertial to the electrical forces). Let us further fix \( L = R_1 \) and introduce the following notation
\[
\eta_R = \frac{R_1}{R_2}, \quad \eta_\beta = \frac{e}{R_1}, \quad \eta_\phi = \frac{\phi_2}{\phi_1}, \quad \eta_\Omega = \frac{\Omega_2}{\Omega_1}.
\]
(5.3)
Moreover, since we are interested in the situation where models (5.2a) and (5.2b) coincide in Poiseuille flow and cylindrical Couette flow, we fix \( \beta = \frac{\varepsilon_1}{2\mu_0} \). Hereafter we only use dimensionless variables, and, for the sake of clarity, we omit the star denoting the dimensionless quantities.

5.2. Bipolar coordinates. The bipolar coordinate system is the natural co-ordinate system within which to study the problem, see for example Jeffery (1921, 1922). Let us recall some basic formulae for the bipolar coordinate system. (See Figure 11 for the notation.) Bipolar coordinates \((\alpha, \beta)\) are related to the Cartesian coordinates \((x, y)\) by the following formula
\[
\alpha + i\beta = \ln \frac{y + i(x + a)}{y + i(x - a)}.
\]
(5.4)
In what follows we call \( \alpha \) the radial variable and \( \beta \) the azimuthal variable, similarly, components of vectors are referred to as the radial (component in the direction of \( e_\alpha \)) and the azimuthal (component in the direction of \( e_\beta \)) components. (See Section 5.2.1 for the formulae for normalized base vectors \( e_\alpha \) and \( e_\beta \).) Using (5.4) it is easy to show that
\[
\begin{align*}
x &= \frac{a \sin \alpha}{\cosh \alpha - \cos \beta}, \quad y = \frac{a \sin \beta}{\cosh \alpha - \cos \beta},
\end{align*}
\]
(5.5)
and consequently \((x - a \coth \alpha)^2 + y^2 = \frac{a^2}{\sinh \alpha} \), \( x^2 + (y - a \cot \beta)^2 = \frac{a^2}{\sin \beta} \). Obviously, the curves of constant \( \alpha \) and \( \beta \) are circles. Knowing the radii of the cylinders and the eccentricity, one can find parameter \( a \), \( a > 0 \), and parameters \( \alpha_{in}, \alpha_{out}, 0 < \alpha_{out} < \alpha_{in} \), such that the curves of constant \( \alpha = \alpha_{in} \) and \( \alpha = \alpha_{out} \) are the corresponding circles \((\alpha = \alpha_{in} \) is the boundary of the inner circle and \( \alpha = \alpha_{out} \) is the boundary of the outer circle\). The formulae for the parameters are the following
\[
a = \frac{1}{2 \eta_\beta} \sqrt{\left( 1 + \frac{1}{\eta_R} \right) - \left( \eta_\beta \right)^2} - \frac{4}{\eta_R}, \quad \alpha_{in} = \arcsin \frac{a}{\eta_\beta}, \quad \alpha_{out} = \arcsin \left( \eta_R \arcsin \frac{a}{\eta_\beta} \right).
\]
(5.6)
The gap between the circles is thus mapped to a rectangular domain \([\alpha, \beta] \in [\alpha_{out}, \alpha_{in}] \times [0, 2\pi]\). If we express the boundary conditions for the velocity and the potential in the bipolar coordinates, we get
\[
\phi|_{\alpha = \alpha_{in}} = 1, \quad \phi|_{\alpha = \alpha_{out}} = \eta_\phi,
\]
(5.7a)
and for components of the velocity we get
\[
\begin{align*}
v^\alpha|_{\alpha = \alpha_{in}} &= 0, \quad v^\beta|_{\alpha = \alpha_{in}} = 1, \quad v^\alpha|_{\alpha = \alpha_{out}} = 0, \quad v^\beta|_{\alpha = \alpha_{out}} = \frac{\eta_\Omega}{\eta_R},
\end{align*}
\]
(5.7b)
Let us recall that we study the problem with rotating inner cylinder and fixed outer cylinder, hence \( \eta_\Omega = 0 \).

5.2.1. Tensor calculus. The tangent vectors are
\[
\mathbf{g}_\alpha = \frac{\partial x_i}{\partial \alpha} e_i = \frac{a}{(\cosh \alpha - \cos \beta)^2} \begin{bmatrix} 1 - \cosh \alpha \cos \beta \alpha \cosh \alpha \sin \beta \end{bmatrix}, \quad \mathbf{g}_\beta = \frac{\partial x_i}{\partial \beta} e_i = \frac{a}{(\cosh \alpha - \cos \beta)^2} \begin{bmatrix} - \sinh \alpha \sin \beta \cosh \alpha \cos \beta - 1 \end{bmatrix},
\]
and consequently the components of covariant and contravariant metric tensors are
\[
[g] = \frac{a^2}{(\cosh \alpha - \cos \beta)^2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad [g^{ij}] = \frac{\cosh \alpha - \cos \beta)^2}{a^2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.
\]
(5.8)
The Christoffel symbols are
\[
\Gamma^{\alpha}{}_{\alpha \alpha} = -\frac{\sinh \alpha}{\cosh \alpha - \cos \beta}, \quad \Gamma^{\beta}{}_{\beta \beta} = -\frac{\sin \beta}{\cosh \alpha - \cos \beta},
\]
(5.9)
\footnote{In some published papers, see for example Dai et al. (1992), there are misprints in the formulae for various differential operators in the bipolar coordinates. Also, we need to use some of these formulae and therefore we feel the need to record the basic formulae.}
and for the remaining nonzero Christoffel symbols we have $\Gamma^\alpha_{\beta\alpha} = \Gamma^\alpha_{\alpha\alpha} \ast \Gamma^\beta_{\alpha\alpha} = -\Gamma^\alpha_{\alpha\alpha}$, $\Gamma^\beta_{\alpha\alpha} = -\Gamma^\beta_{\beta\alpha}$, $\Gamma^\alpha_{\beta\beta} = \Gamma^\beta_{\beta\beta}$. The relations of the physical components of $\mathbf{v} = v^\alpha \mathbf{e}_\alpha + v^\beta \mathbf{e}_\beta$ and components with respect to the non-normed basis $\mathbf{v} = v^\alpha \mathbf{g}_\alpha + v^\beta \mathbf{g}_\beta$ are the following

