Boundary value problems for the Stokes equations with jumps in open sets of $\mathbb{R}^n$

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Abstract: A boundary value problem for the Stokes system is studied in a cracked domain in $\mathbb{R}^n$, $n > 2$, where the Dirichlet condition is specified on the boundary of the domain. The jump of the velocity and the jump of the stress tensor in the normal direction are prescribed on the crack. We construct a solution of this problem in the form of appropriate potentials and determine unknown source densities via integral equation systems on the boundary of the domain. The solution is given explicitly in the form of a series. As a consequence, a maximum modulus estimate for the Stokes system is proved.

Keywords: Stokes system; layer potential; integral equation; crack

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1 Introduction

In the present paper we construct a solution of the Stokes equations

\[-\Delta u + \nabla p = 0 \quad \text{in} \quad \Omega, \quad \nabla \cdot u = 0 \quad \text{in} \quad \Omega, \quad u = b \quad \text{on} \quad \partial \Omega \setminus S, \quad (1)\]

\[u^+ - u^- = f, \quad (T^+_{qN}V^+)^+ - (T^-_{qN}V^-)^- = h \quad \text{on} \quad S \cap G \quad (2)\]

using methods of hydrodynamical potential theory. Here \( \Omega = G \setminus S \), where \( G \subset \mathbb{R}^n \) \((n \geq 3)\) is a domain with bounded boundary \( \partial G \) which is empty (i.e. \( G = \mathbb{R}^n \)) or of class \( C^{1,\alpha} \), \( \alpha > 0 \). The domain \( G \) may be bounded or unbounded, and \( \partial G \) need not to be connected. The crack \( S \) is a closed subset (empty or nonempty) of a surface of class \( C^{2+\alpha} \), and might reach the boundary. The Dirichlet condition \( u = b \) is prescribed on the boundary of \( G \). The jump of the velocity and the jump of the stress tensor in the normal direction are prescribed on the crack \( S \). We look for a classical solution of this problem (for a non-cracked domain see \([15]\)). We also study the class of \( L^q \)-solutions for \( 2 \leq q < \infty \), where we assume \( b \in L^q(\partial G) \), \( f \in W^{1,q}(S) \), \( h \in L^q(S) \). In this case, the boundary conditions are satisfied in the sense of a non-tangential limit and a solution satisfies some regularity condition characterized by the fact that the non-tangential maximal function of \( u \) belongs to \( L^q(\partial \Omega) \) (see below). These classes of strong solutions have been studied for boundary value problems of the Laplace equation for decades (see \([16], [4]\)). Recently these classes of solutions have been studied also for boundary value problems of the Stokes system. Fabes, Kenig, Verchota studied \( L^2 \)-solutions for the Dirichlet problem of the Stokes system in case of a non-cracked domain with connected boundary (see \([3]\)). They developed necessary and sufficient conditions for the existence of a solution and proved a uniqueness theorem. Shen studied \( L^q \)-solutions \((2 \leq q < \infty)\) for the Dirichlet problem in case of a non-cracked bounded domain in \( \mathbb{R}^3 \) with connected boundary (see \([13]\)). Here also necessary and sufficient conditions for the existence of a solution are given. The same problem has also been studied by Deuring (see \([1]\)).

In the present paper we develop necessary and sufficient conditions for the existence of the solution. We prove a uniqueness theorem, a maximum modulus estimate, and construct the solution. The construction we use is based on an explicit representation of the fundamental tensor for \((1)\). It includes a detailed study of the corresponding boundary layer potentials and ends up with a boundary integral equation method reducing \((1)\) to a system
of uniquely solvable second kind Fredholm boundary integral equations. We construct a solution of the integral equations for a non-cracked domain in the form of a Neumann series, where we extend the method used in [8] for the Dirichlet problem of the Laplace equation. Then we apply this result for the construction of a solution for the boundary value problem of the Stokes system in case of a cracked domain. This investigation is motivated by a similar problem for the Laplacian solved in [5].

As in classical potential theory we need the Green formulas to start with. To formulate these identities we define the formally adjoint differential operators $S, S'$ by

$$S: \begin{pmatrix} u \\ p \end{pmatrix} \rightarrow S \begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} -\Delta u + \nabla p \\ \nabla \cdot u \end{pmatrix},$$

$$S': \begin{pmatrix} u \\ p \end{pmatrix} \rightarrow S' \begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} -\Delta u - \nabla p \\ -\nabla \cdot u \end{pmatrix}.\tag{3}$$

The corresponding formally adjoint stress tensors are denoted by

$$T: \begin{pmatrix} u \\ p \end{pmatrix} \rightarrow T \begin{pmatrix} u \\ p \end{pmatrix} := -2Du + pI,$$

$$T': \begin{pmatrix} u \\ p \end{pmatrix} \rightarrow T' \begin{pmatrix} u \\ p \end{pmatrix} := -2Du - pI,\tag{5}$$

where the deformation tensor is given by

$$Du := \frac{1}{2}(\nabla u + (\nabla u)^T)\tag{6}$$

with $(\nabla u)^T$ as the matrix transposed to $\nabla u := (\partial_i u_k)_{k,i=1,...,n}$, and where $I$ denotes the identity matrix. For $a, b \in \mathbb{R}^n$ and matrices $C, D \in \mathbb{R}^{n \times n}$ with $C = (C_{ij}), D = (D_{ij})$ we use

$$a \cdot b := \sum_{i=1}^n a_i b_i, \quad C : D := \sum_{i,j=1}^n C_{ij} D_{ij}.$$

With these notations, for solenoidal vector functions $u \in C^2(\bar{G}), v \in C^1(\bar{G})$ and scalar functions $p \in C^1(\bar{G}), q \in C^0(\bar{G})$ in a bounded open set $G \subset \mathbb{R}^n$. 

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\[ \mathbb{R}^n \ (n \geq 3) \] with boundary \( \partial G \) of class \( C^1 \) we have Green’s first and second formula

\[ \int_G \left( S_{ij}^u \right) \cdot \left( \nu \right) \, dy = \int_{\partial G} (T_{ij}^u N_G) \cdot v \, d\sigma + 2 \int_G Du : Dv \, dy, \tag{7} \]

\[ \int_G \left\{ \left( S_{ij}^u \right) \cdot \left( \nu \right) - \left( u \right)_p \cdot \left( S_{ij}^u \right) \right\} \, dy = \int_{\partial G} \left\{ (T_{ij}^u N_G) \cdot v - u \cdot (T_{ij}^u \nu G) \right\} \, d\sigma. \tag{8} \]

Here and in the following, \( N_G = N_G(y) \) denotes the exterior (with respect to the open set \( G \)) unit surface normal vector in \( y \in \partial G \) and \( \overline{G} \) denotes the closure of \( G \). If \( u, p \) is a solution of the Stokes system in \( G \), then (7) holds under the weaker assumption \( u \in C^1(\overline{G}), \ p \in C^0(\overline{G}) \). If \( G \) is an unbounded open set with bounded boundary of class \( C^1 \), and \( u, p \) is a solution of the Stokes system satisfying

\[ |(T_{ij}^u N_G(x)) \cdot v(x)| = o(|x|^{-n}) \]

as \( |x| \to \infty \), then (7) also holds.

With help of Green’s second formula (8) a representation of the solution \( u, p \) of the Stokes equations can be obtained, if the corresponding fundamental tensor \( E = (E_{jk})_{j,k=1,\ldots,n+1} \) is known. This tensor can be determined as the solution of

\[ SE = \delta I \tag{9} \]

in the sense of distributions, where \( SE = (SE_1, \ldots, SE_{n+1}) \) means the application of \( S \) to the column vectors \( E_k := (E_{jk})_{j=1,\ldots,n+1} \) for each \( k = 1, \ldots, n+1 \), and where \( \delta \) is Dirac’s distribution in \( \mathbb{R}^n \). In the present case \( E = (E_{jk}(x)) \) has the following form:

\( n \geq 3 \quad (j, k = 1, \ldots, n) \):

\[ E_{jk}(x) = \frac{1}{2\omega_n} \left\{ \delta_{jk} \frac{|x|^{2-n}}{n-2} + \frac{x_j x_k}{|x|^n} \right\}, \]

\[ E_{n+1,k}(x) = E_{k,n+1}(x) = \frac{x_k}{\omega_n |x|^n}, \]

\[ E_{n+1,n+1}(x) = \delta(x) \tag{10} \]
Here and in future, $\omega_n$ always stands for the surface area of the $(n-1)$-dimensional unit sphere in $\mathbb{R}^n$ and $|x| = \sqrt{x_1^2 + \cdots + x_n^2}$.

