Navier-Stokes-Fourier system for incompressible fluids with temperature dependent material coefficients

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Abstract

We consider a complete thermodynamic model for unsteady flows of incompressible homogeneous Newtonian fluids in a fixed bounded three-dimensional domain. The model comprises evolutionary equations for the velocity, pressure and temperature fields that satisfy the balance of linear momentum and the balance of energy on any (measurable) subset of the domain, and is completed by the incompressibility constraint. Finding a solution in such a framework is tantamount to looking for a \textit{weak solution} to the relevant equations of continuum physics. If in addition the entropy inequality is required to hold on any subset of the domain, the solution that fulfills all these requirements is called \textit{suitable weak solution}. In our setting, both the viscosity and the coefficient of the thermal conductivity are functions of the temperature. We deal with Navier’s slip boundary conditions for the velocity that yield a globally integrable pressure, and we consider zero heat flux across the boundary. For such a problem, we establish the large-data and long-time existence of weak as well as suitable weak solutions.

\textsuperscript{*} M. Bulíček thanks the Nečas Center for Mathematical Modeling, the project LC06052 financed by MSMT for its support. The contribution of E. Feireisl to this work is a part of the general research program of the Academy of Sciences of the Czech Republic, Institutional Research Plan AV0Z10190503. The contribution of J. Málek to this work is a part of the research project MSM 0021620839, financed by MSMT. E. Feireisl and J. Málek thank the Czech Science Foundation, project GACR 201/05/0164, for its support.
1. A brief characterization of the result

This paper concerns the mathematical analysis of a model that describes mechanical and thermal changes exhibited by unsteady flows of incompressible homogeneous Newtonian fluids in fixed bounded three-dimensional domains. Such motions are described by the system of equations representing the constraint of incompressibility, the balance of linear momentum and balance of energy over any open subset of the domain (Reynold’s formulation at the Eulerian description). We can also add the entropy inequality over any open subset of the domain to the above set of equations.

Since the works by Oseen [20] and Leray [15] (see also [9] or Appendix A) it has been known that such formulations of the balance equations are "equivalent" to the so-called weak formulations of the balance equations of continuum physics, and there are systems of continuum physics that possess such weak solution globally in time and for any data. From this perspective the notion of a weak solution is a very natural concept to problems of continuum physics, particularly those studied in this paper.

As regards the constitutive equations, we assume that the fluid is Newtonian (i.e. the viscous part of the Cauchy stress depends linearly on the velocity gradient), the internal energy is proportional to the temperature and the heat flux is proportional to the temperature gradient. We treat the case where both the viscosity and the heat conductivity coefficient depend on the temperature.

As regards the boundary conditions, Navier’s slip for the velocity and zero heat flux for the temperature are considered. The reason why Navier’s slip boundary conditions are incorporated is because we have the possibility to introduce the pressure globally as an integrable function. At this time it is not known whether it is possible to introduce pressure globally as an integrable function for no-slip boundary conditions (see [13] or [25]), the case that is not investigated in this work.

Prescribing arbitrarily the initial velocity and the temperature in suitable spaces, we establish long-time and large-data existence of the relevant solutions. Usually, one subtracts the equations for the kinetic energy from the equation for the balance of energy. Thus, the equation for the internal energy (it means for the temperature if the internal energy is proportional to it) is achieved. These apparently equivalent forms of the balance of energy are indeed equivalent only for smooth velocity fields. Our approach is based on the observation that the original form of the balance of energy shares better mathematical features in comparison with the equation for the internal energy (temperature). More precisely, while the equation for the internal energy includes in general a nonlinear term that is just $L^1$-integrable and in which it is not known how to take the limit with weakly

\footnote{Smooth velocity fields are those that can be used as the test function in the weak formulation of the balance of linear momentum. If the velocity is free of divergence, this corresponds to the case where the relevant velocity field satisfies so-called Prodi-Serrin’s condition.}
converging sequences, dealing with original formulation of the balance of energy we completely avoid such difficulties as all terms are weakly compact. Thus, we obtain a new result for flows in general three-dimensional domains (the analysis of the same model is based on the previous studies in the spatially periodic setting presented by Feireisl and Málek in [10]) that strengthens significantly the results obtained in previous studies. We shall briefly comment upon these previous studies.

In [16, Section 3.4], Lions studied the case where the viscosity and the heat conductivity are positive constants (temperature independent) and provides two approaches how the problem can be investigated in order to establish long-time and large-data existence theory. Naturally none of these approaches is extendable to the model with temperature dependent material coefficients studied here.

Naumann [19] studied the model with the temperature dependent viscosity and the heat conductivity. He establishes the existence of the velocity and the temperature solving the balance of linear momentum and the entropy inequality, and satisfying the balance of total energy. Thus, in his approach the balance of energy is omitted, as clarified also below.

The additional dependence of the viscosity on the shear rate may help, particularly if the velocity is smooth enough in order to be used as a test function in balance of linear momentum. Such feature was exploited in work Consiglieri [7] (see also [6]). Note however that this study does not cover the Newtonian model treated here.

Although the system that is being investigated shares common features with the Boussinesq approximation and its generalizations, the relevant systems of PDEs are of completely different origin. Here, we deal with incompressible fluids and are interested in understanding (mathematical properties) of a quite general class of mechanical and thermal processes associated with them. The Boussinesq approximation and its generalizations are approximations of the complete Navier-Stokes-Fourier system for compressible fluids for flows (regimes) that are isochoric (these are volume preserving flows, i.e., the divergence of the velocity is zero). We completely skip the literature relevant to the Boussinesq approximations and related topics.

The structure of the paper is the following. In the next section, a detailed formulation of the problem is discussed. In section 3, the notion of solution is made precise and the main result is formulated. The section also includes some suitable auxiliary statements needed in the existence proof. The proof is based on a quasicompressible approximation where the pressure is determined as a solution of the Neumann problem, namely \( p = -\frac{1}{\varepsilon} \Delta^{-1} \text{div} \mathbf{v} \), and we also smooth out the velocity \( \mathbf{v} \) in the convective term by a suitable divergenceless mollifier \( \mathbf{v}_\eta \). The existence of \((\varepsilon, \eta)\)-approximation, performed via two-level Galerkin method that might be of independent interest, is for the sake of completeness and clarity included in Appendix B. Key \((\varepsilon, \eta)\)-uniform estimates and the limits as \( \varepsilon \to 0 \) and \( \eta \to 0 \) are presented in Section 4.
2. Formulation of the problem

We assume that flows we are interested in take place in a fixed container identified with a bounded open connected set $\Omega \subset \mathbb{R}^3$ with the boundary $\partial \Omega$.

We deal with incompressible homogeneous fluids. Consequently, their density $\rho$ is equal, at any place $x$ and any time $t$, to a constant value, say $\rho^* \in (0, \infty)$. Motions of such materials are supposed to be well-described through the velocity field $v = (v_1, v_2, v_3) : [0, \infty) \times \Omega \to \mathbb{R}^3$, the pressure $p : [0, \infty) \times \Omega \to \mathbb{R}$ and the internal energy $e : [0, \infty) \times \Omega \to \mathbb{R}$, for which we however assume very simple constitutive equation, namely

$$e(t, x) = c_v \theta(t, x) \quad \text{with} \quad c_v \in (0, \infty) \quad (2.1)$$

where $\theta$ denotes the temperature. Consequently, see Appendix A, the entropy $\eta$ satisfies

$$\eta = c_v \ln \theta. \quad (2.2)$$

The constraint of incompressibility is captured through the equation

$$\text{div } v = 0 \quad \text{in} \quad (0, \infty) \times \Omega. \quad (2.3)$$

In addition, we assume that the fluids of interest are Newtonian and thus the Cauchy stress $T$ takes the form

$$T = -\tilde{p} I + \tilde{\nu}(\theta) D(v), \quad (\tilde{\nu} > 0) \quad (2.4)$$

where $\tilde{\nu}$ is the viscosity and $D(v) = \frac{1}{2}(\nabla v + (\nabla v)^T)$ is the symmetric part of the velocity gradient. Note that the pressure $\tilde{p}$ coincides, due to (2.3), with the mean normal stress, i.e. $\tilde{p} = -\frac{1}{3} \text{tr } T$. Finally, the heat flux $q$ fulfills Fourier’s relation

$$q = -\tilde{k}(\theta) \nabla \theta, \quad (2.5)$$

where the coefficient of heat conductivity $\tilde{k}$ depends on the temperature.

Changes in the velocity field $v$, the pressure $\tilde{p}$ and the internal energy $e$ (or due to (2.1) temperature $\theta$) are supposed to be captured by the balance of linear momentum and the balance of energy over any open subset $O \subset \Omega$, completed by the incompressibility constraint (2.3). Thus, in our setting for any $O \subset \Omega$ we have (see also Appendix A)

$$\frac{d}{dt} \int_O \rho^* v \, dx = -\int_{\partial O} \rho^* v (v \cdot n) \, dS + \int_{\partial O} (T n) \, dS + \int_O \rho b \, dx \quad (2.6)$$

and

$$\frac{d}{dt} \int_O \rho^* \left( e + \frac{|v|^2}{2} \right) \, dx = -\int_{\partial O} \rho^* \left( e + \frac{|v|^2}{2} \right) v \cdot n \, dS$$

$$+ \int_{\partial O} (T n \cdot v) \, dS - \int_{\partial O} q \cdot n \, dS + \int_O \rho^* b \cdot v \, dx. \quad (2.7)$$
In addition, taking the entropy inequality into account we obtain

$$\frac{d}{dt} \int_{\Omega} \rho^* \eta \, dx = - \int_{\partial \Omega} \rho^* \eta v \cdot n \, dS + \int_{\partial \Omega} \frac{q}{\theta} \, n + \int_{\Omega} \rho^* \xi \, dx,$$

with $\rho^* \xi \geq \frac{\tilde{v}(\theta)}{\theta} |D(v)|^2 + \frac{\tilde{k}(\theta)}{\theta^2} |\nabla \theta|^2$.

Following Oseen [20], Leray [15] and Feireisl [9], see also Appendix A, we observe that the equations (2.6) and (2.7) are tantamount to the weak formulation of the system that we write for reader’s convenience in their classical forms, namely

$$(\rho^* v)_t + \text{div}(\rho^* v \otimes v) - \text{div} T = \rho^* b,$$

$$(\rho^* (e + |v|^2/2))_t + \text{div}(\rho^* v \left(e + |v|^2/2\right)) = \text{div}(Tv - q) + \rho^* b \cdot v;$$

similarly, (2.8) leads to a weak formulation of the inequality

$$(\rho^* \eta)_t + \text{div}(\rho^* \eta v) + \text{div} \left(\frac{q}{\theta} \right) \geq \frac{\tilde{v}(\theta)}{\theta} |D(v)|^2 + \frac{\tilde{k}(\theta)}{\theta^2} |\nabla \theta|^2.$$  (2.12)

Taking the constitutive equations (2.1), (2.4) and (2.5) into account, dividing (2.9) and (2.10) by $\rho^*$, and relabelling $\frac{\tilde{v}}{\rho^*}$, $\frac{\tilde{k}}{\rho^*}$ and $\frac{\tilde{p}}{\rho^*}$ by $p, \nu, k$, we obtain from (2.3), (2.9) and (2.10) the system

$$(c v \theta + \frac{|v|^2}{2})_t + \text{div} \left(\left(c v \theta + \frac{|v|^2}{2} + p\right) v\right) - \text{div}(k(\theta) \nabla \theta) = \text{div}(\nu(\theta) D(v)v) + b \cdot v.$$  (2.14)

From (2.12) and (2.2) we obtain similarly

$$c v \theta_t + \text{div}(c v \theta v) - \text{div}(k(\theta) \nabla \theta) \geq \nu(\theta) |D(v)|^2.$$  (2.15)

It is worth of noticing that (2.14) and (2.15) imply

$$\left(\frac{|v|^2}{2}\right)_t + \text{div} \left(\left(\frac{|v|^2}{2} + p\right) v\right) - \text{div}(\nu(\theta) D(v)v) \geq \nu(\theta) |D(v)|^2 \leq b \cdot v.$$  (2.16)

Thus, the system (2.13), (2.14) and (2.15) is equivalent to (2.13), (2.14) and (2.16). See Remark 3.2 below for further comments.

