On steady flows of an incompressible fluids with implicit power-law-like rheology*

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Abstract

We consider steady flows of incompressible fluids with power-law-like rheology given by an implicit constitutive equation relating the Cauchy stress and the symmetric part of the velocity gradient in such a way that it leads to a maximal monotone (possibly multivalued) graph. Such a framework includes Bingham fluids, Herschel-Bulkley fluids, and shear-rate dependent fluids with discontinuous viscosities as special cases. We assume that the fluid adheres to the boundary.

Using tools such as the Young measures, properties of spatially dependent maximal monotone operators and Lipschitz approximations of Sobolev functions, we are able to extend the results concerning large data existence of weak solutions to those values of the power-law index that are of importance from the point of view of engineering and physical applications.

1 Problem formulation

We consider the following problem associated to a fixed, yet arbitrary parameter \(q \in (1, \infty)\) that is connected to its dual exponent \(q'\) through the relation \(q' = \frac{1}{q-1}\).

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\( q/(q - 1) \). For a given \( \Omega \subset \mathbb{R}^d \) bounded open connected set with Lipschitz boundary \( \partial \Omega \) (\( \Omega \subset C^{0,1} \) in short), \( b \in (W^{1,q}(\Omega)^d)\ast =: W^{-1,q'}(\Omega)^d \), and \( A : x \mapsto A(x) \) an inhomogeneous (spatially dependent) maximal monotone \( q \)-graph (specified later), we set

\[
\tilde{q} = \min\{q', q/2\}, \quad \text{where } q'^* = \begin{cases} \frac{dq}{d - q} & \text{if } 1 \leq q < d, \\ +\infty & \text{otherwise}, \end{cases}
\]

and we find \((v, p, S) : \Omega \to \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}_{\text{sym}}^{d \times d}\) such that

\[
v \in W_0^{1,q}(\Omega)^d, \quad p \in \tilde{L}^q(\Omega) \quad \text{and} \quad S \in L^{q'}(\Omega)^{d \times d},
\]

\[
\text{div} (v \otimes v + pI - S) = b \quad \text{in } \mathcal{D}'(\Omega)^d,
\]

\[
(Dv(x), S(x)) \in A(x) \quad \text{for all } x \in \Omega_{a.e.},
\]

where \( \Omega_{a.e.} = \Omega \setminus E \) with \( |E|_d = 0 \) (\( | \cdot |_d \) denotes the \( d \)-dimensional Lebesgue measure). \( Dv \in \mathbb{R}_{\text{sym}}^{d \times d} \) is the symmetric part of the gradient of \( v \), i.e., \( Dv = \frac{\nabla v + (\nabla v)^T}{2} \), and \( \mathbb{R}_{\text{sym}}^{d \times d} \) stands for the space of symmetric \( d \times d \) matrices.

We use standard notations for the Lebesgue and Sobolev spaces\(^1\) \( L^r(\Omega) \) and \( W^{1,r}(\Omega) \). We define \( \tilde{L}^r(\Omega) := \{ f; f \in L^r(\Omega), \int_{\Omega} f \, dx = 0 \} \), \( W^{1,r}_0(\Omega) := \{ u; u \in W^{1,r}(\Omega), u|_{\partial \Omega} = 0 \} \) and \( W^{1,r}_0(\Omega) := \{ v; v \in W^{1,r}_0(\Omega)^d, \text{div} v = 0 \} \). If \( u \in L^r(\Omega) \) and \( v \in L^s(\Omega) \) for some \( s \in [1, +\infty) \), then \((u, v)_2\) denotes \( \int_{\Omega} u(x)v(x) \, dx \). The same symbol is used for vector- or tensor-function as well. By \( \cdot \) we mean the scalar product in \( \mathbb{R}^d \) or \( \mathbb{R}_{\text{sym}}^{d \times d} \).

We assume that for all \( x \in \Omega_{a.e.} \), \( A \) shares the following properties:

(A1) \( (0, 0) \in A(x) \);

(A2) For all \((D_1, S_1), (D_2, S_2) \in A(x)\)

\[
(S_1 - S_2) \cdot (D_1 - D_2) \geq 0 \quad (A \text{ is a monotone graph}).
\]

Moreover, if \( D_1 \not= D_2 \) and \( S_1 \not= S_2 \) then the inequality is strict. \( A \) is a strictly monotone graph in a generalized sense);

(A3) If \((D, S) \in \mathbb{R}_{\text{sym}}^{d \times d} \times \mathbb{R}_{\text{sym}}^{d \times d}\) fulfills

\[
(S - S) \cdot (D - D) \geq 0 \quad \text{for all } (D, S) \in A(x),
\]

then \((D, S) \in A(x) \quad (A \text{ is maximal monotone graph});

(A4) There is a non-negative \( m \in L^1(\Omega) \), and a \( c > 0 \) such that for all \((D, S) \in A(x)\)

\[
S \cdot D \geq -m(x) + c(|D|^q + |S|^q') \quad (A \text{ is a } q \text{-graph}); \quad (1.2)
\]

\(^1\) If \( X \) denotes a space of scalar functions then \( X^d \) or \( X^{d \times d} \) denotes a corresponding space of vector-valued or tensor-valued functions, respectively.
(A5) (i) for all $D \in \mathbb{R}^{d \times d}_{\text{sym}}$ the set $\{S \in \mathbb{R}^{d \times d}_{\text{sym}} : (D, S) \in \mathcal{A}(x)\}$ is closed,

(ii) for any closed $C \subset \mathbb{R}^{d \times d}_{\text{sym}},$

$$\{(x, D) \in \Omega \times \mathbb{R}^{d \times d}_{\text{sym}} : \text{there exists } S \in C \text{ s.t. } (D, S) \in \mathcal{A}(x)\}$$

is measurable with respect to the $\sigma$-algebra $\mathcal{L}(\Omega) \otimes \mathcal{B}(\mathbb{R}^{d \times d}_{\text{sym}})$, where $\mathcal{L}(\Omega)$ denotes the $\sigma$-algebra of Lebesgue measurable subsets of $\Omega$ and $\mathcal{B}(\mathbb{R}^{d \times d}_{\text{sym}})$ that of all Borel subsets of $\mathbb{R}^{d \times d}_{\text{sym}}$.

The problem (1.1) describes steady flows of an incompressible fluid in a bounded container $\Omega$ where the fluid is supposed to adhere to the boundary $\partial \Omega$ and (1.1)$_3$ reflects the rheological properties of the fluid. The assumptions (A1)-(A5) characterize those constitutive relationships between the part $S$ of the Cauchy stress$^2$ $T$ and the symmetric part of the velocity gradient $Dv$ that are admissible. Before we provide examples of such admissible constitutive relations, we formulate the main result of this paper.

**Theorem 1.1** Let $q > \frac{2d}{d+2}$, $\Omega \in C^{0,1}$ and $b \in W^{-1,q'}(\Omega)^d$. Assume that $A$ fulfils (A1)-(A5). Then there is $(v, p, S)$ satisfying (1.1).

The proof of Theorem 1.1 forms the main portion of this paper.

In Section 2, we first summarize the mathematical tools involved in the analysis of the problem. Thus, in particular, we recall a very useful characterization of maximal monotone graphs fulfilling (A1)-(A5) in terms of 1-Lipschitz continuous mappings and strictly monotone operators (this is due to Dal Maso as clarified in Francfort, Murat and Tartar [17]). We also list several weak convergence tools (Young measures, biting lemma), and introduce approximations to discontinuous functions and survey their properties. Finally, we recall a strengthened version of the Lemma on Lipschitz approximations of Sobolev functions due to Diening, Málek and Steinhauer [13] based on the results of Frehse, Málek and Steinhauer [18] and Acerbi and Fusco [1]. In Section 3, the approximate problem to (1.1) is introduced and the existence of its solution is established. Finally, in Section 4 we take the limit as the parameter of approximations vanishes and prove Theorem 1.1. Moreover, in Section 5, we conclude that the Young measure associated to a weakly converging sequence in the nonlinear graph is Dirac on a suitably chosen set $B \subset \Omega$, implying that the sequence is strongly converging in $L^1(B)^{d \times d}$.

We complete this introductory section by discussing what are admissible constitutive relations, what are (in our opinion) strong features of our result and how the result is related to earlier studies dealing with mathematical analysis of similar models.