With help of the fundamental tensor $E$ now we obtain a representation of a solution $u, p$ of the Stokes equations

$$S^u_p = \binom{u}{p} \text{ in } G,$$

where here $G \subset \mathbb{R}^n$ ($n \geq 3$) is some bounded (not necessarily connected) open set with boundary $\partial G$ of class $C^{1,\alpha}$, $\alpha > 0$. This representation has the form (compare [14])

$$\int_{\partial G} E^{(c)}(x - y) T^u_p N_G(y) \, dy - \int_{\partial G} D_G(x, y) u(y) \, dy = \begin{cases} - \begin{pmatrix} u \\ p \end{pmatrix}(x), & x \in G, \\ 0, & x \notin G. \end{cases}$$

(11)

Here the $(n+1) \times n$ matrix $E^{(c)}(x - y)$ is obtained from $E(x - y)$ by eliminating the last column, and the $(n+1) \times n$ double layer tensor $D_G(x, y)$ is defined by

$$D_G(x, y) := \left( (-TzE_k(x - y))_{ij} (N_G)_j(y) \right)_{ki}$$

using the column vectors $E_k := (E_{jk})_{j=1,\ldots,n+1}$ for $k = 1, \ldots, n+1$. The tensor $D_G = (D_{ki}(x, y))_{k=1,\ldots,n+1, i=1,\ldots,n}$ has the following form (for abbreviation we set $z := x - y$, $N_G := N_G(y)$):

$n \geq 3$ ($k, i = 1, \ldots, n$):

$$D_{ki}(x, y) = -\frac{n}{\omega_n} \frac{z_k z_i z \cdot N_G}{|z|^{n+2}},$$

$$D_{n+1,i}(x, y) = -\frac{2}{\omega_n} \left\{ n \frac{z_i z \cdot N_G}{|z|^{n+2}} - \frac{(N_G)_i}{|z|^n} \right\}.$$

(12)

If $G$ is an unbounded open set with bounded boundary $\partial G$ of class $C^{1,\alpha}$, $\alpha > 0$, we again define the tensor $D_G$ by the prescription (12). The validity of the representation (11) for unbounded domains follows from the decay properties of the potentials at infinity.
2 The Surface Potentials

Starting from now, throughout the paper $G$ denotes an open set (bounded or unbounded) with compact boundary $\partial G$ of class $C^{1,\alpha}$, $\alpha > 0$, and $G^* := \mathbb{R}^n \setminus \overline{G}$ denotes its complement with $\partial G^* = \partial G$. With help of the tensors $E$ and $D_G$ calculated above now the surface potentials with vector-valued source densities $\Psi \in C^0(\partial G)$ are constructed for all $n \geq 3$. We need the single layer potential

$$
(E_G \Psi)(x) = \int_{\partial G} E^{(c)}(x - y) \Psi(y) \, dy, \quad x \notin \partial G,
$$

(13)

and the double layer potential

$$
(D_G \Psi)(x) = \int_{\partial G} D_G(x, y) \Psi(y) \, dy, \quad x \notin \partial G.
$$

(14)

The $n$–componential velocity parts of these potentials are supported with a dot, to obtain

$$
(E^\bullet_G \Psi)(x) = \int_{\partial G} E^{(c,\bullet)}(x - y) \Psi(y) \, dy, \quad x \notin \partial G,
$$

(15)

$$
(D^\bullet_G \Psi)(x) = \int_{\partial G} D^{(\bullet)}_G(x, y) \Psi(y) \, dy, \quad x \notin \partial G.
$$

(16)

Here the $n \times n$ matrix $E^{(c,\bullet)}(x - y)$ is obtained from $E(x - y)$ by eliminating the last row and the last column. Moreover, we need the normal stresses of the single layer potential $E_G \Psi$, defined in a neighborhood $U \subset \mathbb{R}^n$ of the surface $\partial G$ by

$$
(H^\bullet_G \Psi)(x) = \int_{\partial G} T_x \left( E^{(\bullet)}(x - y) \Psi(y) \right) N_G(\tilde{x}) \, dy
$$

$$
= \int_{\partial G} H_G(x, y) \Psi(y) \, dy, \quad x \notin \partial G.
$$

(17)

Here $\tilde{x} \in \partial G$ is the projection of $x \in U$ onto $\partial G$, and for the $n \times n$ kernel matrix $H_G(x, y)$ in $x, y \in \partial G$ the following identity holds:

$$
H_G(x, y) = \left( D^{(\bullet)}_G(y, x) \right)^T = \left( D^{(\bullet)}_G(y) \right).
$$
Here $D_G^{(r)}$ is the $n \times n$ kernel matrix resulting from $D_G$ by cancelling the last row.

We need further statements regarding the continuity behavior of some surface potentials with special densities. In particular, we consider the velocity part of the double layer potential with constant density and the single layer potential having the unit normal field as density.

**Lemma 2.1.** Let $G$ be a bounded open set.

1. For the double layer potential $D_G^* b$ (see (16)) with some constant density $b \in \mathbb{R}^n$ we have

   $$(D_G^* b)(x) = \begin{cases} 
   b, & x \in G, \\
   \frac{1}{2}b, & x \in \partial G, \\
   0, & x \in G^*. 
   \end{cases} \quad (18)$$

2. For the single layer potential $E_G N_G$ (see (13)) with the exterior (with respect to $G$) unit normal field $N_G$ as density we have

   $$(E_G N_G)(x) = \int_{\partial G} E^{(c)}(x - y) N_G(y) \, dy = \begin{cases} 
   -\begin{pmatrix} 0 \\ 1 \end{pmatrix}, & x \in G, \\
   -\frac{1}{2}\begin{pmatrix} 0 \\ 1 \end{pmatrix}, & x \in \partial G, \\
   \begin{pmatrix} 0 \\ 0 \end{pmatrix}, & x \in G^*. 
   \end{cases} \quad (19)$$

Consequently, $E_G^* N_G = 0$ in $\mathbb{R}^n$.

Lemma 2.1 does not hold if $G$ is unbounded. For the calculation of these potentials in this case we can use the fact that $E_G \Psi = E_G^* \Psi$, $D_G \Psi = -D_G^* \Psi$.

The continuity and jump relations of the Stokes surface potentials on the boundary $\partial G$ are described in the next proposition. Here we need to define the following limiting values:

$$
\begin{align*}
   w^+(z) &= \lim_{G \ni x \to z \in \partial G} w(z), \\
   w^-(z) &= \lim_{G^* \ni x \to z \in \partial G} w(z).
\end{align*}
$$

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Proposition 2.2. Let $\Psi \in C^0(\partial G)$ and let $E_G^\bullet \Psi$, $D_G^\bullet \Psi$, $H_G^\bullet \Psi$ denote the surface potentials defined in (15), (16), (17), respectively. Then on the boundary $\partial G$ the following continuity and jump relations are satisfied:

$$
(E_G^\bullet \Psi)^+ = E_G^\bullet \Psi = (E_G^\bullet \Psi)^-,
$$

$$
(D_G^\bullet \Psi)^+ - D_G^\bullet \Psi = +\frac{1}{2}\Psi = D_G^\bullet \Psi - (D_G^\bullet \Psi)^-,
$$

$$
(H_G^\bullet \Psi)^+ - H_G^\bullet \Psi = -\frac{1}{2}\Psi = H_G^\bullet \Psi - (H_G^\bullet \Psi)^-,
$$

hence

$$
(D_G^\bullet \Psi)^+ - (D_G^\bullet \Psi)^- = \Psi = (H_G^\bullet \Psi)^- - (H_G^\bullet \Psi)^+.
$$

A detailed proof of the relations (20) in the case $n = 3$ can be found in [2].

Finally, we need some statements concerning the decay properties of the surface potentials at infinity.

Lemma 2.3. For the single layer potential $E_G^\bullet \Psi$ and the double layer potential $D_G^\bullet \Psi$ (see (15) and (16)) we have the following decay behavior as $|x| \to \infty$:

$$
(E_G^\bullet \Psi)(x) = \mathcal{O}(|x|^{2-n}),
$$

$$
[E_G \Psi]_{n+1}(x), |(\nabla E_G^\bullet \Psi)(x)|, (D_G^\bullet \Psi)(x) = \mathcal{O}(|x|^{1-n}),
$$

$$
[(\nabla D_G^\bullet \Psi)(x)], [D_G \Psi]_{n+1}(x), |\nabla [E_G \Psi]_{n+1}(x)| = \mathcal{O}(|x|^{-n}),
$$

$$
|\nabla [D_G \Psi]_{n+1}(x)| = \mathcal{O}(|x|^{-n-1}).
$$

Here $E_G^\bullet \Psi$, $D_G^\bullet \Psi$ denote the velocity parts of the potentials, and $[E_G \Psi]_{n+1}(x)$, $[D_G \Psi]_{n+1}(x)$ the pressure parts.

