The Caffarelli-Kohn-Nirenberg Theorem -
A Direct Proof by Campanato’s Method

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Abstract. In the present paper we give a generalized notion of a suitable weak solution to the Navier-Stokes equations in any domain by the aid of a pressure representation of the form $p = p_0 + \partial \tilde{p}_h/\partial t$ ($p \in L^{4/3}(0, \infty, L^2(\Omega))$; $\tilde{p}_h$ is harmonic with respect to the spatial variable $x \in \Omega$). Then we prove that for any initial velocity $a \in L^2(\Omega)$ there exists a (generalized) suitable weak solution $(u, p)$ in $\Omega \times [0, \infty]$ satisfying the following property of partial regularity: The velocity field $u$ is continuous in a neighbourhood of a point $(x_0, t_0) \in \Omega \times [0, \infty]$ if

$$\limsup_{R \to 0^+} \frac{1}{R} \int_{Q_R(x_0, t_0)} |\text{rot} \ u \times \frac{u}{|u|}|^2 \, dx \, dt \leq \varepsilon_*$$

for a sufficiently small $\varepsilon_* > 0$.

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1. Introduction. Statement of the Main Result

The aim of the present paper is the study of the local regularity of weak solutions to an incompressible Newtonian fluid in a general three-dimensional domain $\Omega$ which is governed by the following two equations, first the by the conservation of volume

$$\text{div} \ u = 0 \quad \text{in} \quad \Omega \times [0, \infty], \quad (1.1)$$

and second by the conservation of momentum

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u - \Delta u = -\nabla p \quad \text{in} \quad \Omega \times [0, \infty], \quad (1.2)$$
where \( \mathbf{u} = (u_1, u_2, u_3) \) denotes the unknown velocity field and \( p \) the unknown pressure. This two equations are completed by the condition of adherence

\[
\mathbf{u} = 0 \quad \text{on} \quad \partial \Omega \times [0, \infty],
\]

and the initial condition

\[
\mathbf{u}(0) = \mathbf{a} \quad \text{on} \quad \Omega \times \{0\},
\]

where \( \mathbf{a} = (a_1, a_2, a_3) \) denotes a given distribution of the velocity at the initial time \( t = 0 \).

The equations (1.1)-(1.4) are called the Navier-Stokes equations which where introduced first by Navier [14] and later rederived by Stokes [20]. The rigorous mathematical treatment started with Leray [11], where he studied the existence of weak solutions to (1.1)-(1.4) for the case \( \Omega = \mathbb{R}^3 \) (for the notion of a weak solutions see below). Later Hopf [8] proved the existence of weak solutions in a general domain by using a Galerkin approximation. For more details and further approaches we refer to the monographs of Temam [23] and Sohr [18].

However despite of many efforts until now one is unable to construct a classical solution to the Navier-Stokes equations for an arbitrarily given smooth initial velocity, which has been considered as one of the seven Millennium problems introduced by the Clay institute. Concerning the issue of regularity the best result which is known is the so-called Caffarelli-Kohn-Nirenberg theorem (cf. [4]), which states that the velocity field \( \mathbf{u} \) of a suitable weak solution to the Navier-Stokes equations is bounded on a neighbourhood of \( (x_0, t_0) \in Q \), where \( Q := \Omega \times [0, \infty] \), if

\[
\limsup_{R \to 0^+} \frac{1}{R} \int_{Q_R(x_0, t_0)} |\nabla \mathbf{u}|^2 \, dx \, dt \leq \varepsilon_0
\]

for a sufficiently small \( \varepsilon_0 > 0 \). This shows that the singular set \( \Sigma \) of all points \( (x_0, t_0) \in Q \), where (1.5) fails is closed such that

\[
\mathcal{P}_1(\Sigma) = 0.
\]

In fact, the first result of partial regularity is due to Scheffer. In his pioneering paper [15] he has introduced the notion of a suitable weak solution to (1.1)-(1.4) that is a pair \((\mathbf{u}, p)\) with

\[
\mathbf{u} \in L^\infty(0, \infty; L^2(\Omega)^3) \cap L^2(0, \infty; \dot{W}^{1,2,\sigma}_\Omega),
\]

\[
p \in L^{3/2}(\Omega \times [0, \infty])
\]

satisfying (1.2) in sense of distribution, such that the local energy inequality

\[
\int_\Omega |\mathbf{u}(t)|^2 \phi \, dx + 2 \int_0^t \int_\Omega |\nabla \mathbf{u}|^2 \phi \, dx \, dt
\]

\[
\leq \int_0^t \int_\Omega |\mathbf{u}|^2 \left( \frac{\partial \phi}{\partial t} + \Delta \phi \right) \, dx \, dt + \int_0^t \int_\Omega (|\mathbf{u}|^2 + 2p) \mathbf{u} \cdot \nabla \phi \, dx \, dt
\]

\[
(1.7)
\]

\[\text{1 Throughout by bold letters we denote vector valued functions.}\]

\[\text{2 For the definition of } \dot{W}^{1,2,\sigma}_\Omega(\Omega) \text{ see below.}\]
holds for all $\phi \in C_0^\infty(Q)$, for almost all $t \in ]0, \infty[$. For such suitable weak solutions the author defined a set $\hat{\Sigma}$ of all possible singular points by a condition different then (1.5), proving that

$$\mathcal{H}_{5/3}(\hat{\Sigma}) = 0.$$  \hfill (1.8)

(for details see [15], [16]).

Later Lin [13] reproved the Caffarelli-Kohn-Nirenberg theorem by using pressure estimates obtained by Sohr and von Wahl [19]. Moreover, Ladyzenskaya and Seregin [9] have given a more detailed proof of the partial regularity of a suitable weak solution to the Navier-Stokes equations including the case when a force $f$ is added to the right of (1.2). Recently, Vasseur [24] established a new proof of the Caffarelli-Kohn-Nirenberg theorem using a direct approach based on a Moser iteration.

In contrast to the results mentioned above the partial regularity of weak solutions to the Navier-Stokes equations in a general domain $\Omega$ is not known. Since in this case the pressure does exist only in a distributional sense one cannot use the classical concept of a suitable weak solution. However as we will see below we are able to treat even this case using a pressure representation of the form

$$p = p_0 + \frac{\partial \hat{p}_h}{\partial t},$$

where $p_0$ denotes a Lebesgue measurable function in $Q$, while $\hat{p}_h$ is a harmonic function with respect to $x \in \Omega$. This representation has been introduced first in [25].

Weak solutions. First let us introduce the function spaces and some notations which we use in what follows. By $W^{m,q}(\Omega)$, $W_0^{m,q}(\Omega)$ ($m = 1, 2, \ldots; 1 \leq q < \infty$) we denote the usual Sobolev space. By $\mathcal{D}_\sigma(\Omega)$ we denote the vector space of all $\phi \in C_0^\infty(\Omega)^3$ with $\text{div} \phi = 0$. Then define

$$L^2_\sigma(\Omega) := \text{closure of } \mathcal{D}_\sigma(\Omega) \text{ in } L^2(\Omega)^3,$$

$$\dot{W}^{1,2}_\sigma(\Omega) := \text{closure of } \mathcal{D}_\sigma(\Omega) \text{ in } W^{1,2}(\Omega)^3.$$

Given a normed vector space $X$ with norm $\| \cdot \|$, we denote by $L^s(a,b; X)$ ($1 \leq s \leq \infty; -\infty \leq a < b \leq \infty$) the vector space of all Bochner measurable functions $z : [a,b[ \to X$ such that

$$\int_a^b \| z(t) \|^s \, dt < \infty \quad \text{if } 1 \leq s < \infty, \quad \text{ess sup}_{[a,b]} \| z(t) \| < \infty \quad \text{if } s = \infty.$$  

(see, e.g., [18; Chap. IV,1] for details).

**Definition 1.1.** Let $a \in L^2_\sigma(\Omega)$. A vector function $u : Q \to \mathbb{R}^3$ is called a weak solution to (1.1) - (1.4) if

$$u \in L^2(0,\infty; \dot{W}^{1,2}_\sigma(\Omega)) \cap L^\infty(0,\infty; L^2_\sigma(\Omega))$$

and there holds the following integral identity

$$\int_Q \left\{ -u \cdot \partial_t \phi + (u \cdot \nabla)u \cdot \phi + \nabla u : \nabla \phi \right\} \, dx \, dt = \int_\Omega a \cdot \phi(0) \, dx$$  \hfill (1.9)
for all \( \varphi \in C_0^\infty([0, \infty]; \mathcal{D}_\sigma(\Omega))^3 \).

Remark 1.1. Let \( u : Q \to \mathbb{R}^3 \) be a weak solution to (1.1) - (1.4). Then \( u \) can be redefined on a set of Lebesgue measure 0 such that \( u \in C_w([0, \infty]; L^2_\sigma(\Omega)) \), i.e. for all \( t > 0 \) there holds \( u(t) \in L^2_\sigma(\Omega) \).

In addition, we have

\[
\lim_{s \to t} \int_{\Omega} u(s) \cdot \xi \, dx = \int_{\Omega} u(t) \cdot \xi \, dx \quad \forall \xi \in L^2_\sigma(\Omega).
\]

Remark 1.2. As we will see in Section 2 for every weak solution \( u \) to the Navier-Stokes equations there exists a pressure \( p_0 \in L^{4/3}(0, \infty; L^2(\Omega)) \) and \( \tilde{p}_h \in C(Q) \) with \( \Delta \tilde{p}_h = 0 \), such that the function \( v := u + \nabla \tilde{p}_h \) satisfies the identity

\[
\int_Q \left\{ - v \cdot \partial \varphi \over \partial t + (u \cdot \nabla)u \cdot \varphi + \nabla v \cdot \nabla \varphi \right\} \, dx \, dt = \int_Q p_0 \text{div} \varphi \, dx \, dt \quad \text{(1.10)}
\]

for all \( \varphi \in C_0^\infty(Q)^3 \) together with the local energy inequality

\[
\int_0^t \int_{\Omega} |v|^2 \phi(t) \, dx + 2 \int_0^t \int_{\Omega} |\nabla v|^2 \phi \, dx \, ds \\
\leq \int_0^t \int_{\Omega} |v|^2 \left\{ \frac{\partial \phi}{\partial t} + \Delta \phi \right\} \, dx \, ds + \int_0^t \int_{\Omega} |u|^2 + 2p_0 |v \cdot \nabla \phi| \, dx \, ds \quad \text{(1.11)}
\]

for almost all \( 0 < t < \infty \) and for all nonnegative functions \( \phi \in C_0^\infty(Q) \).

Remark 1.2. As we will see in Section 2 for every weak solution \( u \) to the Navier-Stokes equations there exists a pressure \( p_0 \in L^{4/3}(0, \infty; L^2(\Omega)) \) and \( \tilde{p}_h \in C(Q) \) being harmonic with respect to \( x \in \Omega \). Setting \( \hat{p} := \frac{|u|^2}{2} + p_0 \) using

\[
(u \cdot \nabla)u = \frac{1}{2} \nabla |u|^2 - \text{rot} u \times u \text{ the identity (1.10) turns into}
\]

\[
\int_Q \left\{ - v \cdot \partial \varphi \over \partial t + (u \cdot \nabla)u \cdot \varphi + \nabla v \cdot \nabla \varphi \right\} \, dx \, dt \\
= \int_Q \left\{ \hat{p} \text{div} \varphi + \text{rot} u \times u \right\} \, dx \, dt \quad \text{(1.12)}
\]
for all $\varphi \in C_0^\infty(Q)^3$. In addition, by virtue of Sobolev’s embedding theorem along with multiplicative inequalities one finds

$$L^2(0, \infty; W^{1, 2}_2(\Omega)) \cap L^\infty(0, \infty; L^2_\sigma(\Omega)) \subset L^{8/3}(0, \infty; L^4(\Omega)^3).$$

Consequently,

$$\dot{p} \in L^{4/3}(0, \infty; L^2(\Omega)).$$

Our first main result is related to the existence of a suitable weak solution to (1.1)-(1.4).

**Theorem 1.3 (Main Theorem 1).** For every initial velocity $a \in L^2_\sigma(\Omega)$ there exists a suitable weak solution $u$ to the Navier-Stokes equations (1.1)-(1.4) with a pressure $p = p_0 + \frac{\partial \tilde{p}}{\partial t}$ according to Definition 1.2.

The proof Theorem 1.3 is given in Section 7 below.

**Notations**

Let $x_0 \in \mathbb{R}^3$, $t_0 \in \mathbb{R}$ and $0 < R < \infty$. We define the ball

$$B_R(x_0) := \{ x \in \mathbb{R}^3 \mid |x - x_0| < R \}$$

and the parabolic cylinder

$$Q_R(x_0, t_0) := B_R(x_0) \times [t_0 - R^2, t_0].$$

Our second main result concerns the partial regularity of suitable weak solutions to the Navier-Stokes equations.

**Theorem 1.4 (Main Theorem 2).** There exists an absolute constant $\varepsilon_* > 0$, such that every suitable weak solution $u$ to (1.1)-(1.4) with corresponding pressure $p = p_0 + \frac{\partial \tilde{p}}{\partial t}$ is continuous in $Q \setminus \text{sing}(u)$, where

$$\text{sing}(u) := \left\{ (x_0, t_0) \in Q \mid \lim_{R \to 0^+} \frac{1}{R} \int_{Q_R(x_0, t_0)} \left| \text{rot} \times \frac{u}{|u|} \right|^2 \, dx \, dt > \varepsilon_* \right\}$$

is a closed subset of $Q$ with $\mathcal{P}_1(\text{sing}(u)) = 0$.