$$v^\alpha = \frac{a}{\cosh \alpha - \cos \beta} v^\alpha, \quad v_\alpha = \frac{\cosh \alpha - \cos \beta}{a} v^\alpha, \quad \hat{v}^\alpha = v^\alpha,$$  \hspace{1cm} (5.10)

and the same formulae hold also for $v^\beta$ since $g_{\alpha\alpha} = g_{\beta\beta}$. Moreover, since the metric tensor is a multiple of the identity tensor, see (5.8), it is easy to see that for 1-1 tensors we have $A_k^i = A_k^i$.

5.2.2. Differential operators. Using formulae derived in Section 5.2.1 one can find expressions for the differential operators in the bipolar coordinates. The gradient of a scalar function is

$$\nabla \phi = \frac{\cosh \alpha - \cos \beta}{a} \left[ \frac{\partial \phi}{\partial g^\alpha} \right] e_\alpha,$$  \hspace{1cm} (5.11)

The gradient of a vector field $\mathbf{v}$ and the symmetric part of the gradient of a vector field $\mathbf{v}$ are given by

$$\nabla \mathbf{v} = \frac{1}{a} \left[ \frac{\cosh \alpha - \cos \beta}{a} \left( \frac{\partial \mathbf{v}}{\partial g^\alpha} \right) + \frac{\cosh \alpha - \cos \beta}{a} \left( \frac{\partial \mathbf{v}}{\partial g^\beta} \right) \right] = \frac{1}{2} \left( \nabla \mathbf{v} + \nabla \mathbf{v}^T \right),$$  \hspace{1cm} (5.12)

Consequently, the expression for the term $[\nabla \mathbf{v}] \mathbf{v}$ reads

$$[\nabla \mathbf{v}] \mathbf{v} = \frac{1}{2} \left( \frac{\cosh \alpha - \cos \beta}{a} \left( \frac{\partial \mathbf{v}}{\partial g^\alpha} \right) + \frac{\cosh \alpha - \cos \beta}{a} \left( \frac{\partial \mathbf{v}}{\partial g^\beta} \right) \right) \mathbf{v} \left( \frac{\cosh \alpha - \cos \beta}{a} \left( \frac{\partial \mathbf{v}}{\partial g^\alpha} \right) + \frac{\cosh \alpha - \cos \beta}{a} \left( \frac{\partial \mathbf{v}}{\partial g^\beta} \right) \right).$$  \hspace{1cm} (5.14)

The divergence of a vector field $\mathbf{v}$ is

$$\text{div} \mathbf{v} = \text{Tr} \nabla \mathbf{v} = \cosh \alpha - \cos \beta \left( \frac{\partial v^\alpha}{\partial \alpha} + \frac{\partial v^\beta}{\partial \beta} \right) - \left( \frac{\sin \beta}{a} \hat{v}^\beta + \frac{\sin \alpha}{a} \hat{v}^\alpha \right).$$  \hspace{1cm} (5.15)

Using (5.11) and (5.14) it can be shown that

$$\Delta \phi = \left( \frac{\cosh \alpha - \cos \beta}{a} \right)^2 \left( \frac{\partial^2 \phi}{\partial \alpha^2} + \frac{\partial^2 \phi}{\partial \beta^2} \right).$$  \hspace{1cm} (5.16)

The divergence of a 1-1 tensor field $\mathbf{A}$ is

$$\text{div} \mathbf{A} = \frac{\cosh \alpha - \cos \beta}{a} \left[ \frac{\partial A^\alpha}{\partial \alpha} + \frac{\partial A^\beta}{\partial \beta} + \frac{\sinh \alpha}{\cosh \alpha - \cos \beta} \left( A^\beta - A^\alpha \right) \right] + \frac{\sin \beta}{a} \hat{v}^\beta + \frac{\sin \alpha}{a} \hat{v}^\alpha,$$  \hspace{1cm} (5.17)

in particular

$$\Delta \mathbf{v} = \text{div} \nabla \mathbf{v} = \left[ \frac{\cosh \alpha - \cos \beta}{a} \right]^2 \left( \frac{\partial^2 \mathbf{v}}{\partial \alpha^2} + \frac{\partial^2 \mathbf{v}}{\partial \beta^2} \right) + \frac{\cosh \alpha - \cos \beta}{a} \left( \frac{\partial^2 \mathbf{v}}{\partial \alpha \partial \beta} \right) \left( \frac{\partial^2 \mathbf{v}}{\partial \alpha \partial \beta} \right) + \frac{\cosh \alpha - \cos \beta}{a} \left( \frac{\partial \mathbf{v}}{\partial \alpha} \right) \left( \frac{\partial \mathbf{v}}{\partial \beta} \right) + \frac{\sin \beta}{a} \hat{v}^\beta + \frac{\sin \alpha}{a} \hat{v}^\alpha,$$  \hspace{1cm} (5.18)

