Strong Solutions to the Stokes-Equations of a Flow Around a Rotating Body in Weighted $L^q$-Spaces

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Abstract

We consider the motion of a fluid in the exterior of a rotating obstacle. This leads to a modified version of the Stokes system which we consider in the whole space $\mathbb{R}^n$, $n = 2$ or $n = 3$ and in an exterior domain $D \subset \mathbb{R}^3$. For every $q \in (1, \infty)$ we prove existence of solutions and estimates in function spaces with weights taken from a subclass of the Muckenhoupt class $A_q$. Moreover, uniqueness is shown modulo a vector space of dimension 3.

1 Introduction

In the present paper we study the following modified Stokes system in an exterior domain $D \subset \mathbb{R}^3$ of class $C^{1,1}$

\[-\nu \Delta u - (\omega \wedge x) \cdot \nabla u + \omega \wedge u + \nabla p = f \quad \text{in } D \]
\[\text{div } u = 0 \quad \text{in } D \]
\[u|_{\partial D} = 0.\]

(1.1)

In particular, we are interested in solutions $u$ which decay at infinity, i.e., they fulfill

$u(x) \to 0$, for $|x| \to \infty$. \hspace{1cm} (1.2)

These equations arise from the motion of a fluid in the exterior of a rigid body rotating with angular velocity $\omega = (0,0,1)^T$. We assume that the complement of this body is filled with a viscous incompressible fluid modeled by the Navier-Stokes equations. This means, given the coefficient of viscosity $\nu > 0$ and an external force $\tilde{f} = \tilde{f}(y,t)$, we are

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looking for the velocity $v = v(y, t)$ and the pressure $q = q(y, t)$ solving the nonlinear system

$$
\begin{align*}
v_t - \nu \Delta v + v \cdot \nabla v + \nabla q &= \hat{f} & \text{in } D(t), \ t > 0, \\
\text{div } v &= 0 & \text{in } D(t), \ t > 0, \\
v(y, t) &= \omega \wedge y & \text{on } \partial D(t), \ t > 0, \\
v(y, t) &\to 0 & \text{as } |y| \to \infty.
\end{align*}
$$

(1.3)

Here the time-dependent exterior domain $D(t)$ is given, due to the rotation with the angular velocity $\omega$, by

$$
D(t) = O(t)D,
$$

where $D \subset \mathbb{R}^3$ is a fixed exterior domain and $O(t)$ denotes the orthogonal matrix

$$
O(t) = \begin{pmatrix} \cos t & -\sin t & 0 \\
\sin t & \cos t & 0 \\
0 & 0 & 1 \end{pmatrix} \quad \text{or} \quad O(t) = \begin{pmatrix} \cos t & -\sin t \\
\sin t & \cos t \end{pmatrix} \quad \text{if } n = 2.
$$

(1.4)

After the change of variables $x = O(t)^T y$ and passing to the new functions $u(x, t) = O(t)^T v(y, t) - \omega \wedge x$, $p(x, t) = q(y, t)$, as well as to the force term $f(x, t) = O(t)^T \hat{f}(y, t)$ we arrive at the modified Navier-Stokes system

$$
\begin{align*}
u_t - \nu \Delta u + u \cdot \nabla u - (\omega \wedge x) \cdot \nabla u + \omega \wedge u + \nabla p &= f & \text{in } D, \\
\text{div } u &= 0 & \text{in } D, \\
u(x, t) &= \omega \wedge x & \text{on } \partial D, \\
u(x, t) + \omega \wedge x &\to 0 & \text{at } \infty,
\end{align*}
$$

(1.5)

for all $t > 0$ in the exterior time-independent domain $D$. Note that because of the new coordinate system attached to the rotating body (1.5) contains two new linear terms, the classical Coriolis force term $\omega \wedge u$ (up to a multiplicative constant) and the term $(\omega \wedge x) \cdot \nabla u$ which is not subordinate to the Laplacian in unbounded domains. As an important step in the study of problem (1.5) one considers its linearized and steady version in the whole space $\mathbb{R}^n$, i.e., the Stokes system

$$
-\nu \Delta u - (\omega \wedge x) \cdot \nabla u + \omega \wedge u + \nabla p = f \quad \text{in } \mathbb{R}^n, \\
\text{div } u &= 0 \quad \text{in } \mathbb{R}^n,
$$

(1.6)

where $n = 2$ or $n = 3$; in the two-dimensional case plainly $\omega \wedge x \overset{\equiv}{=} (-x_2, x_1)$ for $x = (x_1, x_2)$ and $\omega \wedge u \overset{\equiv}{=} (-u_2, u_1)$ for $u = (u_1, u_2)$.

The linear system (1.6) has been analyzed in $L^q$-spaces, $1 < q < \infty$, by Farwig, Hishida and Müller in [5], proving the a priori-estimate

$$
\|\nu \nabla^2 u\|_q + \|(\omega \wedge x) \cdot u - \omega \wedge u\|_q + \|\nabla p\|_q \leq c\|f\|_q.
$$

(1.7)

Similar results have been obtained by Farwig [2], [3] in the case of a rotating body with constant translational velocity $u_\infty$ parallel to $\omega$, leading to an Oseen-like version of (1.6) in which the additional term $u_\infty \cdot \nabla u$ has to be added in the equation of the balance of momentum. For related results on weak solutions in $L^q$- and in Lorentz spaces we refer to Farwig and Hishida [4] and Hishida [16], for the investigation of several auxiliary
linear problems to Neˇ casov´a [21], [22], and for weak solutions to an Oseen system of type (1.6) in \( L^2 \) with anisotropic weights see Krˇ cmar, Neˇ casov´a and Penel [19]. An \( L^q \)-approach to weak solutions by Krˇ cmar, Neˇ casov´a and Penel may be found in [20] and to strong solutions with anisotropic weights by Farwig, Krbec and Neˇ casov´a in [7]. Recent results have also been shown by Galdi and Silvestre [13], Galdi [11], [12], Farwig [1], Hishida [17], [18], Hieber, Geissert and Heck, [14], Farwig, Krbec and Neˇ casov´a [6].

In this paper we study strong solutions to (1.1) in weighted function spaces. More precisely we prove existence and uniqueness results of strong solutions to (1.1) in weighted Lebesgue and Sobolev spaces, i.e., we integrate with respect to the measure \( w \, dx \). The weight function \( w \) is taken from the class of Muckenhoupt weights, see Definition 2.1 below.

