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Preprint no. 2008-007

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Mathematical Results Concerning The Unsteady Flows of Chemically Reacting Incompressible Fluids

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Summary. We investigate the mathematical properties of unsteady three-dimensional internal flows of chemically reacting incompressible shear-thinning (or shear-thickening) fluids. Assuming that we have Navier’s slip at the impermeable boundary we establish the long-time existence of a weak solution when the data are large.

Key words: generalized Navier-Stokes system, incompressible fluid, viscosity depending on concentration, shear-rate dependent viscosity, unsteady flows, Navier’s slip boundary conditions

AMS subject classifications: 35Q30,35Q72,76D03,76A05

1 Introduction

Even though 150 years have elapsed since Darcy [1856] published his celebrated study, it or a minor modification to it remains the equation of choice to describe the flow of fluids through porous media due to a pressure gradient. While the equation that Darcy provided in his study is referred to a "law" it is merely an approximation, and a very simple one at that for the flow of a fluid through porous media. The original equation due to Darcy can be shown to be an approximation of the equations governing the flow of a fluid through

* The contribution of M. Bulíček to this work was supported by the Jindřich Nečas Center for Mathematical Modeling, the project LC06052 financed by MSMT. J. Málek’s contribution is a part of the research project MSM 0021620839 financed by MSMT; the support of GACR 201/08/0315 and GACR 201/06/0352 is also acknowledged. K. R. Rajagopal thanks the National Science Foundation for its support.
a porous solid within the context of the theory of mixtures by appealing to numerous assumptions (see Atkin and Craine [1976a,b], Bowen [1975], Green and Naghdi [1969], Adkins [1963a,b]). Hassanizadeh [1986] and Gray [1983] have also shown that Darcy’s equation can be obtained using an averaging technique, but not within the context of mixtures. To obtain Darcy’s equation within the context of the theory of mixtures, one ignores the balance of linear momentum for the solid (which is assumed to be rigid and thus the stress is whatever it needs to be in order for the flow to take place), assumes that the only interaction between the fluid and the solid is the frictional resistance at the pores of the solid, and that this resistance is proportional to the difference in the velocity between the fluid and solid. The frictional effects within the fluid and thus the dissipation within the fluid are ignored. If the frictional resistance at the pore, between the solid and the fluid, is not assumed to be proportional to the difference in velocity between the fluid and the solid, but depends, in another way, in a non-linear manner on the difference, one obtains the Darcy-Forchheimer equation (see Forchheimer [1901]). If the viscosity of the fluid is not neglected, i.e., the viscous dissipation in the fluid is not ignored, and if it is assumed that it is like that in the classical Navier-Stokes fluid, then one obtains the equation developed by Brinkman [1947a,b] (see Rajagopal [2007]).

An interesting counterpart presents itself when we consider a complex fluid such as blood which is maintained in a state of delicate balance in virtue of a myriad of chemical reactions that take place, some that cause the blood to coagulate, others that cause the lysis, etc. In fact, even the development of a simple model for blood requires one to model dozens of equations and these govern the biochemical reactions that have to be coupled to the balance equations (see Anand et al. [2003, 2005]). As blood involves constituents that can be modeled by the Navier-Stokes model (for the Plasma for instance) or a purely elastic model (say for platelets) and others that are viscoelastic solids (cells) or viscoelastic fluids, one would have a system of equations that would be totally intractable. Thus, it is absolutely necessary to simplify the model while capturing the quintessential feature of the mechanical response characteristics. Instead of keeping track of all the constituents of blood, even though it might be an oversimplification, one could consider blood as a homogeneous fluid whose properties change due to a chemical state variable, which we shall refer to as the concentration $c$, which is a consequence of all chemical reactions that take place. Thus in essence, we are replacing a plethora of chemical reactions by a single equation that has the same effect, on an average, as the system of reactions that actually occur.

