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Abstract

We completely and explicitly describe Young measures generated by matrix-valued mappings $\{Y_k\}_{k\in\mathbb{N}} \subset L^p(\Omega;\mathbb{R}^{n\times n})$, $\Omega \subset \mathbb{R}^n$, such that $\{Y_k^{-1}\}_{k\in\mathbb{N}} \subset L^p(\Omega;\mathbb{R}^{n\times n})$ is bounded, too. Moreover, the constraint $\det Y_k > 0$ can be easily included and is reflected in a condition on the support of the measure. These results allow us to relax minimization problems for functionals $J(Y) := \int_\Omega W(Y(x))\,\mathrm{d}x$, where $W(F)$ tends to infinity if the determinant of F converges to zero. This phenomenon typically occurs in problems of nonlinear-elasticity theory for hyperelastic materials if $Y := \nabla y$ for $y \in W^{1,p}(\Omega;\mathbb{R}^n)$, for instance. We touch this particular situation with the additional condition $\det \nabla y > 0$, as well.

Key Words: Orientation-preserving mappings, relaxation, Young measures
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1 Introduction

In this paper, we investigate a new tool to study minimization problems for integral functionals defined over matrix-valued mappings that take values *only* in the set of regular matrices. Typical examples are found, e.g., in non-linear elasticity where static equilibria are minimizers of the elastic energy, i.e., one is led to solve

$$\text{minimize } J(y) := \int_\Omega W(\nabla y(x))\,\mathrm{d}x \ , \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ denotes the reference configuration of the material, $y \in W^{1,p}(\Omega;\mathbb{R}^n)$ is the deformation, $1 < p < +\infty$, $y = y_0$ on $\partial\Omega$, and $W\colon \mathbb{R}^{n\times n} \to \mathbb{R}$ is the stored energy density, i.e., the potential of the first Piola-Kirchhoff stress tensor. Usually in elasticity, one demands even that $\det \nabla y > 0$ in order to preserve orientation of y .

If W is polyconvex, i.e., $W(A)$ can be written as a convex function of all minors of A, then the existence of minimizers to (1.1) was proved by J.M. Ball in his pioneering paper [3]. We refer, e.g., to [6, 7] for various results in this direction. Namely, the existence theory for polyconvex materials can even cope with the important physical assumption, namely,

$$W(A) \to +\infty \text{ whenever } \det A \to 0_+ \ . \tag{1.2}$$

On the other hand, there are *many materials that cannot be modeled by polyconvex stored energies*, prominent examples are materials with microstructure, like shape-memory materials [5, 20]; in this situation, however, minimizers to J in (1.1) do not necessarily exist. If we give up (1.2) and suppose that W has polynomial growth at infinity, e.g. for $c, \tilde{c} > 0$

$$c(-1 + |A|^p) \le W(A) \le \tilde{c}(1 + |A|^p) \ , \tag{1.3}$$

the existence of a solution to (1.1) is guaranteed if W is quasiconvex [19], which means that for all $\varphi \in W_0^{1,\infty}(\Omega;\mathbb{R}^n)$ and all $A \in \mathbb{R}^{n\times n}$ it holds that

$$|\Omega|W(A) \le \int_\Omega W(A + \nabla\varphi(x))\,\mathrm{d}x \ . \tag{1.4}$$

Yet, quasiconvexity is a very complicated property difficult to verify in many cases. Moreover, stored energy densities of materials with microstructure *do not possess this property either*. As a result, solutions to (1.1) might not exist. Various relaxation techniques were developed [7, 20, 22] to overcome this drawback. One is to *extend the notion of solutions from Sobolev mappings to parameterized measures called Young measures* [4, 10, 21, 22, 24, 25, 28]. The idea is to describe limit behavior of $\{J(y_k)\}_{k\in\mathbb{N}}$ along a minimizing sequence $\{y_k\}_{k\in\mathbb{N}}$. Nevertheless, the *growth condition* (1.3) *is still a key ingredient* in these considerations.

Our goal is to tailor the Young-measure relaxation to functions satisfying (1.2). Our key new idea is that we allow W to depend on the inverse of its argument, more precisely we suppose that W is continuous on regular matrices and that there exist positive constants $c, \tilde{c} > 0$ such that

$$c(-1 + |A|^p + |A^{-1}|^p) \le W(A) \le \tilde{c}(1 + |A|^p + |A^{-1}|^p) \ . \tag{1.5}$$

Notice that (1.5) implies (1.2) and that W has polynomial growth in $|A|$ and $|A^{-1}|$ at infinity.

This motivates the idea to perform relaxation in terms of Young measures generated by sequences of matrix-valued mappings $\{Y_k\}_{k\in\mathbb{N}} \subset L^p(\Omega;\mathbb{R}^{n\times n})$ such that $\{Y_k^{-1}\}_{k\in\mathbb{N}} \subset L^p(\Omega;\mathbb{R}^{n\times n})$ is also bounded. We show that, in this case, the Young measures are necessarily supported on regular matrices and satisfy a certain integral condition, cf. (2.1). If, additionally, $\det Y_k > 0$ almost everywhere in Ω for all $k \in \mathbb{N}$ the resulting Young measure is supported on matrices with positive determinant, cf. Theorem 2.2 and Proposition 4.2. Contrary to the general theory of Young measures generated by L^p-maps [23, 22], where only the behavior of test functions at infinity is important, Young measures supported on regular matrices are also sensitive to the asymptotics of test functions as the argument approaches a singular matrix. *However, they allow for a larger class of test functions, namely, those with growth specified in* (1.5). In particular, our test functions are not necessarily continuous on $\mathbb{R}^{n\times n}$. The precise condition is stated in Theorems 2.1, 2.2. We refer to [15] for another refinement of Young measures involving discontinuous integrands.

Although the characterization of Young measures generated by *vector-valued* mappings $\{Y_k\}_{k\in\mathbb{N}} \subset L^p(\Omega;\mathbb{R}^n)$ and such that $\{|Y_k|^{-1}\}_{k\in\mathbb{N}} \subset L^p(\Omega)$, with Ω an interval in $\mathbb{R}$ is already due to Freddi and Paroni [12], our manuscript presents, to the authors' knowledge, the first explicit and complete characterization of Young measures generated by *matrix-valued* mappings $\{Y_k\}_{k\in\mathbb{N}} \subset L^p(\Omega;\mathbb{R}^{n\times n})$ such that $\{Y_k^{-1}\}_{k\in\mathbb{N}} \subset L^p(\Omega;\mathbb{R}^{n\times n})$ is also bounded. Moreover, we examine the support of the generated Young measures.

The plan of the paper is as follows. After introducing Young measures we state our main results Theorems 2.1, 2.2 in Section 2. Their proofs are left, however, to Section 3 together with a few auxiliary results. In particular, Propositions 3.5, 3.7 are of special interest as they form an L^∞ version of our main theorems. Finally, Section 4 is devoted to two relaxation examples. One of them is related to relaxation in nonlinear elasticity.

Throughout the paper, we use standard notation for Lebesgue L^p and Sobolev $W^{1,p}$ spaces. We say that $\{u_k\}_{k\in\mathbb{N}} \subset L^1(\Omega)$ is equi-integrable if we can pick up a subsequence weakly converging in $L^1(\Omega)$. We refer e.g. to [9, 10] for details about equi-integrability and relative weak compactness in L^1. Finally, C denotes a generic positive constant which may change from line to line.

1.1 Young measures

For $p \geq 0$ we define the following subspace of the space $C(\mathbb{R}^{n\times n})$ of all continuous functions on $\mathbb{R}^{n\times n}$:

$$C_p(\mathbb{R}^{n\times n}) := \left\{ v \in C(\mathbb{R}^{n\times n});\ \lim_{|s|\to\infty} \frac{v(s)}{|s|^p} = 0 \right\} .$$