Regarding the boundary conditions we assume that the boundary is impermeable, i.e.,

$$v \cdot n = 0 \quad \text{on } (0, T) \times \partial \Omega,$$  (2.17)
and the fluid does not completely adhere to the boundary, but rather exhibits a partial slip according to the equation $(\tilde{z}_\tau := z - (z \cdot n)n)$

$$\tilde{\alpha} v_\tau + (Tn)_\tau = 0 \quad \text{where } \tilde{\alpha} \in [0, \infty) \quad (2.18)$$

or equivalently, using (2.4) divided by $\rho^*$,

$$\alpha v_\tau + \nu(\theta)D(v)n_\tau = 0, \quad \alpha \in [0, \infty). \quad (2.19)$$

Note that the case $\alpha = 0$ in (2.19) represents the (complete or perfect) slip condition

$$Tn \times n = \nu(\theta)D(v)n \times n = 0.$$ 

On the contrary, letting $\alpha \to +\infty$ in (2.19) we obtain the no-slip condition:

$$v_\tau = 0,$$

the case that is not included in our analysis here. Condition (2.19) is usually called Navier's slip boundary condition.

There has been a long-lasting debate on suitable type of the equations that capture the flows of a fluid along solid impermeable boundary. Instead of opening the debate on the validity of no-slip (usually accepted for Newtonian fluids and laminar flows with small or medium velocity) versus Navier’s slip-type boundary conditions (used for flows over rough walls, or through porous media, or for flows of polymers etc.), we refer the reader to two review articles [21] and [14], and add several recent interesting studies. Jäger and Mikelić in [11] and [12] deal with the rough boundary of thickness $\varepsilon$ and find (for fixed and small $\varepsilon$) that in comparison with the no-slip, Navier’s slip (considered on flat boundary) approximates better the flows fulfilling no-slip boundary conditions prescribed on the oscillatory boundary. On the other hand, Bueur et al. [2] and [1] study the behavior of $v^\varepsilon$, a solution of the (incompressible) Navier-Stokes system on $\Omega_\varepsilon$ as $\varepsilon \to 0$; $\Omega_\varepsilon$ being perturbations (with $\varepsilon$ amplitude) of the domain $\Omega$. If $\Omega$ is perturbed with little cones (or riblets), then the limit velocity solves the Navier-Stokes equations with no-slip on the boundary. However, if the perturbations of the pipe are sinusoidal and again Navier’s slip is considered on the boundary of $\Omega_\varepsilon$ then the velocity component along the channel vanishes (no-slip) while in the orthogonal directions the boundary condition for the limit solution is of Navier’s slip type, the coefficient $\alpha$ has however a different value.

Concerning the temperature, we consider the case where there is no heat exchange between the body and its surrounding, i.e.,

$$\nabla \theta \cdot n = 0 \quad \text{on } (0,T) \times \partial \Omega. \quad (2.20)$$

Finally, we prescribe the initial conditions

$$v(0,x) = v_0(x) \quad \text{and} \quad \theta(0,x) = \theta_0(x) \quad \text{in } \Omega. \quad (2.21)$$
3. Definition of solution, main theorem and auxiliary results

We start repeating the formulation of our problem: we look for \((v, p, \theta) : [0, T] \times \Omega \to \mathbb{R}^3 \times \mathbb{R} \times \mathbb{R}^+\) solving weak formulations of the following set of five equations in \((0, T) \times \Omega\)

\[
\begin{array}{l}
\frac{\partial v}{\partial t} + \text{div}(v \otimes v) - \text{div}(\nu(\theta) D(v)) + \nabla p = b, \quad \text{div} v = 0,
\end{array}
\]

\[
(c_v \theta + \frac{|v|^2}{2}) + \text{div} \left( c_v \theta + \frac{|v|^2}{2} + p \right) v - \text{div}(k(\theta) \nabla \theta) = \text{div}(\nu(\theta) D(v)v) + b \cdot v,
\]
completed by a weak formulation of the inequality

\[
(c_v \theta), t + \text{div}(c_v \theta v) - \text{div}(k(\theta) \nabla \theta) \geq \nu(\theta)|D(v)|^2,
\]

that can be rewritten in an equivalent form (2.16) and satisfying the boundary and initial conditions

\[
\begin{array}{l}
v \cdot n = 0, \quad \alpha \theta_r + \nu(\theta)(D(v)n)_r = 0, \quad \nabla \theta \cdot n = 0 \quad \text{on} \quad (0, T) \times \partial \Omega, \quad (3.3)
v(0, x) = v_0(x), \quad \theta(0, x) = \theta_0(x) \quad \text{for all} \quad x \in \Omega. \quad (3.4)
\end{array}
\]

Before we give a precise formulation of the notion of weak solution we introduce the notation and several lemmas that will be needed in what follows.

3.1. Basic definitions and spaces

Let \(\Omega \subseteq \mathbb{R}^d\) be a bounded domain with Lipschitz boundary \(\partial \Omega\), i.e. \(\Omega \in C^{0,1}\). Moreover, we write \(\Omega \in C^{1,1}\) if in addition the mappings that locally described the boundary \(\partial \Omega\) belong to \(C^{1,1}\).

For an arbitrary \(r \in [1, \infty]\), \(L^r(\Omega)\) denotes the usual Lebesgue spaces equipped with the norm \(\| \cdot \|_r\), and \(W^{1,r}(\Omega)\) denotes the usual Sobolev spaces with the norm \(\| \cdot \|_{1,r}\). We use the notation \(\text{tr} u\) for a trace of a Sobolev function \(u\) on the boundary \(\partial \Omega\). If \(u\) is a vector-valued Sobolev function, i.e. \(u \in (W^{1,r}(\Omega))^d\), then we use \(\text{tr} \ u := (\text{tr} u_1, \ldots, \text{tr} u_d)\) to denote its trace on the boundary. Navier’s boundary condition requires to introduce some further notation of the subspaces of vector-valued Sobolev or Lebesgue functions which have zero normal component on the boundary. We thus define for arbitrary \(r \in [1, \infty]\)

\[
W_{n}^{1,r} := \left\{ v; v \in W^{1,r}(\Omega)^d, \text{tr} v \cdot n = 0 \text{ on } \partial \Omega, \right\},
\]

\[
W_{n,\text{div}}^{1,r} := \left\{ v \in W_{n}^{1,r}; \text{div} v = 0 \right\}, \quad \text{and} \quad L_{n}^{r} := \left\{ v \in W_{n,\text{div}}^{1,r}; \| \cdot \|_r \right\}.
\]

We also introduce the notation for the dual spaces \((r' = r/(r-1))\):

\[
W_{n}^{-1,r'} := \left(W_{n}^{1,r}\right)^* \quad \text{and} \quad W_{n,\text{div}}^{-1,r'} := \left(W_{n,\text{div}}^{1,r}\right)^*.
\]
Note that for $q \geq \frac{2d}{d+2}$ we have the following representation of the duality between $W_{n}^{1,q}, W_{n}^{-1,q'}$ and $W_{n,\text{div}}^{1,q}, W_{n,\text{div}}^{-1,q'}$:

$$W_{n}^{1,q} \text{ densely } \hookrightarrow L^{2}(\Omega)^{d} \cong (L^{2}(\Omega)^{d})^{*} \hookrightarrow W_{n}^{-1,q'},$$

$$W_{n,\text{div}}^{1,q} \text{ densely } \hookrightarrow L_{n}^{2} \cong (L_{n}^{2})^{*} \hookrightarrow W_{n,\text{div}}^{-1,q'}.$$

Consequently, $(W_{n}^{1,q}, L^{2}(\Omega)^{d}, W_{n,\text{div}}^{-1,q'})$ and $(W_{n,\text{div}}^{1,q}, L_{n}^{2}, W_{n,\text{div}}^{-1,q'})$ form the Gelfand triplets and we can easily identify the dualities $\langle \cdot, v \rangle_{W_{n}^{1,q}, W_{n}^{-1,q'}}$ and $\langle \cdot\rangle_{W_{n,\text{div}}^{1,q}, W_{n,\text{div}}^{-1,q'}}$ using the Hilbert space $L^{2}$ and $L_{n}^{2}$ respectively.

We use standard notation for the Bochner spaces, as are for example $\langle \cdot, v \rangle_{L^{2}(\Omega)^{d}, W_{n,\text{div}}^{-1,q'}}$.

The next lemma provides compactness of traces for relevant Bochner function spaces.

**Lemma 3.1 (Korn’s inequality)** Let $\Omega \in C^{0,1}$ and $q \in (1, \infty)$. There exists a positive constant $C$ depending only on $\Omega$ and $q$ such that for all $v \in W_{n}^{1,q}(\Omega)^{d}$ with $\text{tr} \ n v \in L^{2}(\partial \Omega)^{d}$ there holds

$$C \|v\|_{1,q} \leq \|D(v)\|_{q} + \|\text{tr} \ n v\|_{L^{2}(\partial \Omega)}.$$  

**Proof.** For a proof see Lemma 1.3 in [3].

The next lemma provides compactness of traces for relevant Bochner function spaces.
Lemma 3.2 Let $q_1 \geq 1$ and $r, q_2 \in (1, \infty)$. Let $S$ be defined through
\[ S := \{ v; v \in L^\infty(0, T; L^2(\Omega)^d) \cap L^r(0, T; W^{1,r}_n), v_s \in L^b(0, T; W^{-1,dr}_n) \}. \]
If $\{v^i\}_{i=1}^\infty$ is bounded in $S$ and $r \in \left(\frac{2d}{d+2}, 2\right]$, then $\{\text{tr} v^i\}_{i=1}^\infty$ is precompact
in $L^p(0, T; L^q(\partial\Omega)^d)$ for all $p, s \in [1, \infty)$ satisfying
\[ s \in \left(\frac{2d-1}{d}, \frac{r(d-1)}{d-r}\right), \quad p < s \frac{dr + 2r - 2d}{sd - 2d + 2}. \quad (3.10) \]

**Proof.** See Lemma 1.4 in [3] where even more general cases are treated.

By using Lemma 3.2 we can easily verify the following

**Corollary 3.1** Let $\{v^i\}_{i=1}^\infty$ be bounded in $S$ with $d = 3$ and $r = 2$. Then
$\{\text{tr} v^i\}_{i=1}^\infty$ is precompact in $L^2(0, T; L^2(\partial\Omega)^3)$ and also in $L^q(0, T; L^\frac{6}{5}(\partial\Omega)^3)$
for all $q \in [1, \infty)$.