Note first that the constitutive relations for the power-law fluids of the form

$$S = \mu^* |D|^{q-2}D \iff D = \left(\frac{1}{\mu^*}\right)^{\frac{1}{q-1}} |S|^{\frac{2-q}{q-1}}S$$  \hspace{1cm} (1.3)$$

$^2$The Cauchy stress $T$ is supposed to be of the form $T = -pI + S$. 


gives

\[ \mathbf{S} \cdot \mathbf{D} = \frac{1}{2} \mathbf{S} \cdot \mathbf{D} + \frac{1}{2} \mathbf{S} \cdot \mathbf{D} = \mu^* |\mathbf{D}|^q + \frac{1}{2} \left( \frac{1}{\mu^*} \right)^{\frac{1}{q-1}} |\mathbf{S}|^{\frac{q}{q-1}} \]

\[ \geq \min \left\{ \frac{\mu^*}{2}, \frac{1}{2} \left( \frac{1}{\mu^*} \right)^{\frac{1}{q-1}} \right\} \left( |\mathbf{D}|^q + |\mathbf{S}|^q \right), \]

which is (A3) with \( m \equiv 0 \). For this reason, we say that (A3) reflects the fact that we deal with power-law-like fluids.

We wish to remark that if \( q > 2 \) then (1.3) describes the fluid that has the ability to thicken with the shear-rate, while for \( q \in (1, 2) \) the fluid exhibits shear-thinning. The latter phenomena is exhibited by many materials (see [28] for a representative list of relevant references from areas such as glaciology, hemodynamics, food rheology, polymer industry, etc.) and it motivates us to investigate the problem for low values of \( q \). Note that \( \mathbf{S} \) and \( \mathbf{D} \) related by the (explicit) relation (1.3) fulfills the assumptions (A1)-(A5) (see [26] for details if necessary). The class of admissible constitutive relations is however much broader.

First of all, the model includes the so-called Bingham and Herschel-Bulkley fluids described by the constitutive relationship

\[ \text{if } \mathbf{D}v = 0 \text{ then } |\mathbf{T}| \leq \tau^*, \]

\[ \text{if } \mathbf{D}v \neq 0 \text{ then } \mathbf{T} = \tau^* \frac{\mathbf{D}v}{|\mathbf{D}v|} + \overline{T}(p, \mathbf{D}v), \]

where \( \tau^* \) is the so-called yield stress, and

\[ \overline{T}(p, \mathbf{D}) = -pI + \begin{cases} \nu^* \mathbf{D}v & \text{for Bingham fluid,} \\ \nu^* |\mathbf{D}v|^{q-2} \mathbf{D}v & \text{for Herschel-Bulkley fluid.} \end{cases} \]

Next, in Gwiazda et al. [21] the authors clarified in detail that the model under consideration is suitable for including discontinuous constitutive equations of the form \( (d^* > 0) \):

\[ \mathbf{S} = \nu_1(|\mathbf{D}v|^2) \mathbf{D} \text{ if } |\mathbf{D}v| < d^*, \]

\[ \mathbf{S} = \nu_2(|\mathbf{D}v|^2) \mathbf{D} \text{ if } |\mathbf{D}v| > d^*, \]

(1.4)

\[ \mathbf{S} = \nu^* \mathbf{D}v \text{ if } |\mathbf{D}v| = d^*, \]

where \( \nu^* \in [\min \{ \nu_1^-, \nu_2^+ \}, \max \{ \nu_1^-, \nu_2^+ \}] \) with \( \nu_1^- := \lim_{|\xi| \to d^-} \nu_1(|\xi|^2) \) and \( \nu_2^+ := \lim_{|\xi| \to d^+} \nu_2(|\xi|^2) \). Here \( \nu_i : \mathbb{R}_0^+ \to \mathbb{R}_0^+ \) are (continuous) generalized power-law-like viscosities, i.e.,

\[ \nu_i(|\xi|^2) = \nu_{0i}(\alpha_i + \beta_i|\xi|^2)^{\frac{q_i-2}{2}} \text{ for some } \alpha_i \geq 0, \beta_i > 0, q_i \in (1, \infty), \nu_{0i} > 0, i = 1, 2. \]
The setting (1.4) has been inspired by the study due to Anand and Rajagopal [4] (for a full treatment of the modeling of blood that takes into account more details of the biochemistry see Anand et al. [5], [6]) who developed a single-continuum hemodynamical model that is capable of capturing flow changes in blood due to platelet activation. The constitutive equation (1.4) represents a simplified cartoon to the model by Anand and Rajagopal (see [4] and also [21] for details).

The framework (1.1) completed by (A1)-(A5) is however also capable of capturing fully implicit constitutive relations for fluids, recently outlined by Rajagopal (see [31] and [32])\(^3\). It is obvious that the framework given by the relation

\[ \mathbf{F}(\mathbf{D}, \mathbf{S}) = 0 \]

is more general than the framework of explicit constitutive equations

\[ \mathbf{S} = \mathbf{G}(\mathbf{D}), \]

and it can then capture the responses of much larger class of fluid-like materials.

To illustrate it, consider a one-dimensional relationship between the stress and strain rate as depicted in Fig. 1. We note that the stress \( \sigma \) is not a function of the shear rate \( \kappa \). This graph is related to the Bingham-like response of fluids in that for a set of values of the stress, the strain rate remains constant. In the case of a Bingham fluid, the value of this strain rate is zero and until a threshold stress is reached no flow takes place. To motivate a response depicted by Fig. 1 we could think of a biological fluid which at a particular strain rate or stress undergoes a chemical reaction and does not allow the shear rate increase until a threshold stress is reached. Another possible scenario for such a response is the flow of granular material wherein at a certain shear rate the material continues flowing at the shear rate even though the stress is increasing and only after a higher threshold of stress is reached the shear rate starts to increase.

Note that the graph in Fig. 1 is monotone and if \( \kappa_1 < \kappa_2 \) then \( \sigma_1 < \sigma_2 \), and consequently

\[ (\sigma_1 - \sigma_2)(\kappa_1 - \kappa_2) > 0. \quad (1.5) \]

In Fig. 2 we draw a dual picture representing the response of the material at a different point. Then there are \( \kappa_1, \kappa_2, \kappa_1 \neq \kappa_2 \) so that the corresponding \( \sigma_1 \) and \( \sigma_2 \) coincide and (1.5) does not hold. However if we interchange the role of \( \sigma \) and \( \kappa \) we observe that if \( \sigma_1 \neq \sigma_2 \) then \( \kappa_1 \neq \kappa_2 \) (in fact \( \sigma_1 < \sigma_2 \Rightarrow \kappa_1 < \kappa_2 \)) and consequently (1.5) holds again. The reader can compare this simple observation with the assumption (A2). To summarize, we have observed that the assumptions (A1)–(A5) are sufficiently robust to include implicit (monotone) and discontinuous (monotone) constitutive relations between \( \mathbf{S} \) and \( \mathbf{Dv} \). In this paper, we assume that

\(^3\)Implicit constitutive relations have also been introduced recently to describe the non-dissipative response of solids (see Rajagopal [33], Rajagopal and Srinivasa [34]) but such models cannot be studied within the framework developed here.
the response outside of certain critical values for $\mathbf{D} \mathbf{v}$ or $\mathbf{S}$ is similar to a power-law fluid and we are interested in the result for the lowest values of the power-law index $q$ (typically, in engineering applications, $q$ takes values as $2, \frac{3}{2}, \frac{4}{3}, \frac{6}{5},$ etc.)

We build our theoretical analysis on several previous studies. In Gwiazda et al. [21], the authors dealt with fluids with discontinuous power-law-like rheology, and established the existence of weak solutions for $q \geq \frac{3d}{d+2}$. Their approach strongly relies on the validity of energy equality for the limiting equations. The same result is also established in [22] where Gwiazda and Zatorska used Dal Maso, Francfort, Murat, Tartar characterization of maximal monotone graphs; besides (1.1) they also investigate elliptic and parabolic problems with discontinuous coefficients.

