On the Convergence Order of COMSOL Solutions

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Abstract: The convergence of numerical solutions is mainly determined by the convergence order, which quantifies the improvement of the solution, when the mesh is refined.
In the paper we examine various different differential equations and the convergence behavior of their COMSOL Finite Element solutions. In the first part we describe the general procedure, how for irregular meshes the convergence order is determined.
In the main part of the paper several systems (modes, equations) are studied. We start with classical application modes, as the Laplace equation and the Poisson equation, the latter for smooth and non-smooth right hand sides (like Dirac's function). We proceed with more complex Navier-Stokes equations and finally with a (multi-physics) flow and transport example.
The numerically observed convergence rates are compared with theoretical results, as far as these are available. The examples show that for the studied cases the convergence rate of COMSOL numerics is identical to the theoretically derived convergence rates.

Keywords: Finite Elements, convergence order, adaptive meshes, Poisson equation, potential equation

1. Introduction

The convergence of a numerical solution towards the analytical solution of a set-up of one or several partial differential equations generally depends on various characteristics of the problem and of the numerical algorithm. The convergence order is a measure for the improvement of the solution as a consequence of mesh refinement.

Here we examine several test problems in 2D using classical partial differential equations (Poisson- and Laplace equations), and from different application fields, from electrostatics to fluid dynamics. We explore various set-ups with one or two differential equations, with different boundary conditions and right hand sides. We use different Finite Element spaces, and compute the gain from using adaptive meshing. We also compare with theoretical results: for most models the convergence order lies between 2 and 3 (for default quadratic elements), i.e. between a linear and a quadratic increase of accuracy with the spatial grid size.

2. Determination of Convergence Order

The convergence order \( \vartheta \) is defined by the relationship
\[
\|e\| = O(h^\vartheta)
\]
where \( e \) denotes the error, \( \|\cdot\| \) a norm, and \( h \) the typical element size.
In order to determine the convergence order from numerical runs, the errors of runs with different refinement level have to be related. As it is difficult to evaluate the mean size of the elements \( h \), one may alternatively use the degrees of freedom (DOF) for the determination of the convergence rate. Jänicke & Kost (1999) propose the formula
\[
\vartheta = -2 \frac{\ln(\|e_1\|) - \ln(\|e_2\|)}{\ln(\text{DOF}_1) - \ln(\text{DOF}_2)}
\]
where subscripts denote run number (1999). Formula (2) is valid for 2D problems and has to be replaced by corresponding formulae for 1D or 3D problems, because the characteristic grid spacing \( h \) decreases with number of DOF.

3. Poisson Equation with 1st Type Boundary Condition and Non-singular Right Hand Side

3.1 Testcase 1

The first test problem concerns the Poisson equation
\[
-\nabla^2 u = 1 \ \text{within the unit circle}
\]
with boundary condition of Dirichlet type: $u = 0$ at all positions of the circle. The analytical solution is given by:

$$u(x, y) = -(x^2 + y^2 - 1)/4$$

(4)

In case of quadratic Lagrange elements, the COMSOL default setting, one obtains the exact solution in the interior (the solution is a quadratic function); only in the vicinity of the boundaries there are deviations, because the element shape at the boundary follows the geometry and the relationship between local and global coordinates is not linear (see also: COMSOL model library → benchmarks → Poisson unit disk).

Figure 1. COMSOL numerical solution for the testcase 1 Poisson problem; the solution is depicted as surface plot; the color represents the error for linear elements and for a refined grid.

Figure 1 depicts the solution and the error distribution for the testcase. The results in case of linear Lagrange elements, i.e. degrees of freedom (DOF), the error in the $L_2$-norm and the resulting values for the convergence rate are given in Table 1. The theoretical value of 2 (Ciarlet 1991) for that situation is confirmed by the numerical simulations.