We will also frequently use the following interpolation inequalities that
are consequences of Hölder’s (see [18, Corollary 1.2.10]) and embeddings
inequalities:
\[ \|z\|_q \leq \|z\|_2^{\frac{6(q-2)}{6q-2}} \|z\|_6^{\frac{3(q-1)}{6q-2}} \leq C \|z\|_2^{\frac{3(q-2)}{6q-2}} \|z\|_1^{\frac{3(q-1)}{6q-2}} \quad (2 \leq q \leq 6), \quad (3.11) \]
\[ \|z\|_q \leq \|z\|_1^{\frac{6(q-1)}{6q}} \|z\|_6^{\frac{3(q-2)}{6q}} \leq C \|z\|_1^{\frac{6(q-1)}{6q}} \|z\|_2^{\frac{3(q-2)}{6q}} \quad (1 \leq q \leq 6). \quad (3.12) \]

3.3. **Definition of weak and suitable weak solutions and main theorem**

In this paper we assume that $\nu$ and $k$ are bounded continuous functions
defined on $(0, \infty)$ satisfying for certain $C_1, C_2$
\[ 0 < C_1 \leq \nu(\theta) \leq C_2 \quad \text{for all } \theta \in (0, \infty), \quad (3.13) \]
\[ 0 < C_1 \leq k(\theta) \leq C_2 \quad \text{for all } \theta \in (0, \infty). \quad (3.14) \]
We also prescribe relevant assumptions on the initial conditions $v_0, \theta_0$ and
on external body forces $b$, namely
\[ v_0 \in L^2_n, \quad \theta_0 \in L^1(\Omega), \quad b \in L^2(0, T; W^{-1,2}_n). \quad (3.15) \]
In addition we assume that initial temperature is uniformly positive, i.e.
there is a constant $C_3 \in (0, \infty)$ such that
\[ \theta_0(x) \geq C_3 > 0 \quad \text{for a. a. } x \in \Omega. \quad (3.16) \]
For all $\varphi \in C(\overline{\Omega})$, we define the energetic functionals $E(t, \cdot)$ and $E_0$ as
\[ E(t, \varphi) := (\theta(t, \cdot) + \frac{1}{2}|v(t, \cdot)|^2, \varphi) \quad \text{and} \quad E_0(\varphi) := (\theta_0 + \frac{1}{2}|v_0|^2, \varphi). \]

Next we introduce the notion of weak and suitable weak solution to (3.1)-(3.4).
Definition 3.1 Let $\Omega \in C^{1,1}$. Assume that $\nu$ and $k : \mathbb{R} \to [0, \infty)$ are continuous functions satisfying (3.13)-(3.14), and $v_0, \theta_0, b$ satisfy (3.15) and (3.16). We say that a triplet $(v, \theta, p)$ is weak solution to the problem (3.1), (3.3)-(3.4) if

$$
v \in C([0, T]; L^2_{weak}(\Omega)^3) \cap L^2(0, T; W^{1,2}_{n,\text{div}}),$$

$$v_t \in L^2(0, T; W^{-1,2}_n),$$

$$p \in L^2(0, T; L^2(\Omega)),$$

$$\theta \in L^\infty(0, T; L^1(\Omega)) \cap L^n(0, T; W^{1,n}(\Omega)) \text{ for all } n \in [1, 5/4),$$

$$E(t, \varphi) \in C([0, T]) \text{ and } \lim_{t \to 0^+} E(t, \varphi) = E_0(\varphi) \text{ for all } \varphi \in C(\overline{\Omega}),$$

$$\lim_{t \to 0^+} \|v(t) - v_0\|_2 = 0,$$

$$\int_0^T \left( \langle v_t, \varphi \rangle - (v \otimes v, \nabla \varphi) + (\nu(\theta)|D(v)|^2, D(\varphi)) + \alpha(v, \varphi)_{\partial \Omega} \right) dt$$

$$= \int_0^T (p, \text{div } \varphi) + \langle b, \varphi \rangle dt \quad \text{for all } \varphi \in L^\infty(0, T; W^{1,\infty}_n),$$

$$\int_0^T \left( - \left( \frac{|v|^2}{2} + \theta, \varphi_t \right) - \left( v \left( \frac{|v|^2}{2} + \theta + p \right), \nabla \varphi \right) \right.$$  

$$\left. + (k(\theta)\nabla \theta, \nabla \varphi) + \alpha(|v|^2, \varphi)_{\partial \Omega} + (\nu(\theta)|D(v)|^2, \nabla \varphi) \right) dt$$

$$= \int_0^T \langle b, v\varphi \rangle dt + E_0(\varphi) \quad \text{for all } \varphi \in D(-\infty, \infty, C^\infty(\overline{\Omega})),$$

and temperature $\theta$ satisfies

$$\theta(t, x) \geq C_3 > 0 \quad \text{for a.a.} (t, x) \in (0, T) \times \Omega.$$ 

(3.25)

We say that $(v, \theta, p)$ is suitable weak solution to the problem (3.1)-(3.4) if in addition to (3.17)-(3.25), the following inequality is valid:

$$\int_0^T \left( - (\theta, \psi_t) - (v \theta, \nabla \psi) + (k(\theta)\nabla \theta, \nabla \psi) - (\nu(\theta)|D(v)|^2, \psi) \right) dt$$

$$\geq (\theta_0, \psi(0)) \quad \text{for all } \psi \geq 0, \psi \in D(-\infty, \infty, C^\infty(\overline{\Omega})).$$

(3.26)

Before formulating the main result we provide several remarks related to the above definition.

Remark 3.1. We first observe that the weak formulation (3.23) involving $v$ and $p$ can be split into two equations separating $v$ from $p$. More precisely, considering only those $\varphi$ fulfilling $\text{div } \varphi = 0$ in (3.23) we obtain

$$\int_0^T \left( \langle v_t, \varphi \rangle - (v \otimes v, \nabla \varphi) + (\nu(\theta)|D(v)|^2, D(\varphi)) + \alpha(v, \varphi)_{\partial \Omega} \right) dt$$

$$= \int_0^T \langle b, \varphi \rangle dt \quad \text{for all } \varphi \in L^\infty(0, T; W^{1,\infty}_{n,\text{div}}),$$

\$\LaTeX\$: 2007/3/13 \$
and by inserting $\phi := \nabla \psi$ into (3.23), where $\psi$ solves

$$
\triangle \psi = h \quad \text{in} \quad \Omega,
$$

$$
\frac{\partial \psi}{\partial n} = 0 \quad \text{on} \quad \partial \Omega,
$$

$$
\int_{\Omega} \psi \, dx = 0
$$

we conclude the equation for the pressure

$$(p, h) = (\nu(\theta) \mathbf{D}(v) - v \otimes v, \nabla^2 (\triangle^{-1} h)) + \alpha(v, \nabla (\triangle^{-1} h))_{\partial \Omega}
$$

$$
- \langle b, \nabla (\triangle^{-1} h) \rangle \quad \text{for a.a.} \quad t \in (0, T), \quad \text{all} \quad h \in L^\infty(\Omega) : \int_{\Omega} h \, dx = 0.
$$

**Remark 3.2.** Due to the equivalence of the system (2.13), (2.14) and (2.15) to the system (2.13), (2.14) and (2.16) we observe that the entropy inequality written in the form (3.26) can be replaced by

$$
\alpha \int_0^T (|v|^2, \dot{\psi})_{\partial \Omega} \, dt + \int_0^T \int_{\Omega} (\nu(\theta)|\mathbf{D}(v)|^2 \psi) \, dx \, dt \leq \frac{1}{2} (v_0, \psi)
$$

$$
+ \frac{1}{2} \int_0^T (|v|^2, \dot{\psi}) + (v (|v|^2 + 2p), \nabla \psi) - 2 (\nu(\theta)\mathbf{D}(v) v, \nabla \psi) \, dt
$$

$$
+ \int_0^T (b, \nabla \psi) \, dt \quad \text{for all} \quad \psi \in D(-\infty, T, C^\infty(\Omega)),
$$

Note that if $\nu(\theta) = \nu^*$ the application of the integration by parts leads to

$$
\frac{\nu^*}{2} \int_0^T \int_{\Omega} |\nabla v|^2 \psi \, dx \, dt \leq \frac{1}{2} (v_0, \psi) + \int_0^T (b, \nabla \psi) \, dt
$$

$$
+ \frac{1}{2} \int_0^T \left( |v|^2, \dot{\psi} + v \cdot \nabla \psi + \frac{\nu^*}{2} \triangle \psi \right) + 2 (p v, \nabla \psi) \, dt
$$

$$
\text{for all} \quad \psi \in D(-\infty, T, D(\Omega)),
$$

which is the inequality associated with the notion of suitable weak solution introduced by Caffarelli et al. [4].

If $v$ is a suitable test function in (3.23) then the equation for the total kinetic energy (Power theorem) is achieved. Subtracting it from (3.24) one obtains (3.26) with the equality sign. Consequently, (3.26) is redundant. This for example happens in two spatial dimensions, or, in three spatial dimensions, for a generalized Navier-Stokes-Fourier system with the power-law index $r \geq \frac{11}{5}$.

**Remark 3.3.** Let $\varphi^n \in D(-\infty, T, C^\infty(\Omega))$ approximate the characteristic function of the interval $(0, t)$. Inserting them into (3.24) and letting $n \to \infty$ we easily obtain the balance of total energy involved in the considered bulk, namely

$$
E(t, 1) + \alpha \int_0^t \|v\|^2_{L^2(\partial \Omega)} \, dt = \int_0^t \langle b, \dot{v} \rangle \, dt + E_0(1) \quad \text{for all} \quad t \in (0, T).
$$

(3.29)

At this point we can clearly specify what are the differences between the studies in this paper and the former results by Naumann [19] or Consiglieri [7]. Naumann shows that there is a solution $(v, \theta)$ satisfying (3.17)-(3.18),...
Moreover, he completely avoids the equation for energy (3.24). On the other hand, Consiglieri [7] achieved a stronger result assuming however that $\nu := \nu(\theta, \|D(v)\|)$ is of the type $\mu(\theta)(1 + \|D(v)\|^2)^{\frac{r}{2}}$ with $r \geq \frac{11}{5}$. (It means that Newtonian fluid is not included in her studies.) She proved that there exists $(v, \theta)$ solving the relevant weak formulations of the balance of linear momentum and the equation for the internal energy (temperature).

**Remark 3.4.** It is worth of noticing that all integrals in the definition of weak solution are finite. Indeed, Lemma 3.2 together with (3.17) imply that

$$\text{tr } v \in L^2(0, T; L^2(\partial \Omega)^3),$$

thus all boundary integrals are meaningful.

Concerning the volume integrals we observe that the most critical term is the second one in (3.24). Using the standard interpolation inequalities (3.11), it follows from (3.17) that $v \in L^{\frac{11}{2}}(\Omega \times (0, T))^3$. Since $p \in L^{\frac{5}{3}}(\Omega \times (0, T))$, the application of Hölder’s inequality implies that

$$v \left(\frac{|v|^2}{2} + p\right) \in L^{\frac{10}{3}}(\Omega \times (0, T))^3.$$

Similarly, using (3.12), (3.20) and Hölder’s inequality again we obtain that

$$v\theta \in L^q(\Omega \times (0, T))^3 \text{ for all } q \in [1, 10/9].$$

The other terms appearing in the definition of weak solution can be bounded directly from the estimates (3.17)-(3.20) and the assumptions on $\nu$ and $k$.

Now, we formulate the main theorem of this paper.

**Theorem 3.1** Let $\Omega \in C^{1,1}$ be a three-dimensional bounded domain. Assume that $\nu$ and $k : \mathbb{R} \to [0, \infty)$ are continuous functions satisfying (3.13)-(3.14), and $v_0, \theta_0, b$ satisfy (3.15) and (3.16). Then there exists a suitable weak solution to (3.1)-(3.4).

To our best knowledge, this is the first result that solves fully, from the point of view of the existence theory, the complete thermodynamical system for incompressible Newtonian fluids with temperature dependent material coefficients in bounded three-dimensional domains. This theorem generalizes the results presented in [10], where the authors establish the existence of weak solution for a spatially periodic problem under the same assumptions on $\nu$ and $k$. 
4. Proof of Theorem 3.1

The proof of Theorem 3.1 is split into several steps. In Subsection 4.1, we introduce a quasicompressible approximation where the pressure is determined as a solution of the Neumann problem, namely

$$p = -\frac{1}{\varepsilon} \Delta^{-1} \text{div } v,$$

and we also smooth out the velocity $v$ in the convective term by a suitable divergenceless mollifier $v_\eta$. The existence of $(\varepsilon, \eta)$-approximation, performed via two-level Galerkin method, is for the sake of completeness and clarity included in Appendix B. Then in Subsection 4.2 we derive estimates that are uniform with respect to $\varepsilon$ and $\eta$. In Subsection 4.3 we investigate behavior of solutions of $(\varepsilon, \eta)$-approximations if $\varepsilon$ tends to 0. In particular, the limit velocity is free of divergence. In this part the fact that we deal with Navier’s boundary condition together with the assumption $\Omega \in C^{1,1}$ will play an important role. Finally, in Subsection 4.4, letting $\eta \to 0^+$, we study the behavior of $(v^n, \theta^n, p^n)$ in the entropy inequality and in the equations for the balance of linear momentum and the balance of energy.