Herschley-Bulkley fluid model has been recently analyzed by Málek, Růžička and Shelukhin in [27]. Using a local version of fractional higher differentiability techniques (developed earlier in the context of power-law fluids in [24, 25, 26, 30]), and the earlier method developed by Shelukhin [37] to investigate Bingham fluid problems, that is, similar as the approach here, not based on using variational
inequality techniques (see Duvant, Lions [14] for details), the author established the existence of solution again for 
\[ q \geq \frac{3d}{d+2}. \]

In this paper, we extend these earlier results in two directions:

1. we establish the existence results for \( q > \frac{2d}{d+2} \), and
2. we consider more general class of admissible constitutive relations (See (A2) in particular.)

Finally, we wish to note that the existence and regularity of weak solution of Bingham fluids was also studied by G. Seregin, O. A. Ladyzhenskaya and M. Fuchs (see [19, 23, 35, 36]), and the result presented here has also the relevance to studies on the problems with the so-called \((p, q)\)-growth. Such fluid models were mathematically analyzed for example in [2, 3, 10, 16].

2 Mathematical tools

2.1 A characterization of maximal monotone graphs

According to [17], the graphs \( \mathcal{A} \) satisfying (A1)-(A5) (without requiring the second part of (A2)) can be characterized as graphs \( \mathcal{A} \) satisfying: there exists a function \( \varphi: \Omega \times \mathbb{R}^{d \times d} \rightarrow \mathbb{R}^{d \times d} \) such that

\[
\mathcal{A}(x) = \left\{ (D,S) \in \mathbb{R}^{d \times d} \times \mathbb{R}^{d \times d} : S - D = \varphi(x,S + D) \right\},
\]

whereas \( \varphi \) shares the following properties:

1. \( \varphi(\cdot,D) \) is measurable for all \( D \in \mathbb{R}^{d \times d} \);
2. \( \varphi(x,\cdot) \) is a 1-Lipschitz continuous mapping in \( \Omega \text{a.e.} \);
3. if we define functions \( s,d: \Omega_{\text{a.e.}} \times \mathbb{R}^{d \times d} \rightarrow \mathbb{R}^{d \times d} \) as

\[
s(x,\alpha) := \frac{1}{2}(\alpha + \varphi(x,\alpha)), \quad d(x,\alpha) := \frac{1}{2}(\alpha - \varphi(x,\alpha))
\]

then for all \( x \in \Omega_{\text{a.e.}} \) and all \( \alpha \in \mathbb{R}^{d \times d} \)

\[
s(x,\alpha) \cdot d(x,\alpha) \geq -m(x) + c(|d(x,\alpha)|^q + |s(x,\alpha)|^{q'}) \tag{2.8}
\]

4. \( \varphi(x,0) = 0 \) in \( \Omega_{\text{a.e.}} \).

Assume first that a graph \( \mathcal{A}(x) \subset \mathbb{R}^{d \times d} \times \mathbb{R}^{d \times d} \) satisfies (A1)-(A5) (again without requiring the second part of (A2)). Then, according to [17], there exists a (possibly discontinuous) function \( S^*: \Omega \times \mathbb{R}^{d \times d} \rightarrow \mathbb{R}^{d \times d} \) (a selection) such that for all \( \xi \in \mathbb{R}^{d \times d} \) we have \( (\xi,S^*(x,\xi)) \in \mathcal{A}(x) \) and \( S^* \) satisfies the following conditions:
a1. $S^*$ is measurable with respect to the $\sigma$-field generated by $\mathcal{L}(\Omega) \otimes \mathcal{B}(\mathbb{R}_{\text{sym}}^{d \times d})$ (see e.g. [17, Remark 2.2] and [7, Thm. 8.1.4]);

a2. $\text{Dom} S^*(x, \cdot) = \mathbb{R}_{\text{sym}}^{d \times d}$ on $\Omega_{a.e.}$;

a3. $S^*$ is monotone, i.e. for every $\xi_1, \xi_2 \in \mathbb{R}_{\text{sym}}^{d \times d}$ and all $x \in \Omega_{a.e.}$

$$S^*(x, \xi_1) - S^*(x, \xi_2) \cdot (\xi_1 - \xi_2) \geq 0; \quad (2.9)$$

a4. for all $\xi \in \mathbb{R}_{\text{sym}}^{d \times d}$ the function $S^*$ satisfies the following growth and coercivity conditions in $\Omega_{a.e.}$

$$|S^*(x, \xi)| \leq c_1 |\xi|^{q-1} + k(x),$$

$$S^*(x, \xi) \cdot \xi \geq c_2 |\xi|^{q} - m(x), \quad (2.10)$$

where $c_1, c_2 > 0$, $k \in L^q(\Omega)$ and $m \in L^1(\Omega)$ are nonnegative functions.

For a given graph $A$ the relation between a possibly discontinuous selection $S^*$ and $\varphi$ is established by the following lemma:

Lemma 2.1 [22, Lemma 3.1] Let functions $s$ and $d$ be defined by $\varphi$ according to the formula (2.7). If there exists $\xi$ such that $\alpha = S^*(x, \xi) + \xi$ in $\Omega_{a.e.}$, then

$$s(x, \alpha) = S^*(x, \xi) \quad (2.11)$$

and

$$d(x, \alpha) = \xi. \quad (2.12)$$

2.2 Weak convergence tools

The Young measures, in a classical understanding, are the limits of the sequences of single distributed measures. Since in our consideration we prefer to deal with the sequences of general probability measures, the classical results need the extensions. The forthcoming theorem is a modification of fundamental theorem on Young measures, cf. [8]. We use the standard notation: $\mathcal{M}(\mathbb{R}_{\text{sym}}^{d \times d})$ denotes the space of bounded Radon measures, $L^\infty_w(\Omega; \mathcal{M}(\mathbb{R}_{\text{sym}}^{d \times d}))$ the space of essentially bounded, weakly-* measurable functions $\nu$, i.e. $\nu(x) =: \nu_x \in \mathcal{M}(\mathbb{R}_{\text{sym}}^{d \times d})$.

Theorem 2.2 [20] Let $\Omega$ be an open and bounded subset of $\mathbb{R}^d$. Assume that for every $x \in \Omega$ there exists a sequence of probability measures $\nu^j_x$ on $\mathbb{R}_{\text{sym}}^{d \times d}$ such that for every $j$ the mapping $\nu^j : \Omega \rightarrow \mathcal{M}(\mathbb{R}_{\text{sym}}^{d \times d})$ is weak-* measurable. Assume $\nu : \Omega \rightarrow \mathcal{M}(\mathbb{R}_{\text{sym}}^{d \times d})$ to be such that $\nu^j \rightharpoonup \nu$ in $L_w^\infty(\Omega; \mathcal{M}(\mathbb{R}_{\text{sym}}^{d \times d})).$

If the sequence $\{\nu^j\}_{j=1}^\infty$ satisfies the tightness condition, i.e.,

$$\lim_{M \rightarrow \infty} \sup_j \{x \in \Omega : \text{supp}(\nu^j_x) \setminus B(0, M) \neq \emptyset\} \rightarrow 0, \quad (2.13)$$

then
Lemma 2.3 (Biting lemma) [9, 12] Assume that \( \Omega \) is bounded and let \( \{u^n\} \) be a bounded sequence in \( L^1(\Omega) \). Then there exist a function \( u \), a subsequence \( \{u^{n_j}\} \) and a nonincreasing sequence of measurable subsets \( E_k \) with \(|E_k| \to 0\) such that for every \( k \), \( u^{n_j} \) converges weakly to \( u \) in \( L^1(\Omega \setminus E_k) \). We will say that \( u \) is the biting limit of \( \{u^{n_j}\} \).