Table 1: Results for the Poisson testcase 1 with Dirichlet- boundary conditions

<table>
<thead>
<tr>
<th>DOF</th>
<th>$||_1$</th>
<th>$\vartheta$</th>
<th>$||_2$</th>
<th>$\vartheta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1561</td>
<td>0.0017</td>
<td>1.95</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6145</td>
<td>0.00044</td>
<td>2.00</td>
<td></td>
<td></td>
</tr>
<tr>
<td>24383</td>
<td>0.00011</td>
<td>1.98</td>
<td></td>
<td></td>
</tr>
<tr>
<td>97163</td>
<td>0.000028</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

3.2 Testcase 2

Another testcase, tackled in the literature (Köster & Turek 2006), concerns the function

$$u(x, y) = \sin(xy) \sin((1-x)(1-y))$$

(5)

in the unit square. The results for linear and quadratic elements are given in Table 2. The convergence order for linear elements is 2, the convergence order for quadratic elements is 3. For coarse grids the error for quadratic elements is much less than for linear elements, even if compared for the same DOF. Tables 3a and 3b show results for the testcase with adaptive meshing. Default parameters were used for the adaption process. Convergence orders are in the same range as in the fixed mesh case. Moreover the results show clearly that adaptive meshing does not deliver a higher accuracy. That may be attributed to the fact that high resolution is required at all boundaries, where highest errors appear. The small remaining part in the interior of the region with coarser resolution does not count.

Table 2: Results for the Poisson testcase 2 with Dirichlet- boundary conditions; subscripts for error (max.-norm) $\|\|$ and convergence order $\vartheta$ according to element order

<table>
<thead>
<tr>
<th>DOF</th>
<th>$||_1$</th>
<th>$\vartheta_1$</th>
<th>$||_2$</th>
<th>$\vartheta_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>520</td>
<td>4.1 $10^4$</td>
<td>1.81</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2017</td>
<td>1.2 $10^4$</td>
<td>1.97</td>
<td>9.4 $10^4$</td>
<td>3.00</td>
</tr>
<tr>
<td>7945</td>
<td>3.1 $10^3$</td>
<td>1.97</td>
<td>1.2 $10^6$</td>
<td>2.98</td>
</tr>
<tr>
<td>31537</td>
<td>8.0 $10^6$</td>
<td>2.01</td>
<td>1.5 $10^7$</td>
<td>3.11</td>
</tr>
<tr>
<td>1.25 $10^5$</td>
<td>2.0 $10^6$</td>
<td>2.00</td>
<td>1.8 $10^8$</td>
<td>2.97</td>
</tr>
<tr>
<td>5 $10^7$</td>
<td>5.0 $10^7$</td>
<td>2.3 $10^9$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 3a: Results for the Poisson testcase 2 with adaptive meshing; linear elements

<table>
<thead>
<tr>
<th>DOF</th>
<th>$||_1$</th>
<th>$\vartheta_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>11397</td>
<td>2.9 $10^5$</td>
<td>2.09</td>
</tr>
<tr>
<td>42119</td>
<td>7.4 $10^6$</td>
<td>1.83</td>
</tr>
<tr>
<td>151207</td>
<td>2.3 $10^6$</td>
<td>1.95</td>
</tr>
<tr>
<td>537232</td>
<td>6.7 $10^7$</td>
<td>1.95</td>
</tr>
</tbody>
</table>
Table 3b: Results for the Poisson testcase 2 with adaptive meshing; quadratic elements

<table>
<thead>
<tr>
<th>DOF</th>
<th>$|\cdot|_2$</th>
<th>$\varphi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>11033</td>
<td>1.4 $10^{-6}$</td>
<td>2.64</td>
</tr>
<tr>
<td>57053</td>
<td>1.6 $10^{-7}$</td>
<td>3.10</td>
</tr>
<tr>
<td>28805</td>
<td>1.3 $10^{-8}$</td>
<td></td>
</tr>
</tbody>
</table>

4. Poisson Equation with Dirac Right Hand Side

The second test problem concerns the differential equation

$$-\nabla^2 u = \delta(0) \text{ within the unit circle} \quad (6)$$

with Dirac's $\delta$-function on the right hand side. The boundary conditions are again of Dirichlet type: $u = 0$ on all positions of the unit circle. The analytical solution is given by the logarithmic function:

$$u(x, y) = -\ln(r) / 2\pi = -\ln(r^2) / 4\pi \quad (7)$$

(see also: 'Implementing a point source' in the COMSOL Users Guide and the point-source benchmark example). The solution is visualized in Figure 2.