However, the rotating term forces us to deal with a more restricted class of weight functions. Namely, we consider weights \( w \) on \( \mathbb{R}^n \), \( n = 2, n = 3 \) that are independent of the angular variable \( \theta \) in a cylindrical coordinate system attached to the axis of revolution \((0,0,1)^T\). Moreover we assume that \( w \) satisfies the following condition depending on \( q \in (1, \infty) \):

\[
2 < q < \infty : \quad w^r \in A_{rq/2} \quad \text{for some } r \in [1, \infty),
q = 2 : \quad w^r \in A_r, \text{ or } w^{-r} \in A_r \quad \text{for some } r \in [1, \infty),
1 < q < 2 : \quad w^r \in A_{rq/2} \quad \text{for some } r \in \left( \frac{2}{q}, \frac{2}{2-q} \right).
\]

As shown in [7] examples for weights that fulfill (1.8) are given by

(i) Radially symmetric weights:
\[
w(x) = |x|^\alpha \quad \text{and} \quad w(x) = (1 + |x|)^\alpha,
\]
where

\[
-n < \alpha < n \frac{q}{2} \quad \text{if } \quad 2 \leq q < \infty
\]
\[
-\frac{nq}{2} < \alpha < n(q-1) \quad \text{if } \quad 1 < q < 2.
\]

(ii) Anisotropic, axially symmetric weights \( w(x) = (1 + |x|)^\alpha(1 + r)^\beta, r = \sqrt{x_1^2 + x_2^2} \) on \( \mathbb{R}^3 \) provided that

\[
-2 < \beta < q \quad \text{and} \quad -3 < \alpha + \beta < \frac{3q}{2} \quad \text{if } \quad 2 \leq q < \infty
\]
\[
-q < \beta < 2(q-1) \quad \text{and} \quad -\frac{3q}{2} < \alpha + \beta < 3(q-1) \quad \text{if } \quad 1 < q < 2.
\]

As a preparation we start to deal with the whole-space problem (1.6) where \( n = 2 \) or \( n = 3 \) with inhomogeneous data. However, in an exterior domain \( D \) we only consider the case of 3 space dimensions.

In the case of weak solutions that is considered in [16] the external force is contained in the homogeneous Sobolev space \( \hat{H}^{-1,q}(D) \). In this space the multiplication by a cut-off function which vanishes outside \( D \) is not continuous as an operator from \( \hat{H}^{-1,q}(D) \rightarrow \hat{H}^{-1,q}(\mathbb{R}^n) \). This is one reason why one has to restrict the range of \( q \) to permit Sobolev embeddings. Dealing with strong solutions the force term is contained in \( L^q_w(D) \), where this problem does not occur. Thus it is possible to obtain results for every \( q \in (1, \infty) \).
as shown in Theorem 4.1. The solutions are unique modulo a vector space of dimension 3 and there exists at least one solution which fulfills the a priori estimate (4.2). If appropriate embeddings hold, then by Remark 4.7 there exists exactly one solution that decays at infinity.

2 Preliminaries

2.1 Weighted Function Spaces

Definition 2.1. Let \( A_q, 1 < q < \infty \), the set of Muckenhoupt weights, be given by all \( w \in L^1_{\text{loc}}(\mathbb{R}^n) \) for which

\[
A_q(w) := \sup_Q \left( \frac{1}{|Q|} \int_Q w \, dx \right) \left( \frac{1}{|Q|} \int_Q w^{-\frac{1}{q-1}} \, dx \right)^{q-1} < \infty. \tag{2.1}
\]

The supremum is taken over all cubes \( Q \) in \( \mathbb{R}^n \). To avoid trivial cases, we exclude the case where \( w \) vanishes almost everywhere.

For \( q \in (1, \infty) \), \( w \in A_q \), \( k \in \mathbb{N}_0 \) and an open set \( \Omega \) we define

- the Lebesgue space \( L^q_w(\Omega) := \left\{ f \in L^1_{\text{loc}}(\Omega) \mid \|f\|_{q,w} := \left( \int_{\Omega} |f|^q w \, dx \right)^{\frac{1}{q}} < \infty \right\} \),
- the Sobolev space \( H^k,q_w(\Omega) := \left\{ f \in L^1_{\text{loc}}(\Omega) \mid \nabla^j f \in L^q_w(\Omega), j \leq k \right\} \), equipped with the norm \( \|u\|_{k,q,w} := \sum_{j=0}^k \|\nabla^j u\|_{q,w} \).
- the homogeneous Sobolev space \( \hat{H}^k,q_w(\Omega) := \left\{ f \in L^1_{\text{loc}}(\Omega) \mid \nabla^k f \in L^q_w(\Omega) \right\} \),
- the space of smooth and compactly supported functions \( C^\infty_w(\Omega) \) and its divergence free counterpart \( C^\infty_{w,0}(\Omega) := \{ \phi \in C^\infty_w(\Omega) \mid \text{div} \phi = 0 \} \),
- and the spaces \( \hat{H}^k,q_{w,0}(\Omega) := L^q_w(\Omega) \|n^k_w\| \) and \( H^k,q_{w,0}(\Omega) := \tilde{C}^\infty_w(\Omega) \|n^k_w\| \).

It is easily seen that

\[
(L^q_w(\Omega))' = L^{q'}_{w'}(\Omega) \quad \text{with} \quad \frac{1}{q} + \frac{1}{q'} = 1 \quad \text{and} \quad w' = w^{-\frac{1}{q-1}}. \tag{2.2}
\]

Moreover, by [24] for \( 1 < q < \infty \) and \( w \in A_q \) there exists \( 1 \leq s < q \) such that \( w \in A_s \). In addition, if \( \Omega \) is a bounded domain, then it follows from Hölder’s inequality that the weighed Lebesgue space embeds into an unweighed one as follows

\[
L^q_w(\Omega) \hookrightarrow L^r(\Omega) \quad \text{for every} \quad r < \frac{q}{s}. \tag{2.3}
\]

Considering the dual spaces in (2.3) one obtains that for \( q \) and \( w \) as above there exists \( r \in (1, \infty) \) such that \( L^{r}(\Omega) \hookrightarrow L^{q}_w(\Omega) \).

As shown by Fröhlich in [9] there exists the following weighted counterpart of the Sobolev Embedding Theorems.
Theorem 2.2. Let $\Omega$ be a bounded Lipschitz domain and let $1 < q < \infty$ and $w \in A_q$ for some $s \leq q$. Moreover, let $k \in \mathbb{N}$ and $r \in (1, \infty)$ with $\frac{1}{q} \geq \frac{1}{r} - \frac{k}{ns}$. Then there holds the embedding

$$H^{k,r}_w(\Omega) \hookrightarrow L^q_w(\Omega).$$

It has been shown by Fröhlich in [8] that as in the unweighted case the embeddings

$$H^{k+1}_w q_w(\Omega) \hookrightarrow H^k w(\Omega)$$

are compact in a bounded Lipschitz domain $\Omega$.

Lemma 2.3. (The Weighted Poincaré Inequality)

Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain $q \in (1, \infty)$ and $w \in A_q$.

1. There exists a constant $c = c(\Omega, q, w) > 0$ such that

$$\|u\|_{q,w} \leq c\|\nabla u\|_{q,w} \quad \text{for every } u \in \dot{H}^{1,q}_w(\Omega) \quad \text{with} \quad \int_{\Omega} u = 0$$

and for every $u \in \dot{H}^{1,q}_w(\Omega)$ with $u|_{\Gamma} = 0$, (2.5)

where $\Gamma$ is a connected component of $\partial \Omega$.

2. There exists a constant $c = c(\Omega, q, w) > 0$ such that

$$\|u\|_{2,q,w} \leq c\|\nabla^2 u\|_{q,w} \quad \text{for every } u \in \dot{H}^{2,q}_w(\Omega) \quad \text{with} \quad u|_{\Gamma} = 0,$n$$

Proof. Both assertions are proved in [8]. There the second assertion is only shown for $u|_{\partial \Omega} = 0$, however, the same proof applies to show the slightly more general assertion.