Young measures on a bounded domain $\Omega \subset \mathbb{R}^n$ are weakly* measurable mappings $x \mapsto \nu_x : \Omega \to \mathrm{rca}(\mathbb{R}^{n\times n})$ with values in probability measures; and the adjective "weakly* measurable" means that, for any $v \in C_0(\mathbb{R}^{n\times n})$, the mapping $\Omega \to \mathbb{R} : x \mapsto \langle \nu_x, v\rangle = \int_{\mathbb{R}^{n\times n}} v(s)\nu_x(\mathrm{d}s)$ is measurable in the usual sense. Let us remind that, by the Riesz theorem, $\mathrm{rca}(\mathbb{R}^{n\times n})$, normed by the total variation, is a Banach space which is isometrically isomorphic with $C_0(\mathbb{R}^{n\times n})^*$, where $C_0(\mathbb{R}^{n\times n})$ stands for the space of all continuous functions $\mathbb{R}^{n\times n} \to \mathbb{R}$ vanishing at infinity. Let us denote the set of all Young measures by $\mathcal{Y}(\Omega;\mathbb{R}^{n\times n})$. It is known that $\mathcal{Y}(\Omega;\mathbb{R}^{n\times n})$ is a convex subset of $L^\infty_{\mathrm{w}}(\Omega;\mathrm{rca}(\mathbb{R}^{n\times n})) \cong L^1(\Omega;C_0(\mathbb{R}^{n\times n}))^*$, where the subscript "w" indicates the aforementioned property of weak* measurability. Let $S \subset \mathbb{R}^{n\times n}$ be a compact set. A classical result [24, 27] is that for every sequence $\{Y_k\}_{k\in\mathbb{N}}$ bounded in $L^\infty(\Omega;\mathbb{R}^{n\times n})$ such that $Y_k(x) \in S$ there exists its subsequence (denoted by the same indices for notational simplicity) and a Young measure $\nu = \{\nu_x\}_{x\in\Omega} \in \mathcal{Y}(\Omega;\mathbb{R}^{n\times n})$ satisfying

$$\forall v \in C(S) : \quad \lim_{k\to\infty} v \circ Y_k = v_\nu \qquad \text{weakly* in } L^\infty(\Omega) , \tag{1.6}$$

where $[v \circ Y_k](x) = v(Y_k(x))$ and

$$v_\nu(x) = \int_{\mathbb{R}^{n\times n}} v(s)\nu_x(\mathrm{d}s) . \tag{1.7}$$

Moreover, ν_x is supported on S for almost all $x \in \Omega$. On the other hand, if $\mu = \{\mu_x\}_{x\in\Omega}$, μ_x is supported on S for almost all $x \in \Omega$ and $x \mapsto \mu_x$ is weakly* measurable then there exist a sequence $\{Z_k\}_{k\in\mathbb{N}} \subset L^\infty(\Omega;\mathbb{R}^{n\times n})$, $Z_k(x) \in S$ and (1.6) holds with μ and Z_k instead of ν and Y_k, respectively.

Let us denote by $\mathcal{Y}^\infty(\Omega;\mathbb{R}^{n\times n})$ the set of all Young measures which are created in this way, i.e., by taking all bounded sequences in $L^\infty(\Omega;\mathbb{R}^{n\times n})$. A generalization of this result was formulated by Schonbek [23] (cf. also [4]): if $1 \leq p < +\infty$ then for every sequence $\{Y_k\}_{k\in\mathbb{N}}$ bounded in $L^p(\Omega;\mathbb{R}^{n\times n})$ there exists its subsequence (denoted by the same indices) and a Young measure $\nu = \{\nu_x\}_{x\in\Omega} \in \mathcal{Y}(\Omega;\mathbb{R}^{n\times n})$ such that

$$\forall v \in C_p(\mathbb{R}^{n\times n}) : \quad \lim_{k\to\infty} v \circ Y_k = v_\nu \qquad \text{weakly in } L^1(\Omega) . \tag{1.8}$$

We say that $\{Y_k\}_{k\in\mathbb{N}}$ generates ν if (1.8) holds. Let us denote by $\mathcal{Y}^p(\Omega;\mathbb{R}^{n\times n})$ the set of all Young measures which are obtained through the latter procedure, i.e., by taking all bounded sequences in $L^p(\Omega;\mathbb{R}^{n\times n})$. It was shown in [17] that if $\nu \in \mathcal{Y}(\Omega;\mathbb{R}^{m\times n})$ satisfies the bound

$$\int_\Omega \int_{\mathbb{R}^{n\times n}} |s|^p \nu_x(\mathrm{d}s)\,\mathrm{d}x < +\infty \tag{1.9}$$

then $\nu \in \mathcal{Y}^p(\Omega;\mathbb{R}^{n\times n})$.

2 Main results

Let us denote $\mathbb{R}^{n\times n}_{\mathrm{reg}}$ the set of regular matrices in $\mathbb{R}^{n\times n}$ and $\mathbb{R}^{n\times n}_{\mathrm{reg}+}$ as the set of matrices in $\mathbb{R}^{n\times n}$ with positive determinant. We write inv for the continuous function defined on $\mathbb{R}^{n\times n}_{\mathrm{reg}}$ by $\mathrm{inv}(s) := s^{-1}$, i.e., creating the inversion. Further, we denote by $\mathcal{Y}^{p,-p}(\Omega;\mathbb{R}^{n\times n})$ and by $\mathcal{Y}^{p,-p}_+(\Omega;\mathbb{R}^{n\times n})$ the following subsets of $\mathcal{Y}^p(\Omega;\mathbb{R}^{n\times n})$:

$$\mathcal{Y}^{p,-p}(\Omega;\mathbb{R}^{n\times n}) := \Big\{ \nu \in \mathcal{Y}^p(\Omega;\mathbb{R}^{n\times n});\ \int_\Omega \int_{\mathbb{R}^{n\times n}_{\mathrm{reg}}} (|s|^p + |s^{-1}|^p)\nu_x(\mathrm{d}s)\,\mathrm{d}x < +\infty ,$$
$$\nu_x(\mathbb{R}^{n\times n}_{\mathrm{reg}}) = 1 \text{ for a.a. } x \in \Omega \Big\} , \tag{2.1}$$

$$\mathcal{Y}^{p,-p}_+(\Omega;\mathbb{R}^{n\times n}) := \left\{\nu \in \mathcal{Y}^{p,-p}(\Omega;\mathbb{R}^{n\times n});\ \nu_x(\mathbb{R}^{n\times n}_{\rm reg+}) = 1 \text{ for a.a. } x \in \Omega\right\}\ , \tag{2.2}$$

and the following subspace of the space of continuous functions on $\mathbb{R}^{n\times n}_{\rm reg}$

$$C_{p,-p}(\mathbb{R}^{n\times n}_{\rm reg}) := \left\{v \in C(\mathbb{R}^{n\times n}_{\rm reg});\ \lim_{|s|+|s^{-1}|\to\infty} \frac{v(s)}{|s|^p + |s^{-1}|^p} = 0\right\}\ . \tag{2.3}$$

Our main results are summarized in the following theorems.

Theorem 2.1. *Let $+\infty > p \geq 1$, let $\Omega \subset \mathbb{R}^n$ be open and bounded, and let $\{Y_k\}_{k\in\mathbb{N}}$, $\{Y_k^{-1}\}_{k\in\mathbb{N}} \subset L^p(\Omega;\mathbb{R}^{n\times n})$ be bounded. Then there is a subsequence of $\{Y_k\}_{k\in\mathbb{N}}$ (not relabeled) and $\nu \in \mathcal{Y}^{p,-p}(\Omega;\mathbb{R}^{n\times n})$ such that for every $g \in L^\infty(\Omega)$ and every $v \in C_{p,-p}(\mathbb{R}^{n\times n}_{\rm reg})$ it holds that*

$$\lim_{k\to\infty} \int_\Omega v(Y_k(x))g(x)\,{\rm d}x = \int_\Omega \int_{\mathbb{R}^{n\times n}_{\rm reg}} v(s)\nu_x({\rm d}s)g(x)\,{\rm d}x\ , \tag{2.4}$$

Conversely, if $\nu \in \mathcal{Y}^{p,-p}(\Omega;\mathbb{R}^{n\times n})$ then there is a bounded sequence $\{Y_k\}_{k\in\mathbb{N}} \subset L^p(\Omega;\mathbb{R}^{n\times n})$ such that $\{Y_k^{-1}\}_{k\in\mathbb{N}} \subset L^p(\Omega;\mathbb{R}^{n\times n})$ is also bounded and (2.4) holds for all g and v defined above.

Theorem 2.2. *Let $+\infty > p \geq 1$, let $\Omega \subset \mathbb{R}^n$ be open and bounded, and let $\{Y_k\}_{k\in\mathbb{N}}$, $\{Y_k^{-1}\}_{k\in\mathbb{N}} \subset L^p(\Omega;\mathbb{R}^{n\times n})$ be bounded and for every $k \in \mathbb{N}$ $\det Y_k > 0$ almost everywhere in Ω. Then there is a subsequence of $\{Y_k\}_{k\in\mathbb{N}}$ (not relabeled) and $\nu \in \mathcal{Y}^{p,-p}_+(\Omega;\mathbb{R}^{n\times n})$ such that for every $g \in L^\infty(\Omega)$ and every $v \in C_{p,-p}(\mathbb{R}^{n\times n}_{\rm reg})$ (2.4) holds.*

Conversely, if $\nu \in \mathcal{Y}^{p,-p}_+(\Omega;\mathbb{R}^{n\times n})$ then there is a bounded sequence $\{Y_k\}_{k\in\mathbb{N}} \subset L^p(\Omega;\mathbb{R}^{n\times n})$ such that $\{Y_k^{-1}\}_{k\in\mathbb{N}} \subset L^p(\Omega;\mathbb{R}^{n\times n})$ is also bounded, for every $k \in \mathbb{N}$ $\det Y_k > 0$ almost everywhere in Ω, and (2.4) holds for all g and v defined above.