We chose the default quadratic elements. Table 4 provides the results for the test-case. The theoretical result of first order convergence (Clain, 1995) is confirmed. The quadratic convergence, which is valid for the smooth right hand side, can obviously not be reached for the non-smooth situation.

Table 4: Results for the Poisson problem with Dirac right hand side

<table>
<thead>
<tr>
<th>DOF</th>
<th>$|\cdot|_2$</th>
<th>$\varphi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1561</td>
<td>0.0021</td>
<td>1.015</td>
</tr>
<tr>
<td>6145</td>
<td>0.00010</td>
<td>1.000</td>
</tr>
<tr>
<td>24383</td>
<td>0.00005</td>
<td>1.000</td>
</tr>
<tr>
<td>97163</td>
<td>0.000025</td>
<td></td>
</tr>
</tbody>
</table>

5. Laplace equation with Dirichlet and Neumann Boundary Conditions

An example of a 2D potential problem was suggested by Jänicke & Kost (1999). There are Neumann and Dirichlet-conditions at the boundaries as can be seen in Figure 3 (Neumann at the left and the upper right boundary). The numerical solution is compared to the analytical solution. The latter was programmed using Schwarz-Christoffel transformations applying the Schwarz–Christoffel Toolbox (Driscoll) written in MATLAB®. The COMSOL results are exported to MATLAB®, in order to compare solutions and the calculate error norms.

Figure 2: COMSOL numerical solution for the Poisson problem with Dirac right hand side; the figure shows the solution as surface, streamlines and arrows.

Figure 3. COMSOL numerical solution for the potential problem with Dirichlet- and Neumann boundary conditions

Table 5 shows results for the initial mesh and 2 refinements. The convergence rate of COMSOL...
is better than the one reported for another Finite Element implementation (Jänicke & Kost 1999).

Table 5: Results for the potential problem with Dirichlet- and Neumann boundary conditions, for linear elements

| DOF  | # lin. elements | ||e|| | δ |
|------|-----------------|------|-----|---|
| 527  | 992             | 0.249| 0.95|   |
| 2045 | 3968            | 0.129| 1.07|   |
| 8057 | 15872           | 0.064| 1.05|   |
| 31985| 63488           | 0.0297| 1.02| |
| 127457| 253952         | 0.0146|       | |

Table 6: Results for the potential problem with Dirichlet- and Neumann boundary conditions, for quadratic elements

| DOF  | # quad. elem. | ||e|| | δ |
|------|---------------|------|-----|---|
| 2045 | 992           | 0.1366| 0.96| 1.08|
| 8045 | 3968          | 0.0702|      |   |
| 31985| 15872         | 0.0332| 1.23|   |
| 127457| 63488        | 0.0142|       | |

Table 7: Results for the potential problem with Dirichlet- and Neumann boundary conditions, for linear elements and adaptive meshing

| DOF  | # lin. elements | ||e|| | δ |
|------|-----------------|------|-----|---|
| 2369 | 4603            | 3.3 10^{-2} | 2.22| |
| 9797 | 19318           | 6.8 10^{-3} | 1.86| |
| 38531| 76495           | 1.9 10^{-3} | 1.90| |
| 147906| 294740        | 5.3 10^{-4} |       | |

Table 8: Results for the potential problem with Dirichlet- and Neumann boundary conditions, for quadratic elements and adaptive meshing

| DOF  | # quad. elements | ||e|| | δ |
|------|------------------|------|-----|---|
| 13518| 6659             | 1.3 10^{-3} | 3.72| |
| 74114| 36823            | 5.5 10^{-3} | 3.25| |
| 366399| 182686          | 4.1 10^{-6} |       | |

6. The Navier-Stokes Equations

As an example for a non-linear set of equations we tested the dimensionless Navier-Stokes equations:

\[
(u \cdot \nabla)u = \nabla \left[ -pI + \frac{1}{\text{Re}} \left( \nabla u + (\nabla u)^T \right) \right]
\]

\[
\nabla \cdot u = 0
\]

In the steady state case the Reynolds-number Re remains the only parameter. Dependent variables are the velocity \( u \) and pressure \( p \).

