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EXISTENCE OF WEAK SOLUTIONS TO THE NAVIER–STOKES–FOURIER SYSTEM ON LIPSCHITZ DOMAINS

Lukáš Poul

Mathematical Institute, Academy of Sciences of the Czech Republic
Žitná 25
115 67 Praha 1, Czech Republic

Abstract. We prove existence of a weak solution to the Navier-Stokes-Fourier system on a bounded Lipschitz domain in $\mathbb{R}^3$. The key tool is the existence theory for weak solutions developed by Feireisl for the case of bounded smooth domains. We prove our result by inserting an additional limit passage where smooth domains approximate the Lipschitz one. Results on sensitivity of solutions with respect to the convergence of spatial domains are shortly discussed at the end of the paper.

1. **Introduction.** The immediate state of a viscous, compressible, and heat conducting fluid can be described by a triple of functions $(\rho, \mathbf{u}, \vartheta)$. These functions represent physical quantities of the fluid, density ρ, velocity $\mathbf{u}$, and temperature ϑ. The time-evolution of the system can be caught up by a system of partial differential equations representing basic physical principles. They are: The continuity equation expressing the total balance of mass of the system

$$\partial_t \rho + \operatorname{div}(\rho \mathbf{u}) = 0. \tag{1}$$

The second Newton's law in form of the linear momentum equation

$$\partial_t(\rho \mathbf{u}) + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla p = \operatorname{div} \mathbb{S} + \rho \mathbf{f}, \tag{2}$$

where p denotes the pressure and $\mathbb{S}$ denotes the Cauchy stress tensor. The exact forms of p and $\mathbb{S}$ are given by constitutive relations. External forces are expressed by $\mathbf{f}$.

The first law of thermodynamics specifies internal energy e as a conserved quantity. It is equivalent with the entropy equation.

$$\partial_t(\rho s) + \operatorname{div}(\rho s \mathbf{u}) + \operatorname{div} \frac{\mathbf{q}}{\vartheta} = \sigma, \tag{3}$$

where $\mathbf{q}$ denotes the heat flux and σ stands for the entropy production.

If the state variables ρ, $\mathbf{u}$ and ϑ are smooth, the entropy production σ is equal to $\frac{1}{\vartheta}\mathbb{S} : \nabla \mathbf{u} + \frac{\mathbf{q} \cdot \nabla \vartheta}{\vartheta^2}$. However, for nonsmooth motions only one inequality holds

$$\sigma \geq \frac{1}{\vartheta}\mathbb{S} : \nabla \mathbf{u} + \frac{\mathbf{q} \cdot \nabla \vartheta}{\vartheta^2} \tag{4}$$

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In this case, the system is supplemented by a requirement on the total energy (in)equality.

The constitutive relations describing quantities p, $\mathbb{S}$ and $\mathbf{q}$ are given as follows

$$p = p(\rho, \vartheta) = p_e(\rho) + \vartheta p_\vartheta(\rho) + \frac{d}{3}\vartheta^4 \tag{5}$$

$$\begin{aligned} \mathbb{S} &= \mu(\vartheta)\left(\nabla \mathbf{u} + \nabla \mathbf{u}^T\right) + \lambda(\vartheta)\mathrm{div}\, \mathbf{u}\mathbb{I} \qquad (6) \\ &= \mu(\vartheta)\left(\nabla \mathbf{u} + \nabla \mathbf{u}^T - \frac{2}{3}\mathrm{div}\, \mathbf{u}\mathbb{I}\right) + \zeta(\vartheta)\mathrm{div}\, \mathbf{u}\mathbb{I} \end{aligned}$$

$$\mathbf{q} = -\kappa(\vartheta)\nabla\vartheta. \tag{7}$$

Quantities p, s, and e are interrelated by Gibb's equation $\vartheta Ds = De + pD(\frac{1}{\rho})$, where D represents the total derivative with respect to variables ρ and ϑ. Consequently, assuming moreover that the *specific heat at constant volume* c_v is constant, e and s have the form

$$e(\rho, \vartheta) = P_e(\rho) + d\frac{\vartheta^4}{\rho} + c_v\vartheta,$$

$$s(\rho, \vartheta) = \frac{4}{3}d\frac{\vartheta^3}{\rho} + c_v \log\vartheta - P_\vartheta(\rho)$$

where $P_e(z) = \int_1^z \frac{p_e(s)}{s^2}\,\mathrm{d}s$, and $P_\vartheta(z) = \int_1^z \frac{p_\vartheta(s)}{s^2}\,\mathrm{d}s$.

We assume that there is no slip on the boundary and the system is thermally isolated, i.e.

$$\mathbf{u}|_{\partial\Omega} = 0, \text{ and } (\nabla\vartheta \cdot \mathbf{n})|_{\partial\Omega} = 0.$$

Moreover, we assume that the following structural assumptions hold.

$$p_e(0) = 0, \quad p_e'(\rho) \geq a_1\rho^{\gamma-1} - c_1, \quad p_e(\rho) \leq a_2\rho^\gamma + c_2,$$
$$p_\vartheta(0) = 0, \quad p_\vartheta'(\rho) \geq 0, \quad p_\vartheta(\rho) \leq a_3\rho^\Gamma + c_3,$$

$$0 < \underline{\mu}(1 + \vartheta^\alpha) \leq \mu(\vartheta) \leq \overline{\mu}(1 + \vartheta)^\alpha,$$
$$0 < \underline{\zeta}\vartheta^\alpha \leq \zeta(\vartheta) \leq \overline{\zeta}(1 + \vartheta)^\alpha,$$

$$0 < \underline{\kappa_G} \leq \kappa_G(\vartheta) \leq \overline{\kappa_G}(1 + \vartheta^3), \quad \kappa_R(\vartheta) = \sigma\vartheta^3,$$

where $a_1 > 0, \gamma > \frac{3}{2}, \gamma > \frac{4\Gamma}{3}$, and $\frac{1}{2} \leq \alpha \leq 1$.

