A NOTE ON THE DIMENSION OF THE GLOBAL ATTRACTOR FOR ABSTRACT SEMILINEAR HYPERBOLIC PROBLEM

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Abstract

We study a semilinear hyperbolic problem, written as a second order evolution equation in an infinite-dimensional Hilbert space. Assuming existence of the global attractor, we estimate its fractal dimension explicitly in terms of the data. Despite its elementary character, our technique gives reasonable results. Notably, we require no additional regularity, although the nonlinear damping is allowed.

Key words: Nonlinear hyperbolic problem, dimension of attractor, method of \( \ell \)-trajectories

1 Introduction

There is an ongoing interest in the dynamics of hyperbolic problems with nonlinear damping. Despite the difficulties related to usually minimal information on the regularity of solutions, their large time behavior seems now to be well

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understood. Indeed, the existence of a finite-dimensional attractor has been proved for a large class of abstract problems [1], as well as in a number of particular concrete instances ([2], [3], [4]).

Next natural step of the analysis is an explicit estimate of the attractor dimension; and the literature seems to be less extensive in this respect. Concerning in particular the wave equation with nonlinear damping, we refer to [5], [6], which only deal with the 2d problem under very restrictive assumptions on the nonlinearities, and [7], which handles 3d problem, but the estimates are too large (exponential dependence on the data).

The technique we use in the present paper is similar to [4] or [1]; however, the key observation is that the method works even if \( \ell \) (the length of the trajectories) is kept small. In effect, we obtain estimates that are much lower (polynomial dependence on the data), cf. also [8].

As a special case of our abstract setting, we obtain estimates for the damped wave equation in \( \mathbb{R}^n \). Our results are comparable to those obtained in [9] in the case of linear damping, where a stronger technique of differentiability of solution semigroup and Lyapunov exponents was used.

### 2 Equations and preliminaries

We consider an abstract hyperbolic nonlinear problem

\[
\begin{align*}
\ddot{u} + h(u') + Au + g(u) &= 0, \\
\frac{\partial u}{\partial t}|_{t=0} &= u_0, \quad \frac{\partial u}{\partial x}|_{t=0} = u_1,
\end{align*}
\]  

(1)

for \( u(t) : [0, \infty) \to H \), where \( (H, (\cdot, \cdot)) \) is a separable Hilbert space, \( A : D(A) \subset H \to H \) densely defined selfadjoint operator with compact inverse. If \( \lambda_j, w_j \) are the eigenvalues/eigenvectors to \( A \), we set

\[
H^a = D(A^\frac{a}{2}) = \{ u; \sum \limits_j \lambda_j^a c_j^2 < \infty, \text{ where } c_j = (u, w_j) \}
\]

with the usual norm. Note that \( H^0 = H, \ H^1 = D(A^\frac{1}{2}) \) and \( (Au, u) = \| u \|_{H^1}^2 \).

We assume that

\[
\begin{align*}
(g(u) - g(v), u - v) &\geq \alpha \| u - v \|_{H^0}^2, \\
\| g(u) - g(v) \|_{H^0} &\leq c_0 \| u - v \|_{H^0}, \\
\| f(u) - f(v) \|_{H^0} &\leq \gamma \| u - v \|_{H^s}; \quad s \in [0, 1).
\end{align*}
\]  

(2)

We further assume that for any \( [u_0, u_1] \in E = H^1 \times H^0 \) there exists unique \( u \in C([0, \infty); H^1), \ u' \in C([0, \infty); H^0) \) a solution to (1). The solution semigroup
$S(t) : E \rightarrow E$ is given by $S(t) : [u_0, u_1] \mapsto [u(t), u'(t)]$. It will be convenient to denote $E[u(t)] = \|u(t)\|^2_{H^1} + \|u'(t)\|^2_{H^0}$, i.e., the (square of the) $E$-norm of $u$ at time $t$. Our last assumption is that there exists a global attractor $A \subset E$, i.e., a compact, invariant set that attracts all bounded subsets in $E$.

The purpose of this note is to estimate explicitly its fractal dimension $\dim_F(A)$ (in the metric $E$). Recall that

$$\dim_F(A) = \limsup_{\varepsilon \to 0} \frac{\ln N(A, \varepsilon)}{-\ln \varepsilon},$$

where $N(A, \varepsilon)$ is the smallest number of sets of diameter $\varepsilon$ that cover $A$.