4.1. Definition of $(\varepsilon, \eta)$-approximations and their solutions

Let $\varepsilon$ and $\eta$ be fixed positive in general small numbers. Instead of the equation $\text{div } v = 0$, we consider its quasicompressible approximation given by

$$\varepsilon \Delta p = \text{div } v \text{ in } \Omega \times (0, T), \quad \frac{\partial p}{\partial n} = 0 \text{ on } \partial \Omega \times (0, T), \quad \int_{\Omega} p \, dx = 0. \quad (4.1)$$

Next, we define a divergenceless, smooth $\eta$-approximation of the velocity field. First, we introduce a function $\varphi_\eta$ through

$$\varphi_\eta(x) := \begin{cases} 
0 & \text{if dist}(x, \partial \Omega) \leq 2\eta, \\
1 & \text{elsewhere}.
\end{cases}$$

For $v \in W^{1,q}_n$ we set $v_\eta := ((\varphi_\eta v) * \omega_\eta)_{\text{div}}$, where the symbol $u * \omega_\eta$ denotes the standard regularization of an integrable function $u$ with kernel $\omega_\eta$ having the support in a ball of radius $\eta$. The symbol $(\cdot)_{\text{div}}$ then comes from the Helmholtz decomposition (3.6). Note that if $v^n \to v$ in $L^q(\Omega \times (0, T))$ and $\text{div } v = 0$ then $v^n_\eta \to v$ in $L^q(\Omega \times (0, T))$. We use $v_\eta$ in the convective terms both in the balance of linear momentum and in the equation for the temperature.

**Definition 4.1** Let $\Omega \in C^{1,1}$. Assume that $\nu$ and $k : \mathbb{R} \to [0, \infty)$ are continuous functions satisfying (3.13)-(3.14), and $v_0, \theta_0, b$ satisfy (3.15) and (3.16). Let $\varepsilon, \eta$ be positive numbers. We say that a triplet $(v, \theta, p) :=$
\((v^{\varepsilon,\eta},\theta^{\varepsilon,\eta},p^{\varepsilon,\eta})\) is weak solution to the \((\varepsilon,\eta)\)-approximation of (3.1)-(3.4) if
\[
 v \in C(0,T;L^2(\Omega)^3) \cap L^2(0,T;W^{1,2}_n), \tag{4.2}
\]
\[
 v_t \in L^2(0,T;W^{-1,2}_n), \tag{4.3}
\]
\[
 \theta \in L^\infty(0,T;L^1(\Omega)), \tag{4.4}
\]
\[
 \theta^{1+\lambda} \in L^2(0,T;W^{1,2}(\Omega)) \quad \text{for all } \lambda \in (-1,0), \tag{4.5}
\]
\[
 p \in L^2(0,T;W^{1,2}(\Omega)), \tag{4.6}
\]
\[
 \lim_{t\to 0^+} \|v(t) - v_0\|_2 = 0 \quad \text{and} \quad \lim_{t\to 0^+} \|\theta(t) - \theta_0\|_1 = 0, \tag{4.7}
\]
\[
 -\varepsilon(\nabla p(t),\nabla \varphi) = (\varphi, \text{div } v(t)),
\]
\[
 \text{for all } \varphi \in W^{1,2}(\Omega) \text{ and a.a. } t \in (0,T). \tag{4.8}
\]
\[
 \int_0^T \left( (v_t, \varphi) - (v \otimes v, \nabla \varphi) + (\nu(\theta)D(v),D(\varphi)) + \alpha(v,\varphi)\partial\Omega \right) dt
\]
\[
 = \int_0^T (p, \text{div } \varphi) + (b, \varphi) dt \quad \text{for all } \varphi \in L^2(0,T;W^{1,2}_n), \tag{4.9}
\]
\[
 \int_0^T \left( -(\theta, \psi_t) - (\psi \theta, \nabla \varphi) + (k(\theta)\nabla \theta, \nabla \psi) - (\nu(\theta)D(v)^2, \psi) \right) dt
\]
\[
 = (\theta_0, \psi(0)) \quad \text{for all } \psi \in D(-\infty,T;C^\infty(\Omega)). \tag{4.10}
\]
and
\[
 \theta(t,x) \geq C_3 > 0 \quad \text{for a.a. } (t,x) \in (0,T) \times \Omega. \tag{4.11}
\]

**Lemma 4.1** Let \(\varepsilon\) and \(\eta\) be positive numbers. Let \(\Omega \in C^{1,1}\) be a three-dimensional bounded domain. Assume that \(\nu\) and \(k : \mathbb{R} \to [0,\infty)\) are continuous functions satisfying (3.13)-(3.14), and \(v_0,\theta_0, b\) satisfy (3.15) and (3.16). Then there exists a weak solution to the \((\varepsilon,\eta)\)-approximation.

**Proof.** See Appendix B.

### 4.2. Uniform estimates

For \((v,\theta, p) = (v^{\eta,\varepsilon},\theta^{\eta,\varepsilon}, p^{\eta,\varepsilon})\) we derive estimates that are uniform w.r.t both \(\eta\) and \(\varepsilon\) (the relevant quantities are then bounded by \(C\)), and also estimates uniform only w.r.t \(\varepsilon\) (the quantities are bounded \(C(\eta^{-1})\)).

First, we take \(\varphi = v\) in (4.9) and \(\varphi = p\) in (4.8), add the resulting equations and obtain for all \(t \in (0,T)\)
\[
 \frac{1}{2} \|v(t)\|_2^2 + \int_0^t \int_\Omega \nu(\theta)D(v)^2 dx + \varepsilon\|\nabla p\|_2^2 d\tau
\]
\[
 + \alpha \int_0^t \|v\|_{L^2(\partial\Omega)}^2 d\tau = \int_0^t (b, v) d\tau + \frac{1}{2} \|v_0\|_2^2. \tag{4.12}
\]
Using the assumption on \( \nu \), see (3.13), and Korn’s inequality, see Lemma 3.1, we easily conclude from (4.12) that

\[
\sup_{t \in (0,T)} \|v(t)\|_2^2 + \int_0^T \int_\Omega \nu(\theta) |D(v)|^2 \, dx \, dt + \int_0^T \|v\|^2_{W^{1,2}_\eta} + \varepsilon \|\nabla p\|^2_2 \, dt \leq C. \tag{4.13}
\]

Then, we set \( \varphi \equiv 1 \) in (4.10) and with help of (4.13) we conclude that

\[
\sup_t \|\theta(t)\|_1 \leq C. \tag{4.14}
\]

Next, we take \( \varphi = \theta^\lambda \) with \(-1 < \lambda < 0\) in (4.10). Note that (4.11) implies that \( \theta^\lambda \in L^\infty(0,T;L^\infty(\Omega)) \). Proceeding step by step as in Appendix B, Subsect. 6.5, we obtain, with help of (4.13) and (4.14), the estimate

\[
\int_{Q \cap (0,T)} |\nabla (\theta^{\frac{\lambda+1}{2}})|^2 \, dx \, dt \leq C. \tag{4.15}
\]

As a consequence of (4.13), (4.15) and the interpolation inequalities (3.11)-(3.12), we also derive (see Appendix B, Subsect. 6.5 for details) that

\[
\int_0^T \|v\|^{\frac{\alpha}{\beta}} dx + \|\nabla \theta\|^n_m \, dt \leq C \quad n \in [1,5/4), m \in [1,5/3). \tag{4.16}
\]

To find uniform estimates on the pressures \( p \), we consider, for some \( \beta \in (1,2] \) specified later, the following auxiliary Neumann problem for \( h = h^{\varepsilon,\eta} \)

\[
\triangle h = |p|^{\beta-2} p - \frac{1}{|\Omega|} \int_{\Omega} |p|^{\beta-2} p \quad \text{in} \quad \Omega,
\]

\[
\nabla h \cdot n = 0 \quad \text{on} \quad \partial \Omega, \quad \int_{\Omega} h = 0. \tag{4.17}
\]

Refering to the standard \( L^q \) theory for the Neumann problem we recall that there is a \( C \) independent of \( \varepsilon, \eta \) such that

\[
\|\nabla h\|^\beta_{W^{1,\beta}(\Omega)} \leq C \|p\|^\beta_\beta. \tag{4.18}
\]

Taking \( \varphi := \nabla h \) in (4.9) leads to

\[
\int_0^T \|p\|^\beta_\beta \, dt := I_1 + I_2 + I_3 + I_4 + I_5, \tag{4.19}
\]

where \( I_k, k = 1, \ldots, 5, \) are defined and estimated as follows. Using the notation \( Q := \Omega \times (0,T) \), the fact that \( \beta \leq 2 \) and Hölder’s and Young’s
inequalities in a natural way, we have

\[ I_1 := \int_Q \nu(\theta) \mathbf{D}(\mathbf{v}) \cdot \mathbf{D}(\nabla h) \, dx \, dt \leq \begin{cases} (4.18) & C \int_Q |\mathbf{D}(\mathbf{v})|^{\beta} \, dx \, dt + \frac{1}{8} \int_0^T \|p\|_\beta^2 \, dt, \\ (3.13) & \end{cases} \]

\[ I_2 := -\int_0^T \langle b, \nabla h \rangle \, dt \leq C \int_0^T \|b\|_{W^{1,2}}^2 \, dt + C + \frac{1}{8} \int_0^T \|p\|_\beta^2 \, dt, \]

\[ I_3 := \alpha \int_0^T (\mathbf{v}_\eta \otimes \mathbf{v}) \cdot \nabla^2 h \, dx \, dt \leq C \int_Q |\mathbf{v}_\eta \otimes \mathbf{v}|^\beta \, dx \, dt + \frac{1}{8} \int_0^T \|p\|_\beta^2, \]

and we observe that the term containing \( \mathbf{v}_t \) is nonpositive. More precisely, we have

\[ I_5 := \int_0^T \langle \mathbf{v}_t, \nabla h \rangle \, dt = -\int_0^T \langle (\text{div} \mathbf{v})_t, h \rangle \, dt \]

\[ \begin{aligned} &\stackrel{(4.17)}{=} -\varepsilon \int_0^T \left( p_t, |p|^{\beta-2} p - \frac{1}{\Omega} \int_\Omega |p|^{\beta-2} p \, dx \right) \, dt \\ &\stackrel{(4.8)}{=} -\frac{\varepsilon}{\beta} \|p(t)\|_\beta^\beta + \frac{\varepsilon}{\beta} \|p(0)\|_\beta^\beta \leq 0, \end{aligned} \]

where the last inequality\(^2\) follows from the fact that \( p(0) = -\frac{1}{\varepsilon} \text{div} \mathbf{v}_0 = 0. \)

From (4.19) and the estimates of \( I_k, k = 1, \ldots, 5, \) and from the established uniform estimates (4.13) we observe that

\[ \int_0^T \|p\|_\beta^2 \, dt \leq C + C \int_Q |\mathbf{v}_\eta \otimes \mathbf{v}|^\beta \, dx \, dt. \]

Since \( \|\mathbf{v}_\eta \otimes \mathbf{v}\|_2 \leq \|\mathbf{v}_\eta\|_\infty \|\mathbf{v}\|_2 \leq C(\eta^{-1})\|\mathbf{v}\|_2^2, \) the last inequality implies

\[ \int_0^T \|p\|_\beta^2 \, dt \leq C(\eta^{-1}). \]

(4.20)

If however the estimate (4.16)\(^1\) is used we conclude that

\[ \int_0^T \|p\|_\beta^2 \, dt \leq C. \]

(4.21)

Finally, it follows from the equations (4.9) and (4.10), and the estimates (4.13), (4.14), (4.16), and (4.20) that for \( q \) sufficiently large

\[ \|\mathbf{v}_t\|_{L^2(0,T;W^{1,2})} + \|\theta_t\|_{L^1(0,T;W^{-1,q}(\Omega))} \leq C(\eta^{-1}). \]

(4.22)

Proceeding similarly but using (4.21) instead of (4.20) we also have

\[ \|\mathbf{v}_t\|_{L^2(0,T;W^{1,2})} + \|\theta_t\|_{L^1(0,T;W^{1,q}(\Omega))} \leq C. \]

(4.23)

\(^1\)Note that the estimate \( I_5 \) is not rigorous; it can be however made fully correct by approximating both \( \mathbf{v}_t \) and \( h \) by smooth functions (see [3] for some details).