The notion of a weak (or variational) solution can be seen as an approach where one replaces the pointwise values of physical quantities by their integral averages around the given point. This concept, being started by Leray [7] for the case of incompressible fluids, leads to the following definition.

Definition 1. Let $(\rho, \mathbf{u}, \vartheta)$ be a triple of measurable functions, ρ being nonnegative. We say that $(\rho, \mathbf{u}, \vartheta)$ is a weak solution to the Navier–Stokes–Fourier system on the domain $(0, T) \times \Omega$

- $\rho, \mathbf{u}$ solve the *renormalized continuity equation*

$$\partial_t b(\rho) + \mathrm{div}\,(b(\rho)\mathbf{u}) + (b'(\rho)\rho - b(\rho))\mathrm{div}\, \mathbf{u} = 0 \text{ in } \mathcal{D}'((0, T) \times \mathbb{R}^3) \tag{1*}$$

  provided ρ and $\mathbf{u}$ are extended to be zero outside Ω.
- $\rho, \mathbf{u}\vartheta$ solve the linear momentum equation (2) in $\mathcal{D}'((0, T) \times \Omega)$,
- $\rho, \mathbf{u}, \vartheta$ solve the entropy inequality (3) in $\mathcal{D}'((0, T) \times \overline{\Omega})$, and

- $\rho, \mathbf{u}, \vartheta$ satisfy the total energy equality

$$\int_\Omega \left(\frac{1}{2}\rho|\mathbf{u}|^2 + \rho e(\rho,\vartheta)\right)(t)\,\mathrm{d}\mathbf{x} = \int_\Omega \left(\frac{1}{2}\frac{|\mathbf{m}_0|^2}{\rho_0} + \rho_0 e(\rho_0,\vartheta_0)\right)\mathrm{d}\mathbf{x} + \int_0^t \int_\Omega \rho\mathbf{f}\cdot\mathbf{u}\,\mathrm{d}\mathbf{x}\,\mathrm{d}s \text{ for } t \in (0,T).$$

We introduce the following concept of convergence of domains.

Definition 2. Let Ω be a bounded domain in $\mathbb{R}^N$ with Lipschitz continuous boundary. We say, that the sequence of domains Ω_n converges to Ω if the following holds:

- for any ball $B \subset \mathbb{R}^3 \setminus \Omega$ there exists n_0 such that $B \subset \mathbb{R}^3 \setminus \Omega_n$ for all $n \geq n_0$, and
- for any compact $K \subset \Omega$ there exists n_0 such that $K \subset \Omega_n$ for all $n \geq n_0$.
- $\mathrm{cap}_2(\Omega \setminus \Omega_n) \to 0$ as n tends to infinity.

Note that for any bounded set Ω with Lipschitz continuous boundary there exists a sequence of domains Ω_n with smooth boundary being uniformly Lipschitz continuous with respect to n, that converge to Ω in the sense of our definition. Moreover, one can take $\Omega \subset \Omega_n$. In what follows, we will consider the sequence Ω_n with these properties granted.

Since for domains with Lipschitz boundary there exists an extension domain and trace operator (cf. Stein [9]), one can discover the following lemma.

Lemma 1. *Let Ω be a bounded domain in $\mathbb{R}^N$ with Lipschitz continuous boundary and let Ω_n be a sequence of domains that approximate Ω in the sense of Definition 2. Assume, that u_n is a sequence of functions from $W^{1,2}(\mathbb{R}^N)$ and $u_n \in W_0^{1,2}(\Omega_n)$ for each n. If u_n converge weakly in $W^{1,2}(\mathbb{R}^N)$ to u, then $u \in W_0^{1,2}(\Omega)$.*

Main Theorem. *Let Ω be a bounded domain in $\mathbb{R}^3$ with Lipschitz continuous boundary. Moreover, let the assumptions on terms $p_e, p_\vartheta, \kappa, \lambda, \mu$ hold, and let $\mathbf{f} \in L^\infty((0,T)\times\Omega)$. Then for any initial conditions $\rho(0) = \rho_0 \geq 0$, $\rho_0 \in L^\gamma(\Omega)$, $(\rho\mathbf{u})(0) = \mathbf{m}_0 \in L^1(\Omega;\mathbb{R}^3)$, $\frac{|\mathbf{m}_0|^2}{\rho_0} \in L^1(\Omega)$, $\vartheta(0) = \vartheta_0 \in L^\infty(\Omega)$, $\frac{1}{\vartheta_0} \in L^\infty(\Omega)$, $\vartheta_0 > 0$, there exists a weak solution to the Navier–Stokes–Fourier system on Ω. Moreover, there exists a weak solution $(\rho, \mathbf{u}, \vartheta)$ satisfying the initial conditions above and enjoying the following properties: $\mathbf{u} \in L^r(0,T;W_0^{1,r}(\Omega)^3)$ for some $r > 1$; $\vartheta, \log\vartheta \in L^2(0,T;W^{1,2}(\Omega))$; $\rho \in C([0,T];L^1(\Omega)) \cap L^\infty(0,T;L^\gamma(\Omega))$; $\rho\mathbf{u} \in C([0,T];L^{\frac{2\gamma}{\gamma+1}}_{weak}(\Omega;\mathbb{R}^3))$; the quantities $\rho\mathbf{u}\otimes\mathbf{u}$, $\mathbb{S} : \nabla\mathbf{u}$, p, $\rho\mathbf{f}$ are integrable on $(0,T)\times\Omega$.*