It is then possible to model the flow of blood through a coupled system of equations: the balance of mass, linear and angular momentum for the homogeneous single fluid, and a diffusion-convection equation for the chemical state variable $c$, the concentration. It is also possible to think of blood as a single homogeneous fluid that is co-occupying the flow domain with another fluid that is capable of reacting with the homogeneous fluid and thereby changing
its properties. This reacting fluid is carried along by the flowing fluid, and in the spirit of the development of Darcy's equation, we can choose to ignore the balance equation for the reacting fluid (similar to ignoring the balance equations for the porous solid). As the second fluid moves with the same velocity as the carrier, we do not have an interactive force like the "Drag Force" that is a consequence of the relative velocity between the two fluids, acting on the fluids. Bridges and Rajagopal [2006] associate the concentration with the ratio of the density of the reactant to the sum of the density of the reactant and the homogeneous fluid. While a concentration defined in such a manner tends to zero when the density of the second fluid tends to zero, it cannot tend to unity as the density of the carrier fluid (which they assume to be incompressible) is not zero and the density of the reacting fluid is finite. Though Bridges and Rajagopal [2006] motivate the concentration through such a ratio, as far as their study is concerned the concentration $c$ is merely treated as a variable that can change (one could view it as an internal variable with a clear physical underpinning, namely the concentration of a second fluid that is undergoing a reaction.).

As blood is a multi-constituent material, with the constituents distributed in the vessel in an inhomogeneous manner, it would be more appropriate to approximate it as a single constituent fluid, that is inhomogeneous, that is the homogenization of the multi-constituent body leads to an inhomogeneous body. It is important to recognize that while referring to the "homogenization" of the body we are referring to an averaging procedure of the multi-constituents, while when describing the body as inhomogeneous we are referring to the fact that the averaged body has properties that change from one material point to another. It is important to keep this distinction in mind. In the study that is being carried out, we are able to capture this inhomogeneity by allowing the material properties to change due to the presence of chemical reactions due to the concentration of reactants that are carried along by the homogeneous fluid (see the more detailed explanation that follows). It is also important to recognize that the properties of a fluid varying in a particular configuration of the fluid does not imply that the fluid is inhomogeneous as one merely needs a configuration (some configuration in which the body can be placed) in which the properties of a body are the same for the body to be homogeneous. (See Truesdell [1991] for general discussion of homogeneity and Anand and Rajagopal [2004] for an example of flows of inhomogeneous fluids where the material properties other than just the density being non-constant are considered. There are numerous studies concerning fluids with non-constant density, going back to the seminal work of Rayleigh [1883], and the books by Yih [1965, 1980] are devoted to the study of the flows of such fluids.)

Bridges and Rajagopal [2006] studied the pulsating flow of a chemically reacting incompressible fluid in terms of the balance equations for a homogenized fluid (as explained above, the equations considered by Bridges and Rajagopal [2006] is an averaged equation for a single "average" constituent, the
averaged single constituent not being necessarily a homogeneous body in that its properties are the same at every material point) and a diffusion-convection equation for the concentration $c$.

If the fluid (reactant) that is being carried along and the carrier fluid (which is the fluid obtained by "homogenizing" the multi-constituent fluid such as blood) are of comparable density, then assigning the notion of the ratio of the density of the reactant to the total density would not be appropriate as the balance of linear momentum for the fluid that is carried, whose properties are changing due to the reaction, cannot be expressed merely in terms of its density as the inertial term would have a contribution due the density of the fluid that is also carried along. We are primarily interested in the fluid that is carried along and reacting with our fluid of interest having associated with it a much smaller and in fact ignorable density. Thus, as mentioned earlier in the work of Bridges and Rajagopal [2006] the concentration has to be interpreted in the sense of a variable that is a measure of the reaction rather than a ratio of densities. This point is not made clear in their work though they do mention that $c$ could be an internal variable, which we refer to as a chemical state variable. Here, we shall choose to think of $c$ as a chemical state variable than the ratio of densities, and we shall suppose that the reactant that is carried is not of comparable density to the fluid of interest.

While it would be preferable to consider the problem of multi-constituent materials such as blood within the context of mixture theory such an approach is not without serious difficulties. Not only is it necessary to keep track of all the individual constituents and provide constitutive relations for them, as well as model all the interactions between the constituents, we have a far deeper problem, that of providing boundary conditions for each of the constituents. Usually, one is able to ascertain only the boundary conditions for the mixture as a whole and this is a basic problem that is inherent to mixture theory (see Rajagopal and Tao [1995] for a detailed discussion of the same).

