3D STEADY COMPRESSIBLE NAVIER–STOKES EQUATIONS

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Abstract. We study the steady compressible Navier–Stokes equations in a bounded smooth three-dimensional domain, together with the slip boundary conditions. We show that for a certain class of the pressure laws, there exists a weak solution with bounded density (in $L^\infty$ up to boundary).

1. Introduction; main result. Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with sufficiently smooth connected boundary. We consider the system of PDE’s describing the steady flow of a compressible Newtonian gas in $\Omega$

$$\text{div}(\varrho v) = 0,$$

$$\varrho v \cdot \nabla v - \text{div} \left( 2\mu D(v) + \nu (\text{div} v) I - \pi(\varrho) I \right) = \varrho F,$$ (1)

together with the boundary conditions on $\partial \Omega$

$$v \cdot n = 0,$$

$$n \cdot T(v, \pi(\varrho)) \cdot \tau_k + f v \cdot \tau_k = 0, \quad k = 1, 2.$$ (2)

Here, $v : \Omega \rightarrow \mathbb{R}^3$ represents the sought velocity field, $\pi(\cdot) : \mathbb{R}_0^+ \rightarrow \mathbb{R}$, a given function of the sought density $\varrho$, is the pressure. The density is non-negative and the total mass of the fluid is given $\int_\Omega \varrho dx = M > 0$. The external volume force is denoted by $F$. The stress tensor is given by

$$T(v, \pi(\varrho)) = 2\mu D(v) + \nu (\text{div} v) I - \pi(\varrho) I.$$ (3)

Here, $D(v) = \frac{1}{2}(\nabla v + (\nabla v)^T)$ stands for the symmetric part of the velocity gradient and $\mu$ and $\nu$ are the viscosity coefficients. From the thermodynamical considerations it follows that $\mu > 0$, $2\mu + 3\nu > 0$.

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Remark 2. A pair \((\gamma, v)\) of \((1)\)–\((3)\) iff continuous. Already in the book [2] it can be found that if \(\gamma > 1\) in two space dimensions and \(\gamma > 2/3\) in three space dimensions, in both cases for \(\Omega \in C^2\). Next, based on the Feireisl's approach for the evolutionary problem, Novo and Novotný, see [6], proved the existence in three space dimensions for \(\gamma > 2/3\), however only for potential external forces with a small non potential perturbation. Moreover, they allowed also less regular domains (Lipschitz continuous). Already in the book [2] it can be found that if \(\gamma > 1\) for \(N=2\) or \(\gamma > 3\) for \(N=3\) the constructed solution belongs to \(L_{\text{loc}}^\infty(\Omega)\) and \(v \in W_{\text{loc}}^{1,p}(\Omega)\) for all \(p < \infty\); the proof requires quite heavy technique; see also [5] for similar results.

The slip boundary condition was rigorously studied in the paper [4]. However, only the simple form \(\pi(\gamma) = \gamma\) was considered. Here we present similar results for three space dimensions, for more general pressure functions. Our method is based on a special approximation scheme introduced in [4], see next section. The presented technique enables to control uniformly the \(L^\infty\)-norm of approximative densities and it reduces a number of necessary tricks which appear in the classical theory [2].
2. Construction of the approximation. Let \( \varepsilon > 0 \). Let \( m > 0 \) (sufficiently large). Let \( K : \mathbb{R}_+^1 \to [0,1] \) be a smooth non-increasing function such that

\[
K(t) = \begin{cases} 
1 & \text{for } t \leq m - 1 \\
0 & \text{for } m \leq t.
\end{cases}
\]

Moreover, \( K'(t) < 0 \) in \( (m - 1, m) \). Denote

\[
P(t) = \int_0^t K(s) \pi'(s) ds;
\]

(in particular \( P(t) \geq K(t) \pi(t) \)). Then we consider

\[
\varepsilon \varrho + \frac{1}{2} \text{div}(K(\varrho) \omega v) + \frac{1}{2} K(\varrho) \omega v \cdot \nabla v
\]

\[
- \text{div}(2\mu D(v) + \nu(\text{div} v)I - P(\varrho)I) = K(\varrho) F
\]

\[
 = \varepsilon \varrho + \text{div}(K(\varrho) \omega v) - \varepsilon \Delta \varrho = \varepsilon h K(\varrho)
\]

\[
\frac{\partial \varrho}{\partial n} = 0, \quad \frac{\partial v}{\partial n} = 0, \quad n \cdot v = 0, \quad k = 1, 2
\]

Proposition 1. Let \( \varepsilon > 0, h = \frac{M}{|M|} : K(\cdot) \) satisfy (5). Let assumptions of Theorem 1 be satisfied. Then there exists a solution to (6) such that \( \varrho_\varepsilon \in W^{2,p}(\Omega) \), \( \varrho_\varepsilon \in W^{2,p}(\Omega) \) for all \( p < \infty \). Moreover, \( 0 \leq \varrho_\varepsilon \leq m, \int_{\Omega} \varrho_\varepsilon\,dx \leq M \).