Notice that $C(\mathbb{R}^{n\times n}) \cap C_{p,-p}(\mathbb{R}^{n\times n}_{\rm reg}) = C_p(\mathbb{R}^{n\times n})$, so we allow for a larger class of test function in Theorems 2.1, 2.2 compared with the original result by Schonbek [23] mentioned in (1.8). In particular, our test functions are not necessarily continuous on the whole $\mathbb{R}^{n\times n}$.

Remark 2.3. *For simplicity, we formulated Theorems 2.1 and 2.2 as well as definitions* (2.1), (2.2) symmetrically *in p in the sense that both the generating sequence as well as its inverse are bounded in $L^p(\Omega;\mathbb{R}^{n\times n})$. We could, however, also define for $\infty > p, q \geq 1$*

$$\mathcal{Y}^{p,-q}(\Omega;\mathbb{R}^{n\times n}) := \Big\{\nu \in \mathcal{Y}^{p}(\Omega;\mathbb{R}^{n\times n});\ \int_\Omega \int_{\mathbb{R}^{n\times n}_{\rm reg}} (|s|^p + |s^{-1}|^q)\nu_x({\rm d}s)\,{\rm d}x < +\infty\ ,$$
$$\nu_x(\mathbb{R}^{n\times n}_{\rm reg}) = 1 \textit{ for a.a. } x \in \Omega\Big\}\ , \tag{2.5}$$

$$\mathcal{Y}^{p,-q}_+(\Omega;\mathbb{R}^{n\times n}) := \left\{\nu \in \mathcal{Y}^{p,-q}(\Omega;\mathbb{R}^{n\times n});\ \nu_x(\mathbb{R}^{n\times n}_{\rm reg+}) = 1 \textit{ for a.a. } x \in \Omega\right\}\ . \tag{2.6}$$

and then Theorems 2.1 and 2.2 hold with the single modification that $\left\{Y_k^{-1}\right\}_{k\in\mathbb{N}}$ is bounded $L^q(\Omega;\mathbb{R}^{n\times n})$.

3 Auxiliary results and Proofs of Theorems 2.1, 2.2

Let us start by defining

$$R_\varrho^{n\times n} := \{A \in \mathbb{R}^{n\times n}_{\rm reg};\ \max(|A|, |A^{-1}|) \le \varrho\}\ , \tag{3.1}$$

$$R_{\varrho+}^{n\times n} := \{A \in R_\varrho^{n\times n};\ \det\, A > 0\}\ . \tag{3.2}$$

Then the following holds:

Lemma 3.1. *$R_\varrho^{n\times n}$ is compact in $\mathbb{R}^{n\times n}$ for every $\varrho > 0$. Moreover, the set $R_{\varrho+}^{n\times n}$ is also compact for every $\varrho > 0$.*

Proof. Clearly, $R_\varrho^{n\times n}$ is bounded. Consider a sequence $\{A_k\}_{k\in\mathbb{N}} \subset R_\varrho^{n\times n}$ such that $A_k \to A$. We must show that $A \in R_\varrho^{n\times n}$. If $\det A = 0$ then by continuity $\det A_k \to 0$ as $k \to \infty$ and due to the bound $|\det B| \le C|B|^n$, $C > 0$ for all $B \in \mathbb{R}^{n\times n}$ we would have

$$|1/\det A_k| = |\det A_k^{-1}| \le C|A_k^{-1}|^n \to \infty\ .$$

Hence, $A_k \notin R_\varrho^{n\times n}$ if $k \ge k_0$ which is a contradiction. Therefore, $A \in \mathbb{R}^{n\times n}_{\rm reg}$. The continuity of the matrix inverse $A_k^{-1} \to A^{-1}$ and $|A_k^{-1}| \to |A^{-1}|$ yields, in consequence, that $A \in R_\varrho^{n\times n}$ and $R_\varrho^{n\times n}$ is bounded and closed. Compactness of $R_{\varrho+}^{n\times n}$ follows by continuity of the function $A \mapsto \det A$. □

Remark 3.2. *We have $\mathbb{R}^{n\times n}_{\rm reg} = \cup_{\varrho\in\mathbb{N}} R_\varrho^{n\times n}$ and $\mathbb{R}^{n\times n}_{\rm reg+} = \cup_{\varrho\in\mathbb{N}} R_{\varrho+}^{n\times n}$, i.e., the open sets $\mathbb{R}^{n\times n}_{\rm reg}$ and $\mathbb{R}^{n\times n}_{\rm reg+}$ are both σ-compact.*

For every $v : \mathbb{R}^{n\times n} \to \mathbb{R}$ we define $\hat v : \mathbb{R}^{n\times n} \to \mathbb{R}$:

$$\hat v(s) := \begin{cases} v(s^{-1}) & \text{if } s \in \mathbb{R}^{n\times n}_{\rm reg}, \\ 0 & \text{otherwise.} \end{cases} \tag{3.3}$$

We define the following subspace of $C_0(\mathbb{R}^{n\times n})$:

$$C_{0,\rm reg}(\mathbb{R}^{n\times n}) := \{v \in C_0(\mathbb{R}^{n\times n});\ v(s) = 0 \text{ if } \det s = 0\} \tag{3.4}$$

equipped with the supremum norm. Notice, that $v = \hat{\hat v}$ for every $v \in C_{0,\rm reg}(\mathbb{R}^{n\times n})$.

Lemma 3.3. *$C_{0,\rm reg}(\mathbb{R}^{n\times n})$ is a* separable Banach space *with respect to the standard maximum norm for continuous functions. Moreover, $C_{0,\rm reg}(\mathbb{R}^{n\times n}) = \overline{\bigcup_{\varrho>0} C_{R_\varrho^{n\times n}}(\mathbb{R}^{n\times n})}$, where $C_{R_\varrho^{n\times n}}(\mathbb{R}^{n\times n}) := \{\varphi \in C_0(\mathbb{R}^{n\times n}),\ \mathrm{supp}\,\varphi \subset R_\varrho^{n\times n}\}$.*

Proof. First of all, notice that $C_{0,\rm reg}(\mathbb{R}^{n\times n})$ is closed in $C_0(\mathbb{R}^{n\times n})$. Indeed, take a sequence $\{\phi_k\}_{k\in\mathbb{N}} \subset C_{0,\rm reg}(\mathbb{R}^{n\times n})$ such that $\phi_k \to \phi$ in $C_0(\mathbb{R}^{n\times n})$. Then, in particular, $\phi_k(A) \to \phi(A)$ for all $A \in \mathbb{R}^{n\times n}$. Hence, also $\phi(A) = 0$ for every singular A meaning that $\phi \in C_{0,\rm reg}(\mathbb{R}^{n\times n})$. Therefore, $C_{0,\rm reg}(\mathbb{R}^{n\times n})$ is also a Banach space.

Clearly, any ϕ in $C_{R_\varrho^{n\times n}}(\mathbb{R}^{n\times n})$ is also in $C_{0,\rm reg}(\mathbb{R}^{n\times n})$ for any $\varrho > 0$. Hence also $\bigcup_{\varrho>0} C_{R_\varrho^{n\times n}}(\mathbb{R}^{n\times n}) \subset C_{0,\rm reg}(\mathbb{R}^{n\times n})$ and, because $C_{0,\rm reg}(\mathbb{R}^{n\times n})$ is closed, the same holds for the closure. On the other hand, take $\phi \in C_{0,\rm reg}(\mathbb{R}^{n\times n})$ and define for every ϱ a *smooth cut-off function* Φ_ϱ which is 1 on $R_\varrho^{n\times n}$ and 0 on $\mathbb{R}^{n\times n}_{\rm reg} \setminus R_{\varrho+1}^{n\times n}$. (Note that Φ_ϱ can be found as follows: Define Θ_ϱ a smooth function which is 1 inside the ball $B(0,\varrho) \subset \mathbb{R}^{n\times n}$ and equals 0 on $\mathbb{R}^{n\times n} \setminus B(0, \varrho+1)$. Now we may set $\Phi_\varrho(s) := \Theta_\varrho(s)\hat\Theta_\varrho(s)$. Note that, since $B(0,\varrho+1)$ is a strict subset of $\mathbb{R}^{n\times n}$, Φ_ϱ is indeed smooth.) Then ϕ can be approximated by the set of functions $\{\phi\cdot\Phi_\varrho\}_{\varrho>0}$ if we can show that

$$\forall\epsilon>0\ \exists\varrho_0>0:\quad |\phi(A)|<\epsilon\quad \forall A\in\mathbb{R}^{n\times n}\setminus R^{n\times n}_\varrho. \tag{3.5}$$

To see this, suppose for contradiction that (3.5) does not hold and that there exists $\epsilon>0$ and $\{A_\varrho\}\subset\mathbb{R}^{n\times n}\setminus R^{n\times n}_\varrho$ such that $\phi(A_\varrho)\ge\epsilon$. Clearly, $\{A_\varrho\}$ must be bounded since $\phi\in C_0(\mathbb{R}^{n\times n})$. Therefore, pick a subsequence of $\{A_\varrho\}$ (not relabeled) such that $A_\varrho\to A$. Then, also $\phi(A)\ge\epsilon$ from which it follows that A is a regular matrix. But since $\bigcup_{\varrho\ge0}R^{n\times n}_\varrho=\mathbb{R}^{n\times n}_{\rm reg}$ there has to exist $\tilde\varrho$ such that $A\in R^{n\times n}_{\tilde\varrho}$. Yet, from the construction, for ϱ large enough A_ϱ are *not* elements of $R^{n\times n}_{\tilde\rho}$, a contradiction.