\(^2\)Note that the estimate \( I_5 \) is not rigorous; it can be however made fully correct by approximating both \( \mathbf{v}_t \) and \( h \) by smooth functions (see [3] for some details).
4.3. Limit $\varepsilon \to 0$

In this subsection we write $(v^\varepsilon, \theta^\varepsilon, p^\varepsilon)$ instead of $(v^{\varepsilon, n}, \theta^{\varepsilon, n}, p^{\varepsilon, n})$.

It follows from the estimates (4.13), (4.16), (4.20), (4.22) and a generalized version of Aubin-Lions lemma (see [23] for example) that there are a subsequence (not relabeled) of $(v^\varepsilon, \theta^\varepsilon, p^\varepsilon)$ and a triplet $(v, \theta, p)$ such that

$$
v^\varepsilon_t \rightharpoonup v_t \quad \text{weakly in } L^2(0, T; W^{-1,2}_n),
$$
$$
v^\varepsilon \rightharpoonup v \quad \text{weakly in } L^2(0, T; W^{1,2}),
$$
$$
v^\varepsilon \rightharpoonup^* v \quad \text{weakly* in } L^\infty(0, T; L^2(\Omega)^3),
$$
$$
v^\varepsilon \rightharpoonup^* v \quad \text{strongly in } L^q(0, T; L^q(\Omega)^3) \quad \text{for } q \in [1, 10/3),
$$
$$
\theta^\varepsilon \rightharpoonup \theta \quad \text{weakly in } L^n(0, T; W^{1,n}(\Omega)) \quad \text{for } n \in [1, 5/4),
$$
$$
\theta^\varepsilon \rightharpoonup \theta \quad \text{strongly in } L^m(0, T; L^m(\Omega)) \quad \text{for } m \in [1, 5/3),
$$

$$(\theta^\varepsilon)^{\frac{\lambda+1}{\lambda}} \rightharpoonup (\theta)^{\frac{\lambda+1}{\lambda}} \quad \text{weakly in } L^2(0, T; W^{1,2}(\Omega)) \quad \text{for } \lambda \in (-1, 0),
$$

$$
\nu(\theta^\varepsilon)D(v^\varepsilon) \rightharpoonup \nu(\theta)D(v) \quad \text{weakly in } L^2(0, T; L^2(\Omega)^{3 \times 3}),
$$

$$
p^\varepsilon \rightharpoonup p \quad \text{weakly in } L^2(0, T; L^2(\Omega)).
$$

Moreover, using Corollary 3.1, we have

$$
\text{tr } v^\varepsilon \rightharpoonup \text{tr } v \quad \text{strongly in } L^2(0, T; L^2(\partial \Omega)^3).
$$

We also observe that for $\varphi \in L^2(0, T; W^{1,2}(\Omega))$ we have

$$
\left| \int_0^T (\text{div } v, \varphi) \, dt \right| = \lim_{\varepsilon \to 0} \left| \int_0^T (\text{div } v^\varepsilon, \varphi) \, dt \right| = \lim_{\varepsilon \to 0} \left| \varepsilon \int_0^T \nabla \varphi \cdot \nabla p^\varepsilon \, dt \right|
$$

$$
\leq \lim_{\varepsilon \to 0} \sqrt{\varepsilon} \left( \int_Q |\nabla \varphi|^2 \, dx \, dt \right)^\frac{1}{2} \left( \int_Q \varepsilon |p^\varepsilon|^2 \, dx \, dt \right)^\frac{1}{2}
$$

$$
\leq \frac{1}{C} \varepsilon \int_Q |\nabla \varphi|^2 \, dx \, dt \quad \text{for all } \varphi \in L^2(0, T; W^{1,2}_n),
$$

which implies $\text{div } v = 0$ a.e. in $(0, T) \times \Omega$.

The above established convergences are clearly sufficient for taking the limit in (4.9) and to conclude that $(v, \theta, p) = (v^n, \theta^n, p^n)$ satisfy

$$
\int_0^T ((v_t, \varphi) - (v_n \otimes v, \nabla \varphi) + (\nu(\theta)D(v), D(\varphi)) + \alpha(v, \varphi)_{\partial \Omega}) \, dt
$$

$$
= \int_0^T (p, \varphi) + \langle b, \varphi \rangle \, dt \quad \text{for all } \varphi \in L^2(0, T; W^{1,2}_n),
$$

Similarly, one can show that $(v, \theta) = (v^n, \theta^n)$ fulfill

$$
\int_0^T (- (\theta, \varphi_t) - (\theta v_n, \nabla \varphi) + (k(\theta)\nabla \theta, \nabla \varphi) - (\nu(\theta)D(v))^2, \varphi)) \, dt
$$

$$
= (\theta_0, \varphi(0)) \quad \text{for all } \varphi \in D(-\infty, T; W^{1,q}(\Omega)),
$$

$$
\text{for all } \varphi \in D(-\infty, T; W^{1,q}(\Omega)),
$$
where $q$ is sufficiently large, provided that we are able to show that
\[
\int_Q \nu(\theta^\varepsilon)|\mathbf{D}(v^\varepsilon)|^2 \, dx \xrightarrow{\varepsilon \to 0} \int_Q \nu(\theta)|\mathbf{D}(v)|^2.
\] (4.26)

To prove (4.26), we first observe that
\[
\sqrt{\nu(\theta^\varepsilon)}\mathbf{D}(v^\varepsilon) \rightharpoonup \sqrt{\nu(\theta)}\mathbf{D}(v)
\] weakly in $L^2(0, T; L^2(\Omega)^3 \times 3)$. (4.27)

Consequently,
\[
\int_Q \nu(\theta)|\mathbf{D}(v)|^2 \leq \liminf_{\varepsilon \to 0} \int_Q \nu(\theta^\varepsilon)|\mathbf{D}(v^\varepsilon)|^2
\]
\[
\leq \liminf_{\varepsilon \to 0} \int_Q \nu(\theta^\varepsilon)|\mathbf{D}(v^\varepsilon)|^2 + \varepsilon|\nabla p^\varepsilon|^2 \, dx \, dt
\]
\[
= \liminf_{\varepsilon \to 0} \left( - \frac{\|v^\varepsilon(T)\|^2}{2} + \int_0^T \langle b, v^\varepsilon \rangle \, dt - \alpha \|v^\varepsilon\|_{L^2(\partial\Omega)}^2 \, dt \right)
\]
\[
+ \frac{\|v_0\|^2}{2} - \frac{\|v_0\|^2}{2} + \int_0^T \langle b, v \rangle \, dt - \alpha \|v\|_{L^2(\partial\Omega)}^2 \, dt
\]
\[
(4.24) \text{ with } \varphi = v \int_Q \nu(\theta)|\mathbf{D}(v)|^2,
\]
and (4.26) follows.

4.4. Limit $\eta \to 0$

We know that $(v^n, \theta^n, p^n)$ satisfying the weak formulations (4.24) and (4.25) fulfil the uniform estimates (4.13) with $\varepsilon = 0$, (4.15), (4.16), (4.21) and (4.23). These estimates can be for example obtained if we proceed step by step as in Subsection 4.2. Recall also that $\text{div} \, v^n = 0$ in $Q$. These estimates, the Aubin-Lions compactness lemma and Corollary 3.1 imply the existence of $(v, \theta, p)$ such that (modulo a subsequence)

$\begin{align*}
\nu_n &\to v, \quad \text{weakly in } L^2(0, T; W^{1, 2}(\Omega)), \\
n &\to v, \quad \text{weakly in } L^2(0, T; L^2(\Omega)^{3 \times 3}), \\
v_n &\to v, \quad \text{weakly}^* \text{ in } L^\infty(0, T; L^2(\Omega)^3), \\
\text{tr } v_n &\to \text{tr } v, \quad \text{strongly in } L^2(0, T; L^2(\partial\Omega)), \\
\theta^n &\to \theta, \quad \text{weakly in } L^s(0, T; W^{1, s}(\partial\Omega)), \quad s \in [1, 5/4], \\
\theta^n &\to \theta, \quad \text{strongly in } L^m(0, T; L^m(\Omega)), \quad m \in [1, 5/3], \\
(\theta^n) &\rightharpoonup (\theta), \quad \text{weakly in } L^2(0, T; W^{1, 2}(\Omega)), \quad \lambda \in (-1, 0), \\
\nu(\theta^n) \mathbf{D}(v^n) &\rightharpoonup \nu(\theta) \mathbf{D}(v), \quad \text{weakly in } L^2(0, T; L^2(\Omega)^{3 \times 3}), \\
p_n &\to p, \quad \text{weakly in } L^2(0, T; L^2(\Omega)).
\end{align*}$
These convergences are sufficient to take the limit in (4.24) and to achieve (3.23).

Next, for \( \varphi \in \mathcal{D}(-\infty, T; \mathcal{C}^1(\overline{\Omega})) \) we take \( \varphi^\eta := v^\eta \varphi \) in (4.24) and add the result to (4.25). Thus, we obtain

\[
\int_0^T -\left( \frac{|v^\eta|^2}{2} + \theta^\eta, \varphi_t \right) - \left( v^\eta_0 \left( \frac{|v^\eta|^2}{2} + \theta^\eta \right) + v^\eta p^\eta, \nabla \varphi \right) + (k(\theta^\eta) \nabla \theta^\eta, \nabla \varphi) + \alpha (|v^\eta|^2, \varphi) dt = \int_0^T \langle b, v^\eta \varphi \rangle dt + E_0(\varphi) \quad \text{for} \ \varphi \in \mathcal{D}(-\infty, T; \mathcal{C}^1(\overline{\Omega})).
\]

As discussed in Remark 3.4, all terms belong to some \( L^q(Q) \) with \( q > 1 \).

Letting \( \eta \to 0 \) in (4.28) and using the above convergences lead easily to (3.24).

In order to prove (3.26) we take an arbitrary \( \psi \in \mathcal{D}(-\infty, T; \mathcal{C}^\infty(\Omega)) \) such that \( \psi \geq 0 \) a.e. in \( Q \) as a test function in (4.25). It is easy to take the limit in all terms appearing on the left-hand side of (4.25). Since

\[
\sqrt{\nu(\theta)} \mathbf{D}(v^\eta) \rightharpoonup \sqrt{\nu(\theta)} \mathbf{D}(v) \quad \text{weakly in} \quad L^2(0, T; L^2(\Omega)^{3 \times 3})
\]

the weak lower-semicontinuity of \( L^2 \) norm implies

\[
\int_0^T \int_\Omega \nu(\theta) |\mathbf{D}(v)|^2 \psi \, dx \, dt \leq \liminf_{\eta \to 0^+} \int_0^T \int_\Omega \nu(\theta^\eta) |\mathbf{D}(v^\eta)|^2 \psi \, dx \, dt
\]

and (3.26) follows.

The statements concerning the attainment of the initial values are proved in a standard way. We refer the reader to [17, Sect. 3.8 - 3.10], for example.

The proof of Theorem 3.1 is complete.