The proof of Theorem 1.4 essentially relies on the following two propositions

**Proposition 1.5.** There exists an absolute constant $k_* > 0$ such that if $u$ is a suitable weak solution to (1.1)-(1.4) with corresponding pressure $p = p_0 + \frac{\partial \tilde{p}}{\partial t}$ then there holds

$$\Xi(x_0, t_0; R) \leq k_* (1 + R^2 M(x_0, t_0; R)) \left[ \tau^2 + \tau^{-3} \tilde{\Theta}(x_0, t_0; R) \right] \Xi(x_0, t_0; R)$$

(1.13)
for every $0 < \tau < \frac{1}{4}$, for all $(x_0, t_0) \in Q$ and $0 < R < \min\{\text{dist}(x_0, \partial \Omega), \sqrt{T_0}\}$, where

$$
\Xi(x_0, t_0; \rho) := \frac{1}{\rho} \left( \|\nabla v\|^2_{L^2(Q_\rho(x_0, t_0))} + \|v\|^2_{L^{10/3}(Q_\rho(x_0, t_0))} + \|v\|^2_{L^\infty(t_0 - \rho^2, t_0; L^2(B_\rho(x_0)))} \right) + \frac{1}{\rho} \|\hat{p} - \hat{p}_{B_\rho}\|^2_{L^{4/3}(t_0 - \rho^2, t_0; L^2(B_\rho(x_0)))},
$$

$$
M(x_0, t_0; \rho) := \max_{(x, t) \in \overline{Q}_\rho} |\nabla \hat{p}_\rho(x, t)|^2,
$$

$$
\tilde{\Theta}(x_0, t_0; \rho) := \rho^2 M(x_0, t_0; \rho) + \frac{1}{\rho} \int_{Q_\rho(x_0, t_0)} |\text{rot} \, u| \left| \frac{u}{|u|} \right|^2 \, dx \, dt,
$$

$0 < \rho \leq \min\{\text{dist}(x_0, \partial \Omega), \sqrt{T_0}\}$.

**Proposition 1.6.** There exists an absolute constant $\hat{k}_* > 0$ such that if $u$ is a suitable weak solution to (1.1)–(1.4) with pressure $p = p_0 + \frac{\partial \hat{p}_\rho}{\partial t}$ then there holds

$$
\Xi(x_0, t_0; \tau R) \leq \hat{k}_* \left[ \tau^3 + \tau^{-3} \mathcal{O}(x_0, t_0; R) \right] \Xi(x_0, t_0; R) \tag{1.14}
$$

for every $0 < \tau < \frac{1}{4}$, for all $(x_0, t_0) \in Q$ and $0 < R < \min\{\text{dist}(x_0, \partial \Omega), \sqrt{T_0}\}$, where

$$
\Xi(x_0, t_0; \rho) := \frac{1}{\rho} \|\nabla v\|^2_{L^2(Q_\rho(x_0, t_0))} + \frac{1}{\rho^2} \|\hat{p} - \hat{p}_{B_\rho}\|^2_{L^{4/3}(t_0 - \rho^2, t_0; L^2(B_\rho(x_0)))},
$$

$$
\mathcal{O}(x_0, t_0; \rho) := \rho^2 M(x_0, t_0; \rho) + \frac{1}{\rho} \|v\|^2_{L^4(t_0 - \rho^2, t_0; L^2(B_\rho(x_0)))},
$$

$0 < \rho \leq \min\{\text{dist}(x_0, \partial \Omega), \sqrt{T_0}\}$.

**Remark 1.7.** 1. By Main Theorem 1 and Main Theorem 2 we improve the the known results concerning the interior regularity of weak solutions to the Navier-Stokes equations in the following sense:

- Existence of a weak solution to the Navier-Stokes equations in an arbitrarily given domain $\Omega$ satisfying an analogous property of partial regularity as stated in the Caffarelli-Kohn-Nirenberg theorem;
- The boundeness of $u$ in a neighbourhood of a regular point $(x_0, t_0)$ can be replaced by the continuity of $u$, which seems to be the best possible regularity with respect to the time variable;
- Replacing the condition (1.5) by the more physical condition

$$
\limsup_{R \to 0^+} \frac{1}{R^d} \int_{Q_R(x_0, t_0)} \left| \text{rot} \, u \times \frac{u}{|u|} \right|^2 \, dx \, dt < \varepsilon_*, \tag{1.15}
$$

improves the estimate of the singular set;
The direct method based on Prop. 1.5 and Prop. 1.6 simplifies the proof and in addition enables us to specify the numerical value \( \varepsilon^* > 0 \) for the definition of the singular set (see (1.15) above).

2. The statement of Theorem 1.4 continues to hold if the right hand side of (1.2) is modified by adding a body force \(- \text{div} f\) with \(f \in L^{2, \lambda}(Q)^9, \quad \lambda > 3^3\).

The paper is organized as follows. In Section 2 we prove the existence of a pressure \(p = p_0 + \frac{\partial \tilde{p}}{\partial t}\), which forms the basis for proving the existence of a (generalized) suitable weak solution according to Definition 1.2. In Section 3 we establish fundamental estimates for functions being almost caloric which will be used for both the proof of Prop. 1.5 and Prop. 1.6. In Section 4 we provide present the appropriate model system together with a-priori estimates for weak solutions to such systems in a given cylinder \(Q_R(x_0, t_0)\). The subject of Section 5 is the proof of Proposition 1.5, and the proof of Prop. 1.6. These fundamental estimates will be achieved after having established a Caccioppoli-type inequality combined with a Campanato-type estimates for the pressure \(\hat{p}\) along with the Campanato-type estimates for semi caloric functions and the estimates established for the model system. The proof of the Main Theorem 2 will be completed in Section 6 using a standard iteration procedure. In Section 7 we construct a suitable weak solution to the Navier-Stokes equations according to Definition 1.2 by first solving an approximated Navier-Stokes system and afterwards passing to limit with the parameter of approximation. Finally, in the appendix of the paper we list several lemmas which we have used throughout the paper.

2. Existence of pressure

In the present section we provide the main tool for introducing a pressure written in the form \(p = p_0 + \frac{\partial \tilde{p}}{\partial t}\) for a weak solutions to the Navier-Stokes equations and related systems. Since this method can be applied for more general systems it is for interest itself.

For a given domain \(G \subset \mathbb{R}^3\) define
\[
A^2(G) := \left\{ \Delta u \left| u \in W^{2, 2}_0(G) \right. \right\}, \quad B^2(G) := \left\{ p \in L^2(G) \left| \Delta p = 0 \text{ in } G \right. \right\}.
\]
Hereby \(W^{2, 2}_0(G)\) denotes the closure of \(C_0^\infty(G)\) with respect to the norm
\[
\|\phi\|_{W^{2, 2}_0(G)} := \left( \int_G |\Delta \phi|^2 \, dx \right)^{\frac{1}{2}}.
\]

\(^3\text{Here } L^{2, \lambda}(Q) \text{ stands for the usual parabolic Campanato space (cf. [1]).}\)
Clearly, both \( A^2(G) \) and \( B^2(G) \) are closed subspaces of \( L^2(G) \). Using Weyl’s lemma one sees that \( B^2(G) \) is the orthogonal complement of \( A^2(G) \) in \( L^2(G) \). Hence we have
\[
L^2(\Omega) = A^2(\Omega) \oplus B^2(\Omega).
\]
(2.1)
Consequently, for every \( p \in L^2(G) \) there exist unique \( p_0 \in A^2(\Omega) \) and \( p_h \in B^2(G) \) such that
\[
p = p_0 + p_h.
\]
(2.2)
Next, noticing that
\[
\| \nabla^2 \phi \|_{L^2(G)} = \| \phi \|_{W^{2,2}(\Omega)} \quad \forall \phi \in W^{2,2}(\Omega)
\]
there holds the following

**Lemma 2.1.** Let \( v^* \in W^{-1,2}(\Omega)^3 \) and \( p_0 \in A^2(\Omega) \) such that
\[
\langle v^*, \nabla \phi \rangle = \int_G p_0 \Delta \phi \, dx \quad \forall \phi \in C_0^\infty(G).
\]
Then
\[
\| p_0 \|_{L^2(G)} \leq \| v^* \|_{W^{-1,2}(\Omega)}.
\]

Our main objective of this section is the construction of a representation of the non-stationary pressure given by the following theorem.

**Theorem 2.2.** Let \( u \in C_w([0, \infty[; L^2_{\sigma}(\Omega)) \) and \( h \in L^1_{\text{loc}}([0, \infty[; L^2(\Omega)) \), such that
\[
\int_Q -u \cdot \partial_t \phi + h : \nabla \phi \, dx \, dt = 0 \quad \forall \phi \in C_0^\infty(Q)^3 \quad \text{with} \quad \div \phi = 0.
\]
(2.4)
Let \( B \subset \subset \Omega \). Then there exist functions
\[
p_0 \in L^1_{\text{loc}}([0, \infty[; A^2(\Omega)), \quad p_h \in C(\Omega \times [0, \infty[) \quad \text{with} \quad \Delta p_h = 0 \quad \text{and} \quad (p_h)_B = 0^4,
\]
such that
\[
\int_Q (-u \cdot \partial_t \phi + h : \nabla \phi) \, dx \, dt = \int_Q (p_0 \div \phi + \nabla p_h \cdot \partial_t \phi) \, dx \, dt + \int_Q u(0) \cdot \phi(0) \, dx
\]
(2.5)
for all \( \phi \in C^\infty(Q)^3 \) with \( \text{supp}(\phi) \subset \subset \Omega \times [0, \infty[ \). In addition, there holds
\[
\| p_0(t) \|_{L^2(\Omega)} \leq \| h(t) \|_{L^2(\Omega)}
\]
(2.6)

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4 Let \( A \subset \mathbb{R}^3 \) be a Lebesgue measurable set having finite measure. Then for \( f \in L^1(A) \) we define the mean value
\[
(f)_A = \frac{1}{|A|} \int_A f \, dx.
\]
for almost all \( t \in [0, \infty[ \) and for each Lipschitz domain \( G \subset\subset \Omega \) there exists a constant \( c_G \), such that
\[
\|\tilde{p}_n(t)\|_{L^2(G)} \leq c_G \left\{ \|u(t) - u(0)\|_{L^2(\Omega)} + \left\| \int_0^t h(s) \, ds \right\|_{L^2(\Omega)} \right\} \tag{2.7}
\]
for almost all \( t \in [0, \infty[ \).

The proof of Theorem 2.2 relies essentially on the following lemma

**Lemma 2.3.** Let \( B \subset\subset \Omega \) be a fixed domain. Let \( u \in L^2_q(\Omega) \) and \( v^* \in W^{-1,2}(\Omega)^3 \) such that
\[
\int_{\Omega} u \cdot \varphi \, dx + \langle v^*, \varphi \rangle = 0 \quad \forall \varphi \in D(\Omega). \tag{2.8}
\]
Then there exist unique functions \( p_0 \in A^2(\Omega) \) and \( p_6 \in C^\infty(\Omega) \) with \( \Delta p_6 = 0 \) and \( (p_6)_B = 0 \), such that
\[
\int_{\Omega} u \cdot \varphi \, dx + \langle v^*, \varphi \rangle = \int_{\Omega} (p_0 + p_6) \, \text{div} \varphi \, dx \quad \forall \varphi \in C^\infty_0(\Omega)^3. \tag{2.9}
\]
In addition, we have
\[
\|p_6\|_{L^2(\Omega)} \leq \|v^*\|_{W^{-1,2}(\Omega)}. \tag{2.10}
\]
and for every Lipschitz domain \( G \subset\subset \Omega \) there exists a constant \( c_G \) such that
\[
\|p_6\|_{L^2(G)} \leq c_G \|u\|_{L^2(\Omega)} + \|v^*\|_{W^{-1,2}(\Omega)}. \tag{2.11}
\]

**Proof.** Let \( B \subset \Omega_1 \subset \Omega_2 \subset \ldots \subset \Omega_k \subset\subset \Omega \) be sequence of open subsets with Lipschitz boundary, such that
\[
\bigcup_{k=1}^\infty \Omega_k = \Omega.
\]
For each \( k \in \mathbb{N} \) we define \( w^*_k \in W^{-1,2}(\Omega_k) \) by
\[
\langle w^*_k, \varphi \rangle := \int_{\Omega_k} u \cdot \varphi \, dx + \langle v^*, \varphi \rangle, \quad \varphi \in W^{1,2}(\Omega_k)^3.
\]
Clearly, by (2.8) one easily verifies \( w^*_k \in W^{1,2}(\Omega_k)^\perp \). According to [7; Chap. III, Th. 3.1, Th. 5.2] there exist \( p_k \in L^2(\Omega_k) \), such that
\[
\langle w^*_k, \varphi \rangle = \int_{\Omega_k} p_k \, \text{div} \varphi \, dx \quad \forall \varphi \in C^\infty_0(\Omega_k). \tag{2.12}
\]
Once more consulting [7] shows that for every \( \ell \in \mathbb{N}, \ell \leq k \) there holds
\[
\|p_k - (p_k)_{\Omega_k}\|_{L^2(\Omega_\ell)} \leq c_\ell \sup_{0 \neq \varphi \in C^\infty_0(\Omega_\ell)} \frac{\langle w^*_k, \varphi \rangle}{\|\nabla \varphi\|_{L^2(\Omega_\ell)}} \leq c_\ell \left( \|u\|_{L^2(\Omega)} + \|v^*\|_{W^{-1,2}(\Omega)} \right)^5. \tag{2.13}
\]

\(^5\) Remark that for any bounded domain \( G \subset \mathbb{R}^3 \) with Lipschitz boundary there holds
\[
\|\varphi\|_{L^2(G)} \leq c\|\nabla \varphi\|_{L^2(G)} \quad \forall \varphi \in C^\infty_0(G),
\]
with a constant \( c > 0 \) depending on \( G \) only.
Next, one finds unique functions \( p_{0,k} \in A^2(\Omega_k) \) and \( p_{h,k} \in B^2(\Omega_k) \) such that

\[
p_k = p_{0,k} + p_{h,k} \quad \text{in} \quad \Omega_k.
\]