Lemma 2.4 [29, Corollary 3.2] Suppose that a sequence \( z^j : \Omega \to \mathbb{R} \) of measurable functions generates the Young measure \( \nu : \Omega \to \mathcal{M}(\mathbb{R}) \). Then

\[
z^j \to z \text{ in measure} \iff \nu_x = \delta_{\{z(x)\}} \text{ on } \Omega_{a.e.}.
\]

2.3 Approximations of \( x \)-dependent maximal monotone \( q \)-graphs

Let \( S^* \) be the selection satisfying a1–a4. introduced in Section 2.1. Let \( \eta \in C_0^\infty(\mathbb{R}^{d\times d}_{\text{sym}}) \) be a radially symmetric function with support in a unit ball \( B(0,1) \subset \mathbb{R}^{d\times d}_{\text{sym}} \) and \( \int_{\mathbb{R}^{d\times d}_{\text{sym}}} \eta d\xi = 1 \). For \( n \in \mathbb{N} \) we set \( \eta^n(\xi) = n^{d^2} \eta(n\xi) \) and we define

\[
S^n(x,\xi) = (S^* \eta^n)(x,\xi) = \int_{\mathbb{R}^{d\times d}_{\text{sym}}} S^*(x,\zeta) \eta^n(\xi - \zeta) d\zeta = \int_{\mathbb{R}^{d\times d}_{\text{sym}}} S^*(x,\zeta) d\mu^n_\xi(\zeta). \tag{2.16}
\]

Here, \( \mu^n_\xi \) denotes a probability measure that is absolutely continuous with respect to the Lebesgue measure with density \( \eta^n(\xi - (\cdot)) \). We also set \( \mu^n_x := \mu^n_{\delta_{\nu^n(x)}} \) and recall the properties of \( S^n \).

Lemma 2.5 [21, 22] Let \( S^* \) fulfil (2.9) and (2.10). Then \( S^n \) defined in (2.16) satisfies

\[
(S^n(x,\xi_1) - S^n(x,\xi_2)) \cdot (\xi_1 - \xi_2) \geq 0 \quad \text{for every } \xi_1, \xi_2 \in \mathbb{R}^{d\times d}_{\text{sym}}. \tag{2.17}
\]
Also, there are positive constants \( \tilde{c}_1, \tilde{c}_2 \in \mathbb{R} \) (independent of \( n \)) and nonnegative functions \( \tilde{m} \in L^1(\Omega) \) and \( \tilde{k} \in L^d(\Omega) \) such that
\[
|S^n(x, \xi)| \leq \tilde{c}_1|\xi|^{q-1} + \tilde{k}(x) \quad \text{for all} \quad \xi \in \mathbb{R}^{d \times d},
\]
(2.18)
\[
S^n(x, \xi) \cdot \xi \geq \tilde{c}_2|\xi|^q - \tilde{m}(x) \quad \text{for all} \quad \xi \in \mathbb{R}^{d \times d}.
\]
(2.19)

### 2.4 Lipschitz approximations of Sobolev functions

For the proof of the following theorem we refer to the result in [13], inspired by [18].

**Theorem 2.6** [13] Let \( 1 < q < \infty \) and \( \Omega \in C^{0,1} \). Let \( u^n \in W^1,q_0(\Omega)^d \) be such that \( u^n \rightharpoonup 0 \) weakly in \( W^1,q_0(\Omega)^d \) as \( n \to \infty \). Set
\[
K := \sup_n \|u^n\|_{1,q} < \infty,
\]
(2.20)
\[
\gamma_n := \|u^n\|_q \to 0 \quad (n \to \infty).
\]
(2.21)

Let \( \theta_n > 0 \) be such that (e.g. \( \theta_n := \sqrt{\gamma_n} \))
\[
\theta_n \to 0 \quad \text{and} \quad \frac{\gamma_n}{\theta_n} \to 0 \quad (n \to \infty).
\]

Let \( \mu_j := 2^j \). Then there exists a sequence \( \lambda_{n,j} > 0 \) with
\[
\mu_j \leq \lambda_{n,j} \leq \mu_{j+1},
\]
(2.22)
and a sequence \( u^{n,j} \in W_0^{1,\infty}(\Omega)^d \) such that for all \( j, n \in \mathbb{N} \)
\[
\|u^{n,j}\|_\infty \leq \theta_n \to 0 \quad (n \to \infty),
\]
(2.23)
\[
\|\nabla u^{n,j}\|_\infty \leq c \lambda_{n,j} \leq c \mu_{j+1}
\]
(2.24)

and
\[
\{u^{n,j} \neq u^n\} \subset \Omega \cap \{M u^n > \theta_n\} \cup \{M(\nabla u^n) > 2 \lambda_{n,j}\},
\]
(2.25)

and for all \( j \in \mathbb{N} \) and \( n \to \infty \)
\[
\begin{align*}
    u^{n,j} &\to 0 \quad \text{strongly in } L^s(\Omega)^d \text{ for all } s \in [1, \infty], \\
    u^{n,j} &\to 0 \quad \text{weakly in } W_0^{1,s}(\Omega)^d \text{ for all } s \in [1, \infty], \\
    \nabla u^{n,j} &\rightharpoonup 0 \quad \text{weakly-}^* \text{ in } L^\infty(\Omega)^{d \times d}.
\end{align*}
\]
(2.26)
(2.27)
(2.28)

Furthermore, for all \( n, j \in \mathbb{N} \)
\[
|\{u^{n,j} \neq u^n\}|_d \leq \frac{c\|u^n\|_{1,q}}{\lambda_{n,j}^q} + c \left(\frac{\gamma_n}{\theta_n}\right)^q
\]
(2.29)
and
\[
\|\nabla u^{n,j} \chi_{\{u^{n,j} \neq u^n\}}\|_q \leq c \|\lambda_{n,j} \chi_{\{u^{n,j} \neq u^n\}}\|_q \leq c \frac{\gamma_n}{\theta_n} \mu_{j+1} + c \varepsilon_j,
\]
(2.30)

where \( \varepsilon_j := K 2^{-j/q} \) vanishes as \( j \to \infty \). The constant \( c \) depends on \( \Omega \).

---

\(^4\)The symbol \( M \) denotes the usual Hardy-Littlewood maximal function.
3 Approximation of the original problem, its solvability, apriori estimates

For an arbitrary fix \( n \in \mathbb{N} \) we introduce the following \( n \)-approximative problem:

We look for a weak solution \((v, p) := (v^n, p^n)\) to

\[
\begin{align*}
\text{div} S^n &= \text{div} (v \otimes v) + \nabla p - b + \frac{1}{n} |v|^{2q'-2}v, \\
\text{div} v &= 0, \quad v|_{\partial \Omega} = 0,
\end{align*}
\]

(3.31)

where \( S^n \) denotes the mollification of stress tensor introduced in (2.16). The proof of the existence of solution to (3.31) will be done by using Faedo-Galerkin scheme completed by standard monotone operator theory that is based on Minty method.

Note that due to the presence of the penalty term \( \frac{1}{n} |v|^{2q'-2}v \) in the equation (3.31), and due to continuity of \( S^n \) such a procedure is possible.

Let \( \{w_i\}_{i=1}^\infty \) be a basis of separable space \( W_{1,q}^1(\Omega) \cap L^{2q'}(\Omega)^d \). We construct \( v^N := \sum_{i=1}^N c_i^N w_i \) where \( c_i^N \in \mathbb{R} \) such that they solve the following system of \( N \) equations

\[
\frac{1}{n} (|v^N|^{2q'-2}v^N, w_i)_2 + (S^n(\cdot, Dv^N), Dw_i)_2 - (v^N \otimes v^N, \nabla w_i)_2 = \langle b, w_i \rangle, \quad i = 1, \ldots, N.
\]

(3.32)

The existence of solution to (3.32) follows from Brower fixed point theorem and from the following uniform estimates.

Multiplying the \( i \)-th equation in (3.32) by \( c_i^N \) and taking the sum over \( i = 1, \ldots, N \) we obtain the estimate (using the fact that convective term vanishes since \( \text{div} v^N = 0 \))

\[
\frac{1}{n} \|v^N\|_{2q'}^{2q'} + (S^n(\cdot, Dv^N), Dv^N)_2 = \langle b, v^N \rangle.
\]

(3.33)

Finally, using coercivity and growth properties of \( S^n \) (2.18) and (2.19), \( b \in W^{-1,q'}(\Omega)^d \), Korn and Young inequalities we deduce from (3.33) that

\[
\frac{1}{n} \|v^N\|_{2q'}^{2q'} + \|v^N\|_{1,q}^q + \|S^n(\cdot, Dv^N)\|_{q'} \leq C.
\]

(3.34)

Having apriori estimates (3.34) we can take a (not relabeled) subsequence such that

\[
\begin{align*}
v^N &\rightharpoonup v \quad \text{weakly in } W_{0,\text{div}}^{1,q}(\Omega) \cap L^{2q'}(\Omega)^d, \\
S^n(\cdot, Dv^N) &\rightharpoonup \overline{S}^n \quad \text{weakly in } L^{q'}(\Omega)^{d \times d},
\end{align*}
\]