For the separability we use the classical result that subspaces of separable metric spaces are again separable [18]. □

Notice that if $v\in C_{0,\rm reg}(\mathbb{R}^{n\times n})$ then $\hat v\in C_{0,\rm reg}(\mathbb{R}^{n\times n})$. Indeed, if $s_0\in\mathbb{R}^{n\times n}_{\rm reg}$ then there is a δ-neighborhood of s_0^{-1}, $B(s_0^{-1},\delta)$, such that $B(s_0^{-1},\delta)\subset\mathbb{R}^{n\times n}_{\rm reg}$. The function v is continuous on $B(s_0^{-1},\delta)$, so for every $\varepsilon>0$ we have $|\hat v(s_0)-\hat v(s)|=|v(s_0^{-1})-v(s^{-1})|\le\varepsilon$ if $\delta>0$ is small enough. If s_0 is singular, then $\hat v(s_0)=0$ and $|s^{-1}|$ is arbitrarily large on $B(s_0,\delta)\cap\mathbb{R}^{n\times n}_{\rm reg}$. Hence, $|\hat v(s)|=|v(s^{-1})|<\varepsilon$ on $B(s_0,\delta)\cap\mathbb{R}^{n\times n}_{\rm reg}$ if δ is small. On the other hand, for singular matrices $s_0\in B(s_0,\delta)$ $\hat v(s_0)=0$, anyway.

The following lemma is a simple observation.

Lemma 3.4. *Let $\nu,\mu\in{\rm rca}(\mathbb{R}^{n\times n})$ and let both be supported on $\mathbb{R}^{n\times n}_{\rm reg}$. If for every $v\in C_{0,\rm reg}(\mathbb{R}^{n\times n})$*

$$\int_{\mathbb{R}^{n\times n}_{\rm reg}} v(s)\nu({\rm d}s)=\int_{\mathbb{R}^{n\times n}_{\rm reg}} v(s)\mu({\rm d}s)\ , \tag{3.6}$$

then $\nu=\mu$, i.e., (3.6) *holds even for all $v\in C_0(\mathbb{R}^{n\times n})$.*

Proof. Take $v\in C_0(\mathbb{R}^{n\times n})$. Define, similarly as in the proof of Lemma 3.3, for every ϱ the *smooth cut-off function* Φ_ϱ which is 1 on $R^{n\times n}_\varrho$ and 0 on $\mathbb{R}^{n\times n}_{\rm reg}\setminus R^{n\times n}_{\varrho+1}$ and $v_\varrho(s):=v(s)\Phi_\varrho(s)$ for all $s\in\mathbb{R}^{n\times n}$. Then $v_\varrho\in C_{0,\rm reg}(\mathbb{R}^{n\times n})$ and $|v_\varrho|\le|v|$. The proof is finished by the Lebesgue dominated convergence theorem for which we notice that $v_\varrho\to v$ pointwise everywhere on $\mathbb{R}^{n\times n}_{\rm reg}$. □

Proposition 3.5. *Let $\nu\in\mathcal{Y}(\Omega;\mathbb{R}^{n\times n})$ and suppose that there is $\varrho>0$ such that for almost all $x\in\Omega$ supp $\nu_x\subset R^{n\times n}_\varrho$. Then there exists $\{Y_k\}_{k\in\mathbb{N}}\subset L^\infty(\Omega;\mathbb{R}^{n\times n})$ such that $\{Y_k(x)\}_{k\in\mathbb{N}}\subset R^{n\times n}_\varrho$ for almost all $x\in\Omega$ and $\{Y_k\}_{k\in\mathbb{N}}$ generates ν.*

Proof. This is a classical result mentioned in (1.6). See e.g. [24, Th. 1] for details. □

Remark 3.6. *Proposition 3.5 still holds if we replace $R^{n\times n}_\varrho$ by $R^{n\times n}_{\varrho+}$ which is compact, as well.*

Proposition 3.7. *Let $\varrho>0$ and let $\{Y_k\}\subset L^\infty(\Omega;\mathbb{R}^{n\times n})$, $\{Y_k\}\subset R^{n\times n}_\varrho$ for almost all $x\in\Omega$ and all $k\in\mathbb{N}$. If $\{Y_k\}$ generates $\nu\in\mathcal{Y}^\infty(\Omega;\mathbb{R}^{n\times n})$ and if $\{Y_k^{-1}\}$ generates $\mu\in\mathcal{Y}^\infty(\Omega;\mathbb{R}^{n\times n})$ then for almost all $x\in\Omega$ and every continuous $f:R^{n\times n}_\varrho\to\mathbb{R}$ it holds*

$$\int_{R^{n\times n}_\varrho} f(s)\mu_x({\rm d}s)=\int_{R^{n\times n}_\varrho}\hat f(s)\nu_x({\rm d}s)\ . \tag{3.7}$$

Moreover, $\operatorname{supp}\nu_x\subset R^{n\times n}_\varrho$ *for almost all $x\in\Omega$.*

Proof. First of all, recall that [2, 26] for almost all $x \in \Omega$ ν_x is supported on the set $\cap_{l=1}^{\infty}\overline{\{Y_k(x);\ k \geq l\}}$, i.e., ν_x is supported on $R_\varrho^{n\times n}$. Further, notice that $\{Y_k^{-1}(x)\} \subset R_\varrho^{n\times n}$ for a.a. $x \in \Omega$. If $f : R_\varrho^{n\times n} \to \mathbb{R}$ is continuous, so is $F : R_\varrho^{n\times n} \to \mathbb{R}$, $F(s) := f(s^{-1})$. Then we have for any $g \in L^1(\Omega)$

$$\lim_{k\to\infty} \int_\Omega f(Y_k^{-1}(x))g(x)\,\mathrm{d}x = \int_\Omega \int_{R_\varrho^{n\times n}} f(s)\mu_x(\mathrm{d}s)g(x)\,\mathrm{d}x\ .$$

At the same time,

$$\lim_{k\to\infty} \int_\Omega F(Y_k(x))g(x)\,\mathrm{d}x = \int_\Omega \int_{R_\varrho^{n\times n}} F(s)\nu_x(\mathrm{d}s)g(x)\,\mathrm{d}x = \int_\Omega \int_{R_\varrho^{n\times n}} f(s^{-1})\nu_x(\mathrm{d}s)g(x)\,\mathrm{d}x\ .$$

□

Proposition 3.8. *Let $\{Y_k\}_{k\in\mathbb{N}} \subset L^p(\Omega;\mathbb{R}^{n\times n})$ generate $\nu \in \mathcal{Y}^p(\Omega;\mathbb{R}^{n\times n})$ and let $\int_\Omega |\det Y_k^{-1}|^q \mathrm{d}x \leq C$ for some $C > 0$ and some $q > 0$. Then for almost all $x \in \Omega$ ν_x is supported on $\mathbb{R}^{n\times n}_{\rm reg}$ in the sense that $\nu_x(\mathbb{R}^{n\times n} \setminus \mathbb{R}^{n\times n}_{\rm reg}) = 0$ for almost all $x \in \Omega$. Moreover, if even $\det Y_k > 0$ a.e. in Ω then ν_x is for almost all $x \in \Omega$ supported (in the above sense) on the set of regular matrices with positive determinant.*