Theorem 2.4. (Hörmander-Michlin Multiplier Theorem with Weights)

Let $m \in C^n(\mathbb{R}^n \setminus \{0\})$ fulfill the property

$$|\partial^\alpha m(\xi)| \leq K|\xi|^{-|\alpha|}, \quad \text{for every } \xi \in \mathbb{R}^n \setminus \{0\}, \quad |\alpha| = 0, 1, \ldots, n,$$

for some constant $K > 0$. Then $T$ defined by

$$\hat{Tf} = \hat{m}\hat{f} \quad \text{for } f \in \mathcal{S}(\mathbb{R}^n, \mathbb{R})$$

extends to a continuous operator on $L^q_w(\Omega)$ for every $q \in (1, \infty)$ and $w \in A_q$.

Proof. This is an immediate consequence of [15], Theorem 3.9.

Lemma 2.5. Let $(T_\alpha)_{\alpha \in I}$ be a family of operators such that $T_{\alpha} : L^q_w(\mathbb{R}^n) \rightarrow L^q_w(\mathbb{R}^n)$ is continuous for every $q \geq 2$ and every $w \in A_{2^k}$ and such that $\|T_{\alpha}\|_{\mathcal{L}(L^q_w)}$ is uniformly bounded in $\alpha$. Moreover, we assume for the adjoint operators $T^*_\alpha$ that

$$T^*_\alpha v = T_{\alpha} v \quad \text{for every } v \in L^q_w(\mathbb{R}^n) \cap L^{q'}_{w'}(\mathbb{R}^n)$$

for every $\alpha \in I$.

Then $T_{\alpha} : L^p_w(\mathbb{R}^n) \rightarrow L^p_w(\mathbb{R}^n)$ is continuous for every $v$ fulfilling (1.8) and $\|T_{\alpha}\|_{\mathcal{L}(L^p_w)}$ is uniformly bounded in $\alpha$. 

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Proof. As in the proof of [7][Prop. 3.1] one uses dualization and (2.6) to show that $T_\alpha : L^p_v(\mathbb{R}^n) \rightarrow L^q_v(\mathbb{R}^n)$ is continuous for $p \in (1,2)$, if the weight function $v$ fulfills $v^{\frac{2}{r-1}} \in A_{\frac{q}{r-q}}$. The continuity constant of $T_\alpha$ is equal to $\| T_\alpha \|_{L^q_v(L^p_v)}$ which is uniformly bounded in $\alpha$.

Now one uses the same interpolation procedure as in the proof of [7][Theorem 1.2. (i)] to conclude the continuity of $T_\alpha$ for an arbitrary weight function $w$ fulfilling (1.8). Since the complex interpolation does not increase the continuity constants, the proof of the Lemma is finished.

Corollary 2.6. Let the weight function $w$ be chosen according to (1.8). Then $w \in A_q$.

Proof. Let $\phi$ be nonnegative, radial and radially decreasing with $\int \phi = 1$. Moreover, we set $\phi_\varepsilon(x) := \varepsilon^{-\frac{n}{r}} \phi(\frac{1}{\varepsilon} x)$ and define $T_\varepsilon f := f \ast \phi_\varepsilon$.

By [24, Prop. V.2.1] the operator $T_\varepsilon : L^p_v(\mathbb{R}^n) \rightarrow L^q_v(\mathbb{R}^n)$ is continuous and $(T_\varepsilon)_{\varepsilon > 0}$ is uniformly bounded in $\mathcal{L}(L^q_v(\mathbb{R}^n))$ if and only if $v \in A_r$, in particular, this is true for $r \geq 2$ and $v \in A_2 \subset A_r$.

Thus by Lemma 2.5 we obtain that $T_\varepsilon : L^p_w(\mathbb{R}^n) \rightarrow L^q_w(\mathbb{R}^n)$ is continuous for $q$ and $w$ according to (1.8). Thus again by [24, Prop. V.2.1] we obtain that $w \in A_q$. \hfill $\square$

2.2 The Stokes and the Divergence Equation in Weighted Function Spaces

The following Lemma is a weighted analogue to Bogowski’s theorem. It has been proved in [23].

Lemma 2.7. Let $\Omega \subseteq \mathbb{R}^n$ $(n \geq 2)$ be a bounded domain with Lipschitz boundary $\partial \Omega$ and let $k \in \mathbb{N} \cup \{0\}$, $1 < r < \infty$ and $w \in A_r$. Then for every $f \in H_{w,0}^{k,\delta}(\Omega)$ with $\int_\Omega f(x)dx = 0$, there exists some $u \in H_{w,0}^{k+1,\delta}(\Omega)$ such that

$$\text{div } u = f \text{ and } \| u \|_{H_{w,0}^{k+1,\delta}(\Omega)} \leq C \| f \|_{H_{w,0}^{k,\delta}(\Omega)},$$

where $c = c(\Omega, k, r, w) > 0$.

In the case of weighed Sobolev spaces the solvability of the Stokes equations in bounded domains has been shown by Fröhlich in [8]. More precisely one has the following theorem.

Theorem 2.8. Let $\Omega \subseteq \mathbb{R}^n$ $(n \geq 2)$ be a bounded domain with a boundary of class $C^{1,1}$, let $1 < r < \infty$ and $w \in A_r$. Then for every $f \in L_w^r(\Omega)$ and $g \in H_{w}^{1,r}(\Omega)$ with $\int_\Omega g = 0$ there is a unique $\{ u, p \} \in H_{w,0}^{2,r}(\Omega) \times H_{w}^{1,r}(\Omega)$ with $\int_\Omega p dx = 0$ such that

$$-\Delta u + \nabla p = f, \text{ div } u = g, \text{ in } \Omega, \text{ and } u|_{\partial \Omega} = 0$$

in the sense of distributions in $\Omega$. This solution $\{ u, p \}$ is subject to the inequality

$$\| \nabla^2 u \|_{L_w^p(\Omega)} + \| \nabla p \|_{L_w^p(\Omega)} \leq C \left( \| f \|_{L_w^p(\Omega)} + \| g \|_{H_{w,0}^{1,r}(\Omega)} \right),$$

with a constant $C = C(\Omega, r, w) > 0$. 6
3 The Inhomogeneous Case in $\mathbb{R}^3$

First, we are interested in the whole-space problem

$$
-\nu \Delta u - (\omega \wedge x) \cdot \nabla u + \omega \wedge u + \nabla p = f \quad \text{div } u = g
$$

in $\mathbb{R}^n$. (3.1)

For later use it is important that we can deal with inhomogeneous divergences $g$.

Theorem 3.1. Let $n = 2$ or $n = 3$.

(i) Given $f \in L^q_{w_1}(\mathbb{R}^n)$, and $g \in W^{1,1}_{loc}(\mathbb{R}^n)$ with $\nu \nabla g + (\omega \wedge x)g \in L^q_w(\mathbb{R}^n)$ there exists a solution $(u, p) \in L^1_{loc}(\mathbb{R}^n) \times L^1_{loc}(\mathbb{R}^n)$ of (3.1) that satisfies the estimate

$$
\|\nabla^2 u\|_{q,w} + \| (\omega \wedge x) \cdot \nabla u - \omega \wedge u\|_{q,w} + \| \nabla p\|_{q,w} 
\leq C(\|f\|_{q,w} + \|\nu \nabla g + (\omega \wedge x)g\|_{q,w}),
$$

where $C = C(q, w) > 0$.