We have seen that for $\psi \in C^0(\partial G)$ the velocity part of the single layer potential is continuous in the whole space, i.e. it holds $E_G^\bullet \psi \in C^0(\mathbb{R}^n)$. Since $\partial G \in C^{1,\alpha}$ we even have $E_G^\bullet \psi \in C^\alpha(\partial G)$ and $H_G^\bullet \psi \in C^\alpha(\partial G)$. If, in addition, $\psi \in C^\gamma(\partial G)$ with $\gamma > 0$, then $E_G \psi$ and $\nabla E_G^\bullet \psi$ can be extended to functions in $C^0(\overline{G})$ and in $C^0(\overline{G^*})$, too. Odquist proved this for $n = 3$ see [10]. The proof for general $n \geq 3$ can be carried out analogously.
3 The Method of Integral Equations

First we study the problem for a non-cracked domain. Let $G \subset \mathbb{R}^n$ be a domain with a nonempty bounded boundary $\partial G \in C^{1,\alpha}$, $\alpha > 0$.

We call $u, p$ a classical solution for the Dirichlet problem of the Stokes system with boundary value $b \in C^0(\partial G)$, if $u \in C^2(G) \cap C^1(\overline{G})$, $p \in C^1(G) \cap C^0(\overline{G})$ satisfy

\[ -\Delta u + \nabla p = 0 \text{ in } G, \quad \nabla \cdot u = 0 \text{ in } G, \tag{26} \]

\[ u = b \text{ on } \partial G. \tag{27} \]

If $G$ is unbounded we require moreover

\[ u(x) = O(|x|^{2-n}), \quad |\nabla u(x)| = O(|x|^{-n}), \]

\[ p(x) = O(|x|^{-n}), \quad |\nabla p(x)| = O(|x|^{-n}) \]

as $|x| \to \infty$.

We also study some class of strong solutions for the Dirichlet problem of the Stokes system with boundary values $b \in L^q(\partial G)$.

To do so, let $x \in \partial G$, $\beta > 0$, and denote the non-tangential approach region of opening $\beta$ at the point $x$ by

\[ \Gamma_\beta(x) := \{ y \in G; |x - y| < (1 + \beta) \text{dist}(y, \partial G) \}. \]

If

\[ c = \lim_{y \to x, y \in \Gamma_\beta(x)} u(y) \]

for each $\beta > \beta_0$, we call $c$ the non-tangential limit of $u$ at $x$ in $\partial G$. We fix $\beta > 0$ large enough such that $x \in \overline{\Gamma_\beta(x)}$ for every $x \in \partial G$. If now $u$ is a function defined in $G$, we denote the non-tangential maximal function of $u$ on $\partial G$ by

\[ u^*(x) := \sup\{ |u(y)|; y \in \Gamma_\beta(x) \}. \]

If $b \in L^q(G)$, $2 \leq q < \infty$, then we call $u, p$ an $L^q$-solution of the Dirichlet problem (26), (27), if $u \in C^2(G)$, $p \in C^1(G)$ satisfy (26), if $u^* \in L^q(G)$, and if $b$ is the non-tangential limit of $u$ a.e. on $\partial G$. If $G$ is unbounded, suppose moreover the decay conditions

\[ u(x) = O(|x|^{2-n}), \quad |\nabla u(x)| = O(|x|^{-n}), \]

\[ p(x) = O(|x|^{-n}), \quad |\nabla p(x)| = O(|x|^{-n}) \]
Now let us fix $\eta > 0$. As in [15] we look for a classical solution of the Dirichlet problem in the form $(u,p)^T = D_G \varphi + \eta E_G \varphi$ with an unknown density $\varphi \in C^0(\partial G)$. Using the continuity properties of the potentials we obtain the integral equation $L \varphi = b$, where

$$L := \frac{1}{2} I + D_G^\ast \varphi + E_G^\ast \varphi.$$ 

If $G$ is bounded, then the necessary condition for the solvability of the Dirichlet problem (26), (27) is

$$\int b \cdot N_G \, do = 0,$$

and therefore the operator $L$ is not invertible. To overcome this difficulty, instead of the original integral equation $\frac{1}{2} \varphi + D_G^\ast \varphi + E_G^\ast \varphi = b$, we study the modified integral equation

$$\frac{1}{2} \varphi + D_G^\ast \varphi + E_G^\ast \varphi + a N \varphi = b,$$

where $a$ is a fixed positive constant and

$$N \varphi := \left( \frac{1}{|\partial G|} \int_{\partial G} N_G \cdot \varphi \, do \right) N_G$$

(compare [15]).

If $b \in L^q(\partial G)$, $2 \leq q < \infty$, we look for an $L^q$-solution of the Dirichlet problem (26), (27) in the same form as above, but with $\varphi \in L^q(\partial G)$. The corresponding potentials are defined as in Section 2. The continuity and the jump relations from Proposition 2.2 hold in the sense of non-tangential limits. The decay properties of Lemma 2.3 are true as well. If $\psi \in L^q(\partial G)$, then $E_G \psi$, $D_G \psi$ satisfy (26), $(E_G^\ast \psi)^* \in L^q(G)$ and $(D_G^\ast \psi)^* \in L^q(G)$. Moreover, $E_G^\ast \psi(x)$ is the non-tangential limit of $E_G \psi$ at almost all $x \in \partial G$, and $\frac{1}{2} \psi(x) + D_G^\ast \psi(x)$ is the non-tangential limit of $D_G^\ast \psi(x)$ at almost all $x \in \partial G$ (see [3], [13]).

Now we start to prove a uniqueness theorem in the $L^2$ case.

**Lemma 3.1.** Let $U \subset \mathbb{R}^n$ be a domain, and let $F^+$, $F^-$ be disjoint nonempty closed sets with $F^+ \cup F^- = \mathbb{R}^n \setminus U$ and $F^+$ be bounded. Let $u, p \in C^\infty(U)$ be a solution of the Stokes system (26). Then there are solutions of
the Stokes system \((u^+, p^+) \in C^\infty(U \cup F^+), (u^-, p^-) \in C^\infty(U \cup F^-)\) such that \((u, p) = (u^+, p^+) + (u^-, p^-)\) in \(U\) and

\[
|u^-(x)| = \mathcal{O}(|x|^{2-n}), \quad |\nabla u^-(x)| = \mathcal{O}(|x|^{1-n}), \quad p^- = \mathcal{O}(|x|^{1-n}), \quad |\nabla p^-(x)| = \mathcal{O}(|x|^{-n})
\]
as \(|x| \to \infty\).

Proof. We fix a bounded open set \(V \subset G\) with boundary of class \(C^\infty\) such that \(\mathbb{R}^n \setminus V\) has two components \(K_+, K_-\) with \(F^+ \subset K^+, F^- \subset K^-\). Define

\[
w^-(x) := \int_{\partial K_+} D_V(x - y)u(y) \, dy - \int_{\partial K_+} E^{(c)}(x - y) T^h_p N_V(y) \, dy,
\]

\[
w^+(x) := \int_{\partial K_-} D_V(x - y)u(y) \, dy - \int_{\partial K_-} E^{(c)}(x - y) T^h_p N_V(y) \, dy.
\]

Then \(w^- \in C^\infty(V \cup K_-), w^+ \in C^\infty(V \cup K_+)\) are solutions of the Stokes system. Moreover, we have \((u, p) = w^+ + w^-\) in \(V\).

Now set \((u^+, p^+) = w^+\) in \(V \cup K_+, (u^+, p^+) = (u, p) - w^-\) in \(U \setminus K_+, (u^-, p^-) = w^-\) in \(V \cup K_-, (u^-, p^-) = (u, p) - w^+\) in \(U \setminus K_-\). Since \((u, p) = w^+ + w^-\) in \(V\) these functions are well defined and they are solutions of the Stokes system. Moreover, \(|u^-(x)| = \mathcal{O}(|x|^{2-n}), p^- = \mathcal{O}(|x|^{1-n}), |\nabla u^-(x)| = \mathcal{O}(|x|^{1-n}), |\nabla p^-(x)| = \mathcal{O}(|x|^{-n})\) as \(|x| \to \infty\). Define \((v, r) = (u, p) - (u^+, p^+) - (u^-, p^-)\) in \(U\). Then \((v, r)\) is a solution of the Stokes system in \(U\) and \((v, r) = 0\) in \(V\). Since \(r\) is a solution of the Laplace equation (see [6]), since \(r = 0\) in \(V\) and since \(U\) is connected we have \(r = 0\) in \(U\). Because \((v, r)\) is a solution of the Stokes system we obtain \(\Delta v = 0\) in \(U\). Since \(v = 0\) in \(V\) and since \(U\) is connected, we deduce \(v = 0\) in \(U\). Thus \((u, p) = (u^+, p^+) + (u^-, p^-)\) in \(U\) and the lemma is proved.