In what follows let us extend \( p_k, p_{0,k} \) and \( p_{h,k} \) outside of \( \Omega_k \) by 0 without changing the notations. In particular, we have \( p_{0,k} \in A^2(\Omega) \). Noticing that \( \int_{\Omega_k} u \cdot \nabla \phi \, dx = 0 \) for all \( \phi \in C_0^\infty(\Omega_k) \) from the identity (2.12) it follows

\[
\langle \nu^*, \nabla \phi \rangle = \int_{\Omega_k} p_{0,k} \Delta \phi \, dx \quad \forall \, \phi \in C_0^\infty(\Omega_k).
\]

Then by the aid of Lemma 2.1 one infers

\[
\|p_{0,k}\|_{L^2(\Omega)} \leq \|\nu^*\|_{W^{-1,2}(\Omega)}. \tag{2.14}
\]

Let \( \ell \in \mathbb{N} \) be fixed. Then using Jensen’s inequality along with (2.13) and (2.14) one gets for every \( k \geq \ell \)

\[
\|p_{h,k} - (p_{h,k})_B\|_{L^2(\Omega_k)} \leq 2 \left[ \frac{|\Omega|}{|B|} \|p_{h,k} - (p_{h,k})_B\|_{L^2(\Omega_k)} \right]
\leq c \left[ \frac{|\Omega|}{|B|} \|p_{h,k}\|_{L^2(\Omega)} + \|p_{h,k} - (p_{h,k})_B\|_{L^2(\Omega_k)} \right]
\leq c \left[ \frac{|\Omega|}{|B|} \|u\|_{L^2(\Omega)} + \|\nu^*\|_{W^{-1,2}(\Omega)} \right], \tag{2.15}
\]

where \( c = \text{const independent on } k \in \mathbb{N} \). Hence, observing (2.14) and (2.15) by means of reflexivity using Cantor’s diagonalization principle one can choose a sequence \( \{k_j\} \) in \( \mathbb{N} \) such that

\[
p_{0,k_j} \to p_0 \quad \text{in} \quad A^2(\Omega), \tag{2.16}
\]

\[
p_{h,k_j} - (p_{h,k_j})_B \to p_h \quad \text{in} \quad L^2_{\text{loc}}(\Omega) \quad \text{as} \quad j \to +\infty. \tag{2.17}
\]

By the aid Banach-Steinhaus’ theorem from (2.14) one immediately gets (2.10). On the other hand, given a Lipschitz domain \( G \subset \subset \Omega \) by means of Banach-Steinhaus’ theorem taking into account (2.15) yields

\[
\|p_h\|_{L^2(G)} \leq c_G (\|u\| + \|\nu^*\|_{W^{-1,2}(\Omega)}) \tag{2.18}
\]

with a constant \( c_G = \text{const depending on } G \) only. Whence (2.11).

Let \( \varphi \in C_0^\infty(\Omega)^3 \). There exists \( \mathcal{N}(\varphi) \in \mathbb{N} \) such that \( \text{supp}(\varphi) \in \Omega_k \) for all \( k \geq \mathcal{N}(\varphi) \). Then for \( k_j \geq \mathcal{N}(\varphi) \) we have

\[
\int_{\Omega} u \cdot \varphi \, dx + \langle \nu^*, \varphi \rangle = \langle \nu_{k_j}^*, \varphi \rangle = \int_{\Omega} (p_{0,k_j} + p_{h,k_j}) \text{div} \varphi \, dx.
\]

Using (2.16) and (2.17) carrying out the passage to the limit \( j \to +\infty \) from the identity above one gets (2.9).

**Uniqueness.** Let \( p_0 \in A^2(\Omega) \) and \( p_h \in L^2_{\text{loc}}(\Omega) \) with \( \Delta p_h = 0 \) and \( (p_h)_B = 0 \), such that

\[
\int_{\Omega} (p_0 + p_h) \text{div} \varphi \, dx = 0 \quad \forall \, \varphi \in C_0^\infty(\Omega)^3.
\]
Then, \( p_0 + p_h = C_0 = \text{const in } \Omega \). Hence, \( p_0 = C_0 - p_h \in A^2(\Omega) \cap B^2(\Omega) \), which implies \( 0 = p_0 = C_0 - p_h \). Finally, by virtue of \((p_h)_B = 0\) one finds \( C_0 = 0 \) and therefore \( p_0 + p_h = 0 \).

Proof of Theorem 2.2 Arguing as in \cite{25} one finds

\[
\int_{\Omega} (u(t) - u(0)) \cdot \psi + \tilde{h}(t) : \nabla \psi \, dx = 0 \quad \forall \, \psi \in C^{\infty}_{0, \sigma}(\Omega), \tag{2.19}
\]

where

\[
\tilde{h}(t) := \int_{0}^{t} h(s) \, ds, \quad t \in [0, \infty[.
\]

According to Lemma 2.2 there exists unique functions \( \tilde{\rho}_0(t) \in A^2(\Omega) \) and \( \tilde{\rho}_h(t) \in C^{\infty}(\Omega) \) with \( \Delta \tilde{\rho}_h(t) = 0 \) and \((\tilde{\rho}_h(t))_B = 0\), such that

\[
\int_{\Omega} (u(t) - u(0)) \cdot \psi + \tilde{h}(t) : \nabla \psi \, dx = \int_{\Omega} (\tilde{\rho}_0(t) + \tilde{\rho}_h(t)) \, \text{div} \psi \, dx \tag{2.20}
\]

for all \( \psi \in C^{\infty}_{0, \sigma}(\Omega)^3 \).

Now into \eqref{2.20} replacing \( \psi \) by \( \nabla \phi \), where \( \phi \in C^{\infty}(\Omega) \) one arrives at

\[
\int_{\Omega} \tilde{\rho}_0(t) \Delta \phi \, dx = \int_{\Omega} \tilde{h}(t) : \nabla^2 \phi \, dx \quad \forall \, \phi \in W^{2,2}_0(\Omega). \tag{2.21}
\]

Next, let \( 0 \leq t < s \leq \infty \) be arbitrarily chosen. Then from \eqref{2.21} by the aid of Lemma 2.1 and Minkowski’s inequality one estimates

\[
\|\tilde{\rho}_0(t) - \tilde{\rho}_0(s)\|_{L^2(\Omega)} \leq \|\tilde{h}(t) - \tilde{h}(s)\|_{L^2(\Omega)} \leq \int_{t}^{s} \|h(\sigma)\|_{L^2(\Omega)} \, d\sigma. \tag{2.22}
\]

By means of the absolutely continuity of the Lebesgue integral it follows that \( \tilde{\rho}_0 \in C([0, \infty[; L^2(\Omega)) \).

Let \( G \subset\subset \Omega \) be arbitrarily chosen. Define

\[
\tilde{\rho}_G(t) := \tilde{\rho}_0(t) + \tilde{\rho}_h(t) - (\tilde{\rho}_0(t) + \tilde{\rho}_h(t))_{\mathcal{G}} \quad \text{in } \ G.
\]

Then \eqref{2.20} turns into

\[
\int_{\Omega} (u(t) - u(0)) \cdot \psi + \tilde{h}(t) : \nabla \psi \, dx = \int_{\Omega} \tilde{\rho}_G(t) \, \text{div} \psi \, dx \tag{2.23}
\]

for all \( \psi \in C^{\infty}_{0, \sigma}(G)^3 \).

As in \cite{25} one can verify that \( \tilde{\rho}_G \in C_{\text{loc}}([0, \infty[; L^2(G)) \). Notice that \( \tilde{\rho}_0 \in C([0, \infty[; L^2(\Omega)) \) yields \( \tilde{\rho}_0 - (\tilde{\rho}_0)_{\mathcal{G}} \in C([0, \infty[; L^2(G)) \) and therefore \( \tilde{\rho}_0 - (\tilde{\rho}_0)_{\mathcal{G}} \in C_{\text{loc}}([0, \infty[; L^2(G)) \). By virtue of Lemma A.1 (cf. appendix below) the function \( \tilde{\rho}_0 - (\tilde{\rho}_0)_{\mathcal{G}} \) must be continuous in \( G \times [0, \infty[ \). In addition, recalling \( \tilde{\rho}_B = 0 \) the statement above with \( G = B \) shows that \( \tilde{\rho}_h \) is continuous in \( B \times [0, \infty[ \). According to Lemma A.2 one obtains \( \tilde{\rho}_h \in C(\Omega \times [0, \infty[) \). Since at time \( t = 0 \) the right hand side of \eqref{2.20} vanishes we have \( \tilde{\rho}_0(0) = 0 \).

Since \( \tilde{\rho}_0 \in W^{1,1}_{\text{loc}}([0, \infty[; A^2(\Omega)) \) (cf. \cite{25}) the weak derivative \( p_0 := \partial_t \tilde{\rho}_0 \) belongs to \( L^1_{\text{loc}}([0, \infty[; A^2(\Omega)) \). Let \( \varphi \in C^{\infty}(Q)^3 \) with \( \text{supp}(\varphi) \subset\subset \Omega \times [0, \infty[ \).
Then into (2.20) inserting \( \psi = \partial_t \varphi(t) \) \((0 < t < \infty)\) integrating over \(]0, \infty[\) and using integration by parts implies (2.5).

To verify (2.6) we proceed as follows. Let \( \eta \in C_0^\infty(0, \infty) \) and \( \phi \in C_0^\infty(\Omega) \) be arbitrarily chosen. Then from (2.5) with \( \varphi(x, t) = \eta(t) \nabla \phi(x) \) using integration by parts yields

\[
\int_0^\infty \eta(t) \int_\Omega h(x, t) : \nabla^2(x) \phi \, dx \, dt = \int_0^\infty \eta(t) \int_\Omega p_0(x, t) : \Delta \phi(x) \, dx \, dt.
\]

Now, if \( t \in ]0, \infty[ \) is a Lebesgue point of both \( h \) and \( p_0 \) with respect to the time variable then from the identity above it follows

\[
\int_\Omega h(t) : \nabla^2 \phi \, dx = \int_\Omega p_0(t) : \Delta \phi \, dx \quad \forall \phi \in C_0^\infty(\Omega).
\]

Owing to Lemma 2.1 one gets the desired inequality (2.6). On the other hand, from (2.20) using (2.11) (cf. Lemma 2.2) one easily verifies (2.7). This completes the proof of the lemma. \( \square \)

**Remark 2.4.** The representation of the pressure introduced in Theorem 2.2 is unique in the following sense. If \( q_0 \in L^1_{\text{loc}}([0, \infty]; A^2(\Omega)) \) and \( \tilde{q}_h \in L^1_{\text{loc}}(\Omega \times [0, \infty]) \) with \( \Delta \tilde{q}_h = 0 \) and \( \tilde{q}_h \) \( B = 0 \) are functions fulfilling the identity (2.5) with \( q_0 \) instead of \( p_0 \) and with \( \tilde{q}_h \) instead of \( \tilde{p}_h \), then \( q_0 = p_0 \) and \( \tilde{q}_h = \tilde{p}_h \) almost everywhere in \( Q \). Indeed, taking the difference between this two identities yields

\[
\int_Q (p_0 - \tilde{q}_h) \, \text{div} \varphi \, dx \, dt = - \int_Q \nabla (\tilde{p}_h - \tilde{q}_h) : \partial_t \varphi \, dx \, dt
\]

for all \( \varphi \in C^\infty(\Omega) \) with \( \text{supp}(\varphi) \subset \subset \Omega \times [0, \infty[ \). Arguing as above recalling the definition of the space \( A^2(\Omega) \) shows that \( p_0 = 0 \). Thus, the identity above turns into

\[
\int_0^\infty \eta'(t) \int_\Omega \nabla (\tilde{p}_h(t) - \tilde{q}_h(t)) \cdot \psi \, dx \, dt = 0 \quad \forall \psi \in C_0^\infty(\Omega), \quad \forall \eta \in C_0^\infty([0, \infty]),
\]

which leads to \( \nabla \tilde{p}_h(\cdot, t) = 0 \) for almost all \( t > 0 \). This shows that \( \tilde{p}_h(\cdot, t) = \text{const} \) for almost all \( t > 0 \). Additionally, having \( (\tilde{p}_h - \tilde{q}_h)_B = 0 \) implies \( \tilde{p}_h = 0 \) almost everywhere in \( Q \).

### 3. Fundamental estimates for semi caloric functions

**Definition 3.1** A function \( U \in L^2(Q_1)^3 \) is said to be semi caloric if there exists \( q \in L^1(-1, 0; B^2(B_1)) \), such that

\[
\frac{\partial U}{\partial t} - \Delta U = -\nabla q \quad \text{in} \quad Q_1 \tag{3.1}
\]

in sense of distributions. Then \( q \) is called the caloric pressure related to \( U \).