5. Appendix A: Balance equations of fluid thermodynamics

We consider a fluid-like material that at any time instant \( t \in [0, \infty) \) occupies the same (open) bounded set \( \Omega \subset \mathbb{R}^3 \). One possibility of how to formulate the balance equations of continuum thermodynamics is to use Reynold’s approach at the Eulerian description. In such a case we deal with the velocity field

\( v : (0, \infty) \times \Omega \to \mathbb{R}^3 \)

as it would be a primitive quantity, and we formulate the balance equations for mass, linear momentum and energy, and the entropy inequality (represented through the relevant specific densities \( \rho, \rho v = (\rho v_1, \rho v_2, \rho v_3) \), \( \rho(e + |v|^2/2) \) and \( \rho \eta \)) over any open subset \( \mathcal{O} \subset \Omega \), \( e \) being the internal energy. We introduce \( \theta \) through the relation (see for example [5] or [22] for details)

\[
\frac{1}{\theta} := \frac{\partial \eta}{\partial e} > 0.
\]
Setting

\[ u = \begin{pmatrix} \rho \\ \rho v_1 \\ \rho v_2 \\ \rho v_3 \\ \rho (e + |v|^2/2) \end{pmatrix}, \quad h = \begin{pmatrix} t_1 \\ t_2 \\ t_3 \\ -q \cdot n \\ \rho \eta \end{pmatrix}, \quad f = \begin{pmatrix} \rho b_1 \\ \rho b_2 \\ \rho b_3 \\ \rho \rho \rho \theta + \rho \xi \end{pmatrix}, \] (5.2)

where \( t = (t_1, t_2, t_3) \) is the traction vector, \( q \) the heat flux, \( r \) the source of the energy (as radiation) and \( \xi \) represents the entropy production defined through

\[
\rho \xi := \frac{d}{dt} (\rho \eta) + \text{div} \left( \frac{q}{\theta} \right) - \frac{\rho r}{\theta} \quad (\xi \geq 0),
\]

we can write the balance equations in the compact form

\[
\frac{d}{dt} \int_{\Omega} u_s \, dx = -\int_{\partial \Omega} u_s (v \cdot n) \, dS + \int_{\partial \Omega} h_s \, dS + \int_{\Omega} f_s \, dV
\] (5.3)

In (5.3), the term on the left-hand-side stands for the rate of change of the quantity \( \int_{\Omega} u^s \, dx \) (i.e. the mass of \( \Omega \), the linear momentum over \( \Omega \), the energy of \( \Omega \) and the entropy of \( \Omega \)). The first term on the right-hand-side of (5.3) represents the flux of \( u^s \) through the boundary \( \partial \Omega \) due to the motion, the second term stands for further flux through the boundary \( \partial \Omega \) that contributes to the change of \( \int_{\Omega} u^s \, dx \), and the last term stands for production (source) terms.

Recall that by Cauchy’s theorem the traction is proportional to the normal \( n \) (on \( \partial \Omega \)). Consequently

\[
t_i = T_{ji} n_j \quad \text{or} \quad t = T^T n,
\] (5.4)

where \( T \) is the Cauchy stress (the second order tensor).

Note that we should add the balance of angular momentum to (5.2); we however restrict ourselves to such cases in which these balance equations lead to the conclusion that \( T^T = T \). Consequently, (5.4) takes the form

\[
t = T n.
\] (5.5)

As a consequence of (5.4) resp. (5.5) we observe that \( h \) has the representation

\[
h = Q n,
\] (5.6)

\footnote{In accordance with the second law of thermodynamics \( \xi \) is supposed to be nonnegative.}
where $Q$ is the second order tensor. For example, $Q_{ij} = T_{kj}v_k - q_j$. Thus, (5.3) can be rewritten as

$$\frac{d}{dt} \int_\Omega u^s \, dx = -\int_{\partial \Omega} u^s (v \cdot n) \, dS + \int_{\partial \Omega} Q_{sk} n_k \, dS + \int_\Omega f^s \, dV$$

valid for all $\Omega \subset \Omega$ open and $s = 1, 2, \ldots, 6$.

Next, we follow the approach presented in [9, Chapter 1] that uses the following statement (see [8] for example):

**Lemma 5.1** For any $u \in L^1(\mathbb{R}^d)$ and any $\eta \in D(\mathbb{R}^d), \eta \geq 0$, the function

$$r \mapsto \int_{\{\eta = r\}} u(s) \, ds$$

is measurable and the following representation holds:

$$\int_{\mathbb{R}^d} u |\nabla \eta| \, dx = \int_0^\infty \left( \int_{\{\eta = r\}} u(s) \, ds \right) \, dr.$$  (5.8)

Let $\eta = (\eta^1, \ldots, \eta^6)$ consist of $\eta^s \in D(\Omega), \eta^s \geq 0, s = 1, \ldots, 6$. Consider the $s$-th equation in (5.7) with $\Omega = \Omega_{r,s} = \Omega_r = \{\eta^s > r\} = \{x \in \Omega; \eta^s(x) > r\}$, where $r \in (0, \infty)$ is arbitrary. Then $\partial \Omega_r = \{\eta^s = r\}$ and $n = n^s = -\frac{\nabla \eta^s}{|\nabla \eta^s|}$. Integrating such equations over $r$ from $0$ to $\infty$, observing (no summation over $s$)

$$\int_0^\infty \int_{\partial \Omega_r} u^s (x) \, dS \, dr = \int_0^\infty \int_{\Omega} u^s(x) \text{sgn} \{\eta^s > r\} \, dx \, dr$$

$$= \int_{\Omega} u^s(x) \int_0^\infty \text{sgn} \{\eta^s > r\} \, dr \, dx \quad (5.9)$$

$$- \int_0^\infty \int_{\partial \Omega_r} u^s(x) (v \cdot n) \, dS \, dr = \int_0^\infty \int_{\{\eta^s = r\}} u^s(x) \left( v \cdot \frac{\nabla \eta^s}{|\nabla \eta^s|} \right) \, dS \, dr$$

$$\overset{(5.8)}{=} \int_{\mathbb{R}^3} u^s(x) v \cdot \nabla \eta^s \, dx$$

$$= \int_{\Omega} u^s(x) v \cdot \nabla \eta^s \, dx, \quad (5.10)$$

$$\int_0^\infty \int_{\partial \Omega_r} Q_{sk} (x) n_k^s \, dS \, dr = -\int_0^\infty \int_{\{\eta^s = r\}} Q_{sk} (x) \frac{\partial_x \eta^s}{|\nabla \eta^s|} \, dS \, dr$$

$$\overset{(5.8)}{=} -\int_{\mathbb{R}^3} Q_{sk} (x) \partial_x \eta^s \, dx$$

$$= \int_{\Omega} Q_{sk} (x) \partial_x \eta^s \, dx, \quad (5.11)$$
and noticing that the last term in (5.7) is treated as the term in (5.9), we obtain (relabeling \( \eta_1 \) by \( \psi \), \( \eta_2, \eta_3, \eta_4 \) by \( \varphi^1, \varphi^2, \varphi^3 \), \( \eta_5 \) by \( z \) and \( \eta_6 \) by \( h \))

\[
\frac{d}{dt} \int_{\Omega} \rho \psi \, dx - \int_{\Omega} \rho \mathbf{v} \cdot \nabla \psi \, dx = 0 \quad \text{valid for all } \psi \in \mathcal{D}(\Omega), \quad (5.12)
\]

\[
\frac{d}{dt} \int_{\Omega} \rho \mathbf{v} \cdot \varphi \, dx = \int_{\Omega} (\rho \mathbf{v} \otimes \mathbf{v}) \cdot \nabla \varphi \, dx
- \int_{\Omega} \mathbf{T} \cdot \nabla \varphi + \int_{\Omega} \rho \mathbf{b} \cdot \varphi \quad \text{valid for all } \varphi \in \mathcal{D}(\Omega)^2, \quad (5.13)
\]

\[
\frac{d}{dt} \int_{\Omega} \rho \left( e + \frac{|\mathbf{v}|^2}{2} \right) z \, dx = \int_{\Omega} \rho \left( e + \frac{|\mathbf{v}|^2}{2} \right) \mathbf{v} \cdot \nabla z \, dx
- \int_{\Omega} (\mathbf{T} \mathbf{v}) \cdot \nabla z + \int_{\Omega} \mathbf{q} \cdot \nabla z + \int_{\Omega} \rho r z \, dx
+ \int_{\Omega} \rho \mathbf{b} \cdot \mathbf{v} z \, dx \quad \text{valid for all } z \in \mathcal{D}(\Omega) \quad (5.14)
\]

and

\[
\frac{d}{dt} \int_{\Omega} \rho \eta h \, dx = \int_{\Omega} \rho \eta \mathbf{v} \cdot \nabla h \, dx + \int_{\Omega} \mathbf{q} \cdot \nabla h
- \int_{\Omega} \frac{\rho r}{\theta} h + \int_{\Omega} \rho \xi h \, dx \quad \text{valid for all } h \in \mathcal{D}(\Omega). \quad (5.15)
\]

Noticing that (5.12)-(5.15) are the weak formulations of the equations that in the classical sense take the forms

\[
\rho_{,t} + \text{div}(\rho \mathbf{v}) = 0,
(\rho \mathbf{v})_{,t} + \text{div}(\rho \mathbf{v} \otimes \mathbf{v}) - \text{div} \mathbf{T} = \rho \mathbf{b},
\left[ \rho \left( e + \frac{|\mathbf{v}|^2}{2} \right) \right]_{,t} + \text{div} \left( \rho \mathbf{v} \left( e + \frac{|\mathbf{v}|^2}{2} \right) \right) = \text{div}(\mathbf{T} \mathbf{v}) + \text{div} \mathbf{q} = \rho \mathbf{b} \cdot \mathbf{v} + \rho r,
(\rho \eta)_{,t} + \text{div}(\rho \mathbf{v}) + \text{div} \left( \frac{\mathbf{q}}{\theta} \right) + \frac{\rho r}{\theta} = \rho \xi,
\]

we conclude to say that under the assumption that the integrals in (5.12)-(5.15) are finite, the formulation (5.7) for \( s = 1, \ldots, 6 \) are equivalent to (5.12)-(5.15).

If the fluid is incompressible, i.e. \( \text{div} \mathbf{v} = 0 \), and in addition homogeneous, i.e. \( \rho(t, x) = \rho^* \) where \( \rho^* \in (0, \infty) \) is a constant, (5.12) is automatically met and (5.13) and (5.14) together with the constitutive equations

\[
\mathbf{T} = -\overline{\mathbf{p}} + \bar{\nu}(\theta) \mathbf{D}(\mathbf{v}), \quad (5.16)
\]

and

\[
\mathbf{q} = -\bar{k}(\theta) \nabla \theta, \quad e = c_v \theta, \quad r = 0 \quad (5.17)
\]
lead to
\[
\frac{d}{dt} \int_{\Omega} v \cdot \varphi - \int_{\Omega} (v \otimes v) \cdot \nabla \varphi + \int_{\Omega} \frac{\bar{v}(\theta)}{\rho^*} D(v) \cdot D(\varphi) = \int_{\Omega} \frac{\bar{p}}{\rho^*} \text{div} \varphi + \int_{\Omega} b \cdot \varphi \quad \text{valid for all } \varphi \in D(\Omega)^3,
\]
and
\[
\frac{d}{dt} \int_{\Omega} \left( c_v \theta + \frac{|v|^2}{2} \right) z - \int_{\Omega} \left( c_v \theta + \frac{|v|^2}{2} + \frac{\bar{p}}{\rho^*} \right) v \cdot \nabla z + \int_{\Omega} \frac{\bar{\nu}(\theta)}{\rho^*} D(v) v \cdot \nabla z + \int_{\Omega} \frac{\bar{k}(\theta)}{\rho^*} \nabla \theta \cdot \nabla z = \int_{\Omega} b \cdot vz \quad \text{valid for all } z \in D(\Omega)
\]
(5.18)
It follows from (5.1) and (5.17) that
\[
\eta = c_v \ln \theta.
\]
In our setting, the entropy production term \( \rho \xi \) includes the sum
\[
\frac{\bar{\nu}(\theta)}{\theta^2} |D(v)|^2 + \frac{\bar{k}(\theta)}{\theta^2} |\nabla \theta|^2.
\]
In general, there might be additional entropy production mechanisms that by the second law of thermodynamics should be non-negative. Consequently, it is natural to assume that
\[
\rho \xi \geq \frac{\bar{\nu}(\theta)}{\theta^2} |D(v)|^2 + \frac{\bar{k}(\theta)}{\theta^2} |\nabla \theta|^2.
\]
Finally, using the function \( h := \theta g \) in (5.15), \( (g \in D(\Omega), g \geq 0) \) (5.16), (5.17) and (5.19) we obtain
\[
\frac{d}{dt} \int_{\Omega} c_v \theta g \ dx - \int_{\Omega} \theta c_v \nu \cdot \nabla g \ dx + \int_{\Omega} \frac{\bar{k}(\theta)}{\rho^*} \nabla \theta \cdot \nabla g \geq \int_{\Omega} \frac{\bar{\nu}(\theta)}{\rho^*} |D(v)|^2 g \ dx \quad \text{valid for all } g \in D(\Omega),
\]
(5.20)
that in the classical sense takes the form
\[
c_v \theta_t + \text{div}(c_v \theta v) - \text{div} \left( \frac{\bar{k}(\theta)}{\rho^*} \nabla \theta \right) \geq \frac{\bar{\nu}(\theta)}{\rho^*} |D(v)|^2.
\]
(5.21)