**Notation 3.2.** If \(V\) is an open subset of \(\mathbb{R}^k, k \in N, 1 \leq q < \infty\), denote by \(W^{1,q}(V)\) the space of all functions \(f \in L^q(V)\) such that \(\partial_1 f, \ldots, \partial_k f \in L^q(V)\) in the sense of distributions. Let \(G \subset \mathbb{R}^n\) be an open set with bounded boundary of class \(C^{1,\alpha}\). We say that \(f \in W^{1,q}(\partial G)\) if \(f \in L^q(\partial G)\) and if for each \(x \in \partial G\) there are a neighborhood \(U\) of \(x\), a coordinate system centered in \(x\) and a function \(\varphi\) of class \(C^{1,\alpha}\) defined in \(V = \{y \in \mathbb{R}^{n-1}; |y| < \delta\}\) such that \(U \cap \partial G = \{[y, \varphi(y)]; \varphi \in V\}\), and the function \(h(y) = f(y, \varphi(y))\) is in \(W^{1,q}(V)\).
Proposition 3.3. Let \((u, p)\) be some \(L^2\)-solution of the Dirichlet problem for the Stokes system with the boundary condition \(b \in W^{1,2}(\partial G)\). Then we have \((\nabla u)^*, p^* \in L^2(\partial G)\).

Proof. Suppose first that \(\partial G\) is connected. Then there is a solution \((\tilde{u}, \tilde{p})\) of the problem such that \((\nabla \tilde{u})^*, \tilde{p}^* \in L^2(\partial G)\). For \(G\) bounded this follows from \([3]\), Theorem 4.15 and for \(G\) unbounded see \([3]\), p.791. The uniqueness of an \(L^2\)-solution for the Dirichlet problem in domains with connected boundary (see \([3]\), Theorem 3.9) implies \((u, p) = (\tilde{u}, \tilde{p})\).

Suppose now that \(\partial G\) is not connected. In this case let \(C\) be some component of \(\mathbb{R}^n \setminus G\). Fix some bounded domain \(V \subset G\) such that \(\mathbb{R}^n \setminus V\) has two components \(C\) and \(\tilde{C}\). According to Lemma 3.1 there are solutions \((v, r) \in C^\infty(V \cup \tilde{C}), (\tilde{v}, \tilde{r}) \in C^\infty(V \cup C)\) of the Stokes system satisfying at infinity the decay conditions for the Dirichlet problem such that \((u, p) = (v, r) + (\tilde{v}, \tilde{r})\) in \(V\). Now set \(\Psi = b - \tilde{v}\) on \(\partial C\). Then \((v, r)\) is an \(L^2\) solution of the Dirichlet problem in \(\mathbb{R}^n \setminus C\) with the boundary value \(\Psi \in W^{1,2}(\partial C)\). Since \(\partial C\) is connected we have \((\nabla v)^*, r^* \in L^2(\partial C)\). Finally, due to \((\tilde{v}, \tilde{r}) \in C^\infty(V \cup C)\) and \(|\nabla u(x)| \to 0, p(x) \to 0\) as \(|x| \to \infty\), we obtain \((\nabla u)^*, p^* \in L^2(\partial C)\) and the proposition is proved.

Proposition 3.4. Let \(G\) be connected, and let \((u, p)\) be some \(L^2\)-solution of the Dirichlet problem for the Stokes system with zero boundary condition. Then we have \(u \equiv 0\), and \(p\) is constant. If \(G\) is unbounded, then \(p \equiv 0\).

Proof. According to \([16]\), Theorem 1.12 there is a sequence of \(C^\infty\)-domains \(G_j\) with the following properties:

1. \(\overline{G}_j \subset G\).

2. There are homeomorphisms \(\Lambda_j : \partial G \to \partial G_j\), such that

\[
\sup\{|y - \Lambda_j(y)|; y \in \partial G\} \to 0, \quad \text{as} \quad j \to \infty
\]

and there is some \(\beta > 0\) with \(\Lambda_j(y) \in \Gamma_\beta(y, G)\) for every \(j\) and every \(y \in \partial G\).

3. There are positive functions \(\sigma_j\) on \(\partial G\) bounded away from zero and infinity uniformly in \(j\) such that for any measurable set \(E \subset \partial G\),

\[
\int_E \sigma_j \, d\sigma = \int_{\Lambda_j(E)} 1 \, d\sigma, \quad \text{and so that} \quad \sigma_j \to 1 \text{ pointwise a.e.}
\]

4. The normal vectors \(N^j(\Lambda_j(y))\) to \(G_j\) converge pointwise almost everywhere to \(N_G(y)\).
Now using (7), Proposition 3.3 and Lebesgue’s lemma we obtain

\[ 0 = \lim_{j \to \infty} \int_{\partial G_j} (T_{p}^{u} \cdot N^j) \cdot u \, dy = -2 \lim_{j \to \infty} \int_{G_j} |Du|^2 \, dy = -2 \int_{G} |Du|^2 \, dy \]

Due to \( Du = 0 \) the function \( u \) is a rigid body motion. This means that \( u \in C^0(\overline{G}) \) is a solution of the Laplace equation. Since the non-tangential limit of \( u \) vanishes on \( \partial G \) we find \( u = 0 \) on \( \partial G \). Since \( u(x) \to 0 \) as \( |x| \to \infty \) the maximum principle shows \( u \equiv 0 \). Since \( \nabla p = 0 \) the function \( p \) is constant. If \( G \) is unbounded, then \( p = 0 \) because \( p(x) \to 0 \) as \( |x| \to \infty \).

4 Spectrum of the integral operator

**Notation 4.1.** Let \( T \) be a bounded linear operator in the complex Banach space \( X \). Denote by \( \sigma(T) \) the spectrum of \( T \) and by \( r(T) = \sup \{|\lambda|; \lambda \in \sigma(T)\} \) the spectral radius of \( T \). Denote by \( I \) the identity operator in \( X \).

**Notation 4.2.** If \( X \) is a real Banach space denote by \( \text{compl} X = \{x + iy; x, y \in X\} \) its complexification. For \( z = x + iy \) let \( \overline{z} = x - iy \). Moreover, if \((\varphi_1, \ldots, \varphi_k) \in \text{compl} L^2(\partial G)\) we set

\[ ||(\varphi_1, \ldots, \varphi_k)||_{L^2(\partial G)} = \sqrt{\int_{\partial G} [||\varphi_1(y)||^2 + \cdots + ||\varphi_k(y)||^2] \, dy}. \]

If \( K \) is a nonempty compact subset of \( \mathbb{R}^n \) and \((\varphi_1, \ldots, \varphi_k) \in \text{compl} C^0(K)\), we use

\[ ||(\varphi_1, \ldots, \varphi_k)||_{C^0(K)} = \max_{j=1,\ldots,k} \sup_{x \in K} |\varphi_j(x)|. \]

**Lemma 4.3.** Let \( \gamma > 0 \), \( \varphi \in \text{compl} C^\gamma(\partial G) \). Then we have

\[ \int_{\partial G} \varphi(E_G^*\varphi) \, d\sigma_x = 2 \int_{\mathbb{R}^n \setminus \partial G} |DE_G^*\varphi|^2 \, dx \geq 0. \]

If

\[ \int_{\partial G} \varphi(E_G^*\varphi) \, d\sigma_x = 0, \]

then \( E_G \varphi \) is constant on each component of \( \mathbb{R}^n \setminus \partial G \) and \( E_G^* \varphi = 0 \) in \( \mathbb{R}^n \).
Proof. Let \( \varphi = \psi + i \eta \) where \( \psi, \eta \in C^\infty(\partial G) \). From the symmetry of \( E^{(r,c)} \) and from Fubini's theorem we obtain

\[
\int_{\partial G} \varphi(E_G^* \varphi) \, dx = \int_{\partial G} [\psi E_G^* \psi + \eta E_G^* \eta] \, dx
\]

\[
= -\int_{\partial G} \left[ \left( \frac{1}{2} \psi + H_G^* \psi \right) E_G^* \psi + \left( \frac{1}{2} \psi + H_G^* \eta \right) E_G^* \psi \right. \\
\left. + \left( \frac{1}{2} \eta + H_G^* \eta \right) E_G^* \eta + \left( \frac{1}{2} \eta + H_G^* \eta \right) E_G^* \eta \right] \, dx.
\]

Using (7) for \( G \) and for \( G^* \) we find

\[
\int_{\partial G} \varphi(E_G^* \varphi) \, dx = 2 \int_{\mathbb{R}^n \setminus \partial G} \left[ |D E_G^* \psi|^2 + |D E_G^* \eta|^2 \right] dx = 2 \int_{\mathbb{R}^n \setminus \partial G} |D E_G^* \phi|^2 dx.
\]

Suppose now that

\[
\int_{\partial G} \varphi(E_G^* \varphi) \, dx = 0.
\]

Then \( DE_G^* \phi = 0 \) in \( \mathbb{R}^n \setminus \partial G \), which implies that \( E_G^* \phi \) is linear in each component of \( \mathbb{R}^n \setminus \partial G \). Since \( (E_G^* \varphi)(x) \to 0 \) as \( |x| \to \infty \) we obtain \( E_G^* \varphi = 0 \) in the unbounded component of \( \mathbb{R}^n \setminus \partial G \). Since \( E_G^* \varphi \) is continuous in \( \mathbb{R}^n \), linear in each component of \( \mathbb{R}^n \setminus \partial G \) and \( E_G^* \phi = 0 \) in the unbounded component of \( \mathbb{R}^n \setminus \partial G \) we deduce that \( E_G^* \phi = 0 \) in \( \mathbb{R}^n \). Moreover, since \( SE_G \varphi = 0 \) in \( \mathbb{R}^n \setminus \partial G \) we find that \( E_G^* \varphi \) is constant in each component of \( \mathbb{R}^n \setminus \partial G \).