2. **Existence of δ-approximate solutions on Lipschitz domains.** Following the method for proving existence of solutions developed by Ducomet and Feireisl in [2], one starts solving the modified system of equations namely the continuity equation with the artificial viscosity term, the linear momentum equation with artificial pressure term and equation for internal energy, which is equivalent to the entropy equation. The approximate solutions are constructed so that they satisfy the (approximate) total energy equality. Consider a domain with smooth, at least $C^{2+\nu}$, boundary. Then applying the vanishing-viscosity part of the proof in [2] we obtain solution $(\rho_n, \mathbf{u}_n, \vartheta_n)$ of the δ-approximated system of equations with $\delta > 0$ on the domain Ω_n:

$$\left.\begin{array}{rcll}\partial_t\rho_n + \mathrm{div}\,(\rho_n\mathbf{u}_n) &=& 0 &, \text{ in } \Omega_n\\ \rho_n(0) &=& \rho_{0,n} &, \text{ in } \Omega\end{array}\right\} \tag{8}$$

$$\left.\begin{array}{rcll}\partial_t(\rho_n\mathbf{u}_n) + \mathrm{div}\,(\rho_n\mathbf{u}_n\otimes\mathbf{u}_n) + \nabla p_\delta &=& \mathrm{div}\,\mathbb{S}_n + \rho_n\mathbf{f} &, \text{ in } \Omega_n\\ \mathbf{u}_n &=& 0 &, \text{ on } \partial\Omega_n\\ (\rho_n\mathbf{u}_n)(0) &=& \mathbf{m}_{0,n} &, \text{ in } \Omega_n\end{array}\right\} \tag{9}$$

$$\left.\begin{array}{rcll}\partial_t(\rho_n s_n) + \mathrm{div}\,(\rho_n s_n\mathbf{u}_n) - \mathrm{div}\,\frac{\kappa(\vartheta_n)\nabla\vartheta_n}{\vartheta_n} &=& \sigma_n &, \text{ on } \Omega_n\\ \nabla\vartheta_n\cdot\mathbf{n}_n &=& 0 &, \text{ on } \partial\Omega_n\\ \rho_n(0)s_n(0) &=& \rho_{0,n}s_{0,n} &, \text{ in } \Omega_n\end{array}\right\} \tag{10}$$

where $p_\delta = p_e(\rho_n) + \vartheta_n p_\vartheta(\rho_n) + \frac{d}{3}\vartheta_n^4 + \delta\rho_n^\beta$ represents the pressure term with artificial part $\delta\rho_n^\beta$ and σ_n stands for production of the entropy s_n. Using results of the corresponding part of the existence-proof by Ducomet and Feireisl [2], one can state the following lemma on boundedness of approximate solutions.

Lemma 2. *Let $\Omega\subset\mathbb{R}^3$ be a bounded domain with $C^{2+\nu}, \nu>0$ smooth boundary. Moreover, consider that that the assumptions on constitutive terms hold. Then for any $\delta>0$ there exists a triple $(\rho,\mathbf{u},\vartheta)$ solving the problem* (8), (9) *and* (10) *in the sense of distributions. Moreover, there exists a solution satisfying the total energy equality*

$$\begin{aligned}&-\int_0^T\int_\Omega \partial_t\xi\left(\frac{1}{2}\rho|\mathbf{u}|^2 + \rho P_e(\rho) + \frac{\delta}{\beta-1}\rho^\beta + d\vartheta^4 + c_v\rho\vartheta\right)\mathrm{d}\mathbf{x}\,\mathrm{d}t\\ &= \int_\Omega \frac{|\mathbf{m}_0|^2}{2\rho_0} + \rho_0 P_e(\rho_0) + \frac{\delta}{\beta-1}\rho_0^\beta + d\vartheta_0^4 + c_v\rho_0\vartheta_0\,\mathrm{d}\mathbf{x} + \int_0^T\int_\Omega \xi\mathbf{f}\cdot\mathbf{u}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t\end{aligned} \tag{11}$$

for any $\xi\in C^\infty[0,T], \xi(0)=1, \xi(T)=0$, and enjoying the following estimates indepedently of the smoothness of the boundary:

- $\rho\in L^\infty(0,T;L^\beta(\Omega))$, $\rho|\mathbf{u}|^2\in L^\infty(0,T;L^\beta(\Omega))$, $\rho\vartheta\in L^\infty(0,T;L^1(\Omega))$,
- $\vartheta\in L^\infty(0,T;L^4(\Omega))$, $\rho\log\vartheta\in L^\infty(0,T;L^1(\Omega))$, $\frac{\mathbb{S}:\nabla\mathbf{u}}{\vartheta}\in L^1((0,T)\times\Omega)$,
- $\nabla\log\vartheta\in L^2((0,T)\times\Omega)$, $\nabla\vartheta^{3/2}\in L^2((0,T)\times\Omega)$,
- $\mathbf{u}\in L^r(0,T;W_0^{1,r}(\Omega)), r=\frac{8}{5-\alpha}$.

Now we can benefit from the technique by Ducomet and Feireisl [2]. For domains Ω_n with smooth boundary, that converge to domain Ω with boundary being merely Lipschitz continuous, we obtain solutions $(\rho_n,\mathbf{u}_n,\vartheta_n)$ which satisfy estimates stated in the lemma above. Note that these estimates are independent of n.

First, we use the test function

$$\varphi_n(t,x) = \psi(t)\mathcal{B}\left[\rho_n(t,\cdot) - \fint_\Omega \rho_n(t)\right](x)$$

where to obtain $\rho_n|_\Omega\in L^{\beta+1}((0,T)\times\Omega)$. Here, $\psi\in\mathcal{D}(0,T)$ and $\mathcal{B}$ denotes so called Bogovskii operator on domain Ω (and we consider it is extended by zero outside Ω). It expresses certain kind of inverse to the div operator and its main properties are stated in the following lemma.