Let us denote $h = \frac{\cosh \alpha - \cos \beta}{a}$, the scale factor that frequently occurs in the expressions for differential operators, and let us note that, according to (5.11), we have

$$\nabla h^2 = 2h^2 \left[ \frac{\sinh \alpha}{a} \frac{\sin \beta}{a} \right].$$  \hspace{1cm} (5.19)
5.3. Explicit analytical solution for the electric field. Let us solve (2.3) with boundary conditions (5.7a) reads \( \phi(\alpha, \beta) = \Phi_0 + \Phi_S \), where \( \Phi = \frac{1 - \eta_S}{\alpha_0 - \eta_0} \Phi_S = \frac{1 - \eta_S}{\alpha_0 - \eta_0} \). Consequently, the intensity of the electric field \( E \), its gradient \( \nabla E \), and the force term \( |\nabla E| \) \( E \) in the balance of linear momentum \( \nabla \) are given by

\[
E = -\nabla \phi = -h \Phi \begin{bmatrix} \sinh \alpha \\ \sin \beta \end{bmatrix}, \quad \nabla E = -h \Phi \begin{bmatrix} \sinh \alpha \\ \sin \beta \end{bmatrix}, \quad |\nabla E| \begin{bmatrix} \sinh \alpha \\ \sin \beta \end{bmatrix}.
\]

(5.20a)

Furthermore, using (5.10) it is easy to see that

\[
\nabla |E|^2 = 2 \left(h \Phi^2 \begin{bmatrix} \sinh \alpha \\ \sin \beta \end{bmatrix} \right).
\]

(5.20b)

5.4. Formulae for the divergence of the Cauchy stress tensor. Since we have a formula for the intensity of the electric field, we can substitute for \( E \) in (8.2) and get a convenient formulae for \( \nabla T \) in terms of the coordinates \( \alpha, \beta \) and the fields \( p \) and \( v \). Using (5.13) and (5.20) we see that

\[
\begin{align*}
\mathbb{D}E \otimes E + E \otimes \mathbb{D}E &= (h \Phi)^2 (\mathbb{D} + F),
\quad F = \begin{bmatrix} h \frac{\partial \sinh \alpha}{\partial \alpha} - \sin \beta \frac{\partial \sinh \alpha}{\partial \alpha} & 0 \\
0 & - \left( h \frac{\partial \sin \beta}{\partial \beta} - \sin \alpha \frac{\partial \sin \beta}{\partial \alpha} \right) \end{bmatrix},
\end{align*}
\]

(5.21)

and finally, using (5.21) and (5.19), we obtain

\[
\nabla \cdot (\mathbb{D}E \otimes E + E \otimes \mathbb{D}E) = (h \Phi)^2 \left( 2 \mathbb{D} \left[ \begin{bmatrix} \sinh \alpha \\ \sin \beta \end{bmatrix} \right] + \frac{1}{2} \Delta v \right) + (h \Phi)^2 \left( 2 \mathbb{F} \left[ \begin{bmatrix} \sinh \alpha \\ \sin \beta \end{bmatrix} \right] + \nabla F \right).
\]

(5.22)

Another straightforward calculation yields

\[
\nabla \cdot \left( \begin{bmatrix} 1 + \frac{\mu_1}{2 \mu_0} |E|^2 \end{bmatrix} \mathbb{D} \right) = \frac{1}{2} \left( 1 + \frac{\mu_1}{2 \mu_0} |E|^2 \right) \Delta v + \frac{\mu_1}{\mu_0} (h \Phi)^2 \mathbb{D} \left[ \begin{bmatrix} \sinh \alpha \\ \sin \beta \end{bmatrix} \right].
\]

(5.23)

Divergence of the extra stress tensors (5.22) therefore reads

\[
\begin{align*}
\nabla S_1 &= \hat{\mu}_0 \Delta v + \frac{\mu_1}{2} (h \Phi)^2 \Delta v + 2 \hat{\mu}_1 (h \Phi)^2 \mathbb{D} \left[ \begin{bmatrix} \sinh \alpha \\ \sin \beta \end{bmatrix} \right] + \hat{\mu}_1 (h \Phi)^2 \left( 2 \mathbb{F} \left[ \begin{bmatrix} \sinh \alpha \\ \sin \beta \end{bmatrix} \right] + \nabla F \right),
\end{align*}
\]

(5.24a)

\[
\begin{align*}
\nabla S_2 &= \hat{\mu}_0 \Delta v + \frac{\mu_1}{2} (h \Phi)^2 \Delta v + 2 \hat{\mu}_1 (h \Phi)^2 \mathbb{D} \left[ \begin{bmatrix} \sinh \alpha \\ \sin \beta \end{bmatrix} \right].
\end{align*}
\]

(5.24b)

Obviously, the governing equations for models (3.2a) and (3.2b) are now, unlike in the cases discussed in Section 4, different, and one can expect that the velocity and pressure fields corresponding to the different models are different. This is, however, not completely true, it is possible to prove that the additional term in (5.24a)—compared to (5.24b)—does not have an influence on the velocity field.