For test we chose an application problem: flow through a U-turn, as they are examined in meander studies (Nguyen et al. 2004, Holzbecher 2006). We chose characteristic boundary conditions, as described in the former publication. Figure 4 depicts the geometry. Channel width and also the spacing between the two straight legs was chosen as 1. The Reynolds number is set to 100.

Figure 4 also shows the velocity, represented by colours (red= high, blue=low), streamlines and arrows in flow direction. Moreover the location of the maximum and minimum pressure are indicated.

As there is no analytical solution for the problem, we determined the reference solution, which represents the analytical solution in the error calculations, by COMSOL also. It was produced by using a fine initial mesh and adaptive meshing and has 271351 DOFs.

The numerical models to be checked started with an initial coarse mesh (DOF: 844), which was refined four times (DOFs: 3017, 11359, 44027 and 173299) without adaptive meshing. As Finite Elements we took the default \( P_2-P_1 \), Lagrange elements and did not utilize any artificial diffusion scheme.
Figure 4: COMSOL numerical solution for the Navier-Stokes equations

As post-processing task we computed the errors for pressures ($p$) and $x$-velocities ($u$) in the maximum norm $\|p\|_\infty$ and $L_2$-norm $\|p\|_2$, and derived the convergence rates according to formula (2). The latter are shown in Table 9.

<table>
<thead>
<tr>
<th>Rate</th>
<th>$|p|_\infty$</th>
<th>$|p|_2$</th>
<th>$|u|_\infty$</th>
<th>$|u|_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\vartheta_{11}$</td>
<td>1.13</td>
<td>3.54</td>
<td>0.69</td>
<td>2.54</td>
</tr>
<tr>
<td>$\vartheta_{12}$</td>
<td>1.16</td>
<td>1.80</td>
<td>1.70</td>
<td>2.09</td>
</tr>
<tr>
<td>$\vartheta_{23}$</td>
<td>2.10</td>
<td>1.47</td>
<td>1.25</td>
<td>1.73</td>
</tr>
<tr>
<td>$\vartheta_{34}$</td>
<td>1.55</td>
<td>0.80</td>
<td>3.13</td>
<td>1.60</td>
</tr>
</tbody>
</table>

In contrast to the cases examined before there is no clear tendency concerning the convergence rates. Values are mainly between 1 and 2. The theoretical value of 2 for $\vartheta$ was not reached in the mean, but was even surpassed several times. It turned out that convergence rates are also quite different, concerning the chosen norm and the chosen variable. The example shows that various different effects may be valid or invalid in different stages of grid refinement, as it is characteristic for real application cases. Here the accuracy of the solution is mainly determined by the accuracy near the half circle. Especially the refinement at the two inner corners are crucial: the details of the flow regime in the vicinity of these corners were not captured: even in the most refined grid the pressure minimum is taken at the outlet, while the reference solution shows a minimum near the upstream inner corner (see Figure 4).

7. Connected Flow and Transport

We also tested the coupled system of equations:

$$\nabla^2 \varphi = 4 \quad \nabla^2 T = -\left| \nabla \varphi \right|^2$$

(9)

where the equation for the electric potential $\varphi$ is coupled with the heat transport equation, including heat diffusion and Ohmic losses, the latter related to the gradient of the potential. With zero Dirichlet boundary conditions for both variables, in the unit square of corner $(0,0)$, the solution of system (9) is:

$$\varphi = x^2 + y^2 \quad T = -\frac{1}{3} \left( x^4 + y^4 \right)$$

(10)

The solution is visualized in Figures 5 and 6.
We found a convergence rate of 2 for both the potential and the transport variable $T$ (see Tables 10 and 11). This confirms the theoretical result of Elliot & Larsson (1995).

7. References

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10. Nguyen P.T., Berning T., Djilali N., Computational model of a PEM fuel cell with serpentine gas flow channels, J. of Power Sources 130, 149-157, 2004

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