Lemma 3 (Bogovskii operator, paragraph 3.3 in [6])**.** *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with Lipschitz continuous boundary. Then there exists a bounded linear operator $\mathcal{B} = (B_1, B_2, B_3)$ satisfying the following properties:*

- $\mathcal{B} : \overline{L^p(\Omega)} := \left\{ f \in L^p(\Omega) : \int_\Omega f \,\mathrm{d}\mathbf{x} = 0 \right\} \to W_0^{1,p}(\Omega; \mathbb{R}^3)$ *with*
$$\|\mathcal{B}(f)\|_{W_0^{1,p}(\Omega;\mathbb{R}^3)} \leq c(p) \|f\|_{L^p(\Omega)}$$
for any $1 < p < \infty$.
- *the function* $\mathbf{v} = \mathcal{B}[f]$ *solves the problem*
$$\operatorname{div} \mathbf{v} = f \text{ in } \Omega, \quad \mathbf{v}|_{\partial\Omega} = 0.$$
- *for any* $f \in L^p(\Omega)$ *such that there exists* $\mathbf{g} \in L^q(\Omega; \mathbb{R}^3)$ *satisfying* $f = \operatorname{div} \mathbf{g}$ *and* $\mathbf{g} \cdot \mathbf{n}|_{\partial\Omega} = 0$ *we have*
$$\|\mathcal{B}[f]\|_{L^r(\Omega;\mathbb{R}^3)} \leq c(r) \|\mathbf{g}\|_{L^r(\Omega;\mathbb{R}^3)}$$
for any $1 < r < \infty$.

More precisely, testing (9) with φ_n and employing the estimates given in Lemma 2 we obtain uniform bound in the form
$$\int_0^T \int_\Omega \rho_n^{\beta+1} \,\mathrm{d}\mathbf{x} \,\mathrm{d}t \leq c(\delta) \tag{12}$$
for any $\delta > 0$.

Remark 1. Direct inspection of construction of the Bogovskii operator allows us to claim that for given bounded Lipschitz domain $\Omega \subset \mathbb{R}^3$ and a sequence of bounded smooth domains Ω_n with boundary being uniformly Lipschitz continuous the estimates claimed in Lemma 3 can be made uniform with respect to n for n sufficiently large.

This allows us to obtain $\int_0^T \int_{\Omega_n} \rho_n^{\beta+1} \,\mathrm{d}\mathbf{x} \,\mathrm{d}t \leq c(\delta)$ for any $\delta > 0$ uniformly with respect to n.

2.1. **Strong compactness of the temperature.** Up to this moment, we only have weak compactness of the temperature which follows from the estimates in Lemma 2. In order to strenghten the convergence to the strong one, we shall use the variational formulation of the entropy inequality

$$\partial_t \left(\frac{4d}{3} \vartheta_n^3 + c_v \rho_n \log \vartheta_n \right) + \operatorname{div} \left(\frac{4d}{3} \vartheta_n^3 + c_v \rho_n \log \vartheta_n) \mathbf{u}_n \right) -$$
$$\operatorname{div} \left(\frac{\kappa_G(\vartheta_n) + \sigma \vartheta_n^3}{\vartheta_n} \nabla \vartheta_n \right) \geq -p_\vartheta(\rho_n) \operatorname{div} \mathbf{u}_n + \frac{\mathbb{S}_n : \nabla \mathbf{u}_n}{\vartheta_n} +$$
$$\frac{\kappa_G(\vartheta_n) + \sigma \vartheta_n^3}{\vartheta_n^2} |\nabla \vartheta_n|^2 \text{ in } \mathcal{D}'([0,T) \times \overline{\Omega_n})$$

In order to show relative compactness of the sequence of functions bounded in Bochner spaces, one can utter the following version of the Aubin-Lions lemma (see Lemma 6.3, Chapter 6 by Feireisl [4]).

Lemma 4. *Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$ be a bounded Lipschitz domain. Let $\{v_n\}$ be a sequence of functions bounded in*
$$L^2(0,T; L^q(\Omega)) \cap L^\infty(0,T; L^1(\Omega)), \quad q > \frac{2N}{N+2}.$$

Furthermore, suppose that $\partial_t v_n \geq g_n$ *in* $\mathcal{D}'((0,T)\times\Omega)$*, where distributions* g_n *are bounded in* $L^1(0,T;W^{-m,p}(\Omega))$ *for certain* $m \geq 1, p > 1$*. Then the sequence* $\{v_n\}$ *is relatively compact in the space* $L^2(0,T;W^{-1,2}(\Omega))$*.*

Applying Lemma 4 to the sequence $\left\{\frac{4d}{3}\vartheta_n^3 + c_v\rho_n \log\vartheta_n\right\}_n$ as in the part 5.4 and 6.2 of Ducomet and Feireisl [2] and using $\vartheta_n \rightharpoonup \vartheta$ in $L^2(0,T;W^{1,2}(\Omega))$ we obtain

$$\int_0^T \int_\Omega \left(\frac{4d}{3}\vartheta_n^3 + c_v\rho_n\log\vartheta_n\right)\vartheta_n \,\mathrm{d}\mathbf{x}\,\mathrm{d}t \to \int_0^T \int_\Omega \left(\frac{4d}{3}\overline{\vartheta^3} + c_v\rho_n\overline{\log\vartheta}\right)\vartheta \,\mathrm{d}\mathbf{x}\,\mathrm{d}t. \tag{13}$$

This immediately yields $\vartheta_n \to \vartheta$ in $L^2((0,T)\times\Omega)$.

2.2. **Propagation of density oscillations.** Having proved pointwise convergence of the temperature, the next thing we have to show is convergence in the linear momentum equation. In order to pass, we need to show convergence of the nonlinear pressure term. This can be done by showing pointwise convergence of the density.