It is well established that blood in large blood vessels like the aorta behaves essentially as a Navier-Stokes fluid while in narrower blood vessels it can be approximated as a single constituent fluid that shear thins. In fact, the generalized viscosity associated with such a shear thinning fluid can change by a factor of forty (see Yeleswarapu [1996], Antaki et al. [1998]). This is a consequence of the diameter of the cells becoming significant with respect to the diameter of the blood vessel. In even narrower capillarities and arterioles where the diameter of the blood vessel is comparable or even smaller than the diameter of a cell, we would not be justified in modeling the flowing blood as a continuum. Experiments by Thurston [1972, 1973] also indicate that blood is capable of stress relaxation. This is not surprising as blood contains considerable amount of cells, platelets, etc. The cells have membranes that are elastic or viscoelastic. Thus, even were we to model blood in an averaged sense as a single constituent fluid we would have to take into account its ability for shear thinning and stress relaxation. In this study we shall not concern ourselves with fluids that are capable of stress relaxation.
We shall assume that the homogenized single constituent fluid is incompressible. It means that the fluid can only undergo isochoric motions and thus
\[ \text{div } \mathbf{v} = 0. \] (1)

We shall further suppose that the viscosity of the fluid depends on the concentration \( c \) and the symmetric part of the velocity gradient to allow for the possibility that the properties of the fluid can change due chemical reactions as well as shear-thinning or shear-thickening\(^4\), and thus the Cauchy stress \( \mathbf{T} \) in our fluid of interest is given by
\[ \mathbf{T} = -p \mathbf{I} + 2 \mu(c, |\mathbf{D}(\mathbf{v})|^2) \mathbf{D} =: -p \mathbf{I} + \mathbf{S}(c, \mathbf{D}), \] (2)

where
\[ \mathbf{D} = \frac{1}{2} \left[ (\nabla \mathbf{v}) + (\nabla \mathbf{v})^T \right]. \]

Finally, we assume that flux vector \( \mathbf{q}_c \) related to the chemical reactions is given by
\[ \mathbf{q}_c = \mathbf{q}_c(c, \nabla c, \mathbf{D}) := -K(c, |\mathbf{D}|^2) \nabla c. \] (3)

The specific form of the coefficients \( K_{ij} \) of the matrix \( K \) depends on the specific application (chemical reaction or system of reactions) under consideration. If we are interested in a fluid such as blood and use the equations developed here, we have to replace a host of chemical reactions by one "averaged" reaction and one cannot say much about the form of \( K \) unless we decide on which specific problem we are interested in. For instance, coagulation and lysis have totally different effects on the fluid, one leading to an increase in the viscosity and the other leading to a decrease in viscosity. Similarly, if we are interested in ATIII deficiency or Sickle cell anemia we would have to consider other forms for the coefficient. Also, when dealing with a complex system like blood wherein one has numerous chemical reactions one has to be cognizant of the fact that each of these reactions take place at different rates and blood is maintained in a delicate state of balance while a myriad of biochemical reactions take place. On the other hand if we were dealing with a polymer melt undergoing some reaction we would have a totally different form for the diffusion coefficient. Thus, in this study we shall merely assume a specific form for the coefficient to illustrate our ideas.

Recently, Bulíček et al. [2008] considered an incompressible Navier-Stokes fluid whose density and thermal conductivity depend on the temperature. Under the assumption that the fluid meets Navier’s slip and zero heat flux boundary condition, they established the existence of weak, as well as suitable weak solutions, for long times when the data can be large. Around the same time, Bulíček et al. [2007] considered unsteady flows of incompressible fluids whose viscosity depends on the temperature, shear-rate and pressure. Such

\(^4\) When blood coagulates its viscosity increases while lysis leads to the viscosity of the coagulated blood to decrease.
a fluid model is markedly different from the classical incompressible Navier-Stokes or incompressible Navier-Stokes-Fourier fluid in that the relationship between the stress and the symmetric part of the velocity gradient is implicit. Assuming Navier’s slip at the solid boundary they established the existence of suitably weak solutions for long time, when the data is large.

If we were to ignore the dependence of the viscosity on pressure, and replace the dependence of the viscosity on the temperature by its dependence on the chemical state variable \( c \) (the concentration), we have a problem that bears close relationship to the models studied by Bulíček et al. [2007]. While in the problem wherein the viscosity depends on the temperature we have to satisfy the balance of energy, which leads to an equation for the temperature (or the internal energy), in the problem being considered here, we have a diffusion-convection equation for the chemical state variable \( c \).