Proof: The proof of Proposition 1 will be divided into several steps.

Step 1: We first show the a priori estimates for the density. In order to show the \( L^1 \) bound, we simply integrate (6) over \( \Omega \). Then

\[
\varepsilon \int_{\Omega} \varrho dx + \int_{\partial \Omega} K(\varrho) \omega v \cdot n d\sigma - \varepsilon \int_{\partial \Omega} \frac{\partial \varrho}{\partial n} d\sigma = \varepsilon h \int_{\Omega} K(\varrho) dx.
\]

Due to the regularity of \( \varrho \) and \( v \), together with the boundary conditions, the surface integrals are equal to zero which gives immediately the \( L^1 \) bound for the approximative density. Next, we integrate over the sets \( \{ x \in \Omega; \varrho(x) < 0 \} \) and \( \{ x \in \Omega; \varrho(x) > m \} \), provided these sets are regular. (If this is not the case, we replace \( m \) by \( m^+ \) and \( m^- \), respectively.) Thus

\[
\varepsilon \int_{\Omega^-} \varrho dx + \int_{\partial \Omega^-} K(\varrho) \omega v \cdot n d\sigma - \varepsilon \int_{\partial \Omega^-} \frac{\partial \varrho}{\partial n} d\sigma = \varepsilon h \int_{\Omega^-} K(\varrho) dx,
\]

where \( \Omega^- = \{ x \in \Omega; \varrho(x) < 0 \} \). Again, the boundary terms are zero, which, due to the properties of \( K(\cdot) \) implies \( \| \{ x \in \Omega; \varrho(x) < 0 \} \| = 0 \). Similarly we prove \( \varrho(x) \leq m \); here the fact that \( \frac{\partial \varrho}{\partial n} \leq 0 \) at the boundary of \( \{ x \in \Omega; \varrho(x) > m \} \) is used.

Step 2: Next we consider the approximative continuity equation. We denote for \( p \in [1, \infty] \)

\[
M_p = \{ \varrho \in W^{1,p}(\Omega); \varrho \cdot n = 0 \text{ at } \partial \Omega \}
\]

and define \( S : M_\infty \to W^2_\infty(\Omega), 1 \leq p < \infty, \quad S(\varrho) = \varrho \), where \( \varrho \) solves

\[
- \varepsilon \Delta \varrho + \varepsilon \varrho + \text{div}(K(\varrho) \omega v) = \varepsilon h K(\varrho) \quad \text{in } \Omega
\]

\[
\frac{\partial \varrho}{\partial n} = 0 \quad \text{at } \partial \Omega.
\]

We have
Lemma 1. Let assumptions of Proposition 1 be satisfied. Then $S$ is well defined as the operator from $M^\infty$ to $W^1_p(\Omega)$ for any $p < \infty$. Moreover, if $s = S(v)$ then $0 \leq s \leq m$ in $\Omega$, $\int_\Omega s dx = M$ and if $\|v\|_{1,\infty} \leq L$, $L > 0$ then

$$\|s\|_{2,p} \leq C(\varepsilon, p, \Omega)(1 + L)h, \quad 1 \leq p < \infty. \quad (7)$$

Proof: The a priori bounds $0 \leq s \leq m$ and the $L^1$ bound follow from the estimates presented in Step 1. We have to prove solvability of the approximative continuity equation. To this aim, for $t \in [0,1]$, we consider the following family of problems

$$v_t = T_t(\xi), \quad \xi \in W^{2,p}(\Omega),$$

where

$$-\varepsilon \Delta v_t = -t\varepsilon\xi - t\text{div}(K(\xi)v) + t\varepsilon hK(\xi) \quad \text{in } \Omega$$

$$\frac{\partial v_t}{\partial n} = 0 \quad \text{at } \partial \Omega. \quad (8)$$

First note that we look for solutions to (8) such that $\int_\Omega v_t dx = h \int_\Omega K(v_t) dx$. Thus the solution to (8) is uniquely determined. Especially, for $t = 0$, the solution is $s = A$ with $A$ being the unique solution to

$$A = hK(A).$$

Note that $0 < A < m$ for $h > 0$. Further, we have to verify that there is $C = C(p, L, \varepsilon, \Omega, h)$ such that $\|s\|_{1,p} \leq C$ which implies that $0 \notin (I - T_t)(\partial B_C)$, where

$$B_C = \{ s \in W^{1,p}(\Omega); \|s\|_{1,p} \leq C, \int_\Omega s dx = h \int_\Omega K(s) dx \}.$$

Note that if $v_t = T_t(\xi)$ and $\xi \in W^{1,p}(\Omega)$ with the bound $\|\xi\|_{1,p} \leq C$ then also $\nabla v \in L^1(\Omega)$ and $\|v\|_{2,p} \leq C(p, L, \varepsilon, \Omega, h)$.