5.5. On the influence of various terms on the velocity field. Let us now formulate and prove a proposition concerning the influence of various terms in the governing equations for the velocity field. We show that many terms can be, in our particular problem, rewritten as the gradient of a scalar function and consequently "absorbed in the pressure"—such terms do not have an influence on the velocity field.

Proposition 1. Let \((p_1, v_1, E_1)\) be a solution to (5.21) with \( T \) given by (5.2a), and subject to boundary conditions (5.7). Then \((p_2, v_2, E_2)\),

\[
p_2 = \text{def} \ p_1 - \hat{\mu}_1 (h \Phi)^2 \left( \frac{\partial v_1}{\partial \alpha} - \sin \beta \frac{\partial v_1}{\partial \alpha} \right),
\]

(5.25)

and \( v_2 = \text{def} \ v_1, E_2 = \text{def} \ E_1 \), is a solution to (5.21) with \( T \) given by (5.2a) and subject to boundary conditions (5.7).

Let \( \varepsilon_r = 0 \) and let \((p_0, v_0, E_0)\) be a solution to (5.21) with \( T \) given by (5.24a) or (5.24b), and subject to boundary conditions (5.7). Then \((p_{\varepsilon_r}, v_{\varepsilon_r}, E_{\varepsilon_r})\) where

\[
p_{\varepsilon_r} = \text{def} \ p_0 + \frac{1}{2} \varepsilon_0 \varepsilon_r \Phi^2, \quad h^2,
\]

(5.26)

and \( v_{\varepsilon_r} = \text{def} \ v_0, E_{\varepsilon_r} = \text{def} \ E_0 \), is a solution to (5.21) with \( \varepsilon_r \neq 0 \) and with \( T \) given by (5.24a) or (5.24b) and subject to boundary conditions (5.7).

Proof. Obviously, the governing equations for the fluid (5.2a) and (5.2b) differ in the term

\[
\hat{\mu}_1 (h \Phi)^2 \left( 2 \mathbb{F} \left[ \begin{bmatrix} \sinh \alpha \\ \sin \beta \end{bmatrix} \right] + \nabla F \right).
\]

(5.27)
This term can be however rewritten as a gradient of a scalar function. Using (5.19) we see that (5.27) can be rewritten as 
\[ \mu_1 \Phi^2 (F \nabla h^2) + h^2 \text{div} F \] 
and consequently, (5.27) is equal to \( \mu_1 \Phi^2 \text{div} (h^2 F) \). In virtue of the isochoric condition \( \text{div} \mathbf{v} = 0 \) and its expression in the coordinate system of interest (5.13), we see that \( F_{\alpha}^\alpha - F_{\beta}^\beta = 0 \), and hence
\[ \mu_1 \Phi^2 \text{div} (h^2 F) = \mu_1 \Phi^2 \text{div} \left( h^2 \left( h \frac{\partial \varphi}{\partial \alpha} - \frac{\sin \beta}{a} \psi \right) \right) \] 
\[ = \mu_1 \Phi^2 \nabla \left( h^2 \left( \frac{\partial \varphi}{\partial \alpha} - \frac{\sin \beta}{a} \psi \right) \right). \] (5.28)

Let us now consider the term \( \nabla E F \), recalling formulae (5.19) and (5.20a) it immediately follows that \( \nabla E F = \frac{1}{2} \nabla (\Phi^2 h^2) \).

The balance of linear momentum (5.14) for the model (5.2b) therefore reads
\[ \frac{\text{dv}}{dt} = -\nabla \left( p - \mu_1 \Phi^2 h^2 \left( h \frac{\partial \varphi}{\partial \alpha} - \frac{\sin \beta}{a} \psi \right) - \frac{1}{2} \hat{\varepsilon} \epsilon \Phi^2 h^2 \right) + \hat{\mu}_0 \Delta \mathbf{v} + \frac{\hat{\mu}_1}{2} (h\Phi)^2 \mathbf{v} + 2\hat{\mu}_1 (h\Phi)^2 \mathbf{D} \left[ \frac{\sinh \alpha}{a} \frac{\sin \beta}{a} \right], \quad (5.29a) \]
and for the model (3.2b) we get
\[ \frac{\text{dv}}{dt} = -\nabla \left( p - \frac{1}{2} \hat{\varepsilon} \epsilon \Phi^2 h^2 \right) + \hat{\mu}_0 \Delta \mathbf{v} + \frac{\hat{\mu}_1}{2} (h\Phi)^2 \mathbf{v} + 2\hat{\mu}_1 (h\Phi)^2 \mathbf{D} \left[ \frac{\sinh \alpha}{a} \frac{\sin \beta}{a} \right]. \quad (5.29b) \]
The results (5.29a) and (5.29b) now follow immediately. \( \square \)