### 3 Apriori estimates

Let $u, v$ be solutions to (1). Testing the equation for $w := u - v$ by $w'$ gives, in view of (2),

$$\frac{1}{2} \frac{d}{dt} E[w] + \alpha \|w'\|^2_{H^0} \leq \gamma \|w\|_{H^s} \|w'\|_{H^0} \leq \gamma^2 \|w\|^2_{H^s} + \frac{\alpha}{2} \|w'\|^2_{H^0},$$

hence

$$\frac{d}{dt} E[w] + \alpha \|w'\|^2_{H^0} \leq \frac{\gamma^2}{\alpha} \|w\|^2_{H^s}.$$

Further, since $\|w\|_{H^s} \leq \lambda^{\frac{s+1}{2}} \|w\|_{H^1}$ and $\|w\|_{H^1} \|w\|_{H^0} \leq E[w]/2$, first inequality in (3) implies

$$\frac{d}{dt} E[w] \leq \lambda^{\frac{s+1}{2}} \gamma E[w].$$

It follows from Gronwall’s lemma that

$$E[w(t_2)] \leq cE[w(t_1)], \quad 0 \leq t_2 - t_1 \leq \ell := \lambda^{1-s} \gamma,$$

where $c = \exp(1).$ This is the usual ”bottleneck” of the method of $\ell$–trajectories. One has to keep $\ell$ small to avoid exponentially large estimates.

Secondly, we multiply the equation for $w$ by $w$ to get

$$(w'', w) + \|w\|^2_{H^1} \leq c\alpha \|w'\|^2_{H^0} \|w\|_{H^0} + \gamma \|w\|_{H^s} \|w\|_{H^0}.$$

Noting that $(w'', w) = \frac{d}{dt}(w', w) - \|w'\|^2_{H^0}$, and using estimates $\|w\|_{H^0} \leq \lambda^{\frac{s}{2}} \|w\|_{H^s}$ and $\alpha \|w'\|^2_{H^0} \|w\|_{H^s} \leq \alpha^2 \|w\|^2_{H^0} + \lambda_1^{-s} \|w\|^2_{H^s},$

$$\frac{d}{dt} E[w] \leq \lambda^{\frac{s+1}{2}} \gamma E[w].$$
we conclude
\[
\frac{d}{dt}(w', w) + \|w\|_{H^1}^2 \leq (1 + \alpha^2)\|w'\|_{H^0}^2 + \left(\gamma \lambda_1^{-\frac{3}{2}} + c \lambda_1^{-s}\right) \|w\|_{H^s}^2. \tag{6}
\]

We multiply the last inequality by \(\varepsilon\) and add with (4); having chosen
\[
\varepsilon := \frac{1}{2} \min \left\{ \frac{\alpha}{1 + c\alpha^2}, \lambda_1^{\frac{3}{2}} \right\},
\]
\[
K := \frac{\gamma^2}{\alpha} + \varepsilon \lambda_1^{s}(\gamma \lambda_1^{-\frac{3}{2}} + c),
\]
we arrive at
\[
\frac{d}{dt} E_{\varepsilon} [w] + \varepsilon E[w] \leq K \|w\|_{H^s}^2. \tag{8}
\]
Here, we set \(E_{\varepsilon} [w] := E[w] + \varepsilon (w', w)\). Note that the condition \(\varepsilon \leq \lambda_1^{\frac{3}{2}} / 2\)
implies
\[
c^{-1} E[w] \leq E_{\varepsilon} [w] \leq cE[w]. \tag{9}
\]

4 Smoothing property

By (9) and (5) we have \(E_{\varepsilon} [w(\ell)] \leq cE[w(\ell)] \leq cE[w(s)], \ \forall s \in (0, \ell)\). Hence integrating (8) over \((0, \ell)\) gives
\[
(1 + c\varepsilon\ell) E_{\varepsilon} [w(\ell)] \leq E_{\varepsilon} [w(0)] + K \int_0^\ell \|w\|_{H^s}^2, \nonumber
\]
\[
E_{\varepsilon} [w(\ell)] \leq (1 - \theta) E_{\varepsilon} [w(0)] + K \int_0^\ell \|w\|_{H^s}^2, \tag{10}
\]
where \(\theta \in (0, 1)\) is given by the relation \(1 - \theta = (1 + c\varepsilon\ell)^{-1}\), i.e.,
\[
\theta \geq c\varepsilon\ell. \tag{11}
\]

5 Dimension estimate

The key step now is the following iterated covering argument, deduced from (10). Let \(B \subset E_{\varepsilon}, \ \text{diam } B \leq r\). We will show that \(S(\ell)B\) can be covered by \(N\) sets with diameter \((1 - \theta/2)^{\frac{3}{2}} r\), where \(N\) is independent of \(B\) and \(r\).