**Lemma 4.4.** The operator \( E_G^* : \varphi \mapsto E_G^* \varphi \) is a compact linear operator in \( \text{compl} C^0(\partial G) \) and in \( \text{compl} L^q(\partial G) \) for each \( q \in (1, \infty) \). If we denote

\[
A_0 := \max_{j=1, \ldots, n} \sup_{x \in \partial G} \sum_{k=1}^n |E_{jk}(x - y)| \, dy,
\]

then

\[
\|E_G^*\|_{L^2(\partial G)} \leq A_0 < \infty.
\]

If \( \varphi \in \text{compl} L^2(\partial G) \), then

\[
\int_{\partial G} |E_G^* \varphi|^2 \, dx \leq A_0 \int_{\partial G} \varphi(E_G^* \varphi) \, dx.
\]
If $M$ is a positive constant with
\[ \int_{\{y \in \partial G; |x-y|<r\}} 1 \, dy \leq Mr^{n-1} \quad (31) \]
for each $x \in \partial G$ and $0 < r < \text{diam} \, \partial G$, then
\[ A_0 \leq \frac{n^2}{2\omega_n} M \text{diam} \, \partial G. \]

Proof. $E_G^\bullet$ is a compact linear operator in compl $C^0(\partial G)$ and in compl $L^p(\partial G)$ because it has a weakly singular kernel.

Let now $\|\varphi\| \leq 1$ in compl $L^2(\partial G)$. The symmetry of the matrix $E_{ij}$ and Fubini’s theorem give
\[
\|E^\bullet \varphi\|_{L^2(\partial G)} \leq \left[ \int_{\partial G} \left( \sum_{j=1}^{n} \left( \int_{\partial G} |E_{jk}(x-y)| \right) \right) \left( \sum_{k=1}^{n} |E_{jk}(x-y)| |\varphi_k(y)| \, dy \right) \, dx \right]^{1/2} \\
\leq A_0 \left[ \int_{\partial G} \int_{\partial G} \sum_{j=1}^{n} \sum_{k=1}^{n} |E_{jk}(x-y)| |\varphi_k(y)| \, dy \, dx \right]^{1/2} \\
= A_0 \left[ \int_{\partial G} \sum_{k=1}^{n} |\varphi_k(y)| \left( \int_{\partial G} |E_{jk}(x-y)| \, dy \right) \, dx \right]^{1/2} \leq A_0 \|\varphi\|_{L^2(\partial G)}.
\]
This gives the estimate (29).

Now fix $\gamma > 0$. Since compl $C^\gamma(\partial G)$ is a dense subset of compl $L^2(\partial G)$, Lemma 4.3 implies, that $E_G^\bullet$ is a positive linear operator in compl $L^2(\partial G)$. For $\varphi, \psi \in \text{compl} \, L^2(\partial G)$ let
\[
\langle \varphi, \psi \rangle := \int_{\partial G} \varphi(E_G^\bullet \psi) \, dx
\]
denote a scalar product. Then $\langle \cdot, \cdot \rangle$ is a scalar product. If $\varphi \in \text{compl} \, L^2(\partial G)$, using the Schwartz inequality we get
\[
\int_{\partial G} |E_G^\bullet \varphi|^2 \, dx = \left[ \sup \{ |\langle \psi, \varphi \rangle|; \psi \in \text{compl} \, L^2(\partial G), \|\psi\| = 1 \} \right]^2
\]
\[ \langle \varphi, \varphi \rangle \sup \{ |\langle \psi, \psi \rangle|; \psi \in \text{compl} L^2(\partial G), |\psi| = 1 \} \]
\[ \leq \langle \varphi, \varphi \rangle \| E^*_G \|_{L^2(\partial G)} \leq A_0 \int_{\partial G} \varphi (E^*_G \varphi) \, dx. \]

Suppose now that the inequality (31) is true. Using [17], Lemma 1.5.1 we obtain
\[ A_0 \leq \frac{n + 1}{2 \omega_n} \sup_{x \in \partial G} \int \int_{|y \in \partial G; |x-y|^{2-n}\times t\in \{y \in \partial G; |x-y|^{2-n}\times t\}} 1 \, do_y dt \]
\[ \leq \frac{n + 1}{2 \omega_n} \int_{(\text{diam } G)^{2-n}} Mt^{(n-1)/(2-n)} \, dt \leq \frac{n^2}{2 \omega_n} M \text{ diam } \partial G. \]

Since \( \partial G \) is of class \( C^{1,\alpha} \) there is a constant \( M \) such that the inequality (31) holds true for each \( x \in \partial G \) and \( 0 < r < \text{diam } \partial G \). This proves \( A_0 < \infty \).

**Theorem 4.5.** Suppose that \( G \) is connected and \( a \geq 0, \eta \geq 0 \). Let \( \lambda \) be an eigenvalue of the operator
\[ L^* = \frac{1}{2} I + H^*_G + \eta E^*_G + a N_n \]
in \( \text{compl} C^0(\partial G) \), where
\[ N_n \varphi = \left( \frac{1}{|\partial G|} \int_{\partial G} N_G \cdot \varphi \, do \right) N_G. \]

Then we have \( 0 \leq \lambda \leq 1 + \eta A_0 + a \) where \( A_0 \) is given by (28).

Proof. We can suppose that \( \lambda \neq \frac{1}{2} \). Let \( \varphi = \psi + i \Phi \) be an eigenfunction corresponding to the eigenvalue \( \lambda \). Since \( \partial G \) is of class \( C^{1,\alpha} \), we have \( N_G \in C^\alpha(\partial G) \), \( E^*_G \varphi \in \text{compl} C^\alpha(\partial G) \) and \( H^*_G \varphi \in \text{compl} C^\alpha(\partial G) \). Since \( \lambda \neq \frac{1}{2} \) we obtain \( \varphi \in \text{compl} C^\alpha(\partial G) \).

Suppose first that
\[ \int_{\partial G} \varphi (E^*_G \varphi) \, do_x \neq 0. \]

Since \( \nabla \cdot E^*_G \varphi = 0 \) we get using the divergence theorem and (7)
\[ \int_{\partial G} \lambda \varphi E^*_G \varphi \, do_x = \int_{\partial G} \left( \frac{1}{2} \varphi - H^*_G \varphi + \eta E^*_G \varphi + a N_n \varphi \right) (E^*_G \varphi) \, do_x \]
\[
2 \int_{G^*} |DE_G^* \varphi|^2 dx + \eta \int_{\partial G} |E_G^* \varphi|^2 \, d\sigma.
\]

Using Lemma 4.3 we obtain
\[
\lambda = \frac{\int_{G^*} |DE_G^* \varphi|^2 dx}{\int_{\mathbb{R}^n \setminus \partial G} |DE_G^* \varphi|^2 dx} + \eta \frac{\int_{ \partial G} |E_G^* \varphi|^2 \, d\sigma}{\varphi(\eta \varphi)} \geq 0.
\]

Now Lemma 4.4 implies \( \lambda \leq 1 + \eta A_0 \).

Suppose now that \( \int_{\partial V} \varphi(\eta \varphi) \, d\sigma = 0 \). Then \( E_G^* \varphi = 0 \) in \( \mathbb{R}^n \), and \( E_G \varphi \) is constant in each component of \( \mathbb{R}^n \setminus \partial G \) using Lemma 4.3. Since \( (E_G \varphi)(y) \to 0 \) as \( |y| \to \infty \) we obtain \( (E_G \varphi)(y) = 0 \) on the unbounded component of \( \mathbb{R}^n \setminus \partial G \). If \( V \) is a component of \( G^* \) it follows
\[
\varphi = \left( -\frac{1}{2} \varphi + H_G \varphi \right) + \left( -\frac{1}{2} \varphi + H_G^* \varphi \right) = c_V N_G
\]
on \( \partial V \), where \( c_V \) is some complex constant. Set
\[
b = a \sum \{ c_V |\partial V||\partial G|^{-1}; V \text{ component of } G^* \}. \tag{33}
\]
If \( V \) is an unbounded component of \( G^* \), then
\[
\lambda c_V N_G = L^* \varphi = b N_G \tag{34}
on \partial V. \]
If \( V \) is a bounded component of \( G^* \), then
\[
\lambda c_V N_G = L^* \varphi = (c_V + b) N_G \tag{35}
on \partial V. \]
Let us suppose first \( b = 0 \). Since \( \varphi \) is an eigenfunction there is a component \( V \) of \( G^* \) such that \( c_V \neq 0 \). If \( V \) is unbounded, then \( \lambda = 0 \) by (34). If \( V \) is bounded then \( \lambda = 1 \) by (35). Let suppose now \( b \neq 0 \). Using (35) and (34) we get \( c_V \neq 0 \) for each component \( V \) of \( G^* \). Thus
\[
\lambda = b/c_V \tag{36}
\]
for the unbounded component of $G^*$ and