5.6. On the mean normal stress, the pressure and the Lagrange multiplier \( p \). As we have already noted in the introduction, if the fluid is specified by the constitutive relation (5.24), we cannot in general expect that \( \text{Tr} S = 0 \). Indeed, formula (5.21) gives
\[ \mathbf{T} = -pI + 2\hat{\mu}_0 \mathbf{D} + \hat{\mu}_1 (h\Phi)^2 (\mathbf{D} + F) = -p\mathbf{I} + \left( 2\hat{\mu}_0 + \hat{\mu}_1 (h\Phi)^2 \right) \mathbf{D} + \hat{\mu}_1 (h\Phi)^2 \left( h \frac{\partial \varphi}{\partial \alpha} - \frac{\sin \beta}{a} \psi \right) \left[ \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right], \quad (5.30) \]
and consequently
\[ \text{Tr} \mathbf{T} = -3p + 2\hat{\mu}_1 (h\Phi)^2 \left( h \frac{\partial \varphi}{\partial \alpha} - \frac{\sin \beta}{a} \psi \right) = -3p - 2\hat{\mu}_1 (h\Phi)^2 \left( h \frac{\partial \varphi}{\partial \alpha} - \frac{\sin \beta}{a} \psi \right), \quad (5.31) \]
where the second equality follows from \( F_{\alpha}^\alpha - F_{\beta}^\beta = 0 \), see (5.27) and the following discussion. Note that as a consequence of the boundary conditions (5.7) we have \( \text{Tr} \mathbf{T} \big|_{\alpha = \alpha_{\text{in}}, \alpha = \alpha_{\text{out}}} = -3p \). On the other hand, it is obvious that if we use model (5.20b) we get \( \text{Tr} \mathbf{T} = -3p \). This shows that for electrorheological fluids we need to differentiate between the Lagrange multiplier \( p \) associated to the incompressibility constraint and the mean normal stress \( \pi = \text{def} -\frac{1}{3} \text{Tr} \mathbf{T} \) (the mechanical pressure). If we deal with model (5.20b), then it is not, in general, true that \( \pi = p \), although this equality can hold in special cases as Poiseuille and cylindrical Couette flow, see (4.1) and (4.2). In our particular case we have
\[ \pi_1 = p + \frac{2}{3} \hat{\mu}_1 (h\Phi)^2 \left( h \frac{\partial \varphi}{\partial \alpha} - \frac{\sin \beta}{a} \psi \right), \quad (5.32a) \]
for the mean normal stress in fluid (5.24), and \( \pi_2 = p_2 \) for the mean normal stress in fluid (5.25). Proposition 1 gives
\[ \pi_1 - \pi_2 = \frac{5}{3} \hat{\mu}_1 (h\Phi)^2 \left( h \frac{\partial \varphi}{\partial \alpha} - \frac{\sin \beta}{a} \psi \right) \]
or alternatively \( \pi_1 = \frac{1}{3} (5p_1 - 3p_2) \).

5.7. Numerical solution of the governing equations. Although we are able to simplify the original system (5.1) by giving an explicit formula for \( E \), the remaining equations (5.10) and (5.11b) are still sufficiently complicated that we cannot expect to find an exact solution. (An exact solution is not known even for the classical Navier–Stokes fluid.) Therefore, we need to solve the remaining part of the system numerically. In steady flow, the governing equations read
\[ \left[ \nabla \mathbf{v} \right] \mathbf{v} = -\nabla p + \text{div} S, \quad (5.34a) \]
\[ \text{div} \mathbf{v} = 0. \quad (5.34b) \]
We solve the problem in bipolar coordinates—this allows us to work with a rectangular domain (recall that \( \alpha, \beta \in [\alpha_{\text{in}}, \alpha_{\text{out}}] \times [0, 2\pi] \)) and consequently avoid difficulties with the curved boundary. Formulation in bipolar coordinates further enables a straightforward implementation of the boundary conditions (5.7d) and provides more transparent control of the different length scales in the “narrowest gap” and “widest gap” regions. Once the governing equations are converted to bipolar coordinates (see formulae (5.11)–(5.18) and (5.24)) we can employ a simple linear transformation
\[ \pi = \frac{1}{\alpha_{\text{out}} - \alpha_{\text{in}}} (-2\alpha + (\alpha_{\text{in}} + \alpha_{\text{out}})), \quad (5.35) \]

\[ ^{11} \text{In what follows the third spatial dimension (z coordinate) plays an important role, therefore we write all formulae in their fully three dimensional form.} \]
that maps $[\alpha_{\text{out}}, \alpha_{\text{in}}]$ to the canonical interval $[-1, 1]$. The problem in the rectangular domain $[\alpha, \beta] \in [-1, 1] \times [0, 2\pi]$ can be now solved by a straightforward application of a pseudospectral collocation method. Since the $\beta$ coordinate is periodic, we use Fourier interpolants in the $\beta$ direction, and we use Chebyshev interpolants in the $\alpha$ direction. Consequently the tensor product grid consist of Chebyshev Gauss–Lobatto/Fourier collocation points $[\alpha_i, \beta_j]$ where

$$
\alpha_i = \cos \left( \frac{(k-1)\pi}{M-1} \right), \quad i = 1, \ldots, M; \quad \beta_j = \frac{2\pi}{N}(k-1), \quad k = j, \ldots, N.
$$

See Figure 2a for the tensor product grid in the computational space and Figure 2b for its image in the physical space. In what follows we will use a shorthand notation $w_{i,j} = w(\alpha_i, \beta_j)$ for values of the function $w$ at the given grid point.

Since we are in principle dealing with a boundary value problem with variable coefficients, we work entirely in the physical (not Fourier) space, and for differentiation we use spectral differentiation matrices in the form given by Weideman and Reddy (2000). We use the same grid, both for the velocity and the pressure. The balance of linear momentum is enforced at all the inner collocation points, and the divergence equation is enforced at all the collocation points—in other words we use a $Q_N$–$Q_N$ method, see Canuto et al. (2006, 2007). It is, however, well known—see for example Bernardi et al. (1988) and Schumack et al. (1991)—that although the $Q_N$–$Q_N$ method gives (for the Stokes problem) the correct solution for the velocity field, the method suffers from spurious oscillations in the pressure $p$. This fact is especially inconvenient in our case, where the knowledge of the pressure field (the Lagrange multiplier $p$) constitutes the most important part of the solution to the problem. In order to eliminate the spurious oscillations, we have applied a filtration procedure that was developed by Phillips and Roberts (1993) that is based on detailed knowledge of the form of the possible spurious modes and on the singular value decomposition of the modified pressure Schur complement matrix (Uzawa operator).