Similarly to the part 6.3 by Ducomet and Feireisl [2] we can show that

$$\rho_n\mathbf{u}_n\otimes\mathbf{u}_n \to \rho\mathbf{u}\otimes\mathbf{u} \text{ in } L^1((0,T)\times\Omega)^{3\times 3}$$

Growth assumptions on the pressure term and results of Lemma 2 yield

$$\partial_t(\rho\mathbf{u}) + \operatorname{div}(\rho\mathbf{u}\otimes\mathbf{u}) + \nabla\left(\overline{p_e(\rho)} + \vartheta\overline{p_\vartheta(\rho)} + \frac{d}{3}\vartheta^4 + \delta\overline{\rho^\beta}\right) = \operatorname{div}\mathbb{S} + \rho\mathbf{f} \text{ in } \mathcal{D}'((0,T)\times\Omega)$$

By the Div-Curl lemma (see e.g. Lemma 6.1 by Feireisl [4]), the functions $\rho \in L^\infty(0,T;L^\beta(\Omega))$ and $\mathbf{u} \in L^2(0,T;W_0^{1,2}(\Omega)^3)$ solve the continuity equation in $\mathcal{D}'((0,T)\times\Omega)$ and it is easy to see that, provided we extend them by zero, the equation holds in $\mathcal{D}'((0,T)\times\mathbb{R}^3)$. Moreover, we can take β sufficiently large to recover that ρ and $\mathbf{u}$ solve the renormalized continuity equation on $\mathbb{R}^3$

$$\partial_t b(\rho) + \operatorname{div}(b(\rho)\mathbf{u}) + (b'(\rho)\rho - b(\rho))\operatorname{div}\mathbf{u} = 0 \text{ in } \mathcal{D}'((0,T)\times\mathbb{R}^3) \tag{1*}$$

where the function b satisfies certain growth assumptions (for details, see e.g. Novotný and Straškraba [6], Chapter 6).

Thus we can take $z \mapsto z\log z$ for b and write

$$\partial_t(\rho\log\rho) + \operatorname{div}(\rho\log\rho\mathbf{u}) + \rho\operatorname{div}\mathbf{u} = 0 \text{ in } \mathcal{D}'((0,T)\times\mathbb{R}^3) \tag{14}$$

On the other hand, $\rho_n \in L^\infty(0,T;L^\beta(\Omega))$ and $\mathbf{u}_n \in L^2(0,T;W_0^{1,r}(\Omega)^3)$ satisfy the renormalized continuity equation with $b(z) = z\log z$, passing with n to infinity we get

$$\partial_t(\overline{\rho\log\rho}) + \operatorname{div}(\overline{\rho\log\rho}\mathbf{u}) + \overline{\rho\operatorname{div}\mathbf{u}} = 0 \text{ in } \mathcal{D}'((0,T)\times\mathbb{R}^3) \tag{15}$$

Substracting (14) and (15), and integrating yields

$$\int_\Omega (\overline{\rho\log\rho} - \rho\log\rho)(\tau)\,\mathrm{d}\mathbf{x} = \int_0^\tau\int_\Omega \rho\operatorname{div}\mathbf{u} - \overline{\rho\operatorname{div}\mathbf{u}}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t. \tag{16}$$

As the function $z \mapsto z\log z$ is strictly convex and continuous, we have that the term on the left-hand side is always non-negative and vanishes if and only if $\rho_n \to \rho$ strongly in $L^1((0,T)\times\Omega)$. Therefore our next step is to obtain suitable bounds on the right-hand side. In order to do this, we employ the strategy by Lions [8] to use a test function of the form

$$\varphi_n(t,x) := \psi(t)\eta(x)(\nabla\Delta^{-1})[\rho_n(t,\cdot)](x), \quad \psi \in \mathcal{D}(0,T), \eta \in \mathcal{D}(\Omega)$$

for problem on the set Ω_n. This yields

$$\begin{aligned}
&\int_0^T \int_\Omega \psi\eta \left(p_e(\rho_n) + \vartheta_n p_\vartheta(\rho_n) + \frac{d}{3}\vartheta_n^4 + \delta\rho_n^\beta - \lambda(\vartheta_n)\mathrm{div}\,\mathbf{u}_n \right) \rho_n \,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\
&\quad - 2\int_0^T \int_\Omega \psi\eta\mu(\vartheta_n)\nabla\mathbf{u}_n : (\nabla\Delta^{-1}\nabla)[\rho_n]\,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\
&= \int_0^T \int_\Omega \psi \left[\lambda(\vartheta_n)\mathrm{div}\,\mathbf{u}_n - \left(p_e(\rho_n) + \vartheta_n p_\vartheta(\rho_n) + \frac{d}{3}\vartheta_n^4 + \delta\rho_n^\beta \right) \right] \nabla\eta\cdot(\nabla\Delta^{-1})[\rho_n] \\
&\quad + \int_0^T \int_\Omega \psi \left[\mu(\vartheta_n)(\nabla\mathbf{u}_n + \nabla\mathbf{u}_n^T) - \rho(\mathbf{u}_n\otimes\mathbf{u}_n) \right] \nabla\eta\cdot(\nabla\Delta^{-1})[\rho_n]\,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\
&\quad - \int_0^T \int_\Omega \partial_t\psi\eta\mathbf{u}_n\cdot(\nabla\Delta^{-1})[\rho_n]\,\mathrm{d}\mathbf{x}\,\mathrm{d}t + \int_0^T \int_{\Omega_n} \psi\eta\mathbf{f}\cdot(\nabla\Delta^{-1})[\rho_n]\,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\
&\quad + \int_0^T \int_\Omega \psi\eta\mathbf{u}_n\cdot\big(\rho_n(\nabla\Delta^{-1}\mathrm{div}\,)[\rho_n\mathbf{u}_n] - (\nabla\Delta^{-1}\nabla)[\rho_n](\rho_n\mathbf{u}_n)\big)\,\mathrm{d}\mathbf{x}\,\mathrm{d}t
\end{aligned}$$

where the terms $\nabla\Delta^{-1}\mathrm{div}$ and $\nabla\Delta^{-1}\nabla$ are defined in terms of the Fourier transformation and represent continuous linear operators from $L^p(\mathbb{R}^3)^3$ to $L^p(\mathbb{R}^3)^3$, $L^p(\mathbb{R}^3)$ to $L^p(\mathbb{R}^3)^{3\times3}$ respectively, with $1 < p < \infty$ (see e.g. Stein [9]) for details).