Proof. Assume that the first assertion did not hold, i.e., that there existed a measurable $\omega \subset \Omega$ with positive measure such that $\int_\omega \int_{\mathbb{R}^{n\times n}\setminus\mathbb{R}^{n\times n}_{\rm reg}} \nu_x(\mathrm{d}s)\mathrm{d}x > 0$. Then for any $\epsilon > 0$ define a smooth cut-off $\Phi_\epsilon^{\det,0}$ such that $\Phi_\epsilon^{\det,0}(s) = 1$ on $\mathbb{R}^{n\times n} \setminus \mathbb{R}^{n\times n}_{\rm reg}$ and $\Phi_\epsilon^{\det,0}(s) = 0$ for all $s \in \mathbb{R}^{n\times n}_{\rm reg}$ such that $|\det s| \geq \epsilon$ ($\Phi_\epsilon^{\det,0}$ can be found as follows: first of all find a smooth $\varphi_\epsilon : \mathbb{R} \to \mathbb{R}$ such that $\varphi_\epsilon(0) = 1$ and $\varphi_\epsilon(x) = 0$ for $|x| > \epsilon$. Then define $\Phi_\epsilon^{\det,0}(s) = \varphi_\epsilon(\det s)$). We have by (1.8)

$$\lim_{k\to\infty} \int_\Omega \Phi_\epsilon^{\det,0}(Y_k(x))\,\mathrm{d}x = \int_\Omega \int_{\mathbb{R}^{n\times n}} \Phi_\epsilon^{\det,0}(s)\nu_x(\mathrm{d}s)\,\mathrm{d}x \geq \int_\omega \int_{\mathbb{R}^{n\times n}\setminus\mathbb{R}^{n\times n}_{\rm reg}} \nu_x(\mathrm{d}s)\mathrm{d}x =: \delta > 0\ . \tag{3.8}$$

Hence, there is $k_0 \in \mathbb{N}$ such that $\int_\Omega \Phi_\epsilon^{\det,0}(Y_k(x))\,\mathrm{d}x > \delta/2$ if $k > k_0$. This means that there is always a measurable set $\omega(k) \subset \Omega$, $|\omega(k)| > \delta/2$ such that $|\det Y_k(x)|^q < \varepsilon^q$ if $x \in \omega(k)$. Consequently, $|\det Y_k^{-1}(x)|^q > \varepsilon^{-q}$ if $x \in \omega(k)$. Thus, for every $k > k_0$

$$\int_\Omega |\det\, Y_k^{-1}(x)|^q\,\mathrm{d}x \geq \int_{\omega(k)} |\det\, Y_k^{-1}(x)|^q\,\mathrm{d}x \geq \frac{\delta}{2\varepsilon^q}\ . \tag{3.9}$$

As $\varepsilon > 0$ is arbitrary it contradicts the bound $\int_\Omega |\det Y_k^{-1}|^q \mathrm{d}x \leq C$.

As to the second assertion we proceed analogously only we define instead of $\Phi_\epsilon^{\det,0}$ the smooth cut-off $\Phi_\epsilon^{\det,+}$ which is 1 on all matrices s for which $\det s \leq 0$ and 0 on matrices for which $\det s \geq \epsilon$. Then, if ν_x was not, for almost all $x \in \Omega$, supported on the set of regular matrices with positive determinant, again there would be a measurable subset of Ω with positive measure, such that $\int_\omega \int_{\mathbb{R}^{n\times n}_{\rm reg}\setminus\mathbb{R}^{n\times n}_{\rm reg+}} \nu_x(\mathrm{d}s)\mathrm{d}x > 0$ which analogously to (3.8) means that in some set $\omega(k) \subset \Omega$ $0 \leq \det Y_k \leq \epsilon$. This yields a contradiction for $\epsilon \to 0$ because of (3.9). □

For further notation, let us denote the set $C^{p,-p}(\mathbb{R}^{n\times n}_{\rm reg})$ as

$$C^{p,-p}(\mathbb{R}^{n\times n}_{\rm reg}) = \{f \in C(\mathbb{R}^{n\times n}_{\rm reg});\ |f(s)| \leq C(1 + |s|^p + |s^{-1}|^p)\ \forall s \in \mathbb{R}^{n\times n}_{\rm reg}\}\ .$$

Lemma 3.9. *Let $\nu \in \mathcal{Y}^{p,-p}(\Omega;\mathbb{R}^{n\times n})$, $\mu \in \mathcal{Y}^{p,-p}(\Omega;\mathbb{R}^{n\times n})$. Let $f \in C^{p,-p}(\mathbb{R}^{n\times n}_{\rm reg})$ and $\hat{f}$ be defined as in (3.3) with f instead of v. Let also,*

$$\int_\Omega \int_{\mathbb{R}^{n\times n}_{\rm reg}} f^\varrho(s)\mu_x(\mathrm{d}s)\,\mathrm{d}x = \int_\Omega \int_{\mathbb{R}^{n\times n}_{\rm reg}} \hat{f}^\varrho(s)\nu_x(\mathrm{d}s)\mathrm{d}x \tag{3.10}$$

for all $f^\varrho \in C_{R_\varrho^{n\times n}}(\mathbb{R}^{n\times n})$, *for any* $\varrho > 0$. *Then*

$$\int_\Omega \int_{\mathbb{R}^{n\times n}_{\rm reg}} f(s)\mu_x(\mathrm{d}s)\mathrm{d}x = \int_\Omega \int_{\mathbb{R}^{n\times n}_{\rm reg}} \hat{f}(s)\nu_x(\mathrm{d}s)\mathrm{d}x \ . \tag{3.11}$$

Proof. Take any $f \in C^{p,-p}(\mathbb{R}^{n\times n}_{\rm reg})$ and define (the same was as in the proof of Lemma 3.3 the smooth cut-off Φ_ϱ. Then $f(s)\Phi_\varrho(s) \to f(s)$ pointwise for all $s \in \mathbb{R}^{n\times n}_{\rm reg}$ and similarly (from continuity) $\hat{f}(s)\Phi_\varrho(s) \to \hat{f}(s)$ as $\varrho \to \infty$. Note also that

$$\int_\Omega \int_{\mathbb{R}^{n\times n}_{\rm reg}} |f(s)|\Phi_\varrho(s)\mu_x(\mathrm{d}s)\mathrm{d}x \le \int_\Omega C(1+|s|^p+|s^{-1}|^p)\mu_x(\mathrm{d}s)\mathrm{d}x \le \tilde{C}$$
$$\int_\Omega \int_{\mathbb{R}^{n\times n}_{\rm reg}} |\hat{f}(s)|\Phi_\varrho(s)\nu_x(\mathrm{d}s)\mathrm{d}x \le \int_\Omega C(1+|s^{-1}|^p+|s|^p)\nu_x(\mathrm{d}s)\mathrm{d}x \le \tilde{C}$$

are bounded independently of ϱ. Here $\tilde{C} > 0$ is a constant. Hence we may exploit Lebesgue's dominated convergence theorem to prove the assertion. □

Proposition 3.10. *Let* $p \in [1,\infty)$ *and* $\{Y_k\} \subset L^p(\Omega;\mathbb{R}^{n\times n})$, $\{Y_k^{-1}\} \subset L^p(\Omega;\mathbb{R}^{n\times n})$ *be bounded and* $\{Y_k(x)\} \subset \mathbb{R}^{n\times n}_{\rm reg}$ *for almost all* $x \in \Omega$. *Then there is a subsequence of* $\{Y_k\}$ *(not relabeled) such that this subsequence generates a Young measure* $\nu \in \mathcal{Y}^{p,-p}(\Omega;\mathbb{R}^{n\times n})$.

Moreover, if we denoted μ *the Young measure generated by (a further subsequence of)* $\{Y_k^{-1}\}$ *then* (3.11) *holds for all* $f \in C^{p,-p}(\mathbb{R}^{n\times n}_{\rm reg})$.

Proof. It follows from (1.8) that a (not relabeled) subsequence of $\{Y_k\}$ generates a Young measure $\nu \in \mathcal{Y}^p(\Omega;\mathbb{R}^{n\times n})$ and $\{Y_k^{-1}\}$ generates a Young measure $\mu \in \mathcal{Y}^p(\Omega;\mathbb{R}^{n\times n})$. As $\int_\Omega |\det(Y_k^{-1})|^{p/n}\mathrm{d}x \le C\int_\Omega |Y_k^{-1}|^p\mathrm{d}x \le C$, we know from Proposition 3.8 that ν_x and μ_x are both supported on $\mathbb{R}^{n\times n}_{\rm reg}$ for almost all $x \in \Omega$. We have for all $g \in L^\infty(\Omega)$ and all $v \in C_{0,\rm reg}(\mathbb{R}^{n\times n})$

$$\lim_{k\to\infty}\int_\Omega \hat{v}(Y_k(x))g(x)\,\mathrm{d}x = \int_\Omega \int_{\mathbb{R}^{n\times n}_{\rm reg}} \hat{v}(s)\nu_x(\mathrm{d}s)g(x)\,\mathrm{d}x \ ,$$

$$\lim_{k\to\infty}\int_\Omega v(Y_k^{-1}(x))g(x)\,\mathrm{d}x = \int_\Omega \int_{\mathbb{R}^{n\times n}_{\rm reg}} v(s)\mu_x(\mathrm{d}s)g(x)\,\mathrm{d}x \ ,$$

where $\hat{v}$ given by (3.3) is again in $C_{0,\rm reg}(\mathbb{R}^{n\times n})$. This means that for all $g \in L^\infty(\Omega)$ and all $v \in C_{0,\rm reg}(\mathbb{R}^{n\times n})$

$$\int_\Omega \int_{\mathbb{R}^{n\times n}_{\rm reg}} \hat{v}(s)\nu_x(\mathrm{d}s)g(x)\,\mathrm{d}x = \int_\Omega \int_{\mathbb{R}^{n\times n}_{\rm reg}} v(s)\mu_x(\mathrm{d}s)g(x)\,\mathrm{d}x \ . \tag{3.12}$$

If we define

$$\int_{\mathbb{R}^{n\times n}_{\rm reg}} v(s)\hat{\nu}_x(\mathrm{d}s) := \int_{\mathbb{R}^{n\times n}_{\rm reg}} \hat{v}(s)\nu_x(\mathrm{d}s) \ ,$$

we get by Lemma 3.4 that $\hat{\nu} = \mu$.