6. Appendix B: Solvability of the \((\varepsilon, \eta)\)-approximations - Proof of Lemma 4.1

This appendix is devoted to the proof of Lemma 4.1.
6.1. Galerkin approximations

First, we define a linear mapping $\mathcal{F}$ that appends to any $v \in W^{1,2}_n$ the solution $p \in W^{2,2}_n(\Omega)$ of the problem (4.1). Recalling that $\Omega \in \mathcal{C}^{1,1}$, it follows from the regularity theory for the Neumann problem (4.1) that the mapping $\mathcal{F} : W^{1,2}_n \to W^{2,2}_n(\Omega)$ is continuous.

Next, let $\{w_j\}_{j=1}^\infty$ be an orthogonal basis of $W^{1,2}_n$ that is orthonormal in $L^2(\Omega)$, and in addition, for every $j$, $w_j$ belongs to $W^{1,6}_n$. (Such a basis can be formed by the eigenvectors of the problem $-\Delta u + u = \lambda u$ in $\Omega$, and $u \cdot n = 0$ and $[\nabla u \cdot n]_\tau + \alpha u_\tau = 0$ on $\partial \Omega$. The reader can consult [18, Appendix A.4] for details.) Similarly, let $\{w_j\}_{j=1}^\infty$ be a basis of $W^{1,2}(\Omega)$ which is again orthonormal in the space $L^2(\Omega)$.

We construct the Galerkin approximations $\{v^{N,M}, \theta^{N,M}, p^{N,M}\}_{N,M=1}^\infty$ being of the form

$$v^{N,M} := \sum_{i=1}^N c^{N,M}_i(t)w_i, \quad \theta^{N,M} := \sum_{i=1}^M d^{N,M}_i(t)w_i, \quad p^{N,M} := \mathcal{F}(v^{N,M}),$$

where $c^{N,M} := (c_1^{N,M}, \ldots, c_N^{N,M}), d^{N,M} := (d_1^{N,M}, \ldots, d_M^{N,M})$ solve the system of ordinary differential equations

$$\frac{d}{dt}(v^{N,M}, w_j) - (v^{N,M} \otimes v^{N,M}, \nabla w_j) + (\nu(\theta^{N,M})D(v^{N,M}), \nabla w_j)$$

$$+ \alpha(v^{N,M}, w_j)\mathcal{F}(v^{N,M}), \text{div } w_j = (b, w_j), \quad j = 1, 2, \ldots, N,$$

$$\frac{d}{dt}(\theta^{N,M}, w_k) - (v^{N,M}_0, \theta^{N,M}, \nabla w_k) + (k(\theta^{N,M})\nabla \theta^{N,M}, \nabla w_k)$$

$$= (\nu(\theta^{N,M})D(v^{N,M}))^2, \quad k = 1, 2, \ldots, M.$$ (6.1)

Moreover, we require that $v^{N,M}$ and $\theta^{N,M}$ satisfy the following initial conditions

$$v^{N,M}(\cdot, 0) = v_0^{N,M}, \quad \theta^{N,M}(\cdot, 0) = \theta_0^{N,M},$$

where $v_0^{N,M} := \sum_{j=1}^N c_0^{N,M}w_j$ are the projections of $v_0$ onto linear hulls of $\{w_j\}_{j=1}^N$ and $\theta_0^{N,M}$ has the following meaning. We first define $\theta_0(x) = C_3$ for $x \in \mathbb{R}^3 \setminus \Omega$ and regularize such extended $\theta_0$ with regularization kernel $\omega_{\frac{1}{N}}$ of radii $\frac{1}{N}$, i.e. $\theta_0^{N} := (\omega_{1/N} \ast \theta_0)$. Thus, due to (3.16) we have

$$\theta_0^{N}(x) \geq C_3 \text{ for a.a. } x \in \Omega.$$ (6.3)

Then we apply the projection onto the linear hull of $\{w_j\}_{j=1}^M$. Thus, $\theta_0^{N,M}$ has the form $\theta_0^{N,M} := \sum_{j=1}^M d_0^{M}w_j$. Note that

$$\theta_0^{M,N} \to \theta_0$$ strongly in $L^2(\Omega),$ (6.4)

$$v_0^N \to v_0$$ strongly in $L^2(\Omega)^3,$ (6.5)

$$\theta_0^N \to \theta_0$$ strongly in $L^1(\Omega).$ (6.6)
The classical Caratheodory theory (see [24, Chapter 1] for example) then implies the short-time existence of solutions to (6.1)-(6.2). In the next subsection we derive the estimate on \(c_{N,M}\) that is uniform w.r.t. \(M, N\), and the estimate on \(d_{N,M}\) uniform w.r.t. \(M\). These estimates then imply that solution of (6.1)-(6.2) exists for all \(t \in (0, T)\).

6.2. Uniform estimates

Multiplying the \(j\)-th equation in (6.1) by \(c_{N,M}^j\), summing over \(j = 1, \ldots, M\), integrating over \((0, T)\), using the assumption (3.13) on the viscosity \(\nu\) and the identity

\[
\left( v_{N,M}^j \eta, \nabla v_{N,M}^j \right) = \left( v_{N,M}^j \eta, |\nabla v_{N,M}^j|^2 \right) = -\left( \text{div} v_{N,M}^j \eta, |v_{N,M}^j|^2 \right) = 0,
\]

we observe

\[
\frac{1}{2} \sup_{t \in (0, T)} \| v_{N,M}^j(t) \|_2^2 + \int_0^T \left( C_1 \| D(v_{N,M}) \|_2^2 + \alpha \| v_{N,M} \|_{L^2(\partial \Omega)} \right) + \epsilon \| \nabla p_{N,M} \|_2^2 \, d\tau \leq \int_0^T \langle b, v_{N,M} \rangle \, d\tau + \frac{1}{2} \| v_{N,0}^j \|_2^2.
\]

(6.7)

Applying Korn’s inequality (Lemma 3.1) and Young’s inequality to (6.7) we conclude that

\[
\sup_{t \in (0, T)} \| v_{N,M}^j(t) \|_2^2 + \int_0^T \| v_{N,M} \|_{W^{1,2}}^2 + \epsilon \| \nabla p_{N,M} \|_2^2 \, dt \leq C.
\]

(6.8)

Similarly, multiplying of the \(k\)-th equation in (6.2) by \(d_{N,M}^k\), summing over \(k = 1, \ldots, M\), integrating over time \(t \in (0, T)\), using the assumptions on \(k, \nu\) (3.13)-(3.14) and the cancelation of the convective term we have

\[
\frac{1}{2} \| \theta_{N,M}(t) \|_2^2 + C_1 \int_0^t \| \nabla \theta_{N,M} \|_2^2 \, d\tau \leq \frac{1}{2} \| \theta_{0,N}^M \|_2^2 + C_2 \int_0^t \| D(v_{N,M}) \|^2, |\theta_{N,M}| \, d\tau.
\]

(6.9)

Applying Hölder’s inequality, and using (6.7) and the fact that \(w_j \in W^{1,4}_n\) we observe that the last term in (6.9) is bounded by

\[
\| \theta_{N,M}(t) \|_2^2 + \int_0^t \| \nabla \theta_{N,M} \|_2^2 \, d\tau \leq \| \theta_{0,N}^M \|_2^2 + CC_2N \sup_{j=1,\ldots,N} \| D(w_j) \|_4^2 \left( \int_0^t \| \theta_{N,M} \|_2^2 \, d\tau \right)^{\frac{1}{2}} \leq C(N) \left( 1 + \int_0^t \| \theta_{N,M} \|_2^2 \, d\tau \right)^{\frac{1}{2}}.
\]

(6.10)
Applying Gronwall’s lemma to (6.10), we are led to
\[ \sup_{t \in (0,T)} \| \theta_{N,M}(t) \|_{2}^{2} + \int_{0}^{T} \| \nabla \theta_{N,M} \|_{2}^{2} \, d\tau \leq C(N). \]  
(6.11)

To obtain compactness for the velocity and the temperature, we estimate the norms of their time derivative. Multiplying the \( j \)-th equation in (6.1) by \( \frac{d}{dt} c_{N,M}^{j} \), summing over \( j = 1, \ldots, N \) and integrating the result over time, we easily conclude, using the estimate (6.8), that
\[ \int_{0}^{T} \left( \frac{d}{dt} c_{N,M}^{j} \right)^{2} \, dt \leq C(N). \]  
(6.12)

Moreover, using the fact that \( \langle \theta_{N,M}^{i}, t \varphi(t) \rangle = \langle \theta_{N,M}^{i}, P_{M} \varphi(t) \rangle \), where \( P_{M} \) is the projection of \( W^{1,2}(\Omega) \)-functions onto the linear hull of \( \{ w_{i} \}_{i=1}^{M} \), the equation (6.2) and the estimates (6.7)-(6.11) lead to the conclusion that
\[ \| \theta_{N,M}^{i} \|_{L^{2}(0,T;W^{-1,2}(\Omega))} \leq C(N). \]  
(6.13)

### 6.3. Limit \( M \to \infty \)

Having the estimates (6.8)-(6.13) in hands, we find a (not relabeled) subsequence \( \{ c_{N,M}, \theta_{N,M} \}_{M=1}^{\infty} \) such that for \( M \to \infty \)
\[ c_{t}^{N,M} \to c_{t}^{N} \quad \text{weakly in } L^{2}(0,T), \]  
(6.14)
\[ c^{N,M} \rightharpoonup c^{N} \quad \text{weakly* in } L^{\infty}(0,T), \]  
(6.15)
\[ \theta^{N,M} \rightharpoonup \theta^{N} \quad \text{weakly * in } L^{\infty}(0,T;L^{2}(\Omega)), \]  
(6.16)
\[ \theta^{N,M} \to \theta^{N} \quad \text{weakly in } L^{2}(0,T;W^{1,2}(\Omega)), \]  
(6.17)
\[ \theta_{t}^{N,M} \to \theta_{t}^{N} \quad \text{weakly in } L^{2}(0,T;W^{-1,2}(\Omega)). \]  
(6.18)

Moreover, the interpolation inequality (3.11) with \( q = \frac{10}{3} \) and the Aubin-Lions compactness lemma imply that
\[ \theta^{N,M} \to \theta^{N} \quad \text{strongly in } L^{m}(0,T;L^{m}(\Omega)) \text{ for all } m \in [1,10/3]. \]  
(6.19)

Next, since \( W^{1,2}(0,T) \hookrightarrow C^{1/2}(0,T) \hookrightarrow C(0,T) \), it follows from (6.12) and (6.8) that
\[ c^{N,M} \to c^{N} \quad \text{strongly in } C(0,T). \]  
(6.20)