Let us start with the following Caccioppoli-type inequality.
Lemma 3.1. Let $U \in L^2(Q_1)^3 \times L^1(-1, 0; B^2(B_1))$ be a semi caloric function with its caloric pressure $q$. Then $\nabla^m U \in L^2_{\text{loc}}(B_1 \times (-1, 0))$ for all $m \in \mathbb{N}$ and there exists a constant $c_1$ depending on $m \in \mathbb{N}$ only such that

$$\|\nabla^m U\|_{L^\infty(-1/4, 0; L^2(B_1))} \leq c_1 \left\{ \|U\|_{L^2(Q_1)}^2 + \|q\|_{L^2(-1, 0, L^2(B_1))}^2 \right\}. \quad (3.2)$$

Proof. By using a standard mollification argument one easily proves

$$U \in L^\infty(-\rho^2, 0; W^{m, 2}(B_\rho)^3),$$

$$\frac{\partial U}{\partial t} \in L^1(-\rho^2, 0; W^{m, 2}(B_\rho)^3), \quad m = 1, 2, \ldots$$

for all $0 < \rho < 1$. Furthermore for each multi-index $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ there holds

$$\frac{\partial D^\alpha U}{\partial t} - \Delta D^\alpha U = -\nabla D^\alpha q \quad \text{in} \; Q_1. \quad (3.3)$$

Fix $m \in \mathbb{N}$. Let $\phi \in C_0^\infty(B_1 \times (-1, 0))$ be a cut-off function with $0 \leq \phi \leq 1$ in $Q_1$ and $\phi \equiv 1$ on $Q_{1/2}$. Multiplying (3.3) by $x^{m+2}D^\alpha U$ summing up over $|\alpha| = m$ ($m \in \mathbb{N}$) integrating over $B_1 \times (-1, t) (t \in (-1, 0])$ applying integration by parts along with Cauchy-Schwartz’ s and Young’s inequality yields

$$\begin{align*}
\int_{B_1} \phi^{m+2}(x,t)|\nabla^m U(x,t)|^2 \, dx + 2 \int_{-1}^t \int_{B_1} \phi^{m+2}|\nabla^{m+1} U|^2 \, dx \, ds \\
\leq \int_{-1}^t \int_{B_1} \left( \frac{\partial}{\partial t} + \Delta \right)(\phi^{m+2})|\nabla^m U|^2 \, dx \, ds \\
+ 2 \left\{ \int_{-1}^t \left( \int_{B_1} \phi^{m+2}|\nabla^{m+1} q|^2 \, dx \right)^{1/2} \, ds \right\}^2 \\
+ \frac{1}{2} \text{ess sup }_{s \in (-1, 0]} \int_{B_1} \phi^{m+2}(x,s)|\nabla^m U(x,s)|^2 \, dx.
\end{align*}$$

On the other hand, recalling that $\Delta q = 0$ using integration by parts implies

$$\begin{align*}
\int_{B_1} \phi^{m+2}(x,s)|\nabla^{m+1} q(x,s)|^2 \, dx &= 2^{-|m+1|} \int_{B_1} (\Delta^{m+1} \phi^{m+2})(x,s)q^2(x,s) \, dx \\
f.a.a. \; s \in [-1, 0].
\end{align*}$$

Combining the last two statements shows that

$$\begin{align*}
\text{ess sup }_{t \in (-1, 0]} \int_{B_1} \phi^{m+2}(x,t)|\nabla^m U(x,t)|^2 \, dx + \int_{Q_1} \phi^{m+2}|\nabla^{m+1} U|^2 \, dx \, dt \\
\leq c \int_{Q_1} \phi^m|\nabla^m U|^2 \, dx \, dt + c \left\{ \int_{-1}^0 \left( \int_{B_1} q^2 \, dx \right)^{1/2} \, dt \right\}^2,
\end{align*}$$
where \( c = \text{const depending on } m \) only. Iterating this inequality \( m \)-times gives

\[
\operatorname{ess}
\sup_{t \in [-1,0]} \int_{B_1} \delta^{2m+2}(x,t) |\nabla^m U(x,t)|^2 \, dx \leq c \int_{Q_1} |U|^2 \, dx \, dt + c \left\{ \int_{-1}^0 \left( \int_{B_1} q^2 \, dx \right)^{1/2} \, dt \right\}^2,
\]

with \( c = \text{const depending only on } m \). This completes the proof of (3.2) \( \square \)

We are now in a position to prove the following fundamental estimate

**Theorem 3.2.** Let \( U \in L^2(Q_1) \) be a semi caloric function with its caloric pressure \( q \in L^1(-1,0; L^2(B_1)) \). Then

\[
\|U\|_{L^2(Q_\tau)}^2 \leq c_2 \tau^5 \left\{ \|U\|_{L^2(Q_1)}^2 + \|q\|_{L^1(-1,0; L^2(B_1))}^2 \right\}
\]

for every \( 0 < \tau < 1 \) with an absolute constant \( c_2 > 0 \).

Additionally, if \( q \in L^{4/3}(-1,0; L^2(B_1)) \) then

\[
\|U - U_{Q_\tau}\|_{L^2(Q_\tau)}^2 \leq c_3 \tau^6 \left\{ \|U - U_{Q_1}\|_{L^2(Q_1)}^2 + \|q\|_{L^{4/3}(-1,0; L^2(B_1))}^2 \right\}
\]

for every \( 0 < \tau < 1 \) with an absolute constant \( c_3 > 0 \).

**Proof.** Clearly both fundamental estimates are trivially fulfilled for \( \frac{1}{2} \leq \tau < 1 \).

Now, fix \( 0 < \tau < \frac{1}{2} \). Then using Sobolev’s embedding theorem together with (3.2) (with \( m = 2 \)) yields

\[
\|U\|_{L^2(Q_\tau)}^2 \leq |B_1| \tau^5 \|U\|_{L^\infty(Q_{1/2})}^2 \leq c \tau^5 \operatorname{ess}
\sup_{t \in [-1/4,0]} \int_{B_{1/2}} |\nabla^2 U(x,t)|^2 \, dx \leq c \tau^5 \int_{Q_{1/2}} |U|^2 \, dx \, dt + c \tau^5 \left\{ \int_{-1}^0 \left( \int_{B_1} q^2 \, dx \right)^{1/2} \, dt \right\}^2.
\]

Whence, (3.4).

In order to verify (3.5) we first make use of the Poincaré-type inequality (A.10) (cf. the appendix below) to get

\[
\|U - U_{Q_\tau}\|_{L^2(Q_\tau)}^2 \leq c \tau^2 \|\nabla U_{Q_\tau}\|_{L^2(Q_\tau)}^2 + \tau^2 c \left\{ \int_{-\tau^2}^0 \left( \int_{B_{\tau^2}} |\nabla q|^2 \, dx \right)^{1/2} \, dt \right\}^2.
\]

\[\text{In what follows } c \text{ denotes an absolute positive constant, which value may change from line to line and will be specified if necessary.}\]
As above by using Sobolev’s embedding theorem one infers
\[ \| \nabla U_{Q_r} \|^2_{L^2(Q_r)} \leq c r^5 \text{ess sup}_{t \in [\tau,\tau + 1]} \int_{B_1/2} |\nabla^3 U(x,t)|^2 \, dx. \]
Estimating the right hand side of this inequality by the aid of Lemma 3.1 with orthogonal decomposition
energy inequality (the functions \( p \)) and \( \partial_t p \). Set \( \hat{p} := \frac{|u|^2}{2} + p_0 \). As we have stated in Remark 1.2 the functions \( v \) and \( \hat{p} \) satisfy the identity (1.12). In addition, there holds the local energy inequality (1.11).

Next, thanks to \( \Delta q = 0 \) using Caccioppoli’s inequality and Hölder’s inequality yields
\[ \left\{ \int_{B_r(0)} \left( \int_{B_{r/2}} |\nabla q|^2 \, dx \right)^{1/2} \, dt \right\}^2 \leq c \| \int_{-1}^{0} (\int_{B_1} q^2 \, dx)^{1/2} \, dt \right\}^{3/2} \leq c \| \int_{-1}^{0} (\int_{B_1} q^2 \, dx)^{2/3} \, dt \right\}^{3/2}. \]
Finally estimating the right hand side of (3.6) from above by (3.7) and (3.8) completes the proof of (3.5).

\[ \square \]

4. Estimates for weak solutions to the model system
Let \( u \) be a suitable weak solution to (1.1)–(1.4) with pressure \( p \) represented by \( p_0 + \frac{\partial \tilde{p}_h}{\partial t} \). Set \( v := u + \nabla \tilde{p}_h \) and \( \hat{p} := \frac{|u|^2}{2} + p_0 \). As we have stated in Remark 1.2 the functions \( v \) and \( \hat{p} \) satisfy the identity (1.12). In addition, there holds the local energy inequality (1.11).

Let \((x_0, t_0) \in Q \) and \( 0 < R < \min\{\text{dist}(x_0, \partial \Omega), \sqrt{t_0}\} \) be fixed. Recalling the orthogonal decomposition \( L^2(B_R(x_0)) = A^2(B_R(x_0)) \oplus B^2(B_R(x_0)) \), there exist
\[ p_{0,R} \in L^{4/3}(0, \infty; A^2(B_R(x_0))), \quad p_{h,R} \in L^{4/3}(0, \infty; B^2(B_R(x_0))) \]
such that
\[ \hat{p}(t) - \hat{p}_{B_R(x_0)}(t) = p_{0,R}(t) + p_{h,R}(t) \quad \text{in} \quad B_R(x_0) \quad \text{f.a.a.} \quad t \in [0, \infty[. \]

Clearly, for almost all \( t \in [0, \infty[ \) we have
\[ u(t) \times \nabla u(t) \in L^{6/5}(B_R(x_0))^3, \quad p_{0,R}(t) \in A^2(B_R(x_0)) \]
Letting \( t \in [0, \infty[ \) with (4.1) there holds
\[ \int_{B_R(x_0)} p_{0,R}(t) \Delta \phi \, dx = \int_{B_R(x_0)} u(t) \times \nabla u(t) \cdot \nabla \phi \, dx \]
for all \( \phi \in C_0^\infty (B_R(x_0)) \) (cf. Theorem 2.2 above). Then, by the aid of Lemma A.3 one gets \( p_{0,R} (t) \in W^{1,6/5} (B_R(x_0)) \) together with the estimate
\[
\| \nabla p_{0,R} (t) \|_{L^{6/5} (B_R(x_0))}^{4/3} \leq c \| \text{rot} \, u(t) \times u(t) \|_{L^{6/5} (B_R(x_0))}^{4/3}.
\] (4.3)
In particular, the function \( t \mapsto \| \text{rot} \, u(t) \times u(t) \|_{L^{6/5} (B_R(x_0))}^{4/3} \) belonging to \( L^1 (0, \infty) \) one gets
\[
\nabla p_{0,R} \in L^{4/3} (0, \infty; L^{6/5} (B_R(x_0))^3).
\]
Moreover, the inequality (4.3) implies
\[
\| \nabla p_{0,R} \|_{L^{4/3} (0, \infty; L^{6/5} (B_R(x_0)))} \leq c_4 \| \text{rot} \, u \times u \|_{L^{4/3} (0, \infty; L^{6/5} (B_R(x_0)))} (4.4)
\]
with an absolute constant \( c_4 > 0 \).

In order to estimate the right hand side of (4.4) we will proceed in two different ways.

Firstly, using Hölder’s inequality, recalling that \( u = v + \nabla p_h \) yields
\[
\| \text{rot} \, u(t) \times u(t) \|_{L^{6/5} (B_R(x_0))}^{4/3} \leq \left( \int_{B_R(x_0)} \left| \text{rot} \, u(t) \times u(t) \right| \left( |v| + |\nabla p_h| \right)^{6/5} \, dx \right)^{2/3}.
\]
for almost all \( t \in [t_0 - R^2, t_0] \). Integrating both sides of (4.5) over the interval \([t_0 - R^2, t_0]\) once more using Hölder’s inequality shows that
\[
\| \text{rot} \, u \times u \|_{L^{4/3} (t_0 - R^2, t_0; L^{6/5} (B_R(x_0)))} \leq c_5 R \Theta (x_0, t_0; R) \| v \|_{L^{4/3} (t_0 - R^2, t_0; L^3 (B_R(x_0)))}^2 + c_6 R^2 M(x_0, t_0; R) \| \nabla v \|_{L^2 (Q_R (x_0, t_0))},
\] (4.6)
where \( c_5 \) and \( c_6 \) deonote absolute constants.

Secondly, noticing \( \text{rot} \, u = \text{rot} \, v \) using Hölder’s inequality gives
\[
\| \text{rot} \, u \times u \|_{L^{4/3} (t_0 - R^2, t_0; L^{6/5} (B_R(x_0)))} \leq c_7 \| u \|_{L^1 (t_0 - R^2, t_0; L^3 (B_R(x_0)))} \| \nabla v \|_{L^2 (Q_R (x_0, t_0))},
\] (4.7)
where \( c_7 \) denotes an absolute constant.

In order to fix the model problem we will use the following lemma which for readers convenience will be proved at the end of this section.

**Lemma 4.1.** Given \( f \in L^1 (t_0 - R^2, t_0; L^\sigma (B_R(x_0))) (1 < \sigma \leq 2) \), there exists a unique function \( w \in L^{\frac{\sigma}{2\sigma - 1}} (t_0 - R^2, t_0; W^{1, \frac{\sigma}{2\sigma - 1}} (B_R(x_0))) \cap L^{\frac{\sigma}{\sigma - 1}} (Q_R (x_0, t_0)) \), such that
\[
\int_{Q_R (x_0, t_0)} (-w \varphi + \nabla w \cdot \nabla \varphi) \, dx \, dt = \int_{Q_R (x_0, t_0)} f \varphi \, dx \, dt
\] (4.8)
for all \( \varphi \in C_0^\infty(B_R(x_0) \times [t_0 - R^2, t_0]) \). In addition, there holds
\[
\|u\|_{L^{5/3}(Q_R(x_0, t_0))} \leq c\|f\|_{L^1(t_0 - R^2, t_0; L^6(B_R(x_0)))}
\]  
(4.9)
with an constant \( c > 0 \) depending only on \( \sigma \).