Using as test function in $v_t = T_t(\xi)$ the function $\xi^\beta$ with $0 < \beta \leq 1$ we get

$$\|v_t\|_{1,p+1} \leq C(\beta, \Omega)h$$

for $\beta$ chosen appropriately, for details see [8], Lemma 4.30. Next we rewrite our problem as

$$-\varepsilon \Delta v_t = \text{div } b \quad \text{in } \Omega$$

$$\frac{\partial v_t}{\partial n} = b \cdot n \quad \text{at } \partial \Omega$$

with $b = \varepsilon \beta B(K(v_t)h - v_t) - \xi v_t K(v_t)$, where $B : \{ f \in L^p(\Omega); \int_\Omega f dx = 0 \} \to W^{1,p}_0(\Omega)$ is the solution operator to $\text{div } u = f$ with homogeneous Dirichlet boundary condition. Due to the regularity of the solutions to the Neumann problem (see [8], Lemma 4.27) and the bootstrap argument

$$\|\nabla v\|_p \leq C(\varepsilon, p, \Omega)(1 + L)h, \quad 1 \leq p < \infty$$

which finally gives

$$\|v\|_{1,p} \leq C(\varepsilon, p, \Omega)(1 + L)h;$$

hence (see above)

$$\|s\|_{2,p} \leq C(\varepsilon, p, \Omega)(1 + L)h.$$
Using the fact that (see (4))
we get, after employing the fact that
we use as test function in (10) the solution
the l.h.s. of (12). It remains to estimate the terms on the r.h.s. of (12).
To do this,
the term with the negative sign can be easily controlled by the positive terms on
the l.h.s. of (12). It remains to estimate the terms on the r.h.s. of (12). To do this,
we consider \( \psi \), a suitable solution to
\[
\text{div } \psi = P(\varrho) - \frac{1}{|\Omega|} \int_{\Omega} P(\varrho) dx \quad \text{in } \Omega,
\]
\[
\psi = 0 \quad \text{at } \partial \Omega,
\]
such that
\[
\|\nabla \psi\|_q \leq C(\varrho)\|P(\varrho)\|_q,
\]
see [1] or [8]. Multiplying (10) by \( \psi \) and integrating over \( \Omega \) we get
\[
t \int_{\Omega} |P(\varrho)|^2 dx = t \int_{\Omega} P(\varrho) dx + \varepsilon t \int_{\Omega} v_t \cdot \psi dx + \varepsilon t \int_{\Omega} \psi_t \cdot \psi dx
\]
\[-\frac{1}{2}t \int_{\Omega} K(\varrho)(v_t \otimes v_t) : \nabla \psi dx + \frac{1}{2}t \int_{\Omega} K(\varrho)(v_t \cdot \nabla v_t) \cdot \psi dx + \mu \int_{\Omega} \nabla v_t : \nabla \psi dx
\]
\[+ (\mu + v) \int_{\Omega} (\text{div } v_t)(\text{div } \psi) dx - t \int_{\Omega} K(\varrho) \varrho \mathbf{F} \cdot \psi dx.
\]
Thus, as the most restrictive term is the convective one \((C.T.)\), we get for \( \gamma - 3 \geq 0 \)
\[
|C.T.| \leq t \|v_t\|_{\infty}^2 \|\nabla \psi\|_2 \|K(\varrho)\|_0 \leq C t \|v_t\|_{\infty}^2 \|P(\varrho)\|_2 \left( \int_{\Omega} |P(\varrho)|^\frac{3}{2} dx \right)^{\frac{1}{2}}
\]
\[\leq C(M) t \|v_t\|_{\infty}^2 \|P(\varrho)\|_2^{1 + \frac{6\gamma - 5\gamma - \gamma - 3\gamma}{6\gamma + 5\gamma - \gamma - 3\gamma}}.
\]
Hence
\[
\sqrt{t} \|P(\varrho)\|_2 \leq C_1 \sqrt{t} \|v_t\|_{\infty}^{\frac{6\gamma - 5\gamma - \gamma - 3\gamma}{6\gamma + 5\gamma - \gamma - 3\gamma}} + C_2 \sqrt{t}
\]
and due to the Korn inequality (thus the additional assumptions on \( \Omega \) are needed for \( f = 0 \)) we get
\[
\|v_t\|_{1,2} \leq C_1 \|v_t\|_{\infty}^{\frac{6\gamma - 5\gamma - \gamma - 3\gamma}{6\gamma + 5\gamma - \gamma - 3\gamma}} + C_2
\]
and the desired estimate of \( v_t \) and \( P(\varrho) \) if \( \frac{6\gamma - 5\gamma - \gamma - 3\gamma}{6\gamma + 5\gamma - \gamma - 3\gamma} < 1 \), which is evidently satisfied.

With (11) at hand, using the estimates from Lemma 1, it is not difficult to see that there exists \( C = C(\varepsilon, \Omega, M, F) \), independent of \( t \), such that
\[
\|v_t\|_{1,\infty} \leq C, \quad t \in [0, 1]
\]
and thus
\[
0 \notin (I - \mathcal{T})(\partial B_C)
\]
with \( B_C = \{ u \in M_\infty : \|u\|_{1,\infty} \leq C \} \).