(3.35) \hspace{1cm} (3.36)

using embedding theorem and standard interpolation inequality we also get

\[
\begin{align*}
v^N &\rightarrow v \quad \text{strongly in } L^s(\Omega)^d, \text{ for all } s \in [1, 2q'), \\
|v^N|^{2q'-2}v^N &\rightarrow |v|^{2q'-2}v \quad \text{weakly in } L^{\frac{2q'}{2q'-1}}(\Omega)^d.
\end{align*}
\]

(3.37) \hspace{1cm} (3.38)

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Letting $N \to \infty$ in (3.32) and using the standard argument we obtain the following identity valid for all $\varphi \in W^{1,q}_{0,\text{div}}(\Omega) \cap L^{2q'}(\Omega)^d$

$$
\frac{1}{n}((|v|^{2q'-2}v, \varphi)_2 + (\overline{S}^n, D\varphi)_2 - (v \otimes v, \nabla \varphi)_2 = \langle b, \varphi \rangle. \quad (3.39)
$$

Thus, to complete the proof of existence of solution to (3.31), it remains to identify $\overline{S}^n$. Using the fact that $S^n$ is monotone, see (2.17), we obtain for arbitrary $\varphi \in W^{1,q}_{0,\text{div}}(\Omega) \cap L^{2q'}(\Omega)^d$, that

$$
0 \leq (S^n(\cdot, Dv^N) - S^n(\cdot, D\varphi), Dv^N - D\varphi)_2
$$

$$
\overset{(3.33)}{=} - \frac{1}{n} \|v^N\|^{2q'}_2 + \langle b, v^N \rangle - (S^n(\cdot, Dv^N), D\varphi)_2 - (S^n(\cdot, D\varphi), Dv^N - D\varphi)_2 =: \theta^N.
$$

Using (3.35)-(3.38) and weak lower semicontinuity of $L^{2q'}$-norm, we deduce that

$$
0 \leq \limsup_{N \to \infty} \theta^N \leq - \frac{1}{n} \|v\|^{2q'}_2 + \langle b, v \rangle - (\overline{S}^n, D\varphi)_2 - (\overline{S}^n(\cdot, D\varphi), Dv - D\varphi)_2
$$

$$
\overset{(3.39)}{=} (\overline{S}^n - S^n(\cdot, D\varphi), Dv - D\varphi)_2.
$$

Finally, an admissible choice $\varphi := v \pm \lambda w$ combined with continuity of $S^n$ w.r.t. $D$ and with a standard Minty method then completes the proof of existence of solution to (3.31).

4 Convergence of approximation to the solution of the original problem

In this section, we let $n \to \infty$ and prove the existence of solution to our original problem (1.1). First, note that using lower-semicontinuity of norm and (3.34) we obtain that

$$
\frac{1}{n} \|v^n\|^{2q'}_2 + \|v^n\|^{q}_{1,q} + \|S^n(\cdot, Dv^n)\|_{q'} \leq C. \quad (4.40)
$$

Having (4.40) we can find a subsequence such that

$$
v^n \to v \quad \text{weakly in } W^{1,q}_{0,\text{div}}(\Omega), \quad (4.41)
$$

$$
S^n(\cdot, Dv^N) \to S \quad \text{weakly in } L^{q'}(\Omega)^{d \times d}, \quad (4.42)
$$

$$
v^n \to v \quad \text{strongly in } L^s(\Omega)^d, \text{ for all } s \in [1, q^*). \quad (4.43)
$$

Next passing to the limit in (3.39) we easily derive that for all $\varphi \in W^{1,q}_{0,\text{div}}(\Omega)^d$ with $s := \max\{q, \left(\frac{q^*}{2}\right)\}$ there holds

$$
(S, D(\varphi)) - (v \otimes v, \nabla \varphi) = \langle b, \varphi \rangle. \quad (4.44)
$$
Next, using De Rahm theorem and Nečas theorem on negative norms we can find a pressure \( p \in \hat{L}^q(\Omega) \) such that the triple \((\mathbf{v}, p, \mathbf{S})\) fulfils \((1.1)_2\).

To complete the proof, it remains to show that \( \mathbf{S} \) also satisfies the condition \((1.1)_3\). To prove it, we use several auxiliary assertions. First, we define

\[
a_n(x) := (\mathbf{S}^n(x, D\mathbf{v}^n(x)) - \mathbf{S}^*(x, D\mathbf{v}(x))) \cdot (D\mathbf{v}^n(x) - D\mathbf{v}(x)),
\]

\[
b_n(x) := \int_{\mathbb{R}^d} (\mathbf{S}^*(x, \xi) - \mathbf{S}^*(x, D\mathbf{v}(x))) \cdot (\xi - D\mathbf{v}(x)) d\mu_x^p(\xi),
\]

where \( \mu_x^n := \mu_\mathbf{Dv}^n(x) \) and \( \mu_x^p \) is defined in \((2.16)\), and \( \mathbf{S}^* \) is introduced in Subsection 2.1.

**Lemma 4.1** Let \( \mathbf{S}^n \) be the approximations defined in \((2.16)\). Then

\[
\lim_{n \to \infty} \int_\Omega |a_n(x) - b_n(x)| dx = 0.
\]

**Proof:** It holds

\[
\int_\Omega |a_n(x) - b_n(x)| dx
\]

\[
= \int_\Omega \left| \int_{\mathbb{R}^d} (\mathbf{S}^*(x, \xi) \cdot D\mathbf{v}^n - \mathbf{S}^*(x, D\mathbf{v}) \cdot D\mathbf{v}^n - \mathbf{S}^*(x, \xi) \cdot D\mathbf{v} + \mathbf{S}^*(x, D\mathbf{v}) \cdot D\mathbf{v} \right) dx
\]

\[
= \int_\Omega \int_{\mathbb{R}^d} \left| \mathbf{S}^*(x, \xi) \cdot D\mathbf{v}^n - \mathbf{S}^*(x, \xi) - \mathbf{S}^*(x, D\mathbf{v}) \cdot (\xi - D\mathbf{v}^n) \right| d\mu_x^p(\xi) dx
\]

\[
\leq \int_\Omega \int_{\mathbb{R}^d} \left| \mathbf{S}^*(x, \xi) \cdot D\mathbf{v} - \mathbf{S}^*(x, D\mathbf{v}) \right| d\mu_x^p(\xi) dx
\]

\[
\leq \frac{c}{n} \int_\Omega \left( \sup_{\xi \in B(D\mathbf{v}^n, \frac{1}{n})} |\mathbf{S}^*(x, \xi)| + |\mathbf{S}^*(x, D\mathbf{v})| \right) dx
\]

\[
\leq \frac{c}{n} \int_\Omega \left( (|D\mathbf{v}^n| + \frac{1}{n})^{q-1} + |D\mathbf{v}|^{q-1} + k(x) \right) dx \leq \frac{c}{n}.
\]

\( \blacksquare \)

**Lemma 4.2** The sequence \( a_n(x) \) satisfies

\[
\lim_{n \to \infty} \int_\Omega |a_n(x)| dx = 0, \quad \text{where } A^- = \begin{cases} 0 & \text{if } A \geq 0, \\ -A & \text{if } A < 0. \end{cases}
\]

**Proof:** Notice that for any \( A, B \in \mathbb{R} \) with \( B \geq 0 \) we have

\[
|A^-| \leq |A - B|.
\]

\[
(4.47)
\]

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If $A \geq 0$, then (4.47) is obvious. If $A < 0$, then $A = -A^-$ and $A^- + B \geq A^-$, hence $|A^-| \leq |A^- + B| = |-A^- - B| = |A - B|$.

With the choice $a_n(x) = A$ and $b_n(x) = B$, Lemma 4.1 and the fact that $S$ is monotone yield the assertion.

Before stating the next lemma we will introduce some notation and establish preliminary facts, cf. [13].