(ii) Let $f \in L^q_{w_1}(\mathbb{R}^n) \cap L^q_{w_2}(\mathbb{R}^n)$ such that $(q_1, w_1)$ and $(q_2, w_2)$ satisfy the condition (1.8) and let $u_1, u_2 \in L^1_{loc}(\mathbb{R}^n)$ together with corresponding pressure functions $p_1, p_2 \in L^1_{loc}(\mathbb{R}^n)$ be solutions of (3.1) satisfying (3.2) for $(q_1, w_1)$, $(q_2, w_2)$, respectively. Then there are $\alpha, \beta, \gamma \in \mathbb{R}$ such that $u_1$ coincides with $u_2$ up to an affine linear field $\alpha \omega + \beta \omega \wedge x + \gamma (x_1, x_2, -2x_3)^T$ if $n = 3$ and $u_1 = u_2$ up to the field $\beta(-x_2, x_1)^T$ if $n = 2$.

Moreover, if $\nabla^2 u \in L^q_w(\mathbb{R}^n)$ satisfies (3.1) with $f = g = 0$ then $u$ is a polynomial of the above form. In particular, this implies that every solution $u \in L^1_{loc}(\mathbb{R}^n)$, $\nabla^2 u \in L^q_w(\mathbb{R}^n)$ to (3.1) satisfies the estimate (3.2).

Proof. We follow the approach of [5] and [7].

Due to the geometry of the problem it is reasonable to introduce (polar or) cylindrical coordinates $(r, x_3, \theta) \in (0, \infty) \times \mathbb{R} \times [0, 2\pi)$. Then it turns out that

$$(\omega \wedge x) \cdot \nabla u = \partial_\theta u,$$

using the angular derivative $\partial_\theta$ applied to $u(r, x_3, \theta)$.

As in the case of weak solutions in [7] we solve (3.1) explicitly by the use of Fourier transforms and multiplier operators. Working first in the space $\mathcal{S}(\mathbb{R}^n)$ of tempered distributions we apply the Fourier transform $\mathcal{F}$, denoted by $\hat{\cdot}$, to (3.1). With the Fourier variable $\xi$ and $s = |\xi|$ we get from (3.1)

$$
\nu s^2 \hat{u} - \partial_\theta \hat{u} + \omega \wedge \hat{u} + i\xi \hat{\phi} = \hat{f}, \quad i\xi \cdot \hat{u} = \hat{g}.
$$

(3.3)

As above $(\omega \wedge x) \cdot \nabla \xi = -\xi_2 \partial_1 + \xi_1 \partial_2 + \partial_\theta$ is the angular derivative in the Fourier space when using (polar or) cylindrical coordinates for $\xi \in \mathbb{R}^n$. A straight-forward calculation shows that the unknown pressure $p$ is given by the solution of

$$
\Delta p = \text{div } f + \nu \Delta g + \partial_\theta g = \text{div } G \quad \text{in } \mathbb{R}^n
$$
with $G = f + \nu \nabla g + (\omega \wedge x)g$. Then the Hörmander-Mikhlin multiplier theorem in weighted $L^q$-spaces (Theorem 2.4) yields for every weight $w \in A_q(\mathbb{R}^n)$ the estimate
\[
\|\nabla p\|_{q,w} \leq c \|G\|_{q,w} \leq c (\|f\|_{q,w} + \|\nu \nabla g + (\omega \wedge x)g\|_{q,w}),
\] (3.4)
where $c = c(q,w) > 0$; in particular $\nabla p \in L^q_w(\mathbb{R}^n)$. Hence $u$ may be considered as a solution of the reduced problem
\[
-\nu \Delta u - \partial \theta u + \omega \wedge u = F := f - \nabla p \quad \text{in } \mathbb{R}^n.
\] (3.5)

In the Fourier space $u$ is a solution of the ordinary differential equation
\[
-\partial \phi \hat{u} + \omega \wedge \hat{u} + \nu s^2 \hat{u} = \hat{F}
\]
with respect to $\phi$. As deduced in [5] this equation has a unique solution $\hat{u}(\phi)$ which is $2\pi$-periodic with respect to $\phi$ and it is represented by
\[
\hat{u}(\xi) = \int_0^\infty e^{-\nu s^2 t}O(t)T \hat{F}(O(t)\xi) \, dt.
\] (3.6)

It is also shown in [5] that the function $u$ defined by (3.6) solves the problem (3.1). Since $(2\pi)^{-n/2}e^{-|\xi|^2 t}$ is the Fourier transform of the heat kernel
\[
E_t(x) = \frac{1}{(4\pi t)^{n/2}} e^{-|x|^2/4t},
\]
we get the formal solution
\[
u \Delta u - \partial \theta u + \omega \wedge u = F := f - \nabla p \quad \text{in } \mathbb{R}^n.
\] (3.5)

In other words $u(x) = \int_{\mathbb{R}^n} \Gamma(x,y)F(y) \, dy$ with the fundamental solution
\[
\Gamma(x,y) = \int_0^\infty O(t)^T E_t(O(t)x - y) \, dt.
\]

To guarantee that the integrals in (3.7) converge in $\mathcal{S}'$ in the case of $\mathbb{R}^2$ it is necessary to modify the representation of the solution $u$. Indeed, we set
\[
\langle u, \phi \rangle = \langle \hat{u}, \check{\phi} \rangle = \int_{|\xi|\geq 1} \int_0^\infty e^{-\nu s^2 t}O(t)^T \hat{F}(O(t)\xi) \cdot \check{\phi}(\xi) \, dt \, d\xi
\]
\[
+ \int_{|\xi|< 1} \int_0^\infty e^{-\nu s^2 t} \hat{F}(O(t)\xi) \cdot (\check{\phi}(\xi) - \check{\phi}(0)) \, dt \, d\xi
\]
for all $\phi \in \mathcal{S}(\mathbb{R}^2)$. Here $\check{\phi}$ denotes the inverse Fourier transform. This problem is similar to case of the Poisson equations in 2 dimensions.

To prove Theorem 3.1 note that by virtue of the formula
\[
\partial_j \partial_k u(\xi) = -\xi_j \xi_k \hat{u} = \frac{\xi_j \xi_k}{|\xi|^2} \left(\Delta u(\xi)\right), \quad 1 \leq j, \ k \leq n,
\]
and Theorem 2.4 it suffices to find an estimate of $\|\Delta u\|_{q,w}$, where

$$-\Delta u(x) = \int_{\mathbb{R}^n} K(x, y) F(y) \, dy, \quad K(x, y) = -\Delta_x \Gamma(x, y).$$

(3.8)

To obtain the required estimates we note that by [7, Proposition 3.1] and Corollary 2.6 the operator

$$T : L^q_w(\mathbb{R}^n) \to L^q_w(\mathbb{R}^n), \quad F \mapsto \int_{\mathbb{R}^n} K(x, y) F(y) \, dy$$

is continuous for weight functions that fulfill (1.8). Combining this with (3.4) and (3.5) implies

$$\|\nabla^2 u\|_{q,w} \leq c \|\Delta u\|_{q,w} \leq c \|F\|_{q,w} \leq c (\|f\|_{q,w} + \|\nu \nabla g + (\omega \wedge x) g\|_{q,w})$$

and the proof of (i) is finished.