**Step 5:** To finish the proof of Proposition 1 we recall that for \( t = 0 \) the unique fixed point of \( v_0 = T_0(v_0) \) is \( v_0 = 0 \). Moreover, for any \( t \in [0, 1] \), \( T_t(B) \) is a precompact set in \( M_\infty \), where \( B \) is a ball of center 0 in \( M_\infty \). Finally
\[
\|T_t(u) - T_s(u)\|_{1,\infty} \leq C(\Omega, B, h, F)|t - s|, \quad u \in \overline{B}
\]
and the Leray–Schauder fixed point finished the proof of Proposition 1.

Next we will prove certain \( \varepsilon \)-independent estimates for \( \varrho \) and \( v_\varepsilon \), our solutions to the approximative system. To get them, we basically follow [4], however, due to the general pressure law we will present some more details here. To simplify the notation, throughout this section, we skip writing the subindeces \( \varepsilon \).
Lemma 3. Under the assumptions of Proposition 1 we have for $\gamma_- \geq 3$

$$\|v\|_{1,2}^2 + \|P(\phi)\|_2^2 + \epsilon \|\nabla \phi\|_2^2 \leq C,$$

$$\|v\|_{1,q} + \|P(\phi)\|_q \leq C_1 + C_2 m^{\gamma_- (1 - \frac{2}{q})} \quad \text{for} \quad 2 < q < \infty,$$

where $C$ is independent of $m$, $\epsilon$, however, it depends on $q$, $M$, $\|F\|_{\infty}$ and $\Omega$.

Proof: The proof of

$$\|v\|_{1,2}^2 + \|P(\phi)\|_2^2 \leq C$$

is a consequence of Lemma 2 (with $t = 1$) and was carried out above.

Next, we multiply the approximative continuity equation (6) by $\phi$ and integrate over $\Omega$; it gives

$$\epsilon \int (\phi^2 + |\nabla \phi|^2) dx + \int \text{div} v K(\phi) \phi^2 dx - \int \text{div} v \left( \int_0^\infty K(t) dt \right) dx = \epsilon h \int K(\phi) \phi dx.$$

As $\gamma_- \geq 3$ we get due to estimates proved above

$$\sqrt{2} \|\nabla \phi\|_2 \leq C,$$

independently of $m$ and $\epsilon$.

Finally, we prove the estimates which depend on the approximation parameter $m$. First, due to the standard interpolation we have

$$\|P(\phi)\|_q \leq \|P(\phi)\|_2^{\frac{2}{q}} \|P(\phi)\|_\infty^{\frac{q}{2}} \leq C_1 m^{\gamma_- (1 - \frac{2}{q})} + C_2. \quad (14)$$

Next, we rewrite the approximative momentum equation (6) to the form

$$-\text{div} \left( 2\mu D(v) + \nu (\text{div} v) I \right) = -\frac{1}{2} \text{div} (K(\phi) \phi \otimes v) - \frac{1}{2} K(\phi) \phi \cdot \nabla v$$

$$-\epsilon h \phi - \epsilon \phi v - \phi \cdot \nabla \phi + K(\phi) \phi F$$

$$v \cdot n = 0 \quad \text{at } \partial \Omega$$

and we use the elliptic theory. First for $q \leq 3$ and then for $q > 3$, using

$$\|\phi\|_r \leq C\|P(\phi)\|_\infty^{\frac{r}{q}} \leq C_1 m^{\frac{r}{q} (1 - \frac{2}{q})} + C_2 \quad \text{with} \quad r > 2\gamma_-,$$

we get after standard, however tedious calculations that the most restrictive term is $\nabla P(\phi)$. Thus

$$\|\nabla v\|_q \leq C_1 \|P(\phi)\|_q + C_2 \leq C_1 m^{\gamma_- (1 - \frac{2}{q})} + C_2.$$

The lemma is proved.

3. Limit process — a priori estimates. Next we pass with $\epsilon \to 0^+$. Using the a priori bounds from Lemma 3, together with the $L^\infty$ bound for the density we obtain a subsequence such that:

$$v_\epsilon \to v \quad \text{in } W^{1,p}(\Omega) \quad \text{for} \quad 1 \leq p < \infty,$$

$$v_\epsilon \to v \quad \text{in } L^q(\Omega) \quad \text{for} \quad 1 \leq q \leq \infty,$$

$$\phi_\epsilon \to \phi \quad \text{in } L^\infty(\Omega),$$

$$P(\phi_\epsilon) \to P(\phi) \quad \text{in } L^\infty(\Omega),$$

$$K(\phi_\epsilon) \phi_\epsilon \to K(\phi) \phi \quad \text{in } L^\infty(\Omega).$$

(16)
Passing to the limit, using also the properties of continuity equation in the momentum equation we end up with
\[
K(\eta)\varepsilon \varphi \cdot \nabla \varepsilon v - \text{div} \left( 2\mu D(\varepsilon v) + \nu(\text{div} \varepsilon v)I - \overline{P(\eta)}I \right) = K(\eta)\varepsilon F, \\
\text{div} \left( K(\eta)\varepsilon \varphi \right) = 0,
\] (17)
together with the boundary conditions, satisfied in the weak sense.