Let $u^n := v^n - v$ and $u^{n,j} := (v^n - v)^j$ be the sequence as in Theorem 2.6. Note that the functions $u^{n,j}$ are in general not divergence free on the set $\{u^n \neq u^{n,j}\}$ and we have to correct them in order to use them as a test function in (3.39). Since $\partial \Omega$ is Lipschitz, according to [11], there exists a linear operator $B$ such that for all $\sigma \in (1, \infty)$ we have $B : \dot{L}^\sigma(\Omega) \rightarrow W^{1,\sigma}_0(\Omega)^d$ continuously and $\text{div}(Bh) = h$. In particular for all $\sigma \in (1, \infty)$ and all $h \in L^\sigma(\Omega)$ we have

$$\text{div}(Bh) = h \text{ in } \Omega, \quad Bh = 0 \text{ on } \partial \Omega \quad \text{and} \quad \|Bh\|_{1,\sigma} \leq c\|h\|_\sigma, \quad (4.48)$$

where the constant depends only on $\Omega$ and $\sigma$. We define

$$\psi^{n,j} := B(\text{div } u^{n,j}) = B(\chi_{\{u^n \neq u^{n,j}\}} \text{div } u^{n,j}).$$

Then, by (4.48),

$$\|\psi^{n,j}\|_{1,q} \leq c \|\text{div } u^{n,j} \chi_{\{u^n \neq u^{n,j}\}}\|_q.$$  

Consequently, (2.23)–(2.30) yield for $j \in \mathbb{N}$ and $n \rightarrow \infty$,

$$\psi^{n,j} \rightharpoonup 0 \text{ weakly in } W^{1,\sigma}(\Omega)^d \quad \text{for all } \sigma \in (1, \infty), \quad (4.49)$$

$$\psi^{n,j} \rightarrow 0 \text{ strongly in } L^\sigma(\Omega)^d \quad \text{for all } \sigma \in (1, \infty), \quad (4.50)$$

and

$$\limsup_{n \rightarrow \infty} \|\psi^{n,j}\|_{1,q} \leq c \limsup_{n \rightarrow \infty} \|\text{div } u^{n,j} \chi_{\{u^n \neq u^{n,j}\}}\|_q \leq c \limsup_{n \rightarrow \infty} \|\nabla u^{n,j} \chi_{\{u^n \neq u^{n,j}\}}\|_q \leq c \varepsilon_j. \quad (4.51)$$

Defining $\varphi^{n,j}$ through the relation

$$\varphi^{n,j} = u^{n,j} - \psi^{n,j}, \quad (4.52)$$

we observe that

$$\text{div } \varphi^{n,j} = 0 \quad (4.53)$$

and due to (4.49) and (4.50) we have for $j \in \mathbb{N}, n \rightarrow \infty$

$$\varphi^{n,j} \rightharpoonup 0 \text{ weakly in } W^{1,\sigma}_0(\Omega)^d \quad \text{for all } \sigma \in (1, \infty), \quad (4.54)$$

$$\varphi^{n,j} \rightarrow 0 \text{ strongly in } L^\sigma(\Omega)^d \quad \text{for all } \sigma \in (1, \infty). \quad (4.55)
Lemma 4.3 With notation introduced above and \( \varepsilon_j := K2^{-j/q} \) introduced in Theorem 2.6, there is a positive constant \( c \) such that

\[
\limsup_{n \to \infty} \int_{\{u^n = u^{n,j}\}} a_n(x) dx \leq c \varepsilon_j.
\]

**Proof:** We use the definition of \( \varphi^{n,j} \) (4.52) and observe that

\[
\int_{\{u^n = u^{n,j}\}} a_n(x) dx =
\int_\Omega \nabla^{n} \cdot D\varphi^{n,j} dx + \int_\Omega \nabla^{n} \cdot D\psi^{n,j} dx
- \int_\Omega \nabla \cdot D\psi^{n,j} dx + \int_{\{u^n \neq u^{n,j}\}} (\nabla \cdot Dv) \cdot Du^{n,j} dx
= I_1^{n,j} + I_2^{n,j} + I_3^{n,j} + I_4^{n,j}.
\]

In order to estimate \( I_1^{n,j} \) we take \( \varphi := \varphi^{n,j} \) as a test function in (3.39) and obtain

\[
I_1^{n,j} = (v^n \otimes v^n, D\varphi^{n,j})_2 - (1/n|v^n|^{2q'} - 2v^n, \varphi^{n,j})_2 + \langle b, \varphi^{n,j} \rangle.
\]

According to (4.54)-(4.55)

\[
\lim_{n \to \infty} I_1^{n,j} = 0.
\]

Using (2.27) we conclude

\[
\lim_{n \to \infty} I_3^{n,j} = 0.
\]

To estimate the upper limit of \( I_2^{n,j} \) we observe

\[
\limsup_{n \to \infty} (\nabla \cdot Dv) \cdot Du^{n,j} \leq \limsup_{n \to \infty} \|\nabla \cdot Dv^n\|_q \|Du^{n,j}\|_q
\]

and from (4.51) and (3.34) it follows that

\[
\lim_{n \to \infty} I_2^{n,j} \leq c\varepsilon_j.
\]

Further using (2.30) and (3.34) we deduce

\[
|I_4^{n,j}| \leq \int_{\{u^n \neq u^{n,j}\}} (|\nabla \cdot Dv| + |\nabla \cdot Dv^n|) |Du^{n,j}| \cdot \|Du^{n,j}\|_q \leq c\varepsilon_j.
\]

Lemma 4.4 With the same notation as in Lemma 4.3 it holds

\[
\limsup_{n \to \infty} \int_{\{u^n = u^{n,j}\}} |a_n(x)| dx \leq c\varepsilon_j.
\]
Proof: Since \(|a_n(x)| = a_n^+ + a_n^- = a_n^+ - a_n^- + 2a_n^- = a_n + 2a_n^-|\), then
\[
\int_{\{u^n = u^{n,j}\}} |a_n(x)|dx = \int_{\{u^n = u^{n,j}\}} a_n(x)dx + 2 \int_{\{u^n = u^{n,j}\}} a_n^-(x)dx
\]
and therefore
\[
\limsup_{n \to \infty} \int_{\{u^n = u^{n,j}\}} |a_n(x)|dx
\leq \limsup_{n \to \infty} \int_{\{u^n = u^{n,j}\}} a_n(x)dx + 2 \limsup_{n \to \infty} \int_{\{u^n = u^{n,j}\}} a_n^-dx.
\]
Applying Lemma 4.2 and 4.3, the proof is complete. ■

**Lemma 4.5** Let \(\theta \in (0, 1)\) be arbitrary. Then there holds
\[
\lim_{n \to \infty} \int_{\Omega} |a_n(x)|^\theta dx = 0.
\]

**Proof:** Using Hölder inequality, we easily obtain that
\[
\int_{\Omega} |a_n(x)|^\theta dx = \int_{\{u^n = u^{n,j}\}} |a_n(x)|^\theta dx + \int_{\{u^n \neq u^{n,j}\}} |a_n(x)|^\theta dx
\leq |\Omega|^\frac{1}{1-\theta} \left( \int_{\{u^n = u^{n,j}\}} |a_n(x)|dx \right)^\theta + \left( \int_{\Omega} |a_n(x)|dx \right)^\theta |\{u^n \neq u^{n,j}\}|^\frac{1}{1-\theta}.
\]
Since \(\Omega\) is bounded, Lemma 4.4 implies that the first term on the right hand side is bounded by \(c\varepsilon_j\). Moreover, \(\left( \int_{\Omega} |a_n(x)|dx \right)^\theta\) is uniformly bounded due to (3.34), and \(\limsup_{n \to \infty} |\{u^n \neq u^{n,j}\}| \leq c\varepsilon_j\) by (2.29). Consequently,
\[
0 \leq \limsup_{n \to \infty} \int_{\Omega} |a_n(x)|^\theta dx \leq c\varepsilon_j.
\]
Since \(a_n(x)\) does not depend on \(j\) and \(\varepsilon_j \to 0\), the assertion follows. ■

**Lemma 4.6** \(b_n \to 0\) in measure.

**Proof:** Lemma 4.5 implies that \(|a_n|^\theta \to 0\) strongly in \(L^1(\Omega)\), which implies that \(|a_n|^\theta \to 0\) in measure and also \(|a_n| \to 0\) in measure. Finally, from Lemma 4.1 we conclude that also \(b_n \to 0\) in measure. ■