$$\lambda = 1 + (b/c_V)$$  \hspace{1cm} (37)$$

for a bounded component of $G^*$. If $G^*$ is connected, then the relations (37), (36) and (33) yield $\lambda = a$ or $\lambda = 1 + a$. Let us suppose now that $G^*$ is not connected. The relation (37) gives that there are complex nonzero numbers $c_b, c_u$ such that $c_V = c_b$ for a bounded component $V$ of $G^*$ and $c_V = c_u$ for the unbounded component $V$ of $G^*$. If $G^*$ is bounded, we get from (33) and (37) $\lambda = 1 + a$. Suppose now that $G^*$ is not connected. Denote $c = c_b/c_u$ and $t = |\partial V|/|\partial G|$, where $V$ is the unbounded component of $G^*$. Then we find $0 < t < 1$. We rewrite (37) and (36) using (33) and obtain

$$\lambda = a[t + (1 - t)c]$$,  \hspace{1cm} (38)$$

$$\lambda = 1 + a[t c^{-1} + (1 - t)]$$,  \hspace{1cm} (39)$$

Since the imaginary parts of the expressions (38), (39) have opposite signs we deduce that $\lambda$ is real and therefore $c$ is real. If $c \leq 1$ we get $\lambda \leq a$ by (38). If $c > 1$ we get $\lambda \leq 1 + a$ by (39). Suppose now that $\lambda < 0$. Then we find $t + (1 - t)c < 0$ by (38) and it follows $c < 0$. The relation (39) yields $\lambda = 1 + a[t + (1 - t)c]c^{-1} \geq 1$, which is a contradiction.

**Theorem 4.6.** Suppose that $G$ is connected and $2 \leq q < \infty$. Fix $a \geq 0$, $\eta \geq 0$ and set $L = \frac{1}{2}I + D_n^* + \eta E_G^* + aN_n$. Let us suppose that $L$ is invertible in $C^0(\partial G)$ and fix $\gamma > (1 + \eta A_0 + a)/2$. Then there are constants $d, d_q > 1$ and $t, t_q \in (0, 1)$ such that for each nonnegative $k \in \mathbb{N}_0$

$$\|(I - \gamma^{-1}L)^k\|_{C^0(\partial G)} \leq dt^k$$,  \hspace{1cm} (40)$$

$$\|(I - \gamma^{-1}L)^k\|_{L^q(\partial G)} \leq d_q t_q^k$$  \hspace{1cm} (41)$$

and

$$L^{-1} = \gamma^{-1} \sum_{k=0}^{\infty} (I - \gamma^{-1}L)^k$$  \hspace{1cm} (42)$$

in $C^0(\partial G)$ and in $L^q(\partial G)$.

Proof. The operator $L - \frac{1}{2}I$ is a compact operator in $L^q(\partial G)$. Since the operator $L^*$ given by (32) is the adjoint operator of $L$ in compl $C^0(\partial G)$ and in compl $L^q(\partial G)$, we find $\sigma(L) = \sigma(L^*) \subset (0, 1 + \eta A_0 + a) \subset (0, 2\gamma)$ in compl $C^0(\partial G)$ and in compl $L^q(\partial G)$ using Theorem 4.5 and [11], § 3.7. The spectral mapping theorem gives $\sigma(I - \gamma^{-1}T) \subset (-1, 1)$ in compl $C^0(\partial G)$ and
in compl $L^q(\partial G)$. Since $r(I - \gamma^{-1}L) < 1$ there are constants $d, d_q > 1$ and $s, s_q \in (0, 1)$ satisfying (40), (41). Since $L = \gamma[(I - (I - \gamma^{-1}L)]$, an easy calculation yields (42) in $C^0(\partial G)$ and in $L^q(\partial G)$.

5 Solutions of the Dirichlet problem

Lemma 5.1. Fix $\eta \geq 0, a \geq 0$. If $G$ is not a bounded domain with connected boundary suppose $\eta > 0$. Denote $L^* = \frac{1}{2}I + H_G + \eta E_G + aN_n$ and let $\psi \in C^0(\partial G)$ be such that $L^*\psi = 0$. Then $E_G\psi = 0$ in $\mathbb{R}^n$, and $E_G\psi$ is constant in each component of $\mathbb{R}^n \setminus \partial G$.

Proof. Since $\psi = -2(H_G\psi + \eta E_G + aN_n) \in C^\alpha(\partial G)$, by (7) and the divergence theorem we obtain

$$0 = \int_{\partial G} \left( \frac{1}{2} \psi - H_G\psi + \eta E_G + aN_n \right) (E_G\psi) \, d\sigma$$

$$= 2 \int_{G^*} |DE_G\psi|^2 \, dx + \eta \int_{\partial G} |E_G\psi|^2 \, d\sigma.$$  

It follows $DE_G\psi = 0$ in $G^*$ and $\eta E_G\psi = 0$ on $\partial G$. If $\eta > 0$ then $E_G\psi = 0$ on $\partial G$.

Suppose now that $\eta = 0$. Then $G$ is a bounded domain with connected boundary. Since $DE_G\psi = 0$ in $G^*$ the mapping $E_G\psi(x)$ is affine in $G^*$. Since $E_G\psi(x) \to 0$ as $|x| \to \infty$ we deduce $E_G\psi = 0$ in $G^*$. The continuity of $E_G\psi$ implies $E_G\psi = 0$ on $\partial G$.

Because $E_G\psi = 0$ on $\partial G$ we have

$$\int_{\partial G} \psi(E_G\psi) \, d\sigma = 0.$$  

From Lemma 4.3 we find that $E_G\psi$ is constant on each component of $\mathbb{R}^n \setminus \partial G$ and $E_G\psi = 0$ in $\mathbb{R}^n$.

Theorem 5.2. Let $G$ be unbounded and connected, $2 \leq q < \infty$ and $b \in L^q(\partial G)$. Then there is a unique $L^q$-solution $(u, p)$ of the Dirichlet problem with the boundary value $b$. Let $\eta > 0$ and $L = \frac{1}{2}I + D_G + \eta E_G$. Then $L$ is invertible in $C^0(\partial G)$ and in $L^q(\partial G)$, and

$$(u, p) = D_G \varphi + \eta E_G \varphi$$
where \( \varphi = L^{-1}b \). If \( b \in C^0(\partial G) \), then \( \varphi \in C^0(\partial G) \), and \((u, p)\) is a classical solution of the problem. If we fix \( \gamma > (1 + \eta A_0)/2 \), then \( L^{-1} \) is given by (42).

Proof. Let \( L^* = \frac{1}{2}I + H_G^* + \eta E_G^* \) be the adjoint operator of \( L \) in \( C^0(\partial G) \). Let \( \psi \in C^0(\partial G) \) with \( L^* \psi = 0 \). Then Lemma 5.1 implies that \( E_G \psi \) is constant on each component of \( \mathbb{R}^n \setminus \overline{\partial G} \) and \( E_G^* \psi = 0 \) in \( \mathbb{R}^n \). Since \( G \) is an unbounded domain and \( E_G \psi(x) \to 0 \) as \( |x| \to \infty \), we have \( E_G \psi = 0 \) in \( G \). Since \( 0 = \frac{1}{2} \psi - H_G^*, \psi + \eta E_G^* \psi = -[(E_G \psi)_{n+1}]_N \) we obtain \( E_G \psi = 0 \) in \( \mathbb{R}^n \setminus \partial G \). Now the jump relation (21) implies \( \psi = 0 \).

We see that the operator \( L^* \) is injective. Since \( E_G^*, H_G^* \) are compact operators, the Fredholm theory implies the invertibility of \( L^* \). Since \( L^* \) is invertible, the operator \( L \) is also invertible in \( C^0(\partial G) \) and in \( L^q(\partial G) \) (see [11], § 3.7). Setting \( \varphi = L^{-1}b \) we find that \((u, p) = D_G \varphi + \eta E_G \varphi \) is a solution of the problem. Theorem 4.6 shows that \( L^{-1} \) is given by (42). The uniqueness of a solution of the Dirichlet problem was proved in Proposition 3.4. This proves the theorem.

**Theorem 5.3.** Let \( G \) be bounded and connected, \( 2 \leq q < \infty \), \( b \in L^q(\partial G) \). Then there is an \( L^q \)-solution \((u, p)\) of the Dirichlet problem with boundary value \( b \) if and only if

\[
\int_{\partial G} b \cdot N_G \, d\sigma = 0. \tag{43}
\]

The vector function \( u \) is unique and the function \( p \) is unique up to an additive constant. Fix \( \eta > 0, a > 0 \), and set \( L = \frac{1}{2}I + D_G^* + \eta E_G^* + a N_n \). Then \( L \) is invertible and we have

\[
(u, p) = D_G \varphi + \eta E_G \varphi + (0, \ldots, 0, c),
\]

where \( \varphi = L^{-1}b \) and \( c \) is constant. If \( b \in C^0(\partial G) \), then \( \varphi \in C^0(\partial G) \) and \((u, p)\) is a classical solution of the problem. If we fix \( \gamma > (1 + \eta A_0 + a)/2 \) then \( L^{-1} \) is given by (42).