The model problem. Applying Lemma 4.1 with \( f = f_i = (\text{rot } u \times u)_{i} - \frac{\partial p_{0,R}}{\partial x_i} \) (\( i = 1, 2, 3 \)) and \( \sigma = \frac{6}{5} \) one gets a unique function \( W \in L^{10/7}(t_0 - R^2, t_0; W^{1,10/7}(B_R(x_0)) \cap L^2(Q_R(x_0, t_0))) \) such that
\[
\int_{Q_R(x_0, t_0)} (-W \cdot \varphi + \nabla W : \nabla \varphi) \, dx \, dt
= \int_{Q_R(x_0, t_0)} (\text{rot } u \times u - \nabla p_{0,R}) \cdot \varphi \, dx \, dt
\]
(4.10)
for all \( \varphi \in C_0^\infty(B_R(x_0) \times [t_0 - R^2, t_0]) \). Clearly, using Jensen’s inequality the estimate (4.9) yields
\[
\|W\|_{L^2(Q_R(x_0, t_0))}^2 \leq cR\|\text{rot } u \times u - \nabla p_{0,R}\|_{L^{1/3}(t_0 - R^2, t_0; L^{6/5}(B_R(x_0))))^2.
\]
Now, the right hand side of this inequality can be estimated by (4.4) together with (4.6). Thus,
\[
\|W\|_{L^2(Q_R(x_0, t_0))}^2 \leq c_3 R^4 M(x_0, t_0; R) \|\nabla v\|_{L^2(Q_R(x_0, t_0))}^2
+ c_4 R^2 \Theta(x_0, t_0; R) \|v\|_{L^4(t_0 - R^2, t_0; L^3(B_R(x_0))))^2.
\]
(4.11)
Alternatively, using (4.7) instead of (4.6) one gets
\[
\|W - W_{Q_R}\|_{L^2(Q_R(x_0, t_0))}^2 \leq c_{10} R^{2} \Theta(x_0, t_0; R) \|\nabla v\|_{L^2(Q_R(x_0, t_0))}^2.
\]
(4.12)

Proof of Lemma 4.1 It will be sufficient to prove the assertion of the lemma for the case \( (x_0, t_0) = (0, 0) \) and \( R = 1 \). The general case can be deduced easily by introducing an appropriate transformation of coordinates.

Let \( f \in L^1(-1, 0; L^6(B_1)) \) be given. Clearly there exists a sequence \( \{f_m\} \in L^2(Q_1) \), such that
\[
f_m \to f \quad \text{in} \quad L^1(-1, 0; L^6(B_1)) \quad \text{as} \quad m \to \infty.
\]
Let \( m \in \mathbb{N} \). Consulting [12], there exists a unique function
\[
w_m \in L^2(-1, 0; \dot{W}^{1,2}(B_1)) \cap C([-1, 0]; L^2(B_1)),
\]
satisfying
\[
\int_{Q_1} (-w_m \varphi_t + \nabla w_m \cdot \nabla \varphi) \, dx \, dt = \int_{Q_1} f_m \varphi \, dx \, dt
\]
(4.13)
for all \( \varphi \in C_0^\infty([-1, 0]). \) In particular, \( \frac{dw_m}{dt} \in L^2(-1, 0; W^{-1,2}(B_1)) \), such that
\[
\int_{-1}^{t} \left\langle \frac{dw_m}{dt}(s), \varphi(s) \right\rangle \, ds + \int_{-1}^{t} \int_{B_1} \nabla w_m \cdot \nabla \varphi \, dx \, ds = \int_{-1}^{t} \int_{B_1} f_m \varphi \, dx \, ds
\]
(4.14)
for all \( \varphi \in L^2(-1,0; \dot{W}^{1,2}(B_1)) \), for all \( t \in ]-1,0[ \). Observing \((4.13)\) using integration by parts it follows that \( w_m(-1) = 0 \). Into \((4.14)\) inserting the admissible test function \( \varphi = w_m|w_m|^{\sigma - 2} \) using integration by parts along with Hölder’s inequality gives

\[
\frac{1}{\sigma} \|w_m(t)\|_{L^\sigma(B_1)}^2 + (\sigma - 1) \int_{-1}^t \int_{B_1} |\nabla w_m|^2 |w_m|^{\sigma - 2} \, dx \, ds \\
\leq \int_{-1}^t \int_{B_1} |f_m|^\sigma |w_m|^{\sigma - 1} \, dx \, ds \\
\leq \|f_m\|_{L^1(-1,0; L^\sigma(B_1))} \|w_m\|_{L^\infty(-1,0; L^\sigma(B_1))}^{\sigma - 1}\]

Define \( \theta_m := w_m|w_m|^{\sigma - 2} \). From the inequality above one infers

\[
\theta_m \in L^2(-1,0; \dot{W}^{1,2}(B_1)) \cap L^\infty(-1,0; L^2(B_1))
\]
together with

\[
\|\theta_m\|_{L^\infty(-1,0; L^2(B_1))}^2 + \|\nabla \theta_m\|_{L^2(Q_1)}^2 \\
\leq \frac{\sigma'}{2} \|f_m\|_{L^1(-1,0; L^\sigma(B_1))} \|\theta_m\|_{L^\infty(-1,0; L^2(B_1))}^{2\sigma'}
\]

By means of Sobolev’s embedding theorem together with Hölder’s inequality applying Young’s inequality implies

\[
\|\theta_m\|_{L^{10/3}(Q_1)} + \|\nabla \theta_m\|_{L^2(Q_1)} \leq c \|f_m\|_{L^1(-1,0; L^\sigma(B_1))}^{\sigma/2} 
\]

where \( c = \text{const} > 0 \) depending only on \( \sigma \). Hence, recalling the definition of \( \theta_m \) using Hölder’s inequality shows that

\[
\|w_m\|_{L^{5\sigma/3}(Q_1)} + \|\nabla w_m\|_{L^{5\sigma/3\sigma'}(Q_1)} \leq c \|f_m\|_{L^1(-1,0; L^\sigma(B_1))}^{\sigma/2}
\]

Finally, passing to the limit \( m \to \infty \) using Banach-Steinhaus’ theorem proves the assertion Lemma 3.3. \( \square \)

5. Proof of Propositions 1.5/1.6

As in the previous section by \( u \) we denote a suitable weak solution to \((1.1)-(1.4)\) with pressure \( p \) represented by \( p_0 + \frac{\partial \hat{p}_h}{\partial t} \). As before setting \( v := u + \nabla \hat{p}_h \) and \( \tilde{p} := \frac{|u|^2}{2} + p_0 \) there holds \((1.12)\) together with the local energy inequality \((1.11)\).

For the sake of simplicity we define the following quantities which are invariant under the natural scaling of the Navier-Stokes equations. Let \((x_0, t_0) \in Q\) and
0 < \rho < \min\{\text{dist}(x_0, \partial \Omega), \sqrt{t_0}\}. We define
\begin{align*}
V(x_0, t_0; \rho) &:= \frac{1}{\rho} \left[ \|v\|_{L^\infty(t_0-\rho^2, t_0; L^2(B_{\rho}(x_0)))}^2 \\
&\quad + \|\nabla v\|_{L^{10/3}(Q_{\rho}(x_0, t_0))}^2 + \|\nabla v\|_{L^2(Q_{\rho}(x_0, t_0))}^2 \right], \\
\mathcal{P}(x_0, t_0; \rho) &:= \frac{1}{\rho^2} \|\hat{p} - \hat{p}_{B_{\rho}(x_0)}\|_{L^{4/3}(t_0-\rho^2, t_0; L^2(B_{\rho}(x_0)))}^2, \\
\mathcal{Z}(x_0, t_0; \rho|\varphi) &:= \frac{1}{\rho} \|\varphi\|_{L^2(Q_{\rho}(x_0, t_0))}^2, \quad \varphi \in L^2(Q_{\rho}(x_0, t_0))^3.
\end{align*}

Throughout let \((x_0, t_0) \in Q\) be fixed. For the sake of notational simplicity we will write \(V(\rho), \mathcal{P}(\rho), \mathcal{Z}(\rho|\varphi), \Theta(\rho)\) and \(M(\rho)\) resp. in place of \(V(x_0, t_0; \rho), \mathcal{P}(x_0, t_0; \rho), \mathcal{Z}(x_0, t_0; \rho|\varphi), \Theta(x_0, t_0; \rho)\) and \(M(x_0, t_0; \rho)\) resp.

First let us prove some fundamental estimates which form the bases of the proof of the Prop. 1.5. We start with a Caccioppoli-type inequality which immediately follows from the local energy inequality \(\boxed{1.11}\).

**Lemma 5.1.** There exist absolute positive constants \(c_{11}\) and \(c_{12}\) such that
\begin{equation}
V(\rho) \leq c_{11} \left[ 1 + \rho^2 M(2\rho) \right] \mathcal{Z}(2\rho|v) + c_{12} \mathcal{P}(2\rho) \tag{5.1}
\end{equation}
for all \(0 < \rho < \frac{1}{2} \min\{\text{dist}(x_0, \partial \Omega), \sqrt{t_0}\} \).

**Proof.** Let \(0 < \rho < \frac{1}{2} \min\{\text{dist}(x_0, \partial \Omega), \sqrt{t_0}\}\). Let \(\phi \in C^\infty(\mathbb{R}^4)\) be a cut-off function, such that \(0 \leq \phi \leq 1\) in \(\mathbb{R}^4\), \(\phi \equiv 1 \in \mathbb{R}^4 \setminus B_{2\rho}(x_0) \times ]-\infty, t_0-4\rho^2[, \phi \equiv 0 \text{ in } B_{\rho}(x_0) \times ]t_0-\rho^2, \infty[ \) and
\begin{equation}
|\nabla \phi|^2 + |\nabla^2 \phi| + |\partial_t \phi| \leq \frac{c_0}{\rho^2} \quad \text{in } \mathbb{R}^4.
\end{equation}

From the local energy inequality \(\boxed{1.11}\) (replacing \(\phi\) by \(\phi^2\) therein) using Hölder’s and Young’s inequality one easily estimates
\begin{align*}
\|\phi v\|_{L^2(t_0-4\rho^2, t_0; L^2(B_{2\rho})))}^2 + \|\phi \nabla v\|_{L^2(Q_{2\rho})}^2 &
\leq c \left[ \rho^{-2} + \max_{Q_{2\rho}} |\nabla \hat{p}_{\rho}|^2 \right] \|v\|_{L^2(Q_{2\rho})}^2 + c \rho^{-1} \int_{Q_{2\rho}} |\hat{p} - \hat{p}_{B_{2\rho}}| |v| \phi \, dx \, dt.
\end{align*}

With help of Cauchy-Schwartz’s inequality verifying
\begin{equation}
\int_{Q_{2\rho}} |\hat{p} - \hat{p}_{B_{2\rho}}| |v| \phi \, dx \, dt 
\leq \|p - p_{B_{2\rho}}\|_{L^1(t_0-2\rho^2, t_0; L^2(B_{2\rho})))} \|\phi v\|_{L^\infty(t_0-4\rho^2, t_0; L^2(B_{2\rho})))}
\end{equation}
applying Young’s inequality one arrives at
\begin{align*}
\|\phi v\|_{L^\infty(t_0-4\rho^2, t_0; L^2(B_{2\rho})))}^2 + \|\phi \nabla v\|_{L^2(Q_{2\rho})}^2 &
\leq c \rho \left[ 1 + \rho^2 M(2\rho) \right] \mathcal{Z}(2\rho|v) + c \rho \mathcal{P}(2\rho).
\end{align*}
Finally, by the aid of multiplicative inequality and Sobolev-Poincaré inequality
\[ \|\phi v\|_{L^{10/3}(Q_{2r})}^2 \leq \|\phi v\|_{L^\infty(t_0-4\rho^2,L^\infty(B_{2\rho}))}^{4/5} \|\phi v\|_{L^2(t_0-4\rho^2,L^2(B_{2\rho}))}^{6/5} \leq c \left( \|\phi v\|_{L^\infty(t_0-4\rho^2,L^\infty(B_{2\rho}))} + \|\phi \nabla v\|_{L^2(B_{2\rho})}^2 \right) + c\rho \mathcal{Z}(2\rho|v). \] (5.3)

Combining (5.2) and (5.3) gives (5.1). \[ \square \]

Next we provide a fundamental estimate for the pressure. We have the

**Lemma 5.2.** There exist absolute positive constants \(c_{13}\) and \(c_{14}\) such that for every \(0 < \tau < \frac{1}{2}\), we have
\[ \mathcal{P}(2\tau R) \leq c_{13} \tau \mathcal{P}(R) + c_{14} \tau^{-3} [R^2 M(R) + \Theta(R)] \mathcal{V}(R) \] (5.4)

for all \(0 < R < \min\{\text{dist}(x_0,\partial\Omega),\sqrt{\rho}\}\).

**Proof.** Let \(0 < R < \min\{\text{dist}(x_0,\partial\Omega),\sqrt{\rho}\}\) be arbitrarily chosen. By virtue of the orthogonal decomposition \(L^2(B_R(x_0)) = A^2(B_R(x_0)) \oplus B^2(B_R(x_0))\) one gets unique functions
\[ p_0, R \in L^{4/3}(0,\infty; A^2(B_R(x_0))), \]
\[ p_{h,R} \in L^{4/3}(0,\infty; B^2(B_R(x_0))) \]

such that
\[ \hat{p}(t) - \hat{p}(t)|B_R(x_0) = p_{0,R}(t) + p_{h,R}(t) \quad \text{in} \quad B_R(x_0) \]

for almost all \(t \in [0,\infty]\). With help of the Sobolev-Poincaré inequality, Caccioppoli’s inequality and properties of harmonic functions one easily deduces
\[ \|\hat{p}(t) - \hat{p}(t)|B_{2R}(x_0)\|_{L^2(B_{2R})}^{4/3} \leq 2\|p_{h,R}(t) - p_{h,R}(t)|B_{2R}(x_0)\|_{L^2(B_{2R})}^{4/3} + 2\|p_{0,R}(t) - p_{0,R}(t)|B_{2R}(x_0)\|_{L^2(B_{2R})}^{4/3} \leq c\tau^{10/3} R^{10/3} \max_{x \in B_R(x_0)} |\nabla p_{h,R}(t)|^{4/3} + c\|\nabla p_{0,R}(t)\|_{L^{6/5}(B_R)}^{4/3} \]

for almost all \(t \in [0-R^2, t_0]\). Integrating both sides over the interval \([t_0-4\tau^2 R^2, t_0]\) it follows that
\[ \|\hat{p} - \hat{p}_{B_{2R}}\|_{L^2(A^{1/3}(t_0-R^2,t_0;L^2(B_R(x_0))))} \leq c\tau^5 \|p_{h,R}\|_{L^{4/3}(t_0-R^2,t_0;L^2(B_R(x_0))))} + c\|\nabla p_{0,R}\|_{L^{4/3}(t_0-R^2,t_0;L^{6/5}(B_R(x_0))))}.