The nonlinear term $[\nabla v] v$ is treated by the standard Newton–Raphson iteration, see for example Thomasset (1981), and is always considered in the calculations reported in Section 6, thus we do not treat the problem in the creeping flow approximation. The initial guess for the Newton–Raphson iteration is the solution to the Stokes problem (solution to the governing equations without the nonlinear term).

### 6. Results

The numerical results were obtained for a wide range of parameters. In what follows we mainly report results for a particular case specified by the parameters summarized in Table 1. These parameters correspond to the situation where the diameter of the inner cylinder is “small” and the inner cylinder is placed eccentrically such that the width of the “widest” gap is comparable to the diameter of the inner cylinder. The inner cylinder is rotating and the outer cylinder is kept at rest. In this setting it is possible to clearly observe (for the classical Navier–Stokes fluid) the well known recirculation phenomenon, see for example Szeri (1998).

**Table 1: Parameter values (geometry and boundary conditions).** Note that $a$, $\alpha_{\text{in}}$ and $\alpha_{\text{out}}$ are not independent, they are given by the formulae (5.36).

<table>
<thead>
<tr>
<th>Parameter</th>
<th>$\eta_0$</th>
<th>$\eta^*$</th>
<th>$\eta_R$</th>
<th>$\eta_1$</th>
<th>$a$</th>
<th>$\alpha_{\text{in}}$</th>
<th>$\alpha_{\text{out}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Value</td>
<td>2</td>
<td>0.5</td>
<td>0.6</td>
<td>1.15503</td>
<td>0.986866</td>
<td>0.64694</td>
<td></td>
</tr>
</tbody>
</table>
The remaining parameters \( \hat{v}_0 = \frac{1}{R_0}, \hat{\mu}_1 \) and \( \hat{\varepsilon}_0 \varepsilon_r = \frac{\pi}{M} \) are varied in order to determine the influence of the various terms in the governing equations on the solution. In what follows, we assume that the assumptions that lead us to the model for the flow of electrorheological fluid in between the eccentrically placed cylinders are valid for all possible parameter values, namely we do not take into account the possibility of dielectric breakdown due to a strong electric field, cavitation, and other phenomena. Note that when reporting results for the pressure \( p \) (more precisely the Lagrange multiplier), we in fact report results concerning the difference of \( p \) from a reference level \( p_0 \).

6.1. Convergence of the numerical method. The numerical method converges (as expected) in general very fast, whereas the convergence is slower for electrorheological fluids and faster for the classical Navier–Stokes fluid. For small Reynolds numbers (up to approximately 100) we have found grid \( N \times M = 30 \times 25 \) to be sufficient (with an adequate safety margin) for all the computations with \( \mu_1 \) and \( \hat{\varepsilon}_0 \varepsilon_r \) varying in the ranges specified in Section 6.2 and Section 6.3. Concerning the results reported in Section 6.4 (dependence of the solution on the Reynolds number) we use a finer grid, namely \( N \times M = 50 \times 45 \).

For visualization purposes we reinterpolate all computed raw data to the fine grid \( N \times M = 120 \times 40 \), for reinterpolation we use the method described and implemented by Weideman and Reddy (2000). Note that since we use a pseudospectral spectral method the raw data in principle correspond to coefficients in a Fourier series expansion, and consequently the reinterpolation is a minimal source of further numerical errors.

Concerning the Stokes problem (no nonlinearity due to the inertial term) we have found the residuum for the discrete problem to be of order \( 10^{-9} \) or lower, the residuum in the full nonlinear problem ranges from \( 10^{-6} \) to \( 10^{-8} \), the relative residuum is always less than \( 10^{-9} \). Usually only two or three Newton–Raphson iterations are required to achieve the reported accuracy.

6.2. Dependence on \( \mu_1 \). Let us now discuss the behaviour of the solution with respect to parameter \( \hat{\mu}_1 \). In this section we fix \( \hat{\varepsilon}_0 \varepsilon_r = 0 \) in order to isolate effects due to the term \( \varepsilon_0 \varepsilon_r |\nabla E| \nabla E \). (Note that in view of Proposition 11 the term \( \varepsilon_0 \varepsilon_r |\nabla E| \nabla E \) influences only the pressure field.) Since the governing equations—in this case—contain only the combination \( \hat{\mu}_1 \Phi_0^2 \), increasing \( \hat{\mu}_1 \) in fact either corresponds to increasing the value of material constant \( \mu_1 \) or potential ratio \( \varepsilon_0 \). Recall that \( \mu_1 = 0 \) corresponds to the classical Navier–Stokes fluid. All the results reported in this Section are for \( \Re = 10 \), the described trends are nevertheless to a great extent valid for all Reynolds numbers that do not lead to dominant effects due to the nonlinearity \( |\nabla \Phi| \Phi \).

As we have shown in Proposition 1 the velocity field is the same for both models (3.2), therefore we need to analyze the velocity field only for one of the models. The plots of the velocity field, see Figure 3, show that increasing \( \hat{\mu}_1 \) boosts the backflow, and it only slightly moves the point where the azimuthal velocity changes its orientation (see Figure 3a). The backflow boost is also clearly visible if we compare the “global” view of the velocity field, see Figure 5a (Navier–Stokes fluid) and Figure 5d (electrorheological fluid with \( \hat{\mu}_1 = 10 \)).