Therefore it remains only to prove that $\int_\Omega \int_{\mathbb{R}^{n\times n}_{\rm reg}} (|s|^p + |s^{-1}|^p)\nu_x(\mathrm{d}s)\mathrm{d}x$ is bounded. Boundedness of the first part is guaranteed due to the fact that $\nu \in \mathcal{Y}^p(\Omega;\mathbb{R}^{n\times n})$. To see the second part note that $|\cdot|^p \circ \mathrm{inv}$ is a *continuous, bounded from below* in $\mathbb{R}^{n\times n}_{\rm reg}$ and hence [21]

$$\begin{aligned}\int_\Omega \int_{\mathbb{R}^{n\times n}_{\rm reg}} |s^{-1}|^p\nu_x(\mathrm{d}s)\mathrm{d}x &= \int_\Omega \int_{\mathbb{R}^{n\times n}_{\rm reg}} (|\cdot|^p \circ \mathrm{inv})(s)\nu_x(\mathrm{d}s)\mathrm{d}x \le \liminf_{k\to\infty}\int_\Omega (|\cdot|^p\circ \mathrm{inv})(Y_k)\mathrm{d}x \\ &= \liminf_{k\to\infty}\int_\Omega |Y_k^{-1}|^p\mathrm{d}x < C.\end{aligned} \tag{3.13}$$

Therefore, by Lemma 3.9, (3.11) holds for all $f \in C^{p,-p}(\mathbb{R}^{n\times n}_{\rm reg})$

Proposition 3.11. *Let $\nu \in \mathcal{Y}^{p,-p}(\Omega;\mathbb{R}^{n\times n})$. Then there is a generating sequence $\{Y_k\} \subset L^p(\Omega;\mathbb{R}^{n\times n})$ such that $\{Y_k^{-1}\} \subset L^p(\Omega;\mathbb{R}^{n\times n})$ is bounded. Moreover, $\{|Y_k^{-1}|^p\}$ as well as $\{|Y_k|^p\}$ are equi-integrable.*

Proof. Notice, that inevitably for a.a. $x \in \Omega$ supp $\nu_x \subset \mathbb{R}^{n\times n}_{\rm reg}$ (cf (2.1)). Therefore, define smooth cut-off functions Φ_ϱ as in the proof of Lemma 3.9 and set

$$\nu_x^\varrho \;=\; \Phi_\varrho \nu_x + \left(\int_{\mathbb{R}^{n\times n}_{\rm reg}} (1-\Phi_\varrho(s))\,\nu_x(\mathrm{d}s)\right)\delta_I \;, \tag{3.14}$$

where δ_I denotes the Dirac measure supported at the identity matrix. It is only a simple observation that, for all $\varrho \in \mathbb{N}$ and a.a. $x \in \Omega$, ν_x^ϱ is a probability Radon measure supported on $R^{n\times n}_{\varrho+1}$ and that the mapping $x \mapsto \nu^\varrho : \Omega \to \mathrm{rca}(\mathbb{R}^{n\times n})$ is weakly measurable. Altogether, we see that ν^ϱ defined by (3.14) is a Young measure, i.e. $\nu^\varrho \in \mathcal{Y}(\Omega;\mathbb{R}^{n\times n})$. We have from Propositions 3.5,3.7 that there is $\{Y_k^\varrho(x)\} \subset R^{n\times n}_{\varrho+1}$, $\{(Y_k^\varrho)^{-1}(x)\} \subset R^{n\times n}_{\varrho+1}$ for a.a. $x \in \Omega$ such that they generate ν^ϱ and μ^ϱ, respectively, where

$$\int_{\mathbb{R}^{n\times n}_{\rm reg}} \hat{v}(s)\nu_x^\varrho(\mathrm{d}s) = \int_{\mathbb{R}^{n\times n}_{\rm reg}} v(s)\mu_x^\varrho(\mathrm{d}s) \tag{3.15}$$

holds for all $v \in C_{0,\rm reg}(\mathbb{R}^{n\times n})$.

Now we want to show that, for any $v \in C_{0,\rm reg}(\mathbb{R}^{n\times n})$, it holds $\lim_{\varrho\to\infty} v_{\nu^\varrho} = v_\nu$ weakly in $L^1(\Omega)$, where v_{ν_ϱ} is defined again by (1.7) with ν_ϱ in place of ν. Indeed, for any $g \in L^\infty(\Omega)$ we can estimate

$$\lim_{\varrho\to\infty}\int_\Omega g(x)v_{\nu^\varrho}(x)\mathrm{d}x = \lim_{\varrho\to\infty}\int_\Omega g(x)\int_{\mathbb{R}^{n\times n}_{\rm reg}} v(s)\Phi_\varrho(s)\nu_x(\mathrm{d}s)\mathrm{d}x + \lim_{\varrho\to\infty} v(I)\int_\Omega g(x)\int_{\mathbb{R}^{n\times n}_{\rm reg}}(1-\Phi_\varrho(s))\nu_x(\mathrm{d}s)\mathrm{d}x$$

Now, thanks to the proof of Lemma 3.3, we know that $v\Phi_\varrho$ converges strongly in the C_0-norm to v and hence the first limit converges to v_ν. As for the second limit $\Phi_\varrho(s)$ converges pointwise to 1 for all s in $\mathbb{R}^{n\times n}_{\rm reg}$ and therefore, thanks to Lebesgue's dominated convergence theorem (note that $1-\Phi_\varrho$ is bounded by 1 which is $\nu_x - integrable$), $\int_{\mathbb{R}^{n\times n}_{\rm reg}}(1-\Phi_\varrho(s))\nu_x(\mathrm{d}s)$ converges to 0 for a.a. $x \in \Omega$. Exploiting the dominated convergence once again, the second limit approaches zero. Hence we are in the situation that

$$\lim_{\varrho\to\infty}\lim_{k\to\infty} v(Y_k^\varrho) = v_\nu \text{ weakly in } L^1(\Omega)\ .$$

Further verify that Y_k^ϱ as well as $(Y_k^\varrho)^{-1}$ are bounded in $L^p(\Omega,\mathbb{R}^{n\times n}_{\rm reg})$ independently of ϱ. Indeed, for every $\varrho \in \mathbb{N}$ fixed,

$$\begin{aligned}\lim_{k\to\infty}\int_\Omega |Y_k^\varrho|^p\,\mathrm{d}x &= \int_\Omega\int_{\mathbb{R}^{n\times n}_{\rm reg}} |s|^p\,\nu_x^\varrho(\mathrm{d}s)\,\mathrm{d}x\\ &\le \int_\Omega\int_{B(0,\varrho+1)} |s|^p\nu_x(\mathrm{d}s)\,\mathrm{d}x \le \int_\Omega\int_{\mathbb{R}^{n\times n}_{\rm reg}} |s|^p\,\nu_x(\mathrm{d}s)\,\mathrm{d}x = \|C\|_{L^1(\Omega)} < +\infty,\end{aligned} \tag{3.16}$$

the same calculation could be carried out even $(Y_k^\varrho)^{-1}$ in place of Y_k^ϱ; in this case we need to write s^{-1} in place of s.

Applying the diagonalization argument (as $L^1(\Omega;C_{0,\rm reg}(\mathbb{R}^{n\times n}))$ is separable) we get $\{Y_k\} \subset L^p(\Omega;\mathbb{R}^{n\times n})$ generating ν and thanks to (3.16) also *equi-integrable*; the same holds for the inverse.