Taking into account
\[ \|p_{h,R}\|_{L^{4/3}(t_0-R^2,t_0;L^2(B_R(x_0)))) \leq \|\hat{p} - \hat{p}_{B_{2R}}(x_0)\|_{L^{4/3}(t_0-R^2,t_0;L^2(B_R(x_0))))} \]


using \((4.4)\) along with \((4.6)\) and dividing the result by \(\tau^2 R^2\) applying Hölder’s inequality one infers
\[
\mathcal{P}(2\tau R) \leq c\tau^3 \mathcal{P}(R) + c\tau^{-2} R^{-1} \Theta(R) \|v\|^2_{L^1(t_0-R^2,t_0;L^2(B_R(x_0)))} \\
+ c\tau^{-1} R M(R) \|\nabla v\|^2_{L^2(Q_R(x_0,t_0))} \\
\leq c\tau^3 \mathcal{P}(R) + c\tau^{-2} [R^2 M(R) + \Theta(R)] \mathcal{V}(R).
\]
Whence, \((5.4)\).

**Remark 5.3.** Repeating the proof of Lemma 5.2 while using \((4.7)\) instead \((4.6)\) for the estimation of \(\nabla p_{0,R}\) one immediately gets the following alternative fundamental estimate
\[
\mathcal{P}(2\tau R) \leq \hat{c}_{13} \tau^3 \mathcal{P}(R) + \hat{c}_{14} \tau^{-3} \Theta(R) \frac{1}{R} \int_{Q_R(x_0,t_0)} \|\nabla v\|^2 \, dx \, dt
\]
for all \(0 < R < \min\{\text{dist}(x_0, \partial \Omega), \sqrt{t_0}\}\), with absolute constants \(\hat{c}_{13}\) and \(\hat{c}_{14}\).

Now, we are in a position to complete the proof of Prop. 1.5.

**Proof of Proposition 1.5** Let \(0 < R < \min\{\text{dist}(x_0, \partial \Omega), \sqrt{t_0}\}\) be fixed. According to Lemma 4.1 there exists a unique function
\[
W \in L^{10/7}(t_0-R^2,t_0); \ W^{1,10/7}(B_R(x_0))^3 \cap L^2(Q_R(x_0,t_0))^3
\]
fulfilling the identity \((4.10)\). Furthermore, from \((4.11)\) one easily deduces
\[
Z(R|W) \leq c_{15} [R^2 M(R) + \Theta(R)] \mathcal{V}(R).
\]
Combining \((1.12)\) and \((4.10)\) yields
\[
\int_{Q_R(x_0,t_0)} \{ -V \cdot \varphi_t + \nabla V : \nabla \varphi \} \, dx \, dt = \int_{Q_R(x_0,t_0)} p_{h,R} \text{div} \varphi \, dx \, dt
\]
where \(V = v - W\). Thus, setting
\[
U(y,s) := V(x_0 + Ry, t_0 + R^2 s), \\
qu(y,s) := R p_{h,R}(x_0 + Ry, t_0 + R^2 s), \quad (y,s) \in Q_1
\]
shows that the function \(U\) is semi caloric in \(Q_1\) with corresponding pressure \(q\).
Applying Theorem 3.3 one finds
\[
\|U\|^2_{L^2(Q_2 \cap \Omega)} \leq c R^5 \left\{ \|U\|^2_{L^2(Q_1)} + \|q\|^2_{L^2(|-1,0;L^2(\mathbb{B}_3))} \right\}.
\]
Then using the transformation formula of the Lebesgue integral, noticing that
\[
\|p_{h,R}(t)\|_{L^2(B_R(x_0))} \leq \|\hat{p}(t) - \hat{p}_{B_R(x_0)}(t)\|_{L^2(B_R(x_0))} \quad \text{f.a.a.} \quad t \in |t_0 - R^2, t_0|,
\]
(cf. Lemma 2.1) one arrives at
\[
Z(2\tau R|V) \leq c_{16} \tau^2 [Z(R|V) + \mathcal{P}(R)].
\]
Combining (5.7) and (5.9) yields
\[
Z(2\tau R|v) \leq 2Z(2\tau R|V) + 2Z(2\tau R|W)
\leq c\tau^2[Z(R|V) + \mathcal{P}(R)] + \tau^{-3}Z(R|W)
\leq c\left[\tau^2 + \tau^{-3}\tilde{\Theta}(R)\right]\left(V(R) + \mathcal{P}(R)\right),
\]
(5.10)
where \(\tilde{\Theta}(\rho) := R^2M(\rho) + \Theta(\rho), 0 < \rho < \min\{\text{dist}(x_0, \partial B_R(x_0))\sqrt{t_0}\}\).

Now from (5.11) with \(\rho = \tau R\) using (5.10) together with (5.4) yields
\[
V(\tau R) + \mathcal{P}(\tau R)
\leq c(1 + R^2M(R)Z(2\tau R|v) + c\mathcal{P}(2\tau R)
\leq c(1 + R^2M(R))\left[\tau^2 + \tau^{-3}\tilde{\Theta}(R)\right]\left(V(R) + \mathcal{P}(R)\right),
\]
Whence, (1.13). \(\square\)

**Proof of Proposition 1.6.** Given \(\Lambda \in \mathbb{R}^3\) arbitrarily using (1.10) the identity (1.11) can be modified as follows
\[
\int_\Omega |v(t) - \Lambda|^2\phi(t)\,dx + 2\int_0^t \int_\Omega |\nabla v|^2\phi\,dx\,ds
\leq \int_0^t \int_\Omega |v - \Lambda|^2\left(\frac{\partial \phi}{\partial t} + \Delta \phi\right)\,dx\,ds + \int_0^t \int_\Omega 2\tilde{p}(v - \Lambda) \cdot \nabla \phi\,dx\,ds
\leq 2\int_0^t \int_\Omega (\nabla \tilde{p}_h - \Lambda) \times v - (v - \Lambda)\phi\,dx\,ds
\]
holds for all nonnegative functions \(\phi \in C^\infty_0(Q)\), for almost all \(0 < t < \infty\). Hereby, the set of all \(t > 0\), where (5.11) fails does not depend on \(\Lambda\).

Let \(0 < R < \min\{\text{dist}(x_0, \partial Q)\sqrt{t_0}\}\) be fixed. Arguing as in the proof of Lemma 4.1 using (5.11) with \(\Lambda := v_{Q_R(x_0, t_0)}\) one gets the following alternative Caccioppoli inequality

**Lemma 5.4.** There exists an absolute positive constant \(c_{17}\) such that
\[
\tilde{V}(\rho) \leq c_{17}\left[\tilde{E}(2\rho|v) + O(2\rho)\tilde{V}(2\rho) + \mathcal{P}(2\rho)\right]
\]
for all \(0 < \rho < \frac{1}{2} \min\{\text{dist}(x_0, \partial Q)\sqrt{t_0}\}\), where
\[
\tilde{V}(\rho) = \tilde{V}(x_0, t_0; \rho) := \frac{1}{\rho}\|\nabla v\|_{L^2(Q_R(x_0, t_0))}^2,
\]
\[
\tilde{E}(\rho|\varphi) = \tilde{E}(x_0, t_0; \rho|\varphi) := \frac{1}{\rho^3}\|\varphi - \varphi_{Q_R(x_0, t_0)}\|_{L^2(Q_R(x_0, t_0))}^2,
\]
\(\varphi \in L^2(Q_R(x_0, t_0))\).
Proof of Proposition 1.6. Let $0 < R < \min\{\text{dist}(x_0, \partial \Omega) \sqrt{t_0}\}$ be fixed. Arguing as in the proof of Prop. 1.5 instead of (4.11) using (4.12) for estimating $W$ yields

$$\tilde{Z}(R|W) \leq c_{18} O(R) \tilde{V}(R).$$

(5.13)

where $c_{18} > 0$ denotes an absolute constant.

On the other hand, recalling that $V$ being semi caloric in $Q_R(x_0, t_0)$ applying the fundamental estimate (3.5) (see Theorem 3.3 above) it follows

$$\tilde{Z}(2\tau R|V) \leq c_{19} \tau^3 [\tilde{Z}(R|V) + \mathcal{P}(R)].$$

(5.14)

Then combining (5.7) and (5.9) yields

$$\tilde{Z}(2\tau R|V) \leq 2\tilde{Z}(2\tau R|V) + 2\tilde{Z}(2\tau R|W)$$

$$\leq c \tau^3 [\tilde{Z}(R|V) + \mathcal{P}(R)] + \tau^{-3} \mathcal{Z}(R|W)$$

$$\leq c \left[ \tau^3 + \tau^{-3} O(R) \right] \tilde{V}(R) + \mathcal{P}(R).$$

(5.15)

Finally, into (5.12) putting $\rho = \tau R$, estimating the first term on the right by (5.15) and last term by (5.6) respectively gives

$$\tilde{V}(\tau R) + \mathcal{P}(\tau R) \leq c \left[ \tau^3 + \tau^{-3} O(R) \right] \tilde{V}(R) + \mathcal{P}(R).$$

Whence, (1.14).

\[\square\]

6. Proof of the Main Theorem

To begin with, define

$$\varepsilon_* := 0.18566 k_*^{-2.5}.$$

Set

$$f(\tau) := k_* (\tau^2 + \tau^{-3} \varepsilon_*), \quad \tau > 0.$$  

Let $\tau_* > 0$ such that $\theta_* := f(\tau_*) = \min_{\tau > 0} f(\tau)$. In fact, by an elementary calculus one finds

$$\tau_* = (1.5 \varepsilon_*)^{1/5},$$

$$\theta_* = \frac{5}{3} (1.5 \cdot 0.18566)^{2/5} = 0.999 \ldots < 1.$$  

Next, let $(x_0, t_0) \in Q$, such that

$$\limsup_{R \to 0^+} \Theta(x_0, t_0; R) < \varepsilon_*.$$  

Since $\lim_{R \to 0^+} R^2 M(x_0, t_0; R) = 0$ we may choose $0 < R_0 < \min\{\text{dist}(x_0, \partial \Omega) \sqrt{t_0}\}$ such that

$$\tilde{\Theta}(x_0, t_0; R) < \varepsilon_*,$$

$$(1 + R^2 M(x_0, t_0; R)) f(\tau_*) \leq \frac{\theta_* + 1}{2} \quad \forall 0 < R \leq R_0.$$
Then (1.13) leads to
\[ \Xi(x_0, t_0; \tau_* R) \leq \frac{\theta_* + 1}{2} \Xi(x_0, t_0; R) \quad \forall \ 0 < R \leq R_0. \]
Consequently,
\[ \lim_{R \to 0^+} \Xi(x_0, t_0; R) = 0. \] (6.1)
In particular, having \(|\text{rot } u| \leq 2|\nabla v|\), there exists \(0 < R_1 < R_0\), such that
\[ \tilde{\Theta}(x_0, t_0; R_1) < \frac{\varepsilon_*}{2}, \quad \Xi(x_0, t_0; R_1) < \frac{\varepsilon_*}{4}. \]
By means of the absolute continuity of the Lebesgue integral we may choose \(0 < R_2 \leq R_1\) such that for all \((y, s) \in Q_R(x_0, t_0)\):
\[ \Xi(y, s; R_1) < \frac{\varepsilon_*}{4}, \]
\[ \tilde{\Theta}(y, s; R_1) < \frac{\varepsilon_*}{2}, \]
\[ (1 + R^2 M(y, s; R_1)) f(\tau_*) < \frac{\theta_* + 1}{2}. \]
Now, let \((y, s) \in Q_R(x_0, t_0)\) be fixed. Once more applying (1.13) iteratively replacing \((x_0, t_0)\) by \((y, s)\) and setting \(R = R_1\) therein yields
\[ \Xi(y, s; \tau_*^k R_1) \leq \left( \frac{\theta_* + 1}{2} \right)^k \Xi(y, s; R_1). \]
Choosing \(0 < \alpha < 1\) such that \(\tau_*^\alpha = \frac{\theta_* + 1}{2}\) from the estimate above one gets
\[ \Xi(y, s; R) \leq \tilde{C}_1 R^\alpha \quad \forall \ 0 < R \leq R_1, \] (6.2)
where \(\tilde{C}_1 = \text{const} > 0\) depending neither on \((y, s)\) nor on \(R\).
Next, fix \(0 < \beta < 1\). Choose \(0 < \tau < \frac{1}{2}\) according to
\[ \tau^\beta \tilde{k}_* \leq \frac{1}{2} \] [7].
Furthermore, by virtue of (6.2) one can select \(0 < R_2 \leq R_1\) such that for all \(0 < R \leq R_2\) there holds
\[ \tau^{-3} \mathcal{O}(y, s; R) \leq \frac{\tau^{3-\beta}}{2} \quad \forall \ (y, s) \in Q_R(x_0, t_0). \]
Now, fix \((y, s) \in Q_R(x_0, t_0)\) . Applying (1.14) with \((y, s)\) instead of \((x_0, t_0)\) gives
\[ \Xi(y, s; \tau R) \leq \tau^{3-\beta} \Xi(y, s; R) \quad \forall \ 0 < R \leq R_2. \]
This implies
\[ \int_{Q_R(y, s)} |\nabla v|^2 \, dx \leq \tilde{C}_2 R^{4-\beta} \quad \forall \ 0 < R \leq R_2, \]
[7] Here \(\tilde{k}_*\) denotes the constant, which appears in (1.14).
where the constant $\hat{C}_2 > 0$ depends neither on $(y, s) \in Q_\rho(x_0, t_0)$ nor on $R$. Applying the Poincaré-type inequality (A.10) (cf. Theorem A.1 below) gives

$$v \in L^{6-\beta}(Q_\rho(x_0, t_0)),$$

which, by a well-known theorem of Da Prato [6], shows that $v$ belongs to the Hölder space $C^{(1-\beta)/2, (1-\beta)/4}(Q_\rho(x_0, t_0))^3$. Since $\nabla \hat{p}_h$ is continuous in $Q$ the velocity $u$ is continuous on $Q_\rho(x_0, t_0)$. This completes the proof of the Theorem. \hfill \Box