Finally, we will use the theory of Young measures to the proof of (1.1). First, we characterize the biting limit of sequence \(b_n\) and then use the fact that biting
limit and limit in measures are equal. For this purpose we define (note that $S^*$ is a selection introduced in Subsection 2.1)

$$G_x(\xi) := S^*(x, \xi) + \xi. \quad (4.56)$$

We also define the measures $\nu^n_{x,\xi} \in \mathcal{M}(\text{im } G_x)$ by the formulae

$$\nu^n_{x,\xi}(S) = \mu^n_{\xi}(G^{-1}_x(S)) \quad \text{for } S \subset \text{im } G_x. \quad (4.57)$$

Note that for every $n$ the measure $\nu^n_{x,\xi}$ is a probability measure on $\text{im } G_x$. In what follows, we also set $\nu^n_{x} := \nu^n_{x, Dv(x)}$, and we define $\nu^n_x \equiv 0$ on $\mathbb{R}^{d \times d} \setminus \{\text{im } G_x\}$. By Lemma 2.1, using (2.11)-(2.12) we obtain (by using simple substitution)

$$S^n(x, Dv^n(x)) = \int_{\text{im } G_x} s(x, \alpha) \, d\nu^n_x(\alpha),$$

$$Dv^n(x) = \int_{\text{im } G_x} d(x, \alpha) \, d\nu^n_x(\alpha). \quad (4.58)$$

Similarly, we can deduce that the sequence $b_n$ defined in (4.46) can be rewritten as

$$b_n(x) = \int_{\mathbb{R}^{d \times d}} (s(x, \alpha) - S^*(x, Dv(x))) \cdot (d(x, \alpha) - Dv(x)) \, d\nu^n_x(\alpha).$$

**Lemma 4.7** Let $\nu$ be the weak-$*$ limit of the sequence $\nu^n$. Then

$$b_n \to b \quad \text{in } L^1(\Omega),$$

where $b(x) = \int_{\mathbb{R}^{d \times d}} (s(x, \alpha) - S^*(x, Dv(x))) \cdot (d(x, \alpha) - Dv(x)) \, d\nu_x(\alpha)$.

**Proof:** We will show that the sequence $\{\nu^n\}$ satisfies the assumptions of Theorem 2.2.

**Step 1:** For every $n \in \mathbb{N}$ the mapping $x \mapsto \nu^n_x$ belongs to $L^\infty(\Omega; \mathcal{M}(\mathbb{R}^{d \times d}))$.

For any $f \in L^1(\Omega; C_0(\mathbb{R}^{d \times d}))$ we define $F(x, \xi) := \eta^n(Dv^n(x) - \xi) \cdot f(x, G_x(\xi))$. Since $\eta^n$ are smooth and $f \in L^1(\Omega; C_0(\mathbb{R}^{d \times d}))$, we see that using assumption (A5) and the definition of $G_x$ (4.56) directly implies measurability of $F$ w.r.t. $\mathcal{L}(\Omega) \otimes \mathcal{B}(\mathbb{R}^{d \times d})$. Substitution Theorem also gives the simple relation

$$\int_{\mathbb{R}^{d \times d}} F(x, \xi) \, d\xi = \int_{\mathbb{R}^{d \times d}} f(x, \alpha) \, d\nu^n_x(\alpha).$$

Thus, to finish Step 1, it it enough to show that

$$x \mapsto \int_{\mathbb{R}^{d \times d}} F(x, \xi) \, d\xi$$

is measurable and integrable. But this follows directly from Fubini Theorem and from the properties of $F$. 

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Step 2: The sequence of mappings \( \{ \nu^n \} \) satisfies the tightness condition (2.13).
Define a function

\[
\gamma^n(x) = \max_{\alpha \in \text{supp} \nu^n} |\alpha|.
\]

From the definition of \( \nu^n \)

\[
\max_{\alpha \in \text{supp} \nu^n} |\alpha| = \max_{\xi \in \text{supp} \mu^n} |G_x(\xi)| \leq \max_{\xi \in \text{supp} \mu^n} (|\xi| + |S(x, \xi)|).
\]

Since \( \text{supp} \mu^n \subset B(Dv^n(x), \frac{1}{n}) \), we have \( \|\gamma^n\|_{L^r} \leq C \) for \( r := \min\{q', q\} \) and hence the second step follows.

Step 3: Let \( s(x, \alpha) \) and \( d(x, \alpha) \) be defined by (2.7). Then there exists a non-increasing sequence of measurable subsets \( \{E_k\} \) with \( |E_k| \to 0 \) such that the family of measures \( \{\nu^n\} \) with the function \( f(x, \alpha) := (s(x, \alpha) - S^*(x, Dv(x)) \cdot (d(x, \alpha) - Dv(x)) \) satisfy condition (2.15) on the set \( A = \Omega \setminus E_k \) for each \( k \in \mathbb{N} \).

Since, we have that \( \|Dv^n\|_q + \|Dv\|_q \leq C \), we get that the sequence \( c_n(x) := c \left( |Dv^n(x)|^{q-1} + |Dv(x)|^{q-1} + \frac{1}{q'-1} + m(x) \frac{1}{q'} \right) (|Dv^n(x)| + |Dv(x)| + \frac{1}{q'}) \) is bounded in \( L^1(\Omega) \). Then Lemma 2.3 provides there exists a subsequence \( c_n(x) \) and a non-increasing sequence of measurable subsets \( E_k \) with \( |E_k| \to 0 \) as \( k \to \infty \), such that for every \( k \) the sequence \( c_n(x) \) is relatively weakly compact in \( L^1(\Omega \setminus E_k) \). Then by [15, Theorem 1.3, Chapter 8] this is equivalent to the existence of a non-negative nondecreasing convex function \( \varphi_k : \mathbb{R}_+ \to \mathbb{R}_+ \) satisfying

\[
\lim_{z \to \infty} \varphi_k(z) = \infty
\]

such that

\[
\sup_{n \in \mathbb{N}} \int_{\Omega \setminus E_k} \varphi_k(|c_{n_j}(x)|)dx \leq c.
\]

Let us also define

\[
F(x, \xi) := (S^*(x, \xi) - S^*(x, Dv(x))) \cdot (\xi - Dv(x))
\]

and note that due to condition (1.2) we have the estimate

\[
F(x, \xi) \leq c \left( |\xi|^{q-1} + |Dv(x)|^{q-1} + m(x) \frac{1}{q'} \right) (|\xi| + |Dv(x)|).
\]

Next, we can simply observe that

\[
\int_{\{\alpha \in \mathbb{R}^{d \times d} : |f(x, \alpha)| > R\}} f(x, \alpha) d\nu^n_x(\alpha) = \int_{\{\xi \in \mathbb{R}^{d \times d} : |F(x, \xi)| > R\}} F(x, \xi) d\mu^n_x(\xi).
\]

Therefore, we can estimate

\[
\sup_{n \in \mathbb{N}} \int_{\Omega \setminus E_k} \int_{\{\alpha \in \mathbb{R}^{d \times d} : f(x, \alpha) > R\}} f(x, \alpha) d\nu^n_x(\alpha) dx
\]

\[
= \sup_{n \in \mathbb{N}} \int_{\Omega \setminus E_k} \int_{\{\xi \in \mathbb{R}^{d \times d} : F(x, \xi) > R\}} F(x, \xi) d\mu^n_x(\xi)
\]

\[
\leq \sup_{z \geq R} \varphi_k(z) \sup_{n \in \mathbb{N}} \int_{\Omega \setminus E_k} \int_{\{\xi \in \mathbb{R}^{d \times d} : F(x, \xi) > R\}} \varphi_k(F(x, \xi)) d\mu^n_x(\xi) dx =: J_R
\]
In order to finish the proof we define properties of $\nu_x$ that deduce that for all $G$ the definition of measure imply that $(\ast)$ and the assertion follows.

To show it, we use (4.61) and the fact that $\varphi_k$ is increasing to obtain

$$
\sup_{n \in \mathbb{N}} \int_{\Omega \setminus E_k} \int_{\{\xi \in \mathbb{R}^{d \times d}_{sym} : F(x, \xi) > R\}} \varphi_k(F(x, \xi)) \, d\mu^\alpha_n(\xi) \, dx \leq C.
$$

(4.60)

and the assertion follows.

A direct application of Theorem 2.2 yields the proof of the lemma.