Moreover, if we defined μ as the weak* limit of μ_ϱ, then μ would be generated by $\{Y_k^{-1}\} \subset L^p(\Omega;\mathbb{R}^{n\times n})$ as, due to its definition,

$$\lim_{\varrho\to\infty}\lim_{k\to\infty} v((Y_k^\varrho)^{-1}) = v_\mu \text{ weakly in } L^1(\Omega)\ .$$

Also, by applying $\varrho \to \infty$ in (3.15), it holds that

$$\int_{\mathbb{R}^{n\times n}_{\rm reg}} \hat{v}(s)\nu_x(\mathrm{d}s) = \int_{\mathbb{R}^{n\times n}_{\rm reg}} v(s)\mu_x(\mathrm{d}s), \tag{3.17}$$

for all $v \in C_{0,\rm reg}(\mathbb{R}^{n\times n})$ and hence, by Lemma 3.9 also for all $v \in C^{p,-p}(\mathbb{R}^{n\times n}_{\rm reg})$.

□

Proof of Theorem 2.1. We know from Proposition 3.8 that ν_x is supported on $\mathbb{R}^{n\times n}_{\rm reg}$ for almost all $x \in \Omega$. To show that $\nu \in \mathcal{Y}^{p,-p}(\Omega;\mathbb{R}^{n\times n})$ we use Proposition 3.10. On the other hand, if $\nu \in \mathcal{Y}^{p,-p}(\Omega;\mathbb{R}^{n\times n})$ then the existence of a generating sequence is due to Proposition 3.11.

It remains to prove relation (2.4), which we show analogously to [10, Th. 8.6]. Let $v : \mathbb{R}^{n\times n}_{\rm reg} \to \mathbb{R}$ and $g \in L^\infty(\Omega)$ be as in the theorem.
For clarity, we divide the proof into 3 steps:

Step 1. Define $f(x,s) := g(x)v(s)$. Then $f : \Omega \times \mathbb{R}^{n\times n}_{\rm reg} \to \mathbb{R}$ is a normal integrand [10, Def. 6.27]. Suppose first, that $f \geq -M$ for some $M > 0$. By [10, Th. 8.6(i)]

$$\liminf_{k\to\infty} \int_\Omega f(x, Y_k(x))\,\mathrm{d}x \geq \int_\Omega \int_{\mathbb{R}^{n\times n}_{\rm reg}} f(x,s)\nu_x(\mathrm{d}s)\,\mathrm{d}x\ . \tag{3.18}$$

Step 2. We use [10, Th.8.6(i)] to show that (3.18) also holds if the negative parts of $f(x, Y_k(x))$, $k \in \mathbb{N}$, form an equi-integrable sequence. The proof is the same as the proof of [10, Th.8.6(i)]. We recall that the negative part of $h : \mathbb{R} \to \mathbb{R}$ is defined as $h^-(x) := \max(-h(x), 0)$.

Step 3. Here we prove that if $\{f(x, Y_k(x))\}_{k\in\mathbb{N}}$ is equi-integrable then (3.18) holds as equality. Namely, if $\{f(x, Y_k(x))\}_{k\in\mathbb{N}}$ is equi-integrable then $\{f^-(x, Y_k(x))\}_{k\in\mathbb{N}} = \{(|f(x, Y_k(x))| - f(x, Y_k(x)))/2\}_{k\in\mathbb{N}}$ is equi-integrable, thus by Step 2:

$$\liminf_{k\to\infty} \int_\Omega f(x, Y_k(x))\,\mathrm{d}x \geq \int_\Omega \int_{\mathbb{R}^{n\times n}_{\rm reg}} f(x,s)\nu_x(\mathrm{d}s)\,\mathrm{d}x\ . \tag{3.19}$$

On the other hand, $\{-f^-(x, Y_k(x))\}_{k\in\mathbb{N}}$ is also equi-integrable, hence

$$\liminf_{k\to\infty} \int_\Omega -f(x, Y_k(x))\,\mathrm{d}x \geq \int_\Omega \int_{\mathbb{R}^{n\times n}_{\rm reg}} -f(x,s)\nu_x(\mathrm{d}s)\,\mathrm{d}x\ . \tag{3.20}$$

Altogether, we have

$$\lim_{k\to\infty} \int_\Omega f(x, Y_k(x))\,\mathrm{d}x = \int_\Omega \int_{\mathbb{R}^{n\times n}_{\rm reg}} f(x,s)\nu_x(\mathrm{d}s)\,\mathrm{d}x\ . \tag{3.21}$$

Finally, we show that if $f(x,s) = g(x)v(s)$ for some $g \in L^\infty(\Omega)$ and $v \in C_{p,-p}(\mathbb{R}^{n\times n}_{\rm reg})$ then $\{f(x, Y_k(x))\}_{k\in\mathbb{N}}$ is equi-integrable. To see this, we use [21, Lemma 6.1]. Notice that $v_0(s) := |v(s)|/(|s|^p + |s^{-1}|^p) \leq C$ in $\mathbb{R}^{n\times n}_{\rm reg}$ for some $C > 0$. Moreover, $\lim_{|s|^p+|s^{-1}|^p\to\infty} v_0(s) = 0$. Let $(\|Y_k\|^p_{L^p} + \|Y_k^{-1}\|^p_{L^p}) \leq M$. Take $\varepsilon > 0$ and $K > 0$ large enough so that $|v_0(s)| < \varepsilon/M$ if $|s|^p + |s^{-1}|^p \geq K/C$. Then for all k

$$\begin{aligned}\int_{\{x\in\Omega;\ |v(Y_k(x))|\geq K\}} |v(Y_k(x))|\,\mathrm{d}x &\leq \int_{\{x\in\Omega;\ |Y_k(x)|^p+|(Y_k(x))^{-1}|^p\geq K/C\}} |v(Y_k(x))|\,\mathrm{d}x \\ &\leq \int_{\{x\in\Omega;|Y_k(x)|^p+|Y_k^{-1}(x)|^p\geq K/C\}} |v_0(Y_k(x))|(|Y_k(x)|^p + |Y_k^{-1}(x)|^p)\mathrm{d}x \leq \varepsilon/M \int_\Omega |Y_k(x)|^p + |Y_k^{-1}(x)|^p\mathrm{d}x \leq \varepsilon\ .\end{aligned}$$

□

Proof of Theorem 2.2. It is analogous to the proof of Theorem 2.1. Notice that the measure ν is supported on regular matrices due to Proposition 3.8 The converse implication follows from Remark 3.6. □

4 Applications

Here we briefly show two example applications of the theory presented above concerning minimization of integral functionals.
Example 1: In this example we consider the minimization of an integral functional

$$J(Y) := \int_\Omega W(Y(x)) - F(x) \cdot Y(x)\,\mathrm{d}x\ , \tag{4.1}$$

over the space $\mathcal{A} := \{Y \in L^p(\Omega; \mathbb{R}^{n\times n})\,, Y(x) \in \mathbb{R}^{n\times n}_{\mathrm{reg}}\text{ a.e.}\}$. Here $W \in C^{p,-p}(\mathbb{R}^{n\times n}_{\mathrm{reg}})$ is coercive in the sense

$$W(Y) \geq c(1 + |Y|^p + |Y^{-1}|^p) \tag{4.2}$$

and $F \in L^q(\Omega; \mathbb{R}^{n\times n})$, $q \geq p/(p-1)$.

Since we do not pose any convexity assumptions on W clearly minimizers of $J(Y)$ might not exist, hence the problem demands for relaxation. Yet, relaxation by means of Young measures in the sense of Schonbek [23] is not possible since W is not necessarily continuous on $\mathbb{R}^{n\times n}$. On the other hand, we can relax the problem by means of Young measures supported on $\mathbb{R}^{n\times n}_{\mathrm{reg}}$ as shown in the following Proposition:

Proposition 4.1. *It holds that* $\inf_{\mathcal{A}} J = \inf_{\nu\in\mathcal{Y}^{p,-p}(\Omega;\mathbb{R}^{n\times n})} \bar{J}$ *where* $\bar{J} : \mathcal{Y}^{p,-p}(\Omega; \mathbb{R}^{n\times n}) \to \mathbb{R}$ *with*

$$\bar{J}(\nu) := \int_\Omega \int_{\mathbb{R}^{n\times n}_{\mathrm{reg}}} (W(s) - F \cdot s)\nu_x(\mathrm{d}s)\,\mathrm{d}x\ . \tag{4.3}$$