Concerning the pressure field \( p \), model (3.2a) leads to a significant change in the pressure distribution in the domain. Higher values of \( \hat{\mu}_1 \) lead to formation of a steep pressure drop across the region close to the narrowest gap between the cylinders, see Figure 4. With increasing \( \hat{\mu}_1 \), the difference between the highest and lowest pressure increases and the pressure difference is concentrated in a smaller region; the region of the highest pressure moves towards the narrowest gap, see especially Figure 4e and Figure 4f.

If we consider model (3.2b), the pressure (the mean normal stress) \( \pi \) does not coincide with the Lagrange multiplier \( p \) (see discussion in Section 5.6). The Lagrange multiplier \( p \) exhibits nearly the same behaviour as \( p \) in model (3.2a), the magnitude of the drop across the narrowest gap region remains almost the same, the difference between the models manifests itself mainly in the internal structure of the \( p \) field, compare Figure 5a and Figure 5b. Concerning the mean normal stress \( \pi = \frac{1}{2} \text{Tr} \tau \), the magnitude is about \( \frac{3}{4} \) of \( p \)—this is a consequence of formula (5.33) and the fact that \( p_2 \) (the Lagrange multiplier/pressure for model (3.2b)) is nearly of the same magnitude as \( p_1 \) (the Lagrange multiplier for model (3.2a)). Clearly, the distribution of the mean normal stress is different for models (3.2a) and (3.2b), compare Figure 5a and Figure 5d.

6.3. Dependence on \( \hat{\varepsilon}_0 \varepsilon_r \). The dependence of the solution on \( \hat{\varepsilon}_0 \varepsilon_r \) is illustrated for \( \Re = 10 \) and \( \hat{\mu}_1 = 1 \). Let us first consider model (3.2a). The influence of \( \hat{\varepsilon}_0 \varepsilon_r \) is best seen in the plot of the pressure in the computational space, see Figure 6a and Figure 6b. Initially (for small values of \( \hat{\varepsilon}_0 \varepsilon_r \)) the pressure field shows little variation in the radial direction, at least compared to variation in the azimuthal direction. Further, the minimal and maximal pressure are located on the surface of the inner cylinder. When \( \hat{\varepsilon}_0 \varepsilon_r \) increases the pressure field starts to vary even in the radial direction and finally the maximal pressure is located on the surface of the outer cylinder while the minimal pressure stays located on the surface of the inner cylinder. The plug flow region is shown in Figure 6a and pressure distribution along the cylinder walls is shown in Figure 6c and 6d. Obviously, if \( \hat{\varepsilon}_0 \varepsilon_r \) is small, its influence on the pressure field is also small as expected.

Concerning model (3.2b), the influence of \( \hat{\varepsilon}_0 \varepsilon_r \) on the Lagrange multiplier field \( p \) is similar (not shown).

6.4. Dependence on \( \Re \). All the results given in this section are, in contrast to the previous sections, computed on the grid \( N \times M = 50 \times 45 \) and reinterpolated to the grid \( 120 \times 40 \). The reason is that for computations with higher Reynolds numbers (that are of interest in this section) we need a finer grid in order to achieve sufficient accuracy.

Let us first consider the classical Navier–Stokes fluid without any effects due to the electric field. As it is well known, the increase of the Reynolds number leads to increase of the role of nonlinearity \( |\nabla \Phi| \Phi \). Plots of the velocity and pressure field for various values of the Reynolds number are shown in Figure 7 and Figure 8. Recall that \( \hat{v}_0 = \frac{1}{R_0} \) and consequently
increasing the Reynolds number corresponds to decreasing \( \mu_0 \). The balance of linear momentum for the Navier–Stokes fluid reads \([\nabla v]v = -\nabla p + \mu_0 \Delta v\), and the balance of linear momentum for electrorheological fluid (6.2) is

\[
[\nabla v]v = -\nabla p + \left( \mu_0 + \frac{\mu_1}{2} (h \Phi)^2 \right) \Delta v + 2 \mu_1 (h \Phi)^2 D \left\{ \sinh \frac{\alpha}{h} \frac{\sin \frac{\beta}{h}}{\sinh \frac{\alpha}{h}} \right\} + \varepsilon_0 \varepsilon_r [\nabla E] E, \tag{6.1}
\]

and therefore it is not very surprising that the change of the velocity field due to the decrease in the Reynolds number (see Figure 7) is similar to the changes due to the increase of \( \mu_1 \) as discussed in Section 6.2, see especially Figure 3. In both cases we change the magnitude of the coefficient in front of the Laplace operator and a similar influence on the velocity field can be therefore expected. (Remaining terms in (6.1) either do not have influence on the velocity field, see Proposition 1, or can be understood as a perturbation to the “dominant” Laplace operator.)

The pressure field is however completely different. As documented in Figure 8, increasing the Reynolds number leads to substantial changes in the distribution of the pressure in the domain. For small Reynolds numbers we get almost uniform distribution of the pressure on the radial sections (at least compared to variations in the azimuthal sections), see Figure 8a. Further, the minimal and maximal pressure are located on the surface of the inner cylinder. When Re increases the pressure field starts to vary even in the radial direction and finally the minimal pressure is located on the surface of the outer cylinder.
Figure 4: Model (3.2b), dependence of the pressure field $p$ on $\hat{\mu}_1$. 