To get that the limit is, indeed, weak solution to our problem, we need to verify:

(i) \(K(\eta)\varepsilon \varphi = \eta\) (i.e. \(\eta \leq m - 1\) a.e.)

(ii) \(\overline{P(\eta)} = P(\eta)\).

First, we will concentrate on problem (i). We introduce the Helmholtz decomposition of the velocity
\[
\varepsilon v = \nabla \varphi + \text{curl} \, \varepsilon A,
\]
where the divergence-free part of the velocity satisfies the elliptic problem
\[
\begin{align*}
\text{curl} \, \text{curl} \, \varepsilon A &= \text{curl} \, \varepsilon v = \varepsilon \omega \quad \text{in } \Omega, \\
\text{div} \, \text{curl} \, \varepsilon A &= 0 \quad \text{in } \Omega, \\
\text{curl} \, \varepsilon A \cdot n &= 0 \quad \text{at } \partial \Omega.
\end{align*}
\] (18)

Thus (see e.g. [9])
\[
\|\nabla \text{curl} \varepsilon A\|_q \leq C\|\omega\|_q \quad \text{and} \quad \|\nabla^2 \text{curl} \varepsilon A\|_q \leq C\|\omega\|_{1,q},
\]
for \(1 < q < \infty\). The same can be written for the approximative velocity \(\varepsilon v\), just adding the index \(\varepsilon\) to \(\varphi\), \(\omega\) and \(\varepsilon\).

However, we must be much more careful with the estimates of the vorticity as in the two-dimensional case; the slip boundary conditions are "responsible" for the improved estimates up to the boundary. We have
\[
-\mu \Delta \varepsilon \omega = \text{curl} \left( K(\varepsilon \varphi)\varepsilon \varphi F - K(\varepsilon \varphi)\varepsilon \varphi (\varepsilon v \cdot \nabla \varepsilon v) - \varepsilon h\varepsilon v - \varepsilon \varepsilon \varphi \varepsilon v \right) - \frac{1}{2} \varepsilon \varepsilon hK(\varepsilon \varphi)\varepsilon v + \frac{1}{2} \varepsilon \varepsilon \varphi \varepsilon v - \text{curl} \left( \frac{1}{2} \varepsilon \Delta \varepsilon \omega \right) := H_1 + H_2,
\] (20)
with \(\chi_k\) — main curvatures of the boundary. The boundary relations (20)\(2,3\) follow from properties of slip boundary conditions (2)\(2\) — see [3] or [11]. (It is enough to differentiate (2)\(1\) with respect to tangent directions and apply the suitable condition (2)\(2\).) The last boundary condition is just the consequence of \(\varepsilon \omega = \text{curl} \, \varepsilon v\).

**Lemma 4.** We have \(\varepsilon \omega = \varepsilon \omega^0 + \varepsilon \omega^1 + \varepsilon \omega^2\) with
\[
\|\varepsilon \omega^0\|_{1,q} + \|\varepsilon \omega^1\|_{1,q} \leq C_1 m^{1+\gamma^+\left(\frac{1}{2} - \frac{\lambda}{2}\right)} + C_2 \quad \text{for } q \geq \frac{3}{2},
\]
\[
\|\varepsilon \omega^2\|_q \leq C\varepsilon^{\frac{1}{2}} \quad \text{for } q \leq 2.
\]

**Proof.** We denote by \(\varepsilon \omega^0\) the solution to
\[
-\mu \Delta \varepsilon \omega^0 = 0 \quad \text{in } \Omega,
\]
\[
\varepsilon \omega^0 \cdot \varepsilon \tau_1 = -2\chi_2 - f/\mu \varepsilon v \cdot \varepsilon \tau_2 \quad \text{on } \partial \Omega.
\]
(21)
Let $\alpha^0$ denote the divergence-free extension to the tangent boundary values of $\omega^0$ (constructed e.g. as a solution to the Stokes problem with zero right-hand side and additionally $\alpha^0 \cdot n = 0$ at $\partial\Omega$). Thus we have
\[-\mu \Delta (\omega^0 - \alpha^0) = \mu \Delta \alpha^0 \quad \text{in} \ \Omega,
(\omega^0 - \alpha^0) \cdot \tau_1 = 0
(\omega^0 - \alpha^0) \cdot \tau_2 = 0
\text{div} (\omega^0 - \alpha^0) = 0 \quad \text{at} \ \partial\Omega.
\]
(22)

Solvability of the above system can be found in [9] or [10] (here the assumption of connectness of the boundary is required) with the following estimates for solutions to (22)
\[\|\omega^0\|_{1,q} \leq C\|\alpha^0\|_{1,q} \leq C\|v\|_{1,q} \leq \begin{cases} C & q \leq 2 \\
C_1 m^{\gamma^+(1-\frac{2}{q})} + C_2 & q > 2. \end{cases}\]