Indeed, let \(B_{\ell} \subset L^2(0, \ell; E_{\varepsilon})\) be the \(\ell\)-trajectories starting from \(B\), i.e.,
\[
B_{\ell} = \{\chi; [\chi(0), \chi'(0)] \in B, \ \chi \text{ solves (1) on } [0, \ell]\}.
\]

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It follows from (5) that \( \text{diam } B_\ell \leq c\ell^{1/2}r; \) more precisely, \( B_\ell \subset \chi_0 + \tilde{U}, \) where
\[
\tilde{U} = \{ \chi; \| \chi \|_{L^2(0,\ell;H^1)} \leq \tilde{A}; \| \chi' \|_{L^2(0,\ell;H^0)} \leq \tilde{B} \}.
\]
with \( \tilde{A} = \tilde{B} = c\ell^{1/2}r. \) We need to cover \( B_\ell \) by sets \( F_j^\ell \subset L^2(0,\ell;H^s), j = 1, \ldots, N \) with \( \text{diam } F_j^\ell \leq \eta r, \) where \( \eta \) is given by \( K\eta^2 = \theta/2, \) i.e.,
\[
\eta \geq c \left( \frac{\theta}{K} \right)^{1/2}.
\] (12)

Obviously, the sets \( G_j \subset E_\varepsilon, G_j := \{ [\chi(\ell), \chi'(\ell)]; \chi \in F_j^\ell \} \) cover \( S(\ell)B, \) and it follows from (10) that
\[
(\text{diam } G_j)^2 \leq (1 - \theta)r^2 + K\eta^2r^2 = \left(1 - \frac{\theta}{2}\right)r^2
\]
as required. It remains to determine the covering number \( N. \) Clearly, the problem is equivalent to the covering of
\[
U = \{ \chi; \| \chi \|_{L^2(0,\ell;H^1)} \leq A; \| \chi' \|_{L^2(0,\ell;H^0)} \leq B \}
\]
by 1-balls in \( L^2(0,\ell;H^s), \) where
\[
A = B = c\frac{\tilde{A}}{\eta} = c\ell^{1/2}\theta^{-\frac{1}{2}}K^{\frac{1}{2}}.
\] (13)

It is clear that \( N \) is independent of \( r, \) and if we assume that the behavior of eigenvalues of \( A \) is given as
\[
\lambda_j \sim \lambda_1 j^a,
\] (14)
it follows from [7, Lemma 4.2] that
\[
\ln N \leq c\ell\lambda_1^{-\frac{1}{a}}A^{\left(\frac{a+s}{a}\right)} B \ln(\lambda_1 A + 1).
\]

Now, we can finish the estimate of dim\(_F(A). \) Let diam \( A \leq R. \) Since \( S(\ell)A = A, \) repeated use of the above argument gives \( N(A, (1 - \theta/2)^k) \leq N, \) hence one deduces \( \text{dim}_F(A) \leq \ln N / (-\ln(1 - \theta/2)) \leq c\theta^{-1} \ln N. \) Substitution of the previous quantities gives
\[
\text{dim}_F(A) \leq c\lambda_1^{\frac{1}{a}} \varepsilon^{\left(s - \frac{3}{2} - \frac{1}{a}\right)} \frac{1}{R} K^{\left(\frac{1}{2} + \frac{1}{a}\right)}
\] (15)
where \( \varepsilon, K \) are given in (7).

6 Concluding remarks

Assume that \( A = -\Delta \) in a bounded domain \( \Omega \subset \mathbb{R}^n, \) i.e., (1) corresponds to a damped wave equation. Thus, (14) holds with \( a = 2/n \) (see e.g. [10]). Setting
for simplicity $\lambda_1 = 1$ and assuming $\gamma > 1$, $\alpha < 1$, (7) reduces to $\varepsilon = c\alpha$, $K = c\gamma^2 \alpha^{-1}$ and we come to a more compact estimate

$$\dim_F(\mathcal{A}) \leq c\gamma^{n+1} \alpha^{\frac{n-s-2}{1-s}}.$$  \hspace{1cm} (16)

It is noteworthy that [9, Section 6.2], under stronger assumptions (notably, a linear damping $h(u') = \alpha u'$ and smoothness of $g(\cdot)$) gives a similar estimate $\dim_F(\mathcal{A}) \leq c(\gamma/\alpha)^{\frac{n}{s}}$, using a considerably more involved technique (Lyapunov exponents).

References