Moreover, it is a simple consequence of our choice of basis and (6.20) that
\[ v^{N,M} \to v^{N} \quad \text{strongly in } L^{4}(0,T;W_{n}^{1,4}). \]  
(6.21)
Consequently, as $F$ maps $W^{1,2}_\alpha$ to $W^{1,2}(\Omega)$ continuously, we have
\begin{equation}
 p^{N,M} \rightarrow p^N \quad \text{strongly in } L^2(0,T; W^{1,2}(\Omega)).
 \end{equation}

The convergences established above in (6.14)-(6.22) are sufficient to take the limit in (6.1) and (6.2), and to conclude that $(v^N, \theta^N, p^N)$ satisfies for almost all $t \in (0,T)$
\begin{equation}
 \frac{d}{dt}(v^N, w_j) - (v^N_t \otimes v^N, \nabla w_j) - (p^N, \text{div} w_j) - (b, w_j) + (\nu(\theta^N) D(v^N), \nabla w_j) + \alpha(v^N, w_j)_{\partial \Omega} = 0 \quad \text{for all } j = 1, 2, \ldots, N, \tag{6.23}
\end{equation}
\begin{equation}
 - \varepsilon(\nabla p^N(t), \nabla \psi) = (\psi, \text{div} v^N(t)) \quad \text{for all } \psi \in W^{1,2}(\Omega), \tag{6.24}
\end{equation}
\begin{equation}
 (\theta^N, \varphi) - (v^N_t \theta^N, \nabla \varphi) + (k(\theta^N) \nabla \theta^N, \nabla \varphi) = (\nu(\theta^N) D(v^N)^2, \varphi) \quad \text{for all } \varphi \in W^{1,2}(\Omega). \tag{6.25}
\end{equation}

Due to (6.20) and as $c^{N,M}(0) = c^N_0$ for all $M$, it is clear that
\begin{equation}
 v(\cdot, 0) = v_0^N.
\end{equation}
Moreover, there is a standard procedure to show (see for example [17, Sect. 3.8 - 3.10]) that
\begin{equation}
 \theta^N(0, \cdot) = \theta_0^N.
\end{equation}

\subsection*{6.4. Minimum principle}

In order to show that $\theta^N$ satisfies the lower bound (4.11), we set
\begin{equation}
 \varphi^N := \chi_{[0,T]}(t) \min(0, \theta^N - C_3) \leq 0
\end{equation}
as the test function in (6.25) we obtain (using $\nu(\theta^N) \geq 0$)
\begin{equation}
 \int_0^T (\theta^N_t, \varphi^N) - (v^N_t \theta^N, \nabla \varphi^N) + (k(\theta^N) \nabla \theta^N, \nabla \varphi^N) \, dt \leq 0. \tag{6.26}
\end{equation}
Straightforward computation with help of (6.3) implies that $\|\varphi^N(\tau)\|_2 \leq 0$ for all $\tau \in (0,T)$, which implies (4.11) for $\theta^N$. 
6.5. Estimates independent of $N$

Proceeding as in Subsection 6.2 we first observe that $v^N$ and $p^N$ satisfy for all $t \in (0, T)$
\begin{equation}
\frac{1}{2} \|v^N(t)\|_2^2 + \int_0^t \int_\Omega \nu(\theta^N)|D(v^N)|^2 \, dx \, d\tau + \varepsilon \|\nabla p^N\|_2^2 \, d\tau \\
+ \alpha \int_0^t \|v^N\|_{L^2(\partial \Omega)}^2 \, d\tau = \int_0^t \langle b, v^N \rangle \, dt + \frac{1}{2} \|v_0^N\|_2^2,
\end{equation}
(6.27)
which implies that $v^N$ and $p^N$ also satisfy (6.8).

Then, taking $\varphi \equiv 1$ in (6.25) we easily conclude with help of (3.13) and (6.8)
\begin{equation}
\sup_t \|\theta^N(t)\|_1 \leq C.
\end{equation}
(6.28)
Next, we consider $(\theta^N)^\lambda$ with $-1 < \lambda < 0$ as the test function in (6.25).
Note that $\theta^N \geq C_3$ implies that $0 \leq (\theta^N)^\lambda \leq C_3^{-\lambda}$ almost everywhere in $(0, T) \times \Omega$. After integrating (6.25) (with such test function) over time between 0 and $T$ we obtain
\begin{align*}
\int_0^T \int_\Omega \nu(\theta^N)|D(v^N)|^2(\theta^N)^\lambda - \lambda k(\theta^N)(\theta^N)^{\lambda-1}\|\nabla \theta^N\|_2^2 \, dx \, dt \\
= \int_0^T - (v_0^N, (\nabla \theta^N)^\lambda) + (\theta^N, (\theta^N)^\lambda) \, dt \\
= \frac{\lambda}{1+\lambda} \int_0^T (v_0^N, (\theta^N)^{\lambda+1}) \, dt + \frac{\|((\theta^N(T))^{\lambda+1})\|_1 - \|((\theta^N_0)^{\lambda+1})\|_1}{1+\lambda} \\
= \frac{\|((\theta^N(T))^{\lambda+1})\|_1 - \|((\theta^N_0)^{\lambda+1})\|_1}{1+\lambda} \leq C,
\end{align*}

since $\lambda < 0$ and (6.28) holds. Noticing that the second term on the left hand side is nonnegative and $k \geq C_1 > 0$ by (3.14) one easily concludes from the above estimate that
\begin{equation}
\int_{\Omega \times (0, T)} \|\nabla (\theta^N)^{\frac{\lambda+1}{\lambda}}\|^2 \, dx \, dt \leq C.
\end{equation}
(6.29)
The embedding $W^{1,2}(\Omega) \hookrightarrow L^6(\Omega)$ and (6.28) then imply that $(\theta^N)^{\frac{\lambda+1}{\lambda}}$ is bounded in $L^2(0, T; L^6(\Omega))$ for all $\lambda \in (-1, 0)$. This information together with interpolation inequality (3.12) then leads to the observation that
\begin{equation}
\|\theta^N\|_{L^\infty(0, T; L^6(\Omega))} \leq C \text{ for all } n \in [1, 5/3].
\end{equation}
(6.30)
Further, observing that (for simplicity, we write $Q := \Omega \times (0, T)$)
\begin{align*}
\int_Q |\nabla \theta^N|^s \, dx \, dt &= \int_Q |\nabla \theta^N|^s(\theta^N)(\lambda-1)\left((\theta^N)^{(1-\lambda)}\right)^{\frac{s}{\lambda}} \, dx \, dt \\
&\leq \left(\int_Q |\nabla \theta^N|^2(\theta^N)(\lambda-1) \, dx \, dt\right)^{\frac{s}{2}} \left(\int_Q ((\theta^N)^{(1-\lambda)}\right)^{\frac{s}{\lambda}} \, dx \, dt\right)^{\frac{2-s}{s}},
\end{align*}
(6.31)
we conclude from (6.30) and (6.29) that
\[ \int_0^T \| \theta^N \|_{W^{1,s}(\Omega)}^s \, dt \leq C \quad \text{for all } s \in [1, 5/4). \] (6.32)

Finally, using this estimates and the equations (6.23) and (6.25), we obtain
\[ \| \theta^N \|_{L^1(0,T;W^{-1,q'}(\Omega))} \leq C \quad \text{for } q \text{ being sufficiently large,} \] (6.33)
and
\[ \| v^N_{,t} \|_{L^2(0,T;W^{-1,2}_n)} \leq C. \] (6.34)

6.6. Limit $N \to \infty$

The estimates established in previous subsection and a generalized version of Aubin-Lions lemma (see [23] for example) imply the existence of $(\theta^N, v^N, p^N)$ such that
\[ \theta^N \rightharpoonup \theta \quad \text{weakly in } L^s(0,T;W^{1,s}(\Omega)) \quad s \in [1, 5/4), \] (6.35)
\[ \theta^N \to \theta \quad \text{strongly in } L^m(0,T;L^m(\Omega)) \quad m \in [1, 5/3), \] (6.36)
\[ (\theta^N)^{\frac{1}{m+1}} \to (\theta)^{\frac{1}{m+1}} \quad \text{weakly in } L^2(0,T;W^{1,2}(\Omega)) \quad \lambda \in (-1,0), \] (6.37)
\[ v^N_{,t} \rightharpoonup v_{,t} \quad \text{weakly in } L^2(0,T;W^{-1,2}_n), \] (6.38)
\[ v^N \rightharpoonup^* v \quad \text{weakly* in } L^\infty(0,T;L^2(\Omega)^3), \] (6.39)
\[ v^N \to v \quad \text{weakly in } L^2(0,T;W^{1,2}(\Omega)), \] (6.40)
\[ v^N \to \nu \quad \text{strongly in } L^2(0,T;L^n(\Omega)^3) \quad n \in [1, 10/3), \] (6.41)
\[ p^N \rightharpoonup p \quad \text{weakly in } L^2(0,T;W^{1,2}(\Omega)). \] (6.42)

By Corollary 3.1, (6.8) and (6.34) we also have
\[ \text{tr } v^N \to \text{tr } v \quad \text{strongly in } L^2(0,T;L^2(\partial\Omega)^3). \] (6.43)

We also observe that (6.36) and (6.40) imply the existence of a (not relabeled) subsequence such that
\[ \sqrt{\nu(\theta^N)}D(v^N) \rightharpoonup \sqrt{\nu(\theta)}D(v) \quad \text{weakly in } L^2(0,T;L^2(\Omega)^3). \] (6.44)

These convergence are sufficient to take the limit in (6.23) and (6.24), and to conclude that (4.8) and (4.9) are valid. Consequently, using again the standard procedure (see for example [17, Sect. 3.8 - 3.10] that $v(0) = v_0$.

Also, taking $\varphi = v$ in (4.9) we obtain for all $t \in [0,T]$
\[ \frac{1}{2} \| v(t) \|_2^2 + \int_0^t \int_\Omega \nu(\theta)D(v)^2 dx + \varepsilon \| \nabla p \|_2^2 d\tau \]
\[ + \alpha \int_0^t \| v \|_{L^2(\partial\Omega)}^2 d\tau = \int_0^t \langle b, v \rangle d\tau + \frac{1}{2} \| v_0 \|_2^2. \] (6.45)
To pass to the limit in (6.25), we need to strengthen (6.44) by establishing its strong convergence. For this purpose, we let $N \to \infty$ in (6.27), use the weak lower semicontinuity of $L^2$ norms for the quantities $v(t)$, $\sqrt{\nu(\theta)}D(v)$, $\nabla p$ and the strong convergence (6.5) and (6.43), and compare the result with (6.45). More precisely, we have for almost all $t \in (0,T)$

$$
\int_0^t \int_\Omega \nu(\theta)|D(v)|^2 \, dx \, dt \overset{(6.44)}{=} \liminf_{N \to \infty} \int_0^t \int_\Omega \nu(\theta_N)|D(v_N)|^2 \, dx \, dt
$$

$$
\overset{(6.27)}{=} \liminf_{N \to \infty} \left( \int_0^t -\varepsilon \|\nabla p_N\|_2^2 - \alpha \|v_N\|_{L^2(\partial\Omega)}^2 + \langle b, v_N \rangle \, d\tau \right)
$$

$$
+ \frac{1}{2} \|v_0\|_2^2 - \frac{1}{2} \|v(t)\|_2^2
$$

$$
\overset{(6.45)}{=} \int_0^t \int_\Omega \nu(\theta)|D(v)|^2 \, dx \, d\tau,
$$

which clearly implies that

$$
\sqrt{\nu(\theta_N)}D(v_N) \to \sqrt{\nu(\theta)}D(v) \text{ strongly in } L^2(0,T;L^2(\Omega)^{3\times 3}). \quad (6.47)
$$

Consequently, we can take the limit in (6.25) to obtain (4.10). Finally, using standard techniques we conclude (4.7).

The proof of Lemma 4.1 is complete.

References