Next, we consider problem for $\omega^1$. It solves problem of the type (20) with zero boundary data and right-hand side $H_1$. Thus we get as above (the most restrictive term is the convective one)
\[\|\omega^1\|_{1,q} \leq C(1 + \|q_\varepsilon \cdot \nabla v_\varepsilon\|_q) \leq C_1 m^{1+\gamma^+(\frac{1}{2} - \frac{2}{q})} + C_2.\]

Finally, for the last term, we cannot expect estimate in $W^{1,q}(\Omega)$. However, for $q \leq 2$ we have
\[\|\omega^2\|_q \leq C\|\varepsilon \Delta q_\varepsilon \cdot v_\varepsilon\|_{1,q} \leq C\varepsilon \left(\|\nabla q_\varepsilon \cdot v_\varepsilon\|_q + \|\nabla q_\varepsilon \cdot \varepsilon\|_q\right) \leq C(m)\varepsilon^{\frac{1}{2}}.\]

Next, we introduce the effective viscous flux. We define it (see (18)) as the potential part of the momentum equation
\[G_\varepsilon = \frac{1}{\nu} - (2\mu + \nu)\Delta \varphi_\varepsilon,
G = \frac{1}{\nu} - (2\mu + \nu)\Delta \varphi.\]

Lemma 5. We have for $G$ and $G_\varepsilon$ defined above
(i) $\|G\|_\infty \leq C_1 + C_2(\eta)m^{1+\frac{3}{2}+\gamma^+ + \frac{2}{q}}$, $\eta > 0$, arbitrarily small
(ii) $G_\varepsilon \to G$ in $L^p(\Omega)$, $p \leq 2$.

Proof. As $\nabla G = K(\varphi)\varphi F - K(\varphi)\varphi F \cdot \nabla v_\varepsilon + \mu \Delta \text{curl} A$, we have
\[\|\nabla G\|_q \leq \|K(\varphi)\varphi F\|_q + \|K(\varphi)\varphi F \cdot \nabla v_\varepsilon\|_q + \|\mu \Delta \text{curl} A\|_q \leq C_1 + C_2 m^{1+\gamma^+(\frac{1}{2} - \frac{2}{q})}.\]

(Note that, using proof of Lemma 4)
\[\|\Delta \text{curl} A\|_q \leq C\|\omega\|_{1,q} \leq C\left(\|v\|_{1,q} + \|K(\varphi)\varphi F\|_q + \|K(\varphi)\varphi F \cdot \nabla v_\varepsilon\|_q\right).\]

The corresponding power of $m$ is a consequence of Lemma 4.) Using the estimate above with $q > 3$, sufficiently close to 3, together with
\[\left| \int_\Omega G dx \right| \leq \int_\Omega (\frac{1}{\nu} - (2\mu + \nu)\text{div} v) dx \leq C,
\]
we get by the imbedding theorem ($\frac{1}{2} - \frac{\gamma^+}{2} = \frac{3}{2} + (\frac{3}{2} - \frac{q}{2})$)
\[\|G\|_\infty \leq C\left(\|\nabla G\|_q + \left| \int_\Omega G dx \right|\right) \leq C_1 m^{1+\gamma^+ + \eta} + C_2 \]
with $\eta$ positive, arbitrarily small.
Next, for the difference \( G_\varepsilon - G \) we have
\[
\nabla (G_\varepsilon - G) = (K(\varepsilon \varrho_\varepsilon) \varrho_\varepsilon - K(\varrho) \varrho) F + K(\varrho) \varrho v - K(\varepsilon \varrho_\varepsilon) \varrho_\varepsilon v - \nabla v \\
- \varepsilon h \varrho_\varepsilon v + \frac{1}{2} \varepsilon \varrho_\varepsilon v - \frac{1}{2} \varepsilon \Delta \varrho_\varepsilon v - \frac{1}{2} \varepsilon h K(\varepsilon \varrho_\varepsilon) \varrho_\varepsilon v \\
+ \Delta (\text{curl} A_\varepsilon - \text{curl} A).
\]

Similarly as in the estimate above we prove that \( \nabla (G - G_\varepsilon) = B_\varepsilon^1 + B_\varepsilon^2 + B_\varepsilon^3 \), where \( B_\varepsilon^1 \rightarrow 0 \) in \( L^q(\Omega) \), \( B_\varepsilon^2 \rightarrow 0 \) in \( L^q(\Omega) \) and \( B_\varepsilon^3 \rightarrow 0 \) in \( W^{-1,q}(\Omega) \) for any \( q \leq 2 \). Recalling that the mean value of \( G - G_\varepsilon \) goes to zero, also the second statement of the lemma is proved.

Next we answer on problem (i). We have

**Theorem 2.** Let \( m \) from the definition of function \( K(\cdot) \) – see (5) – be sufficiently large. Then there is a number \( \kappa > 0 \) such that \( \|G\|_{L^\infty} \leq \kappa < m - 2 \) and, up to a subsequence, it holds
\[
\lim_{\varepsilon \to 0^+} |\{ x \in \Omega; \varrho_\varepsilon(x) > \kappa \}| = 0.
\]

Especially, for this subsequence, \( K(\varepsilon \varrho) = \varrho \).