For the proof of (ii) we consider $u - u_k$, $k = 1, 2$, where $u$ is the solution constructed in (i). Then $u$ fulfills the estimate (3.2) for both, $(q_1, w_1)$ and $(q_2, w_2)$. Moreover, $u - u_k$ is a solution with respect to homogeneous data. Thus the uniqueness result follows from the corresponding fact in the case of a homogeneous divergence shown in [7, Theorem 1.2 (ii)] in the proof of which it is shown that any strong solution to (3.1) with respect to zero data is equal to a polynomial of the asserted form.

4 The Exterior Domain

We now turn to the exterior domain, where we consider the problem (1.1). For the convenience of notation we define the operator

$$L := -\Delta - (\omega \wedge x) \cdot \nabla + \omega \wedge,$$

where for simplicity we have put the viscosity coefficient $\nu$ equal to 1. Moreover, throughout this section we assume that $1 < q < \infty$ and that $w$ is a weight function that fulfills (1.8). Note that by Corollary 2.6 every such weight function is contained in the Muckenhoupt class $A_q$. Thus the results from Section 2 hold for this $w$.

Moreover we consider the null space of the problem which is given by

$$K = \left\{ u \in \hat{H}^{2,q}_w(D) \mid \text{div} u = 0, \ u|_{\partial D} = 0, \ Lu + \nabla p = 0, \text{for some } p \in \hat{H}^{1,q}_w(D) \right\}.$$

(4.1)

The following theorem sumarizes our results in the exterior domain.

**Theorem 4.1.** For every $f \in L^q_w(D)$, there exists a strong solution $\{u, p\} \in L^1_{loc}(D) \times L^1_{loc}(D)$ to (1.1) satisfying the estimate

$$\|\nabla^2 u\|_{q,w} + \|((\omega \wedge x) \cdot \nabla) u - \omega \wedge u\|_{q,w} + \|\nabla p\|_{q,w} \leq c \|f\|_{q,w},$$

(4.2)

with some $c > 0$. The solution $u$ is unique in $\hat{H}^{2,q}_w(D)/K$ and $\dim K = 3$. 

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Proof. The idea of this proof is to split up this problem into an unbounded part that is treated by the use of the results from the whole space problem in Theorem 3.1 and into a part on a bounded domain and that can be estimated using the estimates of solutions to the Stokes equations in bounded domains in Theorem 2.8. The assertion about the dimension of $K$ is proved in Proposition 4.3 and Proposition 4.5, the proof of existence is given in Lemma 4.4. See Proposition 4.6 for the proof of the a priori estimate.

Lemma 4.2. Given $f \in L^q_w(D)$, let $\{u, p\} \in H^2_w(D) \times H^1_w(D)$ be a strong solution to the problem (1.1). Fix $\rho > \rho_0 > 0$ so large that $\mathbb{R}^3 \setminus D \subset B_{\rho_0}$, take $\psi \in C_0^\infty(B_\rho; [0, 1])$ such that $\psi = 1$ on $B_{\rho_0}$. Then

$$\|\nabla^2 u\|_{q,w,D} + \|\nabla p\|_{q,w,D} + \|\omega \cdot \nabla u - \omega \wedge u\|_{q,w,D} \leq C\left(\|f\|_{q,w,D} + \|u\|_{1,q,w,D_0} + \|p\|_{q,w,D_0}\right),$$

where by $B_{\rho}$ we denote the ball centered at 0 with radius $\rho$ and $D_{\rho} = B_\rho \cap D$.

Proof. By the use of the cut-off function $\psi$, we decompose the solution $\{u, p\}$ as follows:

$$u = U + V, \quad U = (1 - \psi)u, \quad V = \psi u, \quad p = \sigma + \tau, \quad \sigma = (1 - \psi)p, \quad \tau = \psi p.$$

Then $\{U, \sigma\}$ is a strong solution of

$$LU + \nabla \sigma = Z_1$$
$$\nabla \cdot U = -u \cdot \nabla \psi =: g_1$$

in $\mathbb{R}^3$, where

$$Z_1 = (1 - \psi)f + 2\nabla \psi \cdot \nabla u + [\Delta \psi + (\omega \wedge x) \cdot \nabla \psi]u - (\nabla \psi)p.$$

Similarly, $\{V, \tau\}$ is a strong solution to the Stokes problem

$$-\Delta V + \nabla \tau = Z_2 \quad \text{in} \quad D_{\rho}$$
$$\nabla \cdot V = u \cdot \nabla \psi =: g_2 \quad \text{in} \quad D_{\rho}$$
$$V\big|_{\partial D_{\rho}} = 0,$$

where

$$Z_2 = \psi[f + (\omega \wedge x) \cdot \nabla u - \omega \wedge u] - 2\nabla \psi \cdot \nabla u - (\Delta \psi)u + (\nabla \psi)p.$$

From Theorem 3.1 it follows that

$$\|\nabla^2 U\|_{q,w,\mathbb{R}^n} + \|\omega \cdot \nabla U - \omega \wedge U\|_{q,w,\mathbb{R}^n} + \|\nabla \sigma\|_{q,w,\mathbb{R}^n} \leq C(\|Z_1\|_{q,w,\mathbb{R}^n} + \|\nabla g_1 + (\omega \wedge x)g_1\|_{q,w,\mathbb{R}^n})$$

and from Theorem 2.8 one obtains

$$\|\nabla^2 V\|_{q,w,D_{\rho}} + \|\nabla \tau\|_{q,w,D_{\rho}} \leq C(\|Z_2\|_{q,w,D_{\rho}} + \|g_2\|_{1,q,w,D_{\rho}}).$$

Now it remains to estimate $Z_1$ and $Z_2$.

$$\|Z_1\|_{q,w,\mathbb{R}^n} \leq \|(1 - \psi)f\|_{q,w,\mathbb{R}^n} + 2\|\nabla \psi \cdot \nabla u\|_{q,w,\mathbb{R}^n} + \|\Delta \psi + (\omega \wedge x) \cdot \nabla \psi\|_{q,w,\mathbb{R}^n} + \|\nabla \psi\|_{q,w,\mathbb{R}^n}\|p\|_{q,w,\mathbb{R}^n} \leq C(\|f\|_{q,w,D} + \|\nabla u\|_{q,w,D_{\rho}} + \|u\|_{q,w,D_{\rho}} + \|p\|_{q,w,D_{\rho}}).$$

$$\leq C(\|f\|_{q,w,D} + \|\nabla u\|_{q,w,D_{\rho}} + \|p\|_{q,w,D_{\rho}}),$$
using the weighted Poincaré inequality (2.5) together with $u|_{\partial D} = 0$. Moreover,

$$
\|Z_2\|_{q,w,D_\rho} \leq c(\|f\|_{q,w,D_\rho} + \|(\omega \wedge x) \nabla u - \omega \wedge u\|_{q,w,D_\rho} \\
+ \|\nabla u\|_{q,w,D_\rho} + \|u\|_{q,w,D_\rho} + \|p\|_{q,w,D_\rho})
$$

(4.10)

and

$$
\|g_2\|_{q,w,D_\rho} + \|\nabla g_2\|_{q,w,D_\rho} \leq c\|\nabla g_2\|_{q,w,D_\rho} \leq c\|\nabla u\|_{q,w,D_\rho}
$$

(4.11)

Then from (4.4)–(4.11) it follows

$$
\|\nabla^2 u\|_{q,w} + \|\nabla p\|_{q,w} + \|(\omega \wedge x) \cdot \nabla u - \omega \wedge u\|_{q,w} \leq C\left(\|f\|_{q,w} + \|\nabla u\|_{q,w,D_\rho} + \|p\|_{q,w,D_\rho}\right)
$$

Proposition 4.3. (Uniqueness)

Suppose that

$$
\{u, p\} \in \hat{H}_{w}^{2,q}(D) \times \hat{H}_{w}^{1,q}(D)
$$

is a strong solution to the problem (1.1) with respect to $f = 0$. Then there exists a polynomial of the form

$$
\pi = \alpha \omega + \beta \omega \wedge x + \gamma (x_1, x_2, -2x_3)^T
$$

such that $u - \pi \xrightarrow{|x| \to \infty} 0$ and $u$ is uniquely determined by this polynomial. In particular, $u = 0$, if $\pi = 0$.