(a) Pressure on the inner cylinder wall, $\overline{\alpha} = 1$, ($\overline{\alpha} = \overline{\alpha}_k, k = 1$).

(b) Pressure on the outer cylinder wall, $\overline{\alpha} = 1$, ($\overline{\alpha} = \overline{\alpha}_k, k = \overline{\alpha}$).

(c) Pressure in the physical space, $\hat{\mu}_1 = 0$ (Navier–Stokes fluid). The region circumscribed with the dashed line corresponds to the “plug region” shown in Figure 4c.

(d) Pressure in the physical space, $\hat{\mu}_1 = 0.05$. The region circumscribed with the dashed line corresponds to the “plug region” shown in Figure 4d.

(e) Pressure in the physical space, $\hat{\mu}_1 = 0$ (Navier–Stokes fluid), the plug region is magnified. Pressure isolines are shown in the range from 2 to 2.8 in steps of 0.05.

(f) Pressure in the physical space, $\hat{\mu}_1 = 0.05$, the plug region is magnified. Pressure isolines are shown in the range from 8.5 to 10.5 in steps of 0.25.
while the maximal pressure stays located on the surface of the inner cylinder, see Figure 8c. (Compare with the effect discussed in Section 6.3.) A magnification of the plug region is shown in Figure 8d–f.

The behaviour of the solution with respect to the Reynolds number is dominated by the term \( \bar{\varepsilon} \mu \), which is irrelevant with respect to the velocity field. It influences only the pressure field.

7. Conclusion

We have studied the behaviour predicted by two models for the response of electrorheological fluids, which are expected to be quite different within a fully three dimensional setting. Both models can be used to model the dependence of the viscosity on the intensity of the electric field. We have shown that although the models are not identical, they lead to the same velocity and pressure fields in simple shear flows (plane Poiseuille flow and cylindrical Couette flow). Next, we have considered flows of fluids between eccentric cylinders. Both models can be used to model the dependence of the viscosity on the intensity of the electric field. We have proved that in the case of flow between eccentric cylinders the models lead to the same velocity field but they lead to different pressure fields. Thus, even the flow due to cylinders rotating about non-coincident axes is insufficient to produce a problem wherein both the velocity and pressure fields are different and one would need a more complicated flow geometry to fully discriminate between the three dimensional constitutive models. In the present study, the problem of flow between eccentric cylinders has been solved numerically and we have discussed the behaviour of the solution with respect to various parameters that have an effect on the flow.

Note that if we were working with one material (fixed material constants) and the magnitude of the Reynolds number is adjusted by the choice of the characteristic velocity (increasing/decreasing the angular velocity of the inner cylinder) then the experiment of the type “keep \( \bar{\mu}_1 \) and \( \bar{\varepsilon}_0 \bar{\varepsilon}_r \) constant and vary Re” would have been impossible even if we were allowed to change the characteristic intensity of the electric field, see the definition of the dimensionless parameters in Section 8.1.
The paper points out the necessity to consider fully three dimensional constitutive relations instead of one dimensional constitutive relations, and provides an explicit example and a thorough discussion of two different three dimensional constitutive relations that coincide in simple shear flows but that are not identical in more complicated flows. Moreover, the flow geometry we have used to document the difference between the considered models corresponds to a technologically relevant problem—the problem of flow of an electrorheological fluid in a journal bearing.

**REFERENCES**


Figure 7: Navier–Stokes fluid, dependence of the velocity field on Re.

(a) Radial velocity \( v^r \) in the section through the widest gap, \( \beta = 0 \) \( (\beta = \beta_j, \ j = 1) \).

(b) Azimuthal velocity \( v^\beta \) in the section through the widest gap, \( \beta = 0 \) \( (\beta = \beta_j, \ j = 1) \).

Figure 8: Navier–Stokes fluid, dependence of the pressure field on Re.

(a) Pressure field in the computational space, Re = 0.01.

(b) Pressure field in the computational space, Re = 50.

(c) Pressure field in the computational space, Re = 200.

(d) Pressure field in the physical space, the plug region is magnified, Re = 0.01.

(e) Pressure field in the physical space, the plug region is magnified, Re = 50.

(f) Pressure field in the physical space, the plug region is magnified, Re = 200.
(a) Radial velocity $v^\alpha$ in the section through the widest gap, $\beta = 0$ ($\beta = \beta_j$, $j = 1$).

(b) Azimuthal velocity $v^\beta$ in the section through the widest gap, $\beta = 0$ ($\beta = \beta_j$, $j = 1$).

Figure 9: Model (3.2b), dependence of the velocity field on $Re$, $\varepsilon_0 = 10$.

(a) Pressure field in the computational space, $Re = 0.01$.

(b) Pressure field in the computational space, $Re = 0.1$.

(c) Pressure field in the computational space, $Re = 0.5$.

(d) Pressure field in the computational space, $Re = 1.0$.

Figure 10: Model (3.2b), dependence of the pressure field on $Re$, $\varepsilon_0 = 10$.

(a) Pressure field in the computational space, $Re = 0.01$.

(b) Pressure field in the computational space, $Re = 0.1$.

(c) Pressure field in the computational space, $Re = 0.5$.

(d) Pressure field in the computational space, $Re = 1.0$.

Figure 11: Model (3.2b), dependence of the pressure field on $Re$, $\varepsilon_0 = 0.01$.