Similarly, one can use the test function $\varphi(t,x) = \psi(t)\eta(x)(\nabla\Delta^{-1})[\rho]$, with $\psi \in \mathcal{D}(0,T)$, and $\eta \in \mathcal{D}(\Omega)$ in the limit version of the linear momentum equation. Substracting both equations and passing to the limit, results on the weak continuity of the bilinear forms of singular integrals (Lemma 3.4 in Feireisl, Novotný and Petzeltová [5]) can be used to obtain

$$\begin{aligned}
&\lim_{n\to\infty} \Bigg(\int_0^T \int_{\Omega_n} \psi\eta(p_e(\rho_n) + \vartheta_n p_\vartheta(\rho_n) + \delta\rho_n^\beta - \lambda(\vartheta_n)\mathrm{div}\,\mathbf{u}_n)\rho_n\,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\
&\qquad -2\int_0^T \int_{\Omega_n} \psi\eta\mu(\vartheta_n)\nabla\mathbf{u}_n : (\nabla\Delta^{-1}\nabla)[\rho_n]\,\mathrm{d}\mathbf{x}\,\mathrm{d}t \Bigg) \\
&= \int_0^T \int_\Omega \psi\eta(\overline{p_e(\rho)} + \vartheta\overline{p_\vartheta(\rho)} + \delta\overline{\rho^\beta} - \lambda(\vartheta)\mathrm{div}\,\mathbf{u})\rho\,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\
&\quad - 2\int_0^T \int_\Omega \psi\eta\mu(\vartheta)\nabla\mathbf{u} : (\nabla\Delta^{-1}\nabla)[\rho]\,\mathrm{d}\mathbf{x}\,\mathrm{d}t.
\end{aligned}$$

Our next step is to simplify the integrals in the equation, more precisely, we wish to obtain

$$\begin{aligned}
&\lim_{n\to\infty} \int_0^T \int_\Omega \psi\eta\left(2\mu(\vartheta_n)\nabla\mathbf{u}_n : (\nabla\Delta^{-1}\nabla)[\rho_n] - 2\mu(\vartheta)\nabla\mathbf{u} : (\nabla\Delta^{-1}\nabla)[\rho]\right)\,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\
&= \lim_{n\to\infty} \int_0^T \int_\Omega \psi\eta\left(2\mu(\vartheta_n)\mathrm{div}\,\mathbf{u}_n\rho_n - 2\mu(\vartheta)\mathrm{div}\,\mathbf{u}\rho\right)\,\mathrm{d}\mathbf{x}\,\mathrm{d}t.
\end{aligned}$$

To this end, we employ the commutator theory for singular integrals developed by Coifman and Meyer [1].

Lemma 5 (Commutator Lemma (Proposition 5.1 in [3])). *Let* $\mathbf{v} : (0,T)\times\mathbb{R}^N \to \mathbb{R}^N$ *be a vector field and let* $g : (0,T) \times \mathbb{R}^N \to \mathbb{R}$ *be a scalar function such that*

$$\mathbf{v} \in L^r(0,T; L^r(\mathbb{R}^N;\mathbb{R}^N)), g \in L^p(0,T; W^{1,p}(\mathbb{R}^N)) \cap L^\infty(0,T; (L^q \cap L^1)(\mathbb{R}^N)),$$

where $\frac{Nr}{Nr+r-N} < p < N$, $\frac{1}{q} + \frac{1}{r} < 1$. *Furthermore, assume that*

$$\operatorname*{ess\,sup}_{t\in(0,T)} \left(\|g(t)\|_{L^q(\mathbb{R}^N)} + \|g(t)\|_{L^1(\mathbb{R}^N)} \right) \le K.$$

Then there exist constants $c = c(p,q,r,K) > 0$, $\omega = \omega(p,q,r) > 0$, *and* $s = s(p,q,r) > 1$ *such that*

$$\begin{aligned}\big\|(\partial_i\Delta^{-1}\mathrm{div}\,)[g\mathbf{v}] - g(\partial_i\Delta^{-1}\mathrm{div}\,)[\mathbf{v}]\big\|_{L^s(0,T;W^{\omega,s}(\mathbb{R}^N))}\\ \le c\|g\|_{L^p(0,T;W^{1,p}(\mathbb{R}^N))}\|\mathbf{v}\|_{L^r(0,T;W^{1,r}(\mathbb{R}^N;\mathbb{R}^N))}, \text{ for } i = 1,\dots,N.\end{aligned}$$

Taking $\mathbf{v} = \nabla u^i$ and $g = \eta\mu(\vartheta)$ in the preceding lemma and using the strong convergence of the temperature we see that the identity we claim holds.