7. Existence of a suitable weak solution

Let $\Phi \in C^{\infty}([0, \infty[)$ be a non-increasing function, such that $0 \leq \Phi \leq 1$ in $[0, \infty[$, $\Phi \equiv 1$ on $[0, 1]$, $\Phi \equiv 0$ in $[2, \infty[$ and $0 \leq -\Phi' \leq 2$. For $\varepsilon > 0$ we set

$$\Phi_\varepsilon(\rho) := \Phi(\varepsilon \rho), \quad \rho \in [0, \infty[.$$

Let $a \in L^2_\sigma(\Omega)$ be arbitrarily given. Arguing as in [25] one easily can construct a unique function $u_\varepsilon \in C([0, \infty; L^2_\sigma(\Omega)) \cap L^2(0, \infty; W^{1,2}_\sigma(\Omega))$, such that

$$-\int_Q u_\varepsilon \cdot \partial_t \varphi \, dx \, dt + \int_Q \nabla u_\varepsilon : \nabla \varphi \, dx \, dt$$

$$= \int_Q u_\varepsilon \otimes u_\varepsilon \Phi_\varepsilon(|u_\varepsilon|) : \nabla \varphi \, dx \, dt + \int_\Omega a \cdot \varphi(0) \, dx$$

holds for all $\varphi \in C_0^{\infty}([0, \infty[, D_\sigma(\Omega))$. In addition, there holds the global energy equality

$$\frac{1}{2} \|u_\varepsilon(t)\|_{L^2(\Omega)}^2 + \int_0^t \int_\Omega |\nabla u_\varepsilon|^2 \, dx \, ds = \frac{1}{2} \|a\|_{L^2(\Omega)}^2 \quad \forall t > 0.$$  \hfill (7.2)

From this identity one easily gets the a-priori estimate

$$\|u_\varepsilon\|_{L^\infty(0, \infty; L^2(\Omega))} + \|\nabla u_\varepsilon\|_{L^2(Q)} \leq K.$$  \hfill (7.3)

Using Sobolev’s inequality combined with multiplicative inequalities one infers

$$\|u_\varepsilon\|_{L^{6/5}(0, \infty; L^4(\Omega))} \leq K.$$  \hfill (7.4)

Let $B \subset \subset \Omega$ be a fixed domain with Lipschitz boundary. Applying Theorem 2.2 with $u = u_\varepsilon$ and $h = u_\varepsilon \otimes u_\varepsilon \Phi_\varepsilon(|u_\varepsilon|)$ one gets two functions

$$\hat{p}_0, \varepsilon \in L^1_{\text{loc}}((0, \infty[, A^2(\Omega)),$$

$$\hat{p}_{h, \varepsilon} \in C(Q) \quad \text{with} \quad \Delta \hat{p}_{h, \varepsilon} = 0, \quad \text{and} \quad (\hat{p}_{h, \varepsilon})_B = 0.$$
such that
\[
- \int_Q \mathbf{u}_\varepsilon \cdot \partial_t \varphi \, dx \, dt + \int_Q \nabla \mathbf{u}_\varepsilon : \nabla \varphi \, dx \, dt \\
= \int_Q \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon (|\mathbf{u}_\varepsilon|) : \nabla \varphi \, dx \, dt \\
+ \int_Q (p_{0,\varepsilon} \div \varphi + \nabla \tilde{p}_{h,\varepsilon} \cdot \partial_t \varphi) \, dx \, dt \\
+ \int_\Omega a \cdot \varphi(0) \, dx
\] (7.5)
for all \( \varphi \in C^\infty(\Omega)^3 \). Into (7.5) putting \( \varphi(x, t) = \nabla \phi(x) \eta(t) \) for \( \phi \in C^\infty_0(\Omega) \) and \( \eta \in C^\infty_0(0, \infty) \) using integration by parts yields

\[
\int_\Omega p_{0,\varepsilon}(t) \Delta \phi \, dx = \int_\Omega \mathbf{u}_\varepsilon(t) \otimes \mathbf{u}_\varepsilon(t) \Phi_\varepsilon(|\mathbf{u}_\varepsilon(t)|) : \nabla^2 \phi \, dx
\] (7.6)
for all \( \phi \in C^\infty_0(\Omega) \), for almost all \( t \in [0, \infty[ \). Hence, from the definition of \( A^2(\Omega) \) it follows

\[
\|p_{0,\varepsilon}(t)\|_{L^2(\Omega)} \leq \|\mathbf{u}_\varepsilon(t) \otimes \mathbf{u}_\varepsilon(t)\|_{L^2(\Omega)} \leq \|\mathbf{u}_\varepsilon(t)\|^2_{L^4(\Omega)}.
\]
Thus, observing (7.6) yields

\[
\|p_{0,\varepsilon}\|_{L^{8/3}(0, \infty; L^2(\Omega))} \leq K.\quad (7.7)
\]
Furthermore, by means of (2.7) for every \( G \subset \subset \Omega \) with Lipschitz boundary there exists a constant \( c_G \) such that

\[
\|\tilde{p}_{h,\varepsilon}(t)\|_{L^2(G)} \leq c_G \left\{ \|\mathbf{u}_\varepsilon(t) - a\|_{L^2(\Omega)} + \left\| \int_0^t (|\mathbf{u}_\varepsilon(s)|^2 + |\nabla \mathbf{u}_\varepsilon(s)|) \, ds \right\|_{L^2(\Omega)} \right\}.
\]
Hence using Minkowski’s inequality along with (7.4) and (7.4) yields

\[
\|\tilde{p}_{h,\varepsilon}(t)\|_{L^2(G)} \leq c_G K (1 + \sqrt{t}).\quad (7.8)
\]
By virtue of reflexivity using Cantor’s diagonalization principal there exist a sequence \( \{\varepsilon_j\} \) of positive numbers tending to 0 as \( j \to \infty \) and functions

\[
\mathbf{u} \in L^\infty(0, \infty; L^2_\sigma(\Omega)) \cap L^2(0, \infty; \dot{W}^{1,2}(\Omega)), \\
q_0 \in L^{4/3}(0, \infty; L^2(\Omega)), \\
\tilde{q}_h \in L^2_{\text{loc}}(\Omega \times [0, \infty[) \quad \text{with} \quad \Delta \tilde{q}_h = 0 \quad \text{and} \quad (\tilde{p}_h)_B = 0, \\
such that
\]

\[
u_{\varepsilon_j} \rightharpoonup \mathbf{u} \quad \text{in} \quad L^\infty(0, \infty; L^2_\sigma(\Omega)) \cap L^2(0, \infty; \dot{W}^{1,2}(\Omega)), \\
p_{0,\varepsilon_j} \rightharpoonup q_0 \quad \text{in} \quad L^{4/3}(0, \infty; L^2(\Omega)) \\
\tilde{p}_{h,\varepsilon_j} \rightharpoonup \tilde{q}_h \quad \text{in} \quad L^2_{\text{loc}}(\Omega \times [0, \infty[) \quad \text{as} \quad j \to \infty.
\] (7.9)
Arguing as in [25] using the Lemma of Aubin-Lions we infer that for each subdomain $G \subset\subset \Omega$ and $T > 0$ there holds

$$u_{\varepsilon_j} \to u \quad \text{in} \quad L^2(G \times ]0, T[)^3 \quad \text{as} \quad j \to \infty. \quad (7.10)$$

Using (7.9) and (7.10) we are able to carry out the passage to the limit in both sides of (7.1). Hence, $u$ is a weak solution to (1.1)-(1.4).

Furthermore carrying out the passage to the limit in both sides of (7.5) making use of (7.9) and (7.10) gives

$$- \int_Q u \cdot \partial_t \varphi \, dx \, dt + \int_Q \nabla u : \nabla \varphi \, dx \, dt = \int_Q u \otimes u : \nabla \varphi \, dx \, dt + \int_Q (q_0 \text{div} \varphi + \nabla \tilde{q}_h \cdot \partial_t \varphi) \, dx \, dt \quad (7.11)$$

for all $\varphi \in C^\infty(\Omega)^3$. On the other hand appealing to Theorem 2.2 one gets two functions $p_0 \in L^{4/3}(0, \infty; A^2(\Omega))$ and $\tilde{p}_h \in C(\Omega)$ with $\Delta \tilde{p}_h$ and $(\tilde{p}_h)_T = 0$ fulfilling (7.11) too. Thus, by Remark 2.4 one concludes $q_0 = p_0$ and $\tilde{q}_h = \tilde{p}_h$ almost everywhere in $\Omega$.

As in [25] using the properties of harmonic functions one easily proves that for all $t > 0$

$$\nabla^2 \tilde{p}_{h, \varepsilon_j} \to \nabla^2 \tilde{p}_h \quad \text{in} \quad L^2_{\text{loc}}(\Omega \times [0, \infty[)^9 \quad \text{as} \quad j \to \infty. \quad (7.12)$$

Now, set

$$v_{\varepsilon_j} := u_{\varepsilon_j} + \nabla \tilde{p}_{h, \varepsilon_j}, \quad v := u + \nabla \tilde{p}_h.$$

Combining (7.9), (7.10) and (7.12) implies

$$\nabla v_{\varepsilon_j} \to \nabla v \quad \text{in} \quad L^2_{\text{loc}}(\Omega \times [0, \infty[)^3, \quad (7.13)$$

$$v_{\varepsilon_j} \to v \quad \text{in} \quad L^2_{\text{loc}}(\Omega \times [0, \infty[)^3 \quad \text{as} \quad j \to \infty. \quad (7.14)$$

In addition, we have

$$v_{\varepsilon_j}(t) \to v(t) \quad \text{in} \quad L^2_{\text{loc}}(\Omega)^3 \quad \text{as} \quad j \to \infty. \quad (7.15)$$

for almost all $t > 0$.

In order to complete the proof we only need prove that $u, p_0$ and $\tilde{p}_h$ satisfy the local energy inequality (1.11). In fact, from the identity (7.5) it follows that $\frac{dv_{\varepsilon}}{dt} \in L^2_{\text{loc}}(0, \infty; W^{-1,2}(\Omega)^3)$ and there holds

$$\int_0^t \left( \frac{dv_{\varepsilon}}{dt}(s), \varphi(s) \right) \, ds + \int_Q \nabla v_{\varepsilon} : \nabla \varphi \, dx \, ds = \int_0^t \int_\Omega u_{\varepsilon} \otimes u(\varepsilon) \nabla \varphi \, dx \, ds + \int_0^t \int_\Omega p_0 \text{div} \varphi \, dx \, ds \quad (7.16)$$
for all $\varphi \in L^2(0, \infty; \dot{W}^{1,2}_{\sigma} (\Omega))$ and $t > 0$. Let $\phi \in C_0^\infty (Q)$, $\phi \geq 0$. Then in (7.16) putting $\varphi = u_\varepsilon \phi$ using integration by parts yields

$$
\frac{1}{2} \int_\Omega |v_\varepsilon(t)|^2 \phi(t) \, dx + \int_0^t \int_\Omega |\nabla v_\varepsilon|^2 \phi \, dx \, dt
= \int_0^t \int_\Omega (\Psi_\varepsilon'(|u_\varepsilon|) u_\varepsilon \cdot \nabla \phi + p_{0,\varepsilon} v_\varepsilon \cdot \nabla \phi) \, dx \, dt
+ \int_0^t \int_\Omega u_\varepsilon \otimes u_\varepsilon \phi(\varepsilon) : \nabla (\nabla \tilde{p}_{h,\varepsilon} \phi) \, dx \, dt
$$

(7.17)
for all $t > 0$, where

$$
\Psi_\varepsilon(u) := u^2 \Phi_\varepsilon(u) - \int_0^u \Phi_\varepsilon'(r) \, dr, \quad u \geq 0.
$$

Clearly,

$$
\Psi_\varepsilon(u) \rightarrow \frac{u^2}{2} \quad \text{uniformly on each interval } [0, T] \quad \text{as } \varepsilon \rightarrow 0.
$$

Now, let $t \in [0, \infty]$ such that (7.15) is true. Then into (7.17) passing to the limit with $\varepsilon = \varepsilon_j \rightarrow 0$ as $j \rightarrow \infty$ using (7.12), (7.13), (7.14) and (7.15) along with the lower semi continuity of the functional on the left of (7.17) leads to

$$
\int_\Omega |v(t)|^2 \phi(t) \, dx + 2 \int_0^t \int_\Omega |\nabla v|^2 \phi \, dx \, dt
\leq \int_0^t \int_\Omega (-|u|^2 \nabla \tilde{p}_h \cdot \nabla \phi + 2 \hat{p} \cdot \nabla \phi) \, dx \, dt
+ 2 \int_0^t \int_\Omega u \otimes u : \nabla (\nabla \tilde{p}_h \phi) \, dx \, dt
$$

(7.18)
where $\hat{p} = \frac{|u|^2}{2} + p_0$. Using integration by parts and recalling $\Delta \tilde{p}_h = 0$ one easily calculates

$$
- \int_0^t \int_\Omega |u|^2 \nabla \tilde{p}_h \cdot \nabla \phi \, dx \, dt + 2 \int_0^t \int_\Omega u \otimes u : \nabla (\nabla \tilde{p}_h \phi) \, dx \, dt
= 2 \int_0^t \int_\Omega (u_j \partial_x u_j - u_j \partial_x u_i)(\partial_x \tilde{p}_h) \phi \, dx \, dt
= -2 \int_0^t \int_\Omega u \times \text{rot } u \cdot \nabla \tilde{p}_h \phi \, dx \, dt
= 2 \int_0^t \int_\Omega \nabla \tilde{p}_h \times \text{rot } v \cdot \text{rot } \phi \, dx \, dt.
$$

Whence (1.11).
Appendix A.