**Proof.** Let us take e.g. \( \kappa = m - 3 \) and define a non-increasing function
\[
M(t) = \begin{cases} 
1 & \text{for } t \leq \kappa \\
\in (0,1) & \text{for } \kappa < t < \kappa + 1 \\
0 & \text{for } \kappa + 1 \leq t,
\end{cases}
\]
such that \( M'(t) < 0 \) for \( t \in (\kappa, \kappa + 1) \).

Take \( l \in \mathbb{N} \) (later on we require \( l \) sufficiently large) and multiply the approximative continuity equation (6) by \( M'(\varrho_\varepsilon) \). As
\[
\varepsilon \int_{\Omega} M'(\varrho_\varepsilon) \Delta \varrho_\varepsilon dx = -\varepsilon l \int_{\Omega} M^{l-1}(\varrho_\varepsilon) M'(\varrho_\varepsilon) |\nabla \varrho_\varepsilon|^2 dx \geq 0,
\]
we get
\[
\int_{\Omega} M^l(\varrho_\varepsilon) \text{div} (K(\varrho_\varepsilon) \varrho_\varepsilon v_\varepsilon) dx \geq -\varepsilon \int_{\Omega} \varrho_\varepsilon M^l(\varrho_\varepsilon) dx + \varepsilon h \int_{\Omega} K(\varrho_\varepsilon) M^l(\varrho_\varepsilon) dx := R_\varepsilon,
\]
where \( R_\varepsilon \to 0 \) as \( \varepsilon \to 0^+ \), uniformly in \( l \). Thus
\[
-l \int_{\Omega} M^{l-1}(\varrho_\varepsilon) M'(\varrho_\varepsilon) K(\varrho_\varepsilon) \varrho_\varepsilon v_\varepsilon \cdot \nabla \varrho_\varepsilon dx \geq R_\varepsilon.
\]
As \( M(t) = 0 \) for \( t \geq \kappa + 1 \), we get
\[
l \int_{\Omega} \left( \int_0^{\varrho_\varepsilon} t M^{l-1}(t) M'(t) dt \right) \text{div} v_\varepsilon dx \geq R_\varepsilon. \tag{25}
\]
From the definition of \( G_\varepsilon \) – see (23), recalling that \( M'(t) = 0 \) for \( t < \kappa \) (we know that \( \varrho_\varepsilon \leq m \), inequality (25) yields
\[
- \kappa \int_{\Omega} \left( \int_0^{\varrho_\varepsilon} t M^{l-1}(t) M'(t) dt \right) P(\varrho_\varepsilon) dx \\
\leq \kappa \int_{\Omega} \left( \int_0^{\varrho_\varepsilon} t M^{l-1}(t) M'(t) dt \right) G_\varepsilon dx + (2\mu + \nu) |R_\varepsilon|.
\]
Thus
\[
\frac{\kappa}{m} \int_{\{ \varrho_\varepsilon > \kappa \}} (1 - M'(\varrho_\varepsilon)) P(\varrho_\varepsilon) dx \leq \int_{\{ \varrho_\varepsilon > \kappa \}} (1 - M'(\varrho_\varepsilon)) |G_\varepsilon| dx + \frac{(2\mu + \nu)}{m} |R_\varepsilon|. \tag{26}
\]
Due to (4) inequality (26) yields
\[
\left( \frac{\kappa}{m} \left( \frac{a_2}{\gamma^-} \kappa^{-} + \frac{b_2}{2} \kappa^2 \right) - \|G\|_\infty \right) |\{ \rho_\varepsilon > \kappa \}| \leq \frac{\kappa}{m} \|P(\rho_\varepsilon)\|_{L^2(\{\rho_\varepsilon > \kappa\})} \|M^I(\rho_\varepsilon)\|_{L^2(\{\rho_\varepsilon > \kappa\})} + \|G - G_\varepsilon\|_1 + \frac{2\mu + \nu}{m} |R_\varepsilon|. \tag{27}
\]
Now, under the assumption that \( m \) is sufficiently large, we find, using Lemma 5 and the assumption \( \frac{2}{3} \gamma^+ + 1 < \gamma^- \), number \( \kappa \) such that
\[
\left( \frac{\kappa}{m} \left( \frac{a_2}{\gamma^-} \kappa^{-} + \frac{b_2}{2} \kappa^2 \right) - \|G\|_\infty \right) \geq 1
\]
and therefore (27) implies
\[
|\{ \rho_\varepsilon > \kappa \}| \leq C \left( \|M^I(\rho_\varepsilon)\|_{L^2(\{\rho_\varepsilon > \kappa\})} + \|G - G_\varepsilon\|_1 + |R_\varepsilon| \right). \tag{28}
\]
We estimate the r.h.s. of (28). Fix \( \delta > 0 \). There is \( \varepsilon_0 > 0 \) such that for \( \varepsilon < \varepsilon_0 \)
\[
C \left( \|G - G_\varepsilon\|_1 + |R_\varepsilon| \right) < \frac{\delta}{2}.
\]
Moreover, as sequence \( \{M^I(\rho_\varepsilon)\}_{\varepsilon \in \mathbb{N}} \) for \( \{ x \in \Omega : \rho_\varepsilon(x) < \kappa \} \) tends to zero pointwise monotonely, there is \( l = l(\delta, \varepsilon) \) such that
\[
C \|M^I(\rho_\varepsilon)\|_{L^2(\{\rho_\varepsilon > \kappa\})} < \frac{\delta}{2} \quad \text{and therefore} \quad \lim_{\varepsilon \to 0^+} |\{ \rho_\varepsilon > \kappa \}| \leq \delta.
\]
As \( \delta > 0 \) can be chosen arbitrarily small, the theorem is proved. \( \square \)