Proof. Take a minimizing sequence of J, $\{Y_k\} \subset \mathcal{A}$. As $W \in C^{p,-p}(\mathbb{R}^{n\times n}_{\mathrm{reg}})$, $J \neq +\infty$ on $\mathbb{R}^{n\times n}_{\mathrm{reg}}$ and hence

$$+\infty > J(Y_k) + c \geq c \int_\Omega (|Y_k(x)|^p + |Y_k^{-1}(x)|^p)\,\mathrm{d}x - \|F\|_{L^q}\|Y_k\|_{L^p}\ , \tag{4.4}$$

which implies that $\|Y_k\|_{L^p} + \|Y_k^{-1}\|_{L^p} < C$ for all $k \in \mathbb{N}$. Then, by Proposition 3.10, we may find a $\nu \in \mathcal{Y}^{p,-p}(\Omega; \mathbb{R}^{n\times n})$ that is generated by $\{Y_k\}$. Since W is bounded from below we may write; cf. [21]

$$\inf_{\mathcal{A}} J = \lim_{k\to\infty} J(Y_k) \geq \int_\Omega \int_{\mathbb{R}^{n\times n}_{\mathrm{reg}}} (W(s) - F \cdot s)\nu_x(\mathrm{d}s)\,\mathrm{d}x \geq \inf_{\nu\in\mathcal{Y}^{p,-p}(\Omega;\mathbb{R}^{n\times n}} \bar{J}.$$

On the other hand, if there exists $\mu \in \mathcal{Y}^{p,-p}(\Omega; \mathbb{R}^{n\times n})$ such that $\inf_{\mathcal{A}} J > \bar{J}(\mu)$ then by Proposition 3.10 there is its *equi-integrable* generating sequence $\{X_k\} \subset \mathcal{A}$. Inevitably, $J(X_k) < \inf_{\mathcal{A}} J$ for k large enough, which would yield a contradiction. □

Example 2: In this example, again, we are concerned with minimizing an integral functional, however it depends on gradients in the following sense: Let Ω be a Lipschitz bounded domain in $\mathbb{R}^n$, let $y_0 \in W^{1,p}(\Omega; \mathbb{R}^n)$, then the task is to minimize the following functional $J : W^{1,p}(\Omega; \mathbb{R}^n) \to \mathbb{R}$,

$$J(y) := \int_\Omega W(\nabla y(x))\,\mathrm{d}x\ . \tag{4.5}$$

on the set

$$\mathcal{U} := \{y \in W^{1,p}(\Omega; \mathbb{R}^n);\ y = y_0 \,\mathrm{on}\, \Gamma_0 \subset \partial\Omega,\ \det\,\nabla y > 0\text{ a.e.}\} \neq \emptyset\ , \tag{4.6}$$

with Γ_0 of a positive $(n-1)$-dimensional Lebesgue measure. Here again $W \in C^{p,-p}(\mathbb{R}^{n\times n}_{\mathrm{reg}})$ satisfying (4.2).

Note that this example is of particular interestest in non-linear hyperelasticity, since $W(\nabla y)$ (and in particular also $J(y)$ approaches $+\infty$ if $\det \nabla y \to 0$. On the other hand we do not impose any convexity assumptions on W, so it can have multiple minima which would be suitable to describe materials with microstructure, like shape-memory alloys. For example, one could choose W as

$$W(F) := \min_{i=1,\dots,M} \left(|F^\top F - F_i^\top F_i|^2 + |F^{-1}F^{-\top} - F_i^{-1}F_i^{-\top}|^2 \right) ,$$

where $F_i \in \mathbb{R}^{3\times 3}$, $i = 1, \dots, M$, are positions of the minima of the multiwell energy, e.g. in shape-memory alloys they would represent given stress-free strains of martensitic variants. Notice that, in this case, W satisfies (1.5) with $n = 3$, $p = 4$, and that $W(F) = 0$ precisely if $F = RF_i$ for an arbitrary rotation R

First, let us show that weak limits of minimizing sequences of J are again in $\mathcal{U}$.

Proposition 4.2. *Let $p > n$. If $y_k \rightharpoonup y$ in $W^{1,p}(\Omega;\mathbb{R}^n)$ with $\{y_k\}_{k\in\mathbb{N}} \subset \mathcal{U}$ such that $\{(\nabla y_k)^{-1}\} \subset L^p(\Omega;\mathbb{R}^{n\times n})$ is bounded then $\det \nabla y > 0$ a.e. in Ω. Moreover, every Young measure generated by a subsequence of $\{\nabla y_k\}_{k\in\mathbb{N}}$ is supported on $\mathbb{R}^{n\times n}_{\rm reg+}$.*

Note that this proposition can indeed be applied to minimizing sequences of J since, owing to (4.2), the minimizing sequence itself as well as its inverse have to be bounded in $L^p(\Omega, \mathbb{R}^{n\times n})$.

Proof. By the Mazur theorem $\det \nabla y \geq 0$. Suppose that, by cotradiction, there was a Lebesgue measurable $\omega \subset \Omega$, $|\omega| > 0$ such that $\det \nabla y = 0$ on Ω. We have by the sequential weak continuity of $W^{1,p}(\Omega;\mathbb{R}^n) \to L^{p/n}(\Omega)$: $y \mapsto \det \nabla y$ ([6]) that

$$\int_\omega |\det \nabla y_k(x)| \,\mathrm{d}x = \int_\omega \det \nabla y_k(x) \,\mathrm{d}x \to 0 \text{ as } k \to \infty ,$$

so, it holds for a subsequence (not relabeled) that $0 < \det \nabla y_k \to 0$ a.e. in ω. By the Fatou lemma, we have

$$\int_\omega \liminf_{k\to\infty} \frac{\mathrm{d}x}{\det \nabla y_k(x)} \leq \liminf_{k\to\infty} \int_\omega \frac{\mathrm{d}x}{\det \nabla y_k(x)} \leq C \liminf_{k\to\infty} \int_\omega |(\nabla y_k(x))^{-1}|^n \,\mathrm{d}x ,$$

however, the left-hand side tends to $+\infty$. This contradicts the boundedness of $\{(\nabla y_k)^{-1}\}_{k\in\mathbb{N}}$ in $L^p(\Omega;\mathbb{R}^{n\times n})$ because $p > n$ and Ω is bounded. Hence, $\det \nabla y > 0$ a.e. in Ω. The assertion about the support follows from Proposition 3.8.

□

Now let us define the following subset of $\mathcal{Y}^{p,-p}(\Omega;\mathbb{R}^{n\times n})$:

$$\mathcal{G} = \{\nu \in \mathcal{Y}^{p,-p}(\Omega;\mathbb{R}^{n\times n}), \text{ such that } \nu \text{ can be generated by } \{\nabla w_k\}_{k\in\mathbb{N}} ,\ w_k \in \mathcal{U}\} .$$

Then in analogy with classical gradient Young measure theory (cf. e.g. [21]) one would expect that the relaxation of J should attain its minimum on $\mathcal{G}$. This would be true if a suitable version of the following decomposition lemma for $\mathcal{G}$ held. We state here for reader's convenience the original decomposition lemma for gradient Young measures [11, 16] which is used to relax variational problems with polynomial growth.

Lemma 4.3. *Let $1 < p < +\infty$ and $\Omega \subset \mathbb{R}^n$ be an open bounded set and let $\{y_k\}_{k\in\mathbb{N}} \subset W^{1,p}(\Omega;\mathbb{R}^m)$ be bounded. Then there is a subsequence $\{y_j\}_{j\in\mathbb{N}}$ and a sequence $\{z_j\}_{j\in\mathbb{N}} \subset W^{1,p}(\Omega;\mathbb{R}^m)$ such that*

$$\lim_{j\to\infty} |\{x \in \Omega;\ z_j(x) \neq y_j(x) \text{ or } \nabla z_j(x) \neq \nabla y_j(x)\}| = 0 \tag{4.7}$$

and $\{|\nabla z_j|^p\}_{j\in\mathbb{N}}$ is relatively weakly compact in $L^1(\Omega)$. In particular, $\{\nabla y_j\}$ and $\{\nabla z_j\}$ generate the same Young measure.

Then we have the following simple observation.

Proposition 4.4. *It holds that*

$$\inf_{y\in\mathcal{U}} J(y) \geq \inf_{\nu\in\mathcal{G}} \int_\Omega \int_{\mathbb{R}^{n\times n}_{\rm reg}} W(s)\nu_x(\mathrm{d}s)\,\mathrm{d}x. \tag{4.8}$$

Proof. This can be shown the same way as Proposition 4.1.

□

We would have equality in (4.8) if the following conjecture held true (however, it is still an open problem to confirm or falsify it) :

Let the Young measure from Lemma 4.3 belong to $\mathcal{Y}^{p,-p}(\Omega;\mathbb{R}^{n\times n})$. Then the sequence $\{z_j\}_{j\in\mathbb{N}}$ can be chosen in such a way that $\{|(\nabla z_j)^{-1}|^p\}_{j\in\mathbb{N}}$ is equi-integrable.

If the conjecture did not hold, we would need to resort to generalized Young measures taking into account also concentration effects [8]. We hope to address these issues in the future.

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