Lemma A.1. Let \( \Omega \) be a domain. Let \( p : \Omega \times [0, \infty[ \rightarrow \mathbb{R} \) such that for all \( t \in [0, \infty[ \) the function \( x \mapsto p(x, t) \) is harmonic in \( \Omega \). Assume that for all \( G \subset \subset \Omega \)

\[
t \mapsto \int_G p(x, t) \, dx \text{ is continuous on } [0, \infty[;
\]

\[
\sup_{t \in [0, \infty[} \int_G |p(t)| \, dx < +\infty \quad \forall 0 < T < \infty.
\]

Then \( p \in C(\Omega \times [0, \infty[) \).

Proof. Boundness of \( p \). Let \( x_0 \in \Omega \) and \( 0 < R < \frac{1}{2} \text{dist}(x_0, \partial \Omega) \). Let \( x \in B_R(x_0) \). Clearly, \( B_R(x) \subset B_{2R}(x_0) \). Thus, observing (A.2) using the well-known mean value formula for harmonic functions one estimates

\[
|p(x, t)| \leq \int_{B_R(x)} |p(y, t)| \, dy \leq \int_{B_{2R}(x_0)} |p(y, t)| \, dy \leq CR^{-3}
\]

for all \( (x, t) \in B_R(x_0) \times [0, T] \) (\( 0 < T < +\infty \)). Thus, \( p \) is bounded on \( B_R(x_0) \times [0, T] \) and therefore locally bounded in \( \Omega \times [0, \infty[ \).

Next, let \( x \in \partial B_R(x_0) \). Once more using the mean value formula and integration by parts one gets

\[
|\partial_i p(x, t)| = \left| \int_{B_R(x)} \partial_i p(y, t) \, dy \right| \leq cR^{-3} \int_{\partial B_R(x_0)} |p(y, t)| \, dy \leq CR^{-4}.
\]

(\( i = 1, 2, 3 \)). Therefore, \( \nabla p \) is locally bounded in \( \Omega \times [0, \infty[ \) too.

Continuity of \( p \). Let \( (x_0, t_0) \in Q \). Given \( (x, t) \in B_R \times [0, \infty[ \) using the triangular inequality together with the mean value theorem one gets

\[
|p(x_0, t_0) - p(x, t)| \leq |p(x_0, t_0) - p(x_0, t)| + |p(x_0, t) - p(x, t)|
\]

\[
= \left| \int_{B_R(x_0)} p(y, t_0) - p(y, t) \, dy \right| + \max_{y \in B_R(x_0)} |\nabla p(y, t)| |x_0 - x|.
\]

Observing (A.1) the estimate above immediately implies that \( p \) is continuous at \((x_0, t_0)\). \( \square \)

Lemma A.2. Let \( p : \Omega \times [0, \infty[ \rightarrow \mathbb{R} \) such that for all \( t \in [0, \infty[ \) the function \( x \mapsto p(x, t) \) is harmonic in \( \Omega \). Assume that

\[
p - p_G \text{ is continuous on } G \times [0, \infty[ \quad \forall G \subset \subset \Omega,
\]

\[
\exists x_0 \in \Omega : \quad t \mapsto p(x_0, t) \text{ is continuous on } [0, \infty[.
\]

Then \( p \in C(\Omega \times [0, \infty[) \).

Proof. We define

\[
A := \left\{ x \in \Omega \mid p(x, \cdot) \in C([0, \infty[) \right\}.
\]
A is open. Let $x \in A$. Let $0 < R < \text{dist}(x, \partial \Omega)$. By means of (A.3) we have $p - p_{B_R(x)}$ is continuous in $B_R \times [0, \infty]$. On the other hand by the mean value formula we have $p_{B_R(x)} = p(x, \cdot) \in C([0, \infty])$. Thus, $p$ is continuous in $B_R(x) \times [0, \infty]$, which yields $B_R(x) \subset A$.

A is closed. Let $\{x_m\}$ be a sequence in $A$ converging to $x \in \Omega$ as $m \to +\infty$. Let $0 < R < \text{dist}(x, \partial \Omega)$. Hence, there exists $m \in \mathbb{N}$ such that $|x_m - x| < R/2$. By the triangular inequality it follows that $|x_m - x| < R/2$. Therefore, $B_{R/2}(x_m) \subset \Omega$. As we have proved above there holds $B_{R/2}(x_m) \subset A$ and in particular $x \in A$.

Since $A \neq \emptyset$ (cf. (A.4)) implies $A = \Omega$. Let $x \in \Omega$ and $0 < R < \text{dist}(x, \partial \Omega)$. Then $p = p - p_{B_R(x)} + p(x, \cdot)$ in $B_R(x)$. By means of (A.3) taking into account $p(x, \cdot) \in C([0, \infty])$ proves the claim of the lemma $\square$

**Lemma A.3.** Let $g \in L^r(B_R)^3 (\frac{6}{5} \leq r \leq 2)$ and $g_0 \in A^2(B_R)$ with

$$
\int_{B_R} g \cdot \nabla \phi \, dx = \int_{B_R} g_0 \Delta \phi \, dx \quad \forall \phi \in C_0^\infty(B_R).
$$

(A.5)

Then $g_0 \in W^{1,r}(B_R)$ and there exists a constant $C_r$, such that

$$
\|\nabla g_0\|_{L^r(B_R)} \leq C_r \|g\|_{L^r(B_R)}.
$$

(A.6)

Proof. First let us consider the case $R = 1$. There exists $h \in W_0^{1,r}(B_1)^3$ such that $\text{div } h = g - g_{B_1}$ with

$$
\|\nabla h\|_{L^r(B_1)} \leq c \|g\|_{L^r(B_1)}.
$$

(A.7)

Next, applying integration by parts, from (A.5) it follows that

$$
- \int_{B_1} (h - h_{B_1}) : \nabla^2 \phi \, dx = \int_{B_1} q_0 \Delta \phi \, dx \quad \forall \phi \in W_0^{2,2}(B_1).
$$

By the definition of $A^2(B_1)$ there exists $u \in W_0^{2,2}(B_1)$ such that $q_0 = \Delta u$. Thus, from the identity with $\phi = u$ above using Sobolev-Poincaré’s inequality along with (A.7) yields

$$
\|q_0\|_{L^2(B_1)} \leq c \|h - h_{B_1}\|_{L^2(B_1)} \leq c \|g\|_{L^r(B_1)}.
$$

(A.8)

Next, applying [Simader, Th. 9.11, p.156] with $B[u, \phi] = \int_{B_1} \Delta u \Delta \phi \, dx$ and

$$
F(\phi) = \int_{B_1} g \cdot \nabla \phi \, dx \text{ one gets } q_0 \in W^{1,r}(B_1) \text{ and}
$$

$$
\|\nabla q_0\|_{L^r(B_1)} \leq c (\|g\|_{L^r(B_1)} + \|u\|_{L^r(B_1)}).
$$

Finally, taking into account

$$
\|u\|_{L^r(B_1)} \leq c \|\Delta u\|_{L^2(B_1)} = c \|q_0\|_{L^2(B_1)}
$$

making use of (A.8) shows (A.6).

For the general case $R > 0$ the assertion easily follows from the former case $R = 1$ by using an appropriate transformation of coordinates applying the transformation formula of the Lebesgue integral. $\square$
For reader’s convenience we will present a short proof of the Poincaré-type inequality we have used above.

**Lemma A.4.** Let $u \in L^2(Q_R)$ with $\nabla u \in L^2(Q_R)$ and $h \in L^1(Q_R)^3$, such that

$$-\int_{Q_n} u \partial_t \varphi \, dx \, dt = \int_{Q_n} h \cdot \nabla \varphi \, dx \, dt \quad \forall \varphi \in C_0^\infty(Q_R). \quad \text{(A.9)}$$

Then

$$\int_{Q_n} |u - u_{Q_n}|^2 \, dx \, dt \leq c_0 R^2 \left\{ \int_{Q_n} |u - u_{Q_n}|^2 \, dx \, dt + \left( \int_{Q_n} |h| \, dx \, dt \right)^2 \right\}, \quad \text{(A.10)}$$

where $c_0 = \text{const} > 0$ depends on $n$ only.

**Proof** 1) First, let us prove the assertion for the case $R = 1$.

For a given function $\zeta \in C_0^\infty(B_1)$ with $0 \leq \zeta \leq 1$ in $B_1$ and $\zeta \neq 0$ we define the functional $F \in L^1(B_1)^*$ by means of

$$\langle F, v \rangle := \frac{1}{|B_1|} \int_{B_1} \zeta v \, dx, \quad v \in L^1(B_1).$$

Clearly, $\langle F, 1 \rangle = 1$ and

$$\langle F, v - \langle F, v \rangle \rangle = 0 \quad \forall v \in L^1(B_1(0)).$$

Since

$$|||v||| := \|\nabla v\|_{L^2(B_1)} + |\langle F, v \rangle|$$

defines an equivalent norm on $W^{1,q}(B_1)$. Thus,

$$||v - \langle F, v \rangle||_{L^q\ast(B_1(0))} \leq c \|\nabla v\|_{L^q(B_1)} \quad \forall v \in W^{1,q}(B_1). \quad \text{(A.11)}$$

Now let $u \in L^2(Q_1)$ with $\nabla u \in L^2(Q_1)^3$ and $h \in L^1(Q_1)^3$ fulfilling (A.9). One easily calculates

$$u(x, t) - u_{Q_1} = (u(x, t) - (F, u(t))) + ((F, u(t)) - u_{Q_1})$$

$$= u(x, t) - \langle F, u(t) \rangle + \int_{Q_1} \langle F, u(t) \rangle - u(y, s) \, dy \, ds$$
for almost all \((x, t) \in Q_1\). Moreover, verifying
\[
\int_{Q_1} \langle F, u(t) \rangle - u(y, s) \, dy \, ds
\]
\[
= \frac{1}{\int_{B_1} \zeta \, dy} \int_{Q_1} \int_{B_1} \zeta(y') (u(y', t) - u(y, s)) \, dy' \, dy \, ds
\]
\[
= \int_{-1}^0 \langle F, u(t) - u(s) \rangle \, ds
\]
\[
+ \frac{1}{2 \int_{B_1} \zeta \, dy} \int_{-1}^0 \int_{B_1} \int_{B_1} \zeta(y') (u(y', s) - u(y, s)) \, dy' \, dy \, ds.
\]
for almost all \(-1 < t - 0\) gives
\[
u(x, t) - u_{Q_1} = \nu(x, t) - \langle F, u(t) \rangle + \int_{-1}^0 \langle F, u(t) \rangle - u(y, s) \, ds
\]
\[
+ \frac{1}{2 \int_{B_1} \zeta \, dy} \int_{-1}^0 \int_{B_1} \int_{B_1} \zeta(y') (u(y', s) - u(y, s)) \, dy' \, dy \, ds.
\]
Hence, from the identity above one obtains
\[
\|u(t) - u_{Q_1}\|_{L^2(B_1)} \leq \|u(t) - \langle F, u(t) \rangle\|_{L^2(B_1)}
\]
\[
+ \left\| \int_{-1}^0 |\langle F, u(t) - u(s) \rangle| \, ds \right\|_{L^2(B_1)}
\]
\[
+ \frac{1}{\int_{B_1} \zeta \, dy} \left\| \int_{-1}^0 \int_{B_1} |u(y, s) - u(s)B_1| \, dy \, ds \right\|_{L^2(B_1)}
\]
for almost all \(t \in ]-1, 0[\). Then by the aid of \(A.11\) and Poincaré’s inequality one finds
\[
\|u(t) - u_{Q_1}\|_{L^2(B_1)}^2 \leq c\|\nabla u(t)\|_{L^2(B_1)}^2
\]
\[
+ c \int_{-1}^0 |\langle F, u(t) - u(s) \rangle|^2 \, ds \quad + c\|\nabla u\|_{L^2(Q_1)}^2.
\]
for almost all \(t \in ]-1, 0[\). Next, integrating both sides of this inequality over the interval \([-1, 0]\) yields
\[
\int_{-1}^0 \|u(t) - u_{Q_1}\|_{L^2(B_1)}^2 \, dt
\]
\[
\leq c \int_{-1}^0 \|\nabla u(t)\|_{L^2(B_1)}^2 \, dt \quad + c \int_{-1}^0 \int_{-1}^0 |\langle F, u(t) - u(s) \rangle|^2 \, ds \, dt.
\]
Finally into \(A.9\) inserting \(\varphi(x, t) = \zeta(x)\) using integration by parts implies
\[
|\langle F, u(t) - u(s) \rangle|^2 \leq \frac{1}{\int_{B_1} \zeta \, dy} \int_{B_1} h \cdot \nabla \zeta \, dx \, dt \quad \leq c \left( \int_{Q_1} |h| \, dx \, dt \right)^2
\]
for almost all $s, t \in ] - 1, 0[$. Whence, (A.10).

2) The general case easily follows from above by means of a standard homothety argument using an appropriate transformation of coordinates. □

References


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