4. Convergence of the pressure. In this part we finish the proof of Theorem 1. We show that the convergence of approximative density can be improved to the strong one in any \( L^q \)-space for \( q < \infty \). Thus we solve our problem (ii).

**Lemma 6.** We have
\[
\int_\Omega P(\rho) \, dx \leq \int_\Omega G \, dx.
\]

**Proof.** For a fixed \( \delta > 0 \) we multiply the approximative continuity equation (6)_2 by \( \ln m - \ln(\rho_\varepsilon + \delta) \) and integrate over \( \Omega \) which, after straightforward calculations, gives
\[
\int_\Omega K(\rho_\varepsilon) \rho_\varepsilon \nabla \rho_\varepsilon \cdot \nabla \rho_\varepsilon \, dx \geq \varepsilon \int_\Omega (\rho_\varepsilon K(\rho_\varepsilon) - \rho_\varepsilon)(\ln m - \ln(\rho_\varepsilon + \delta)) \, dx. \tag{29}
\]
We pass with \( \delta \to 0^+ \). Following the idea presented in [4], we get
\[
\int_\Omega K(\rho_\varepsilon) \nabla \rho_\varepsilon \cdot \nabla \rho_\varepsilon \geq \varepsilon C(m)
\]
and using Theorem 2 we conclude that
\[
\int_\Omega \nabla \rho_\varepsilon \, dx \geq R_\varepsilon, \tag{30}
\]
where \( R_\varepsilon \to 0 \) for \( \varepsilon \to 0^+ \). Hence from (23)_1 and (30) we obtain
\[
\int_\Omega P(\rho_\varepsilon) \, dx \leq \int_\Omega G \, dx - (2\mu + \nu) R_\varepsilon. \tag{31}
\]
Passing with \( \varepsilon \to 0^+ \) in inequality (31) we get the result. \( \square \)

Next we prove
Lemma 7. We have
\[ \int_\Omega P(\varrho) \varrho \, dx = \int_\Omega G \varrho \, dx. \quad (32) \]

Proof. Using the fact that we can approximate \( \varrho \) in suitable spaces by a sequence of smooth functions \( \varrho_n \) in such a way that \( 0 \leq \varrho_n \leq m \) and \( \mathbf{v} \cdot \nabla \varrho_n \to \mathbf{v} \cdot \nabla \varrho \) (see [7] for more details – here is hidden Friedrich’s lemma) and the fact that
\[ \int_\Omega (\varrho_n \text{div} \mathbf{v} + \mathbf{v} \cdot \nabla \varrho_n) \, dx = 0, \]
we get, passing with \( n \to \infty \),
\[ \int_\Omega (\varrho \text{div} \mathbf{v} + \mathbf{v} \cdot \nabla \varrho) \, dx = 0. \quad (33) \]

Next, using as a test function for the continuity equation: \( -\ln \frac{\varrho}{\varrho_n + \delta} \) and passing with \( n \to \infty \) and with \( \delta \to 0^+ \) we get
\[ \int_\Omega \mathbf{v} \cdot \nabla \varrho \, dx = 0. \]

Thus from (23)_2 and (33) we conclude the sought identity (32), which finishes the proof of the lemma.

To conclude we recall that the assumptions on \( P(\cdot) \) imply that the function is monotone on \( \mathbb{R}_0^+ \). Thus, (details e.g. Lemma 3.35 from [8]) we have
\[ P(\varrho) \varrho \geq P(\overline{\varrho}) \overline{\varrho} \quad \text{a.e. in } \Omega. \quad (34) \]

This, together with Lemmas 6 and 7 gives
\[ P(\varrho) \varrho = P(\overline{\varrho}) \overline{\varrho} \quad \text{a.e. in } \Omega. \quad (35) \]

Then, using standard methods as Lemma 3.39 from [8], information from (34) and (35) yields \( P(\varrho) = P(\overline{\varrho}) \). Moreover, using the Lebesgue dominated convergence theorem and Theorem 2 we conclude the strong convergence of approximative densities \( \rho_n \to \rho \) in \( L^q(\Omega) \) for any \( q < \infty \). It implies that limit found by (16) satisfies the definition of the weak solution – see Remark 2 – to the original system (1)–(3). Theorem 1 is proved.

REFERENCES


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