Accordingly, the gradient of the pressure function $\nabla p$ is uniquely determined by $\pi$.

Proof. Step 1. As in the proof of Lemma 4.2 one obtains that $\{U, \sigma\}$ with $U := (1 - \psi)u$ and $\sigma = (1 - \psi)p$ is a strong solution to the whole-space problem (3.1) with $f$ replaced by

$$
Z_1 := 2\nabla \psi \cdot \nabla u + [\Delta \psi + (\omega \wedge x) \cdot \nabla \psi]u - (\nabla \psi)p.
$$

Theorem 2.2 implies that $Z_1 \in H_{w}^{1,q}(D_\rho) \hookrightarrow L_{w}^{r}(D_\rho)$ where $\frac{1}{r} \geq \frac{1}{q} - \frac{1}{3q}$. Then by Theorem 3.1 (ii) we obtain $\nabla^2 U \in L_{w}^{r}(D)$. Note that the solutions in the whole space case are only unique up to a polynomial, however, if this polynomial is multiplied by a cut-off function, one still obtains a smooth function with compact support which is contained in $L_{w}^{r}(D)$.

Now we apply the same procedure to the part with bounded support $V := \psi u$ and $\tau = \psi p$. From the uniqueness of solutions to the Stokes equations we thus obtain $\nabla^2 V \in L_{w}^{r}(D)$ and combined with the above we obtain for the solution $u$ that $\nabla^2 u \in L_{w}^{r}(D)$.

A finite number of repetitions of this procedure gives us $Z_1, Z_2 \in L_{w}^{r}(D_\rho)$ for every $r > 1$. In particular by (2.3) we may choose $r$ so large that

$$
L_{w}^{r}(D_\rho) \hookrightarrow L_{w}^{6}(D_\rho)
$$
Then again the above procedure yields $\nabla^2 u \in L^6(D)$. Analogously one obtains $\nabla p \in L^6(D)$.

**Step 2.** Now the weighted version of Bogowski’s Theorem 2.7 in $D_\rho$ yields that there exists $v \in H^{1,2}_0(D_\rho)$ with $\text{div}\, v = \text{div}\, \psi u$. We set $U_1 := (1 - \psi)u + v$.

Then the Sobolev Embedding Theorem yields

$$LU_1 + \nabla (1 - \psi)p \in L^6(D) \hookrightarrow \mathring{H}^{-1,2}(D)$$

**Step 3.** By [16] there exists a weak solution $\{\tilde{U}_1, \tilde{p}\} \in \mathring{H}^{1,2}(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ to the problem

$$LU_1 + \nabla \tilde{p} = LU_1 + \nabla (1 - \psi)p, \quad \text{div}\, \tilde{U}_1 = 0 \text{ in } \mathbb{R}^n \text{ and } \lim_{|x| \to \infty} \tilde{U}_1(x) = 0 \quad (4.12)$$

with an appropriate pressure function $\tilde{p}$. This solution $\tilde{U}_1$ is contained in $\mathring{H}^{1,2}_0(\mathbb{R}^n) \hookrightarrow L^6(\mathbb{R}^n)$ and is given by the formula (3.6). Since the same formula yields a strong solution to the problem (3.1), we obtain $\nabla^2 \tilde{U}_1 \in L^6(D)$.

By the uniqueness of solutions to the whole space problem in [7] or in Theorem 3.1, one obtains

$$U_1 - \tilde{U}_1 = \pi = \alpha \omega + \beta \omega \land x + \gamma(x_1, x_2, -2x_3)^T$$

with appropriate coefficients $\alpha, \beta, \gamma$.

**Step 4.** Now let $g \in H^{1,2}(D)$ be chosen such that

$$g|_{\partial D} = \pi|_{\partial D}, \quad \text{div}\, g = 0 \quad \text{and} \quad \text{supp}\, g \subset D_\rho. \quad (4.13)$$

We show that such a function $g$ exists. Note that

$$\int_D \text{div}\,(\psi \pi) \cdot 1 \, dx = \int_{\partial D_\rho} \psi \pi \cdot N \, dS = \int_{\partial (\mathbb{R}^n \setminus D)} \pi(-N) \, dS = -\int_{\mathbb{R}^n \setminus D} \text{div}\, \pi \, dx = 0.$$ 

Now Bogowski’s Theorem 2.7 gives us a function $\zeta \in W^{1,2}_0(D_\rho)$ with $\text{div}\, \zeta = \text{div}\, (\psi \pi)$. Then $g := \psi \pi - \zeta$ fulfills (4.13).

Then one has for every $\phi \in \mathring{H}^{1,2}_0(D)$

$$|\langle Lg, \phi \rangle_D| \leq |\langle \nabla g, \nabla \phi \rangle| + |\langle \omega \land g - (\omega \land x) \cdot \nabla g, \phi \rangle| \\
\leq \|\nabla g\|_{2,D} \|\nabla \phi\|_{2,D} + \|\omega \land g - (\omega \land x) \cdot \nabla g\|_{2,D_\rho} \|\phi\|_{2,D_\rho} \\
\leq c\|g\|_{H^{1,2}(D)} \|\nabla \phi\|_{2,D}.$$ 

using the Poincaré inequality in $D_\rho$ and the fact that $\phi|_{\partial D} = 0$. We have shown that $Lg \in \mathring{H}^{-1,2}(D)$. Thus by [16] there exists $G \in \mathring{H}^{1,2}_0(D)$ solving the problem

$$LG + \nabla p^* = Lg \text{ in } D, \quad \text{div}\, G = 0 \quad (4.14)$$

for an appropriate pressure function $p^*$.

**Step 5.** Now we set

$$V := \tilde{U}_1 - v + g - G + \psi u = u + g - G - \pi.$$ 

Then one has $V \in \mathring{H}^{1,2}_0(D)$ and it solves the problem

$$LV + \nabla p = Lu + \nabla p = 0, \quad \text{div}\, V = 0,$$
where the pressure function $p$ is composed by the pressures in (4.12) and (4.14). Thus by the uniqueness of weak solutions in [16] we obtain that $V = 0$ which in turn yields

$$u = \pi - g + G.$$ 

Since the functions $g$ and $G$ depend linearly on $\pi$, the function $u$ is taken from some space with dimension 3. The behavior of $u$, i.e., the growth of $\pi$ at infinity determines the particular element.