Proof. If there is a solution of the Dirichlet problem with boundary condition \( b \), (43) follows from the divergence theorem.

Suppose now that (43) holds. We will prove the existence of a solution of the Dirichlet problem as follows: Let \( L^* = \frac{1}{2}I + H_G^* + \eta E_G^* + a N_n \) be the adjoint operator of \( L \) in \( C^0(\partial G) \). Consider \( \psi \in C^0(\partial G) \) with \( L^* \psi = 0 \). By Lemma 5.1 we obtain \( E_G \psi \) is constant on each component of \( \mathbb{R}^n \setminus \overline{\partial G} \), and \( E_G^* \psi = 0 \) in \( \mathbb{R}^n \). The jump relation (21) implies that for each component \( V \) of \( G^* \) there is a constant \( c_V \) with \( \psi = c_V N_G \) on \( \partial V \). Denote by \( \hat{V} \) the unbounded
component of \( G^* \). According to Lemma 2.1 we have \( 0 = L^* \psi = aN_n \psi \) on \( \partial \tilde{V} \). Since \( a > 0 \) we get \( N_n \psi = 0 \). Let now \( V \) be a bounded component of \( G^* \). Since \( N_n \psi = 0 \), Lemma 2.1 gives \( 0 = L^* \psi = (c_V - c_\psi) N^G \). Thus \( \psi = c N_G \), and since \( N_n \psi = 0 \) we obtain \( c = 0 \) and thus \( \psi \equiv 0 \).

We see that the operator \( L^* \) is injective. Since \( E^*_G, H^*_G \) are compact operators, the Fredholm theory implies the invertibility of \( L^* \). As above, this implies the invertibility of \( L \) in \( C^0(\partial G) \) and in \( L^q(\partial G) \) (see [11], §3.7). If we put \( \varphi = L^{-1} b \), then \( (u, p) = D_G \varphi + \eta E_G \varphi + (0, \ldots, 0, c) \) is a solution of the problem. Here \( c \) is an arbitrary constant. Theorem 4.6 implies that \( L^{-1} \) is given by (42) and the uniqueness of a solution was proved in Proposition 3.4.

If we want to calculate \( L^{-1} \) using (42) we need an estimation of \( A_0 \). The estimation of \( A_0 \) might be unpleasant. But we don’t need it in a special case of a bounded domain with connected boundary.

**Theorem 5.4.** Let \( G \) be bounded and \( \partial G \) be connected, and let \( 2 \leq q < \infty \). Fix \( a > 0 \) and set \( L = \frac{1}{2} I + D_G + aN_n \). Then \( L \) is invertible in \( C^0(\partial G) \) and in \( L^q(\partial G) \). If \( b \in L^q(\partial G) \), then \( \varphi = L^{-1} b \) is a classical solution of the Dirichlet problem with boundary value \( b \). If \( b \in C^0(\partial G) \), then \( \varphi \in C^0(\partial G) \) and \( (u, p) \) is a classical solution of the problem. If we fix \( \gamma > (1 + a)/2 \) then \( L^{-1} \) is given by (42).

Proof. Let \( L^* = \frac{1}{2} I + H^*_G + aN_n \) be the adjoint operator of \( L \) in \( C^0(\partial G) \), and let \( \psi \in C^0(\partial G) \) satisfy \( L^* \psi = 0 \). Then Lemma 5.1 gives that \( E_G \psi \) is constant on each component of \( \mathbb{R}^n \setminus \partial G \), and \( E^*_G \psi = 0 \) in \( \mathbb{R}^n \). Since \( G^* \) is an unbounded domain and \( E_G \psi(x) \to 0 \) as \( |x| \to \infty \), we have \( E_G \psi = 0 \) in \( G^* \). The jump relation (21) implies \( \psi = c N_G \), hence \( 0 = L^* \psi = a c N^G \). Since \( a > 0 \) we find \( c = 0 \) and thus \( \psi = 0 \).

We see that the operator \( L^* \) is injective. Since \( E^*_G, H^*_G \) are compact operators, the Fredholm theory implies that \( L^* \) is invertible. This gives the invertibility of \( L \) in \( C^0(\partial G) \) and in \( L^q(\partial G) \) (see [11], §3.7). Using \( \varphi = L^{-1} b \) as above, \( (u, p) = D_G \varphi + (0, \ldots, 0, c) \) is a solution of the problem, and Theorem 4.6 implies that \( L^{-1} \) is given by (42).

Maremonti and Russo proved in [7], if \( (u, p) \) is a classical solution of the Dirichlet problem for the Stokes system with boundary value \( b \) in a bounded planar domain \( G \) with \( C^2 \)-boundary, then a maximum modulus estimate holds:

\[
\sup_{x \in G} |u(x)| \leq K \sup_{x \in \partial G} |b(x)|.
\]

We extend this result to higher dimensions.
Theorem 5.5. Let \((u,p)\) be a classical solution of the Dirichlet problem for the Stokes system with the boundary condition \(b\). Then

\[
\sup_{x \in G} |u(x)| \leq K \sup_{x \in \partial G} |b(x)|,
\]

where the constant \(K\) depends only on \(G\).

Proof. Let \(L\) be the operator defined in Theorem 5.3 if \(G\) bounded and in Theorem 5.2 if \(G\) unbounded. Since \(L\) is invertible there is some positive constant \(M_1\) such that

\[
\| L^{-1} \psi \|_{C^0(\partial G)} \leq M_1 \| \psi \|_{C^0(\partial G)}
\]

for all \(\psi \in C^0(\partial G)\). For \(\psi \in C^0(\partial G)\) define

\[
W \psi(x) = \begin{cases} 
D_G^* \psi(x) + \eta E_G^* \psi(x), & x \in G, \\
(D_G^* \psi + \eta E_G^* \psi)^+(x), & x \in \partial G.
\end{cases}
\]

Then \(W\) is a linear operator from \(C^0(\partial G)\) to \(C^0(\partial G)\). We now show that \(W\) is a closed operator. To do so, let \(\psi_k \to \psi, W \psi_k \to f\) in \(C^0(\partial G)\) as \(k \to \infty\). If \(x \in G\), then \(W \psi_k(x) \to W \psi(x)\) as \(k \to \infty\). This means \(f = W \psi\). Thus \(W\) is closed from the Banach space \(C^0(\partial G)\) to the Banach space \(C^0(\partial G)\). The closed graph theorem (see [12], Theorem 3.10) shows that \(W\) is a bounded linear operator, hence there is a positive constant \(M_2\) with

\[
\| W \psi \|_{C^0(\partial G)} \leq M_2 \| \psi \|_{C^0(\partial G)}
\]

for each \(\psi \in C^0(\partial G)\). Therefore we find

\[
\sup_{x \in G} |u(x)| \leq n \| W L^{-1} b \|_{C^0(\partial G)} \leq n M_1 M_2 \| b \|_{C^0(\partial G)} \leq n M_1 M_2 \sup_{x \in \partial G} |b(x)|.
\]

Remark 5.6. Let \(b \in C^0(\partial G)\) be such that the Dirichlet problem for the Stokes system with boundary value \(b\) is solvable. Using Theorem 5.2, Theorem 5.3 or Theorem 5.4, we can reduce the original problem to the problem \(L \varphi = b\). Let \(\gamma\) be some constant satisfying the condition in the corresponding theorem. To solve \(L \varphi = b\) we can use the method of successive approximations. To do so we write \(L \varphi = b\) as \(\varphi = (I - \gamma^{-1} L) \varphi + \gamma^{-1} b\). We fix an arbitrary \(\varphi_0 \in C^0(\partial G)\) and set

\[
\varphi_{k+1} = (I - \gamma^{-1} L) \varphi_k + \gamma^{-1} b
\]
for \( k = 0, 1, 2, \ldots \). According to the Theorem 4.6 there are constants \( d > 1, q \in (0, 1) \) such that the estimate (40) holds for all \( k \). Since

\[
\varphi_{k+1} - \varphi_k = (I - \gamma^{-1}L)(\varphi_k - \varphi_{k-1}) = \cdots = (I - \gamma^{-1}L)^k(\varphi_1 - \varphi_0)
\]

we find

\[
\|\varphi_{k+1} - \varphi_k\|_{C^0(\partial G)} \leq dq^k\|\varphi_1 - \varphi_0\|_{C^0(\partial G)}.
\]

If \( j < k \), then

\[
\|\varphi_k - \varphi_j\|_{C^0(\partial G)} \leq d(q^j + \ldots q^{k-1})\|\varphi_1 - \varphi_0\|_{C^0(\partial G)} = \frac{dq^j}{1 - q}\|\varphi_1 - \varphi_0\|_{C^0(\partial G)}.
\]

Thus \( \{\varphi_k\} \) is a Cauchy sequence and converges to the solution \( \varphi \) of the equation \( L\varphi = b \). Moreover, we find

\[
\|\varphi - \varphi_j\|_{C^0(\partial G)} \leq \frac{dq^j}{1 - q}\|\varphi_1 - \varphi_0\|_{C^0(\partial G)}.
\]

The same arguments hold true also in \( L^q(\partial G) \).