We have shown, that

$$\begin{aligned}&\overline{p_e(\rho)\rho} + \vartheta\overline{p_\vartheta(\rho)\rho} + \delta\overline{\rho^{\beta+1}} - (2\mu(\vartheta) + \lambda(\vartheta))\overline{\rho\mathrm{div}\,\mathbf{u}}\\ &\quad= \overline{p_e(\rho)}\rho + \vartheta\overline{p_\vartheta(\rho)}\rho + \delta\overline{\rho^\beta}\rho - (2\mu(\vartheta) + \lambda(\vartheta))\rho\mathrm{div}\,\mathbf{u} \text{ in } \mathcal{D}((0,T)\times\Omega).\end{aligned}$$

This relation can be rewritten to the form

$$\rho\mathrm{div}\,\mathbf{u} - \overline{\rho\mathrm{div}\,\mathbf{u}} = \frac{1}{2\mu(\vartheta) + \lambda(\vartheta)}(Q_1 + Q_2 + Q_3)$$

where

$$Q_1 = \overline{p_e(\rho)}\rho - \overline{p_e(\rho)\rho}, \quad Q_2 = \vartheta\left(\overline{p_\vartheta(\rho)}\rho - \overline{p_\vartheta(\rho)\rho}\right), \quad Q_3 = \delta\left(\overline{\rho^\beta}\rho - \overline{\rho^{\beta+1}}\right).$$

As p_ϑ is non-decreasing, we have $Q_2 \le 0$, and similarly $Q_3 \le 0$. What remains is to estimate the term Q_1. We can use the pressure decomposition technique by Feireisl [4] in order to show that $Q_1 \le \overline{p_b(\rho)}\rho - \overline{p_b(\rho)\rho}$, where the term p_b is a bounded part of the pressure $p_e = p_c + p_m + p_b$ with the convex part p_c and the monotone part p_m. Now we can estimate the difference of the bounded pressure parts as it was done by Feireisl [4], and employ (16) in order to obtain existence of $\Lambda < \infty$ such that

$$\int_\Omega \left(\overline{\rho\log\rho} - \rho\log\rho\right)(\tau)\,\mathrm{d}\mathbf{x} \le \frac{\Lambda}{\underline{\mu}}\int_0^\tau\int_\Omega \left(\overline{\rho\log\rho} - \rho\log\rho\right)\,\mathrm{d}\mathbf{x}\,\mathrm{d}t$$

for almost every $\tau \in [0,T]$. Consequently, the Gronwall lemma yields $\overline{\rho\log\rho} = \rho\log\rho$ which is equivalent to

$$\rho_n \to \rho \text{ in } L^1((0,T)\times\Omega).$$

2.3. **Approximate entropy inequality and total energy equality.** As we already know, the limit functions ρ and ϑ satisfy the continuity equation as well as the linear momentum equation, our next task is to verify that also the entropy inequality and energy equality are satisfied. In the previous parts, we have proved convergence of all the terms involved in the energy and entropy formulae except for $\rho P_e(\rho)$ and $\rho P_\vartheta(\rho)$, but this follows as ρ and $\mathbf{u}$ solve the renormalized continuity equation on $(0,T)\times\mathbb{R}^3$.

Passing to the limit in the entropy inequality, we see that the terms $\frac{\mathbb{S}_n:\nabla\mathbf{u}_n}{\vartheta_n}$, $\frac{\kappa_G(\vartheta_n)+\sigma\vartheta_n^3}{\vartheta_n}$ and $\frac{\kappa_G(\vartheta_n)+\sigma\vartheta_n^3}{\vartheta_n^2}|\nabla\vartheta_n|^2$ need some special care.

Lemma 6 (Lemma 5.4 in [2])**.** *Let* $\vartheta_n \to \vartheta$ *in* $L^2((0,T)\times\Omega)$*, and* $\log\vartheta_n \rightharpoonup \overline{\log\vartheta}$ *in* $L^2((0,T)\times\Omega)$*. Then* ϑ *is strictly positive a.e. on* $(0,T)\times\Omega$*, and* $\log\vartheta = \overline{\log\vartheta}$*.*

A direct consequence of the lemma above yields $\log\vartheta_n \to \log\vartheta$ in $L^2((0,T)\times\Omega)$. As $\nabla\log\vartheta_n = \frac{\nabla\vartheta_n}{\vartheta_n}$ is uniformly bounded in $L^2((0,T)\times\Omega)$, we obtain

$$\frac{\kappa_G(\vartheta_n)+\sigma\vartheta_n^3}{\vartheta_n}\nabla\vartheta_n \to \frac{\kappa_G(\vartheta)+\sigma\vartheta^3}{\vartheta}\nabla\vartheta \text{ in } \mathcal{D}'([0,T)\times\overline{\Omega})$$

Convergence in terms $\frac{1}{\vartheta}\mathbb{S}:\mathbf{u}$ and $\frac{\kappa_G(\vartheta)+\sigma\vartheta^3}{\vartheta^2}|\nabla\vartheta|^2$ follows by the weak lower semi-continuity of the norm and formulae

$$\frac{\mathbb{S}:\nabla\mathbf{u}}{\vartheta} = \left|\sqrt{\frac{\mu(\vartheta)}{\vartheta}}\left(\nabla\mathbf{u}+\nabla\mathbf{u}^T-\frac{1}{3}\operatorname{div}\mathbf{u}\mathbb{I}\right)\right|^2 + \left|\sqrt{\frac{\zeta(\vartheta)}{\vartheta}}\operatorname{div}\mathbf{u}\right|^2,$$

$$\frac{\kappa_G(\vartheta)+\sigma\vartheta^3}{\vartheta^2}|\nabla\vartheta|^2 = |\nabla\mathcal{K}_{G,\sigma}(\vartheta)|^2, \text{ where } \mathcal{K}_{G,\sigma}(z) = \int_1^z \frac{\sqrt{\kappa_G(s)+\sigma s^3}}{s}\,\mathrm{d}s.$$

To complete our considerations, it is enough to write for any $\varphi\in\mathcal{D}([0,T)\times\mathbb{R}^3)$, $\varphi|_{(0,T)\times\overline{\Omega}} \geq 0$.