For the pressure $p$ that belongs to $u$ one has $\nabla p = -Lu = Lg - LG = \nabla p^*$ with $\nabla p^*$ defined as in (4.14) depending linearly on $\pi$.

\[\text{Lemma 4.4. Let } f \in L_w^q(D) \text{ then (1.1) has a strong solution in the class } u \in \hat{H}_w^{2,q}(D). \text{ The corresponding pressure is contained in } p \in \hat{H}_w^{1,q}(D).\]

\textbf{Proof.} Let $f_0$ be the extension of $f$ by 0 on $\mathbb{R}^n \setminus D$ and let $(u_1, p_1) \in \hat{H}_w^{2,q}(\mathbb{R}^n) \times \hat{H}_w^{1,q}(\mathbb{R}^n)$ be a solution of

$$Lu_1 + \nabla p_1 = f_0, \quad \text{and } \text{div } u_1 = 0 \quad \text{in } \mathbb{R}^n$$

which exists by Theorem 3.1. Moreover, as in the proof of Proposition 4.3 one shows that there exists a function $g \in H_w^{2,q}(D_\rho)$ with compact support in $D_\rho \cup \partial D$ and $g|_{\partial D} = u_1|_{\partial D}$ and $\text{div } g = 0$.

Then by (2.3) there exists $s > 1$ such that $g$ is contained in the unweighted space $H_0^{2,s}(D)$. We show that $Lg \in \hat{H}_0^{1,r}(D)$ for some $r \in (\frac{3}{2}, 3)$.

\[|\langle Lg, \psi \rangle| \leq \|Lg\|_{L^r(D_\rho)} \|\psi\|_{L^{r'}(D_\rho)} \leq c\|Lg\|_{L^r(D_\rho)} \|\nabla \psi\|_{L^{r'}(D_\rho)},\]

for every $\psi \in \hat{H}_0^{1,r'}(D)$, where $r$ is chosen such that $W^{1,r'}(D_\rho) \hookrightarrow L^{r'}(D_\rho)$. This requires $\frac{1}{r'} \leq \frac{1}{r} + \frac{1}{3}$ and thus $r \in (\frac{3}{2}, 3)$ is possible.

By [16, Theorem 2.2] there exists a weak solution $G \in \hat{H}_0^{1,r}(D)$ to the exterior problem

$$LG + \nabla p_2 = Lg, \quad G|_{\partial D} = 0, \quad \text{div } G = 0.$$ 

Then $u := u_1 - g + G$, $p = p_1 + p_2$ solves the problem

$$Lu + \nabla p = f, \quad u|_{\partial D} = 0, \quad \text{div } u = 0$$

in the sense of distributions in $D$. It remains to show that $\nabla^2 u \in L_w^q(D)$ and $(\omega \wedge x) \cdot \nabla u - \omega \wedge u \in L_w^q(D)$. This follows from a similar cut-off procedure as in the proof of Lemma 4.2 as follows.

Set $r_1 := r$. By construction the function $u$ is contained in $\hat{H}_w^{2,q}(D) + \hat{H}_w^{1,r_1}(D)$. Using the same notation as in the proof of Lemma 4.2 we obtain

$$Z_1 \in L_w^q(\mathbb{R}^n) + L^{r_1}(D_\rho), \quad \text{and } Z_2 \in L_w^q(D_\rho) + L^{r_1}(D_\rho) \quad \text{for } i = 1, 2,$$

i.e., we may write $Z_1 = Z_1^0 + Z_1^1$, $Z_2 = Z_2^0 + Z_2^1$, with $Z_1^0 \in L_w^q(\mathbb{R}^n)$, $Z_1^1 \in L^{r_1}(\mathbb{R}^n)$ with $\text{supp } Z_1^1 \subset D_\rho$, $Z_2^0 \in L_w^q(D_\rho)$ and $Z_2^1 \in L^{r_1}(D_\rho)$. Let $u_1^0 \in H_w^{2,q}(\mathbb{R}^n)$ and $u_1^1 \in H_0^{2,r_1}(\mathbb{R}^n)$ be the solutions to

$$ Lu_1^0 + \nabla p = Z_1^0 \quad \text{and} \quad Lu_1^1 + \nabla p = Z_1^1 \quad \text{in } D_\rho,$$
with vanishing divergence. Then \((1 - \psi)u = u^\circ_1 + u^\prime_1\) up to a polynomial of degree \(\leq 1\). Analogously let \(v^\circ_1 \in H^{2,q}_w(D, \rho_0)\) and \(v^\prime_1 \in H^{2,r_1}(D, \rho_0)\) be the solutions to the Stokes equations

\[-\Delta v^\circ_1 + \nabla p = Z^\circ_2 \quad \text{and} \quad -\Delta v^\prime_1 + \nabla p = Z^\prime_2\]

with vanishing divergence and boundary values. Choose a cut-off function \(\tilde{\psi} \in C^\infty_0(B_{\rho_0})\) with \(\tilde{\psi} \equiv 1\) on \(\text{supp} \, \psi\). Then one has by the uniqueness of solutions to the Stokes equation \(\psi u = \tilde{\psi} \psi u = \tilde{\psi} v^\circ_1 + \tilde{\psi} v^\prime_1 \in \tilde{H}^{2,q}_w(D) + \tilde{H}^{2,r_1}(D)\). Combining the above yields

\[u \in \tilde{H}^{2,q}_w(D) + \tilde{H}^{2,r_1}(D)\]

This in turn shows that \(Z_i \in L^q_\omega(D) + H^{1,r_1}(D) \hookrightarrow L^q_\rho(D) + L^2(D)\) with \(\frac{1}{r_2} \leq \frac{1}{r_1} + \frac{1}{n}\) and again \(u \in \tilde{H}^{2,q}_w(D) + \tilde{H}^{2,r_2}(D)\). If one repeats this procedure a finite number of times one obtains \(u \in \tilde{H}^{2,q}_w(D) + \tilde{H}^{2,r_k}(D)\) with \(r_k\) arbitrarily large. In particular, it can be chosen so large that \(L^r_k(D) \hookrightarrow L^q_\omega(\Omega)\). Then one has \(u \in \tilde{H}^{2,q}_w(D)\) as asserted. \(\square\)

**Proposition 4.5.** We consider the null space \(K\) defined in (4.1). Then \(\dim K = 3\).

**Proof.** In Lemma 4.3 we have already shown that \(\dim K \leq 3\).

Let \(\pi = \alpha \omega + \beta \omega \wedge x + \gamma(x_1, x_2, -2x_3)^T\). Then as in the proof of Lemma 4.4 we find that there exists \(g \in \tilde{H}^{2,q}_w(D)\) such that

\[g|_{\partial D} = \pi|_{\partial D}, \quad \text{div} \, g = 0, \quad \text{supp} \, g \text{ compact in } D \quad \text{and} \quad Lg \in H^{-1,2}(D).\]

Then by [16] there exists \(v \in H^{1,2}_0(D)\) solving \(Lv + \nabla p = Lg\) for some \(p\), \(\text{div} \, u = 0\). Thus if we set \(u := v - g + \pi\) we find as in the proof of Lemma 4.4 that actually \(\nabla^2 u \in L^q_w(D)\) and thus

\[u \in K \quad \text{with} \quad u(x) - \pi(x) \xrightarrow{|x| \to \infty} 0.\]

Thus for every \(\alpha, \beta, \gamma\) there exists \(u \in K\) with \(u(x) - \pi(x) \xrightarrow{|x| \to \infty} 0\), which means \(\dim K \geq 3\). \(\square\)

Similarly to the results in the case of the classical Stokes equations in [10, V.4.] the solutions to (1.1) can be estimated in the factor space modulo the null-space \(K\).