### 6 Cracked domain

Suppose that \( V^+ \) is a nonempty bounded open set with boundary of class \( C^{2+\alpha}, \alpha > 0 \). Let \( G \) be a nonempty domain with bounded boundary of class \( C^{1,\alpha} \), where \( G = \mathbb{R}^n \) is also possible. Let \( S \subset \partial V^+ \cap \overline{G} \) be a nonempty compact set such that \( \text{cl}(G \cap S) = S \). We suppose that \( \partial G \cap \partial V^+ \setminus S = \emptyset \). Denote \( V^- = \mathbb{R}^n \setminus \text{cl} V^+ \) and \( \Omega = G \setminus S \).

We consider the following problem: Find \( (v, q) \in C^\infty(\Omega) \) such that \( v \in C^0(\text{cl} \Omega \setminus S) \) is continuously extendible to \( \text{cl}(V^+ \cap \Omega) \) and \( \text{cl}(V^- \cap \Omega) \), and such that \( \nabla v, q \) are continuously extendible to \( \text{cl} V^+ \cap G \) and \( \text{cl} V^- \cap G \), with

\[
-\Delta v + \nabla q = 0 \quad \text{in } \Omega \quad (44)
\]

\[
\nabla \cdot v = 0 \quad \text{in } \Omega \quad (45)
\]

\[
v = f \quad \text{on } \partial \Omega \setminus S \quad (46)
\]

\[
v^+ - v^- = g \quad \text{on } S \quad (47)
\]

\[
(T_q^vN_{V^+})^+ - (T_q^vN_{V^+})^- = h \quad \text{on } S \cap G. \quad (48)
\]

If \( G \) is unbounded we suppose in addition \( v(x) = \mathcal{O}(|x|^{2-n}), q(x) = \mathcal{O}(|x|^{1-n}) \), \( |\nabla v(x)| = \mathcal{O}(|x|^{1-n}), |\nabla q(x)| = \mathcal{O}(|x|^{-n}) \) as \( |x| \to \infty \).
By the superscripts "+" and "−" we denote the limiting values of functions on $S$ with respect to $V^+$ and $V^−$, respectively.

If $G$ is bounded and $(v,p)$ is a solution of the problem (44)–(48) then the divergence theorem implies

$$0 = \int_{\partial V^+} N_{V^+} \cdot v^+ \, do + \int_{\partial V^−} N_{V^−} \cdot v^− \, do.$$

And from the boundary conditions (46), (47) it follows that

$$\int_{\partial \Omega \setminus S} f \cdot N_\Omega \, do + \int_S N_{V^+} \cdot g \, do = 0 \quad (49)$$

is a necessary condition for the solvability of the problem (44)–(48).

**Theorem 6.1.** Let $(v,q)$ be a solution of the problem (44) – (48) with $g \equiv 0$, $h \equiv 0$. Then $(v,q)$ can be extended to $G$ such that $(v,q)$ is a solution of the Stokes system in $G$. If moreover $f \equiv 0$, then $v \equiv 0$ and $q$ is constant.

If $G$ is unbounded, then $q \equiv 0$.

Proof. Fix $x \in S \cap G$. Choose $r > 0$ small enough such that $\text{dist}(x, \partial G) > r$. Then $\partial V^+$ divides $B_r(x) = \{ y \in \mathbb{R}^n ; |y - x| < r \}$ into 2 components $B^+$, $B^−$ such that $\partial B^+ \cap \partial B^− = \partial V^+ \cap \text{cl} B_r(x)$. Fix arbitrary $z \in B^+$ and choose $\rho > 0$ with $B^\rho(z) \subset B^+$. Now by Green’s formula we get (compare (11))

$$(v,q)(z) = -\int_{\partial B^\rho(z)} E^{(c)}(z - y)T_q^u N_{B^\rho(z)}(y) \, do_y + \int_{\partial B^\rho(z)} D_{B^\rho(z)}(z,y) v(y) \, do_y,$$  

$$0 = -\int_{\partial (B^+ \setminus B^\rho(z))} E^{(c)}(z - y)T_q^u N_{B^+ \setminus B^\rho(z)}(y) \, do_y + \int_{\partial (B^+ \setminus B^\rho(z))} D_{(B^+ \setminus B^\rho(z))}(z,y) v(y) \, do_y, \quad (51)$$

$$0 = -\int_{\partial B^-} E^{(c)}(z - y)T_q^u N_{B^-}(y) \, do_y + \int_{\partial B^-} D_{B^-}(z,y) v(y) \, do_y. \quad (52)$$

Adding (50), (51) and (52) we get

$$(v,q)(z) = -\int_{\partial B_r(x)} E^{(c)}(z - y)T_q^u N_{B_r(x)}(y) \, do_y + \int_{\partial B_r(x)} D_{B_r(x)}(z,y) v(y) \, do_y.$$  

(53)
By a similar method we obtain (53) for \( z \in B^- \). Then we define \((v, q)\) by the limit on \( S \) and (53) holds for each \( z \in B_r(x) \). According to (53), the vector field \((v, q)\) is a solution of the Stokes system in \( B_r(x) \).

Suppose now \( f \equiv 0 \). Since \((v, q)\) is a classical solution of the Dirichlet problem for the Stokes system in \( G \) with zero boundary value, Theorem 6.1. is proved.

Assume now \( g \in C^{1+\gamma}(\partial V^+) \), \( h \in C^\gamma(\partial V^+) \) where \( 0 < \gamma < \alpha \) and \( g = h = 0 \) on \( G \cap \partial V^+ \setminus S \). Suppose, moreover, \( f \in C^0(\partial G \setminus S) \), \( f \) is continuously extendible to \( \text{cl} V^+ \cap \partial G \) and \( \text{cl} V^- \cap \partial G \) and \( f^+-f^- = g \) on \( S \cap \partial G \). We can suppose, moreover, that the supports of \( g \) and \( h \) are compact subsets of \( \partial V^+ \).

Define in \( \Omega \)
\[
(w, r) := -E_{V^+}h + D_{V^+}g. \tag{54}
\]

Then \((w, r)\) is a solution of the Stokes system outside of the crack \( S \), \( w \) is continuously extendible to \( \text{cl} V^+ \cap \partial G \) and \( \text{cl} V^- \cap G \), and \( \nabla w, r \) are continuously extendible to \( \text{cl} V^+ \cap \partial G \) and \( \nabla \cdot \nabla w, r \) are continuously extendible to \( \text{cl} V^+ \cap \partial G \). We look for a solution of the problem \((44)-(48)\) in the form \((v, q) = (u, p) + (w, r)\). Theorem 6.1 implies that \((v, q)\) is a solution of the problem \((44)-(48)\) if and only if \((u, p)\) is a classical solution of the Stokes equations
\[
-\Delta u + \nabla p = 0, \quad \nabla \cdot u = 0 \quad \text{in } G,
\]
\[
u = b \quad \text{on } \partial G,
\]
where \( u(x) = O(|x|^{2-n}) \), \( p(x) = O(|x|^{1-n}) \), \( |\nabla u(x)| = O(|x|^{1-n}) \), \( |\nabla p(x)| = O(|x|^{-n}) \) as \( |x| \to \infty \). The existence of such a solution for unbounded \( G \) is proved above Section 5.

Suppose now that \( G \) is bounded then the condition (49) is necessary for the solvability of the problem \((44)-(48)\). Suppose that this condition is satisfied and set \( \tilde{f} = w \) on \( \partial \Omega \setminus S \). Since \((w, r)\) is a solution of the problem
\[ -\Delta w + \nabla r = 0, \quad \nabla \cdot w = 0 \quad \text{in } \Omega, \]
\[ w = \tilde{f} \quad \text{on } \partial \Omega \setminus S, \]
\[ w^+ - w^- = g \quad \text{on } S, \]
\[ (T_{wV}^u N_V^+) - (T_{wV}^u N_V^-) = h \quad \text{on } S \cap G, \]

we have
\[ \int_{\partial \Omega \setminus S} \tilde{f} \cdot N_\Omega \, do + \int_S N_{V+} \cdot g \, do = 0. \]

Subtracting this and (49) we get (43). Now Theorem 5.3 implies that there is a classical solution \((u, p)\) of the Dirichlet problem (26), (27) which has been constructed in Section 5.

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**References**