$$\begin{aligned}
&\int_0^T\int_\Omega \partial_t\varphi\left(\frac{4d}{3}\vartheta_n^3 + c_v\rho_n\log\vartheta_n - \rho_n P_\vartheta(\rho_n)\right)\mathrm{d}\mathbf{x}\,\mathrm{d}t\\
&\quad+\int_0^T\int_\Omega\left(\frac{4d}{3}\vartheta_n^3 + c_v\rho_n\log\vartheta_n - \rho_n P_\vartheta(\rho_n)\right)\mathbf{u}_n\cdot\nabla\varphi\,\mathrm{d}\mathbf{x}\,\mathrm{d}t\\
&\quad-\int_0^T\int_\Omega\frac{\kappa_G(\vartheta_n)+\sigma\vartheta_n^3}{\vartheta_n}\nabla\vartheta_n\cdot\nabla\varphi\,\mathrm{d}\mathbf{x}\,\mathrm{d}t\\
&\quad+\int_0^T\int_{\Omega_n\setminus\Omega}\partial_t\varphi\left(\frac{4d}{3}\vartheta_n^3 + c_v\rho_n\log\vartheta_n - \rho_n P_\vartheta(\rho_n)\right)\mathrm{d}\mathbf{x}\,\mathrm{d}t\\
&\quad+\int_0^T\int_{\Omega_n\setminus\Omega}\left(\frac{4d}{3}\vartheta_n^3 + c_v\rho_n\log\vartheta_n - \rho_n P_\vartheta(\rho_n)\right)\mathbf{u}_n\cdot\nabla\varphi\,\mathrm{d}\mathbf{x}\,\mathrm{d}t\\
&\quad-\int_0^T\int_{\Omega_n\setminus\Omega}\frac{\kappa_G(\vartheta_n)+\sigma\vartheta_n^3}{\vartheta_n}\nabla\vartheta_n\cdot\nabla\varphi\,\mathrm{d}\mathbf{x}\,\mathrm{d}t\\
&\leq-\int_0^T\int_{\Omega_n}\varphi\left(\frac{\mathbb{S}_n:\nabla\mathbf{u}_n}{\vartheta_n}+\frac{\kappa_G(\vartheta_n)+\sigma\vartheta_n^3}{\vartheta_n^2}|\nabla\vartheta_n|^2\right)\mathrm{d}\mathbf{x}\,\mathrm{d}t\\
&\quad-\int_\Omega\varphi(0)\left(\frac{4d}{3}\vartheta_{0,n}^3 + c_v\rho_{0,n} + \log(\vartheta_{0,n}) - \rho_{0,n}P_\vartheta(\rho_{0,n})\right)\mathrm{d}\mathbf{x}\,\mathrm{d}t\\
&\quad-\int_{\Omega_n\setminus\Omega}\varphi(0)\left(\frac{4d}{3}\vartheta_{0,n}^3 + c_v\rho_{0,n} + \log(\vartheta_{0,n}) - \rho_{0,n}P_\vartheta(\rho_{0,n})\right)\mathrm{d}\mathbf{x}\,\mathrm{d}t\\
&\leq-\int_0^T\int_{\Omega}\varphi\left(\frac{\mathbb{S}_n:\nabla\mathbf{u}_n}{\vartheta_n}+\frac{\kappa_G(\vartheta_n)+\sigma\vartheta_n^3}{\vartheta_n^2}|\nabla\vartheta_n|^2\right)\mathrm{d}\mathbf{x}\,\mathrm{d}t\\
&\quad-\int_\Omega\varphi(0)\left(\frac{4d}{3}\vartheta_{0,n}^3 + c_v\rho_{0,n} + \log(\vartheta_{0,n}) - \rho_{0,n}P_\vartheta(\rho_{0,n})\right)\mathrm{d}\mathbf{x}\,\mathrm{d}t\\
&\quad-\int_{\Omega_n\setminus\Omega}\varphi(0)\left(\frac{4d}{3}\vartheta_{0,n}^3 + c_v\rho_{0,n} + \log(\vartheta_{0,n}) - \rho_{0,n}P_\vartheta(\rho_{0,n})\right)\mathrm{d}\mathbf{x}\,\mathrm{d}t
\end{aligned}$$

where we have taken for φ its nonnegative part, $\varphi(t,x) := (\varphi(t,x))^+$, which is possible by the density argument.

It is now easy to see that all the integrals over $\Omega_n \setminus \Omega$ vanish in the limit. As the weak lower semicontinuity of the first integral on the right-hand side preserves the inequality sign in the limit, we are done.

3. **Vanishing artificial pressure.** As we have proved existence of a solution to the δ-approximate problem on domain Ω with boundary being merely Lipschitz continuous, we are now able to employ the rest of procedures of the proof by Ducomet and Feireisl [2] and obtain solution to the Navier–Stokes–Fourier system on Ω.

4. **Remarks on sensitivity with respect to the boundary.** Throughout our proof we considered approximation of the Lipschitz domain Ω by smooth domains Ω_n, $\Omega \subset \Omega_n$. It was shown, that only some reasonable property that for any ball $B \subset \mathbb{R}^3 \setminus \Omega$ there exists $n(B)$ such that if $n \geq n(B)$, then $B \subset \mathbb{R}^3 \setminus \Omega_n$, is needed. The question is what can one obtain in the case of approximation by "'smaller"' smooth domains, that is $\Omega_n \subset \Omega$. It turns out, that in addition to the rather natural requirement that any compact subset $K \subset \Omega$ is absorbed by Ω_n for all $n \geq n_0(K)$, we have to require even more – $\mathrm{cap}_2(\Omega \setminus \Omega_n) \to 0$ as $n \to \infty$.

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E-mail address: poul@karlin.mff.cuni.cz