**Proposition 4.6.** Suppose \(\{u, p\} \in H^{2,q}_w(D) \times H^{1,q}_w(D)\) is a strong solution to (1.1) with \(f \in L^q_w(D)\). Then the estimate

\[
\inf_{L^v + \nabla p^* = 0} \left( \|u + v\|_{q, w} + \|\omega \wedge x \cdot \nabla (u + v) - \omega \wedge (u + v))\|_{q, w} + \|
\nabla p + \nabla p^*\|_{q, w} \right)
\leq c\|f\|_{q, w}
\]

holds, where the infimum is taken over all \(v \in K\). In particular, since the null space \(K\) has dimension 3, at least one representant of \(u + K\) fulfills the estimate (4.2).

**Proof.** We assume the contrary. Then there exist sequences \((f_k) \subset L^q_w(D)\) and corresponding strong solutions \((u_k, p_k)_k \subset \tilde{H}^{2,q}_w(D) \times \tilde{H}^{1,q}_w(D),\) such that

\[
\inf_{L^v + \nabla p^* = 0} \left( \|\nabla^2 (u_k + v)\|_{q, w, D} + \|\omega \wedge x \nabla (u_k + v) - \omega \wedge (u_k + v))\|_{q, w, D} + \|\nabla p_k + \nabla p^*\|_{q, w, D} \right) = 1
\]  \hspace{1cm} (4.15)

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and \( \|f_k\|_{q,w,D} \to 0 \) as \( k \to \infty \). Since \( K \) has finite dimension, we may assume that
\[
\|\nabla^2 u_k\|_{q,w,D} + \|(\omega \wedge x)\nabla u_k - \omega \wedge u_k\|_{q,w,D} + \|\nabla p_k\|_{q,w,D} = \inf_{L^p + \nabla^q = 0} \left( \|\nabla^2(u_k + v)\|_{q,w} + \|(\omega \wedge x)\nabla (u_k + v) - \omega \wedge (u_k + v)\|_{q,w,D} + \|\nabla p_k + \nabla p^*\|_{q,w,D} \right).
\]
As in Proposition 4.2 we put \( D_\rho = D \cap B_\rho \) and obtain
\[
\|\nabla u_k\|_{1,q,D_\rho,w} \leq \|u_k\|_{2,q,w,D_\rho} \leq c\|\nabla^2 u\|_{q,w,D_\rho} \leq c\|\nabla^2 u\|_{q,w,D} \leq 1
\]
where we have used \( u_k|_{\partial D} = 0 \) and the weighted version of Poincaré’s inequality for second derivatives in Lemma 2.3. Similarly, if we subtract an appropriate constant from each \( p_k \) we may assume that
\[
\|p_k\|_{1,q,w,D} \leq \|\nabla p_k\|_{q,w,D} \leq 1.
\]
We choose subsequences, which we again denote by \( (u_k), (p_k) \) , such that:
\[
\begin{align*}
u_k &\to u \quad \text{in} \quad H^2_w(D_\rho), \\
p_k &\to p \quad \text{in} \quad H^1_w(D_\rho).
\end{align*}
\]
Now we apply that by (2.4) the embeddings \( H^2_w(D_\rho) \hookrightarrow H^1_w(D_\rho) \) and \( H^1_w(D_\rho) \hookrightarrow L^q_w(D_\rho) \) are compact and obtain the strong convergence of \( (u_k) \) in \( H^1_w(D_\rho) \) and \( p_k \) in \( L^q_w(D_\rho) \). From (4.3) it follows that \( (u_k, p_k)_k \) and \( ((\omega \wedge x) \cdot \nabla u_k - \omega \wedge u_k)_k \) are Cauchy sequences in \( \tilde{H}^2_w(D) \times \tilde{H}^1_w(D) \) and \( L^q_w(D) \). This implies that there exists \( \{u, p\} \subseteq \tilde{H}^2_w(D) \times \tilde{H}^1_w(D) \) such that
\[
\begin{align}
\|\nabla^2 u_k - \nabla^2 u\|_{q,w} + \|\nabla p_k - \nabla p\|_{q,w} &\xrightarrow{k \to \infty} 0, \quad (4.16) \\
\|[(\omega \wedge x) \cdot \nabla u_k - \omega \wedge u] - [(\omega \wedge x) \cdot \nabla u - \omega \wedge u]\|_{q,w} &\xrightarrow{k \to \infty} 0. \quad (4.17)
\end{align}
\]
It follows from (4.16), (4.17) that the pair \( \{u, p\} \) is a strong solution to (1.1) with \( f = 0 \). This contradicts our assumption that
\[
\inf_{L^p + \nabla^q = 0} \left( \|\nabla^2(u + v)\|_{q,w} + \|(\omega \wedge x) \cdot \nabla(u + v) - \omega \wedge (u + v)\|_{q,w} + \|\nabla p + \nabla p^*\|_{q,w} \right) = 1.
\]
Thus (4.2) valid.

\[\square\]

**Remark 4.7.** Assume that there exists \( r > 1 \) such that
\[
\|\phi\|_{r,D} \leq \|\nabla^2 \phi\|_{q,w,D}
\]
for every \( \phi \in C_0^\infty(D) \). This holds in particular in the case \( w = 1 \) and \( q \in (1, \frac{3}{2}) \). Let \( f \) be given as in Lemma 4.4. Then there exists a strong solution \( u \in \tilde{H}^2_w(D) \) which fulfills \( \lim_{|x| \to \infty} u(x) = 0 \).

To see this approximate \( f \) by a sequence \( (f_n) \subseteq C_0^\infty(D) \). Then as before one uses the weighted Poincaré inequality in \( B_n \cap D \), supp \( f_n \subseteq B_n \), to show that \( f_n \in H^{-1,2}(D) \). Thus by [16] there exist corresponding weak solutions \( (u_n) \) to (1.1). In particular \( u_n \in \)

[15]
As in the proof of Lemma 4.4 one shows that these solutions also satisfy \( \nabla^2 u_n \in L^q_w(D) \). Thus by the a priori estimate that we have shown in Proposition 4.6 we find that there exists some \( u \in \hat{H}^{2,q}_w(D) \) such that \( u_n \xrightarrow{n \to \infty} u \) in \( \hat{H}^{2,q}_w(D)/K \). Since \( K \) is finite dimensional, there exists \( v_n \) with \( \| \nabla^2 v_n \|_{q,w} = \| u_n \|_{\hat{H}^{2,q}_w}/K \). Thus the sequence of solutions \( v_n \) converges in the Banach space

\[
(\hat{H}^{2,q}_w(D) \cap L^r(D)) + K
\]

and we obtain that there exists \( \eta \in K \) such that \( u - \eta \in L^r(D) \). Then \( u - \eta \) is the solution which decays at infinity.

References