We will characterize the limit of the sequences $\{S^n(x, Dv^n)\}$ with the help of the measure $\nu_x$. Theorem 2.2, relations (4.58) and the fact that weak limit is unique imply that

$$
S^n(\cdot, Dv^n) \rightharpoonup \int_{\mathbb{R}^{d \times d}_{sym}} s(\cdot, \alpha) \, d\nu_x(\alpha) = S \quad \text{weakly in } L^q(\Omega),
$$

$$
Dv^n \rightharpoonup \int_{\mathbb{R}^{d \times d}_{sym}} d(\cdot, \alpha) \, d\nu_x(\alpha) = Dv \quad \text{weakly in } L^q(\Omega).
$$

To complete the proof we need to show that

$$
\left( Dv(x), \int_{\mathbb{R}^{d \times d}_{sym}} s(x, \alpha) \, d\nu_x(\alpha) \right) \in \mathcal{A}(x) \quad \text{in } \Omega_{a.e.}. \quad (4.62)
$$

Combining Lemmas 4.6 and 4.7 we conclude that $b(x) \equiv 0$. Next, using the definition of $G_x$, see (4.56), the definition of $\nu_x^n$, see (4.57), and also $\mu^n_x$, and convergence properties of $\nu^n$ and $\mu^n_x$ we see that for all $\alpha \in \text{supp} \, \nu_x$ there exists $\xi \in \mathbb{R}^{d \times d}$ such that $\alpha = S^*(x, \xi) + \xi$. Lemma 2.1 then implies that for all $\alpha \in \text{supp} \, \nu_x$ there holds $(s(x, \alpha), d(x, \alpha)) \in \mathcal{A}(x)$. Consequently, using (A2) and the fact that $b(x) \equiv 0$ we deduce that for all $x \in \Omega_{a.e.}$

$$
\left\{ \alpha \in \mathbb{R}^{d \times d} : (s(x, \alpha) - S^*(x, Dv(x))) \cdot (d(x, \alpha) - Dv(x)) > 0 \right\} \cap \text{supp} \, \nu_x = \emptyset. \quad (4.63)
$$

In order to prove the proof we define

$$
\Omega^D_{a.e.} := \left\{ x \in \Omega_{a.e.} : \forall (S_1, D_1), (S_2, D_2) \in \mathcal{A}(x) : (S_1 - S_2, D_1 - D_2) = 0 \implies D_1 = D_2 \right\},
$$

$$
\Omega^S_{a.e.} := \left\{ x \in \Omega_{a.e.} : \forall (S_1, D_1), (S_2, D_2) \in \mathcal{A}(x) : (S_1 - S_2, D_1 - D_2) = 0 \implies S_1 = S_2 \right\}.
$$
Note, that (A2) implies that $\Omega_{a.e.} = \Omega_{a.e.}^D \cup \Omega_{a.e.}^S$. Thus, using (4.63) we observe that
\[ x \in \Omega_{a.e.}^D \implies \supp \nu_x \subset \{ \alpha \in \mathbb{R}^{d \times d}_{sym} : \mathbf{d}(x, \alpha) = \mathbf{D}v(x) \}, \tag{4.64} \]
\[ x \in \Omega_{a.e.}^S \implies \supp \nu_x \subset \{ \alpha \in \mathbb{R}^{d \times d}_{sym} : \mathbf{s}(x, \alpha) = S^*(x, \mathbf{D}v(x)) \}. \tag{4.65} \]
Therefore for $x \in \Omega_{a.e.}^S$ we obtain that $\int_{\mathbb{R}^{d \times d}_{sym}} \mathbf{s}(x, \alpha) \, d\nu_x(\alpha) = S^*(x, \mathbf{D}v(x))$ and (4.62) follows. For $x \in \Omega_{a.e.}^D$ we see that to prove (4.62) is equivalent to show that
\[ \left( \mathbf{d}(x, \alpha), \int_{\mathbb{R}^{d \times d}_{sym}} \mathbf{s}(x, \alpha) \, d\nu_x(\alpha) \right) \in A(x) \text{ for all } x \in \Omega_{a.e.}^D. \tag{4.66} \]
If for all $\alpha \in \supp \nu_x$, $\mathbf{s}(x, \alpha)$ belongs to the convex, closed set and $\nu_x$ is a probability measure, then also $\int_{\mathbb{R}^{d \times d}_{sym}} \mathbf{s}(x, \alpha) \, d\nu_x(\alpha)$ is in the same set. Consequently, if $(\mathbf{d}(x, \alpha), \mathbf{s}(x, \alpha)) \in \mathcal{A}(x)$, whose sections are convex closed sets, then also (4.66) holds. The proof of the theorem is now complete. 

\section{Strong convergence}

In this section, we establish some strong convergence properties of sequences $\mathbf{S}^n$ and $\mathbf{D}v^n$ in domains $\Omega_{a.e.}^S$ and $\Omega_{a.e.}^D$.

\begin{lemma}
Let $\mu_x$ and $\lambda_x$ be Young measures corresponding to sequences $\mathbf{D}v^n(x)$ and $\mathbf{S}^n(x)$. Then
\[ \mu_x = \delta_{\{\mathbf{D}v^n(x)\}} \quad \text{on } \Omega_{a.e.}^D, \quad \tag{5.67} \]
\[ \lambda_x = \delta_{\{S^*(x, \mathbf{D}v(x))\}} \quad \text{on } \Omega_{a.e.}^S. \quad \tag{5.68} \]
\end{lemma}

\begin{proof}
First, we observe that
\[ \delta_{\{\mathbf{D}v^n(x)\}} = (\delta_{\{\mathbf{D}v^n(x)\}} - \mu^n) + \mu^n \quad \tag{5.69} \]
Next, for arbitrary fix $f \in L^1(\Omega; C_0(\mathbb{R}^{d \times d}_{sym}))$ we deduce that
\[ |\langle f, \delta_{\{\mathbf{D}v^n\}} - \mu^n \rangle| = \left| \int_{\Omega} f(x, \mathbf{D}v^n(x)) - \int_{\mathbb{R}^{d \times d}_{sym}} f(x, \xi) \, d\mu^n_{\mathbf{D}v^n(x)}(\xi) \, d\xi \, dx \right| \]
\[ = \left| \int_{\Omega} \int_{\mathbb{R}^{d \times d}_{sym}} f(x, \mathbf{D}v^n(x)) - f(x, \xi) \, d\mu^n_{\mathbf{D}v^n(x)}(\xi) \, d\xi \, dx \right| \]
\[ \leq \int_{\Omega} \supp \xi \in B(\mathbf{D}v^n(x), \frac{1}{n}) |f(x, \mathbf{D}v^n(x)) - f(x, \xi)| \, dx \xrightarrow{n \to \infty} 0, \]
where we used the fact that $f$ is equicontinuous in second variable. Combining this with (5.69) we see that to prove (5.67) it is enough to show that $\mu^n \rightharpoonup^* \mu \equiv \delta_{\{\mathbf{D}v\}}$. 

However, from the convergence properties of \( \nu^n \) and from the definition (4.57) we see that the measure \( \nu_x \) satisfies

\[
\mu_x(A) = \nu_x(B) \quad \text{where } B \text{ is such that } A = d(x, B).
\]

Using this relation, we finally observe that (4.64) implies that for \( x \in \Omega^D \text{a.e.} \)

\[
\text{supp } \mu_x \subset \{ \xi \in \mathbb{R}^{d \times d}_{\text{sym}} : \xi = Dv(x) \} = \{Dv(x)\},
\]

which completes the proof of (5.67)

The second part, i.e. (5.68), can be proved similarly. Note that the measure \( \lambda_x \) satisfies

\[
\lambda_x(A) = \nu_x(B) \quad \text{where } B \text{ is such that } A = s(x, B).
\]

\[\square\]

Lemmas 5.1, 2.4, Vitali theorem and apriori estimates (4.40) then directly imply the following Lemma.

**Lemma 5.2** Let \( \{(S^n, v^n)\} \) be a sequence of solutions to the approximate problem (3.31), and \( (S, v) \) be the solution to (1.1). Then for all \( r_1 < q, r_2 < q' \)

\[
Dv^n \rightarrow Dv \quad \text{strongly in } L^{r_1}(\Omega^D_{a.e.})^{d \times d}, \\
S^n \rightarrow S^*(\cdot, Dv) \quad \text{strongly in } L^{r_2}(\Omega^D_{a.e.})^{d \times d}.
\]

**References**


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