As the structure of the diffusion-convection equation is simpler in comparison to the equation representing the balance of energy, the problem studied here might, at the first glance, seem easier than that considered by Bulíček et al. [2007]. On the other hand, the dependence of the material moduli on the pressure considered by Bulíček et al. [2007] allows one to consider only fluids that can shear thin, while in the present study we investigate unsteady flows of both shear-thinning and shear-thickening fluids. As a consequence, we can establish several new solutions and we also are able to make statements concerning the role of the material parameters on the nature of these solutions.

The structure of the paper is the following. In the next section, we formulate the governing equations, and state the appropriate initial and boundary conditions. For the sake of simplicity and in order to make some comparison to the earlier study by Bulíček et al. [2007] Navier’s slip boundary condition and \( C^{1,1} \) domains are considered first. We also state the assumptions concerning the constitutive quantities \( S \) and \( q_c \), define the notion of weak solution to our problem and formulate the result regarding its existence. Section 3 is focused on the proof (we merely provide the main steps and refer the reader to former studies for details). Section 4 contains several extensions of the main result in various directions (no-slip boundary conditions, qualitative properties of the solution, validity of the result for a large range of model parameters, etc).

Here we do not provide any details, but rather refer to studies where results are established to similar problems, so the interested reader can (with some effort) deduce the validity of the results that are presented. Appendix 5 includes several auxiliary assertions used in the proof of main result.
2 Formulation of the problem and the results

2.1 Balance equations, boundary and initial conditions. Structure of $\mathbf{S}$ and $q_c$.

We are interested in understanding the mathematical properties relevant to unsteady flows of chemically reacting fluids whose transport coefficients depend on the concentration and on the shear rate, flowing in a bounded open set $\Omega$ in $\mathbb{R}^3$ with the boundary $\partial \Omega$. We would like to know if there are solutions that exist in the domain $Q := (0,T) \times \Omega$, where $(0,T)$ denotes the time interval of interest, and we are able to establish that such solutions exist. It would be worthwhile to find how these solutions behave with time and in space, i.e., to carry out analysis that discusses the nature of the solution. We shall however restrict ourselves to the question of existence of solutions in this study.

Motions of incompressible fluids which are reacting chemically are described in terms of the velocity field $\mathbf{v}$, the pressure (mean normal stress) $p$ and the concentration $c$ in terms of a system of partial differential equations that are a consequence of the balance of mass, balance of linear and angular momentum, and diffusion equation for $c$. The balance of angular momentum which leads to the Cauchy stress being symmetric is automatically met by virtue of the form chosen for the Cauchy stress $\mathbf{T}$. The system governing the flows of interest take the form

$$\begin{align*}
\text{div} \mathbf{v} &= 0, \\
\mathbf{v}_t + \text{div}(\mathbf{v} \otimes \mathbf{v}) - \text{div} \mathbf{S} &= f - \nabla p, \\
\mathbf{c}_t + \text{div}(\mathbf{c} \mathbf{v}) &= - \text{div} q_c,
\end{align*}$$

(4)

where $f$ represents the specific body forces. In this setting, for given functions $\mathbf{v}_0$ and $c_0$ defined in $\Omega$, we prescribe the initial conditions

$$\begin{align*}
\mathbf{v}(0,x) &= \mathbf{v}_0(x) & \text{and} & \quad c(0,x) = c_0(x) \quad (x \in \Omega).
\end{align*}$$

(5)

We assume that the boundary $\partial \Omega$ is completely described from outside by a finite number of overlapping $C^{0,1}$-mappings, and if this is indeed the case we write $\Omega \in C^{0,1}$. We also set $\Gamma := (0,T) \times \partial \Omega$. We prescribe the following boundary conditions

$$\begin{align*}
\mathbf{v} \cdot \mathbf{n} &= 0, & \mathbf{v}_\tau &= -\frac{1}{\alpha}[\mathbf{T} \mathbf{n}]_\tau = -\frac{1}{\alpha}[\mathbf{S} \mathbf{n}]_\tau \quad \text{on } \Gamma_N, \\
q_c \cdot \mathbf{n} &= 0 \quad \text{on } \Gamma_N & \quad \text{and} & \quad c = c_b \quad \text{on } \Gamma_D,
\end{align*}$$

(6)

where $\mathbf{n}$ is the unit outward normal and $z_\tau$ stands for the projection of the quantity $z$ along the tangent plane, i.e. $z_\tau = z - (z \cdot \mathbf{n})\mathbf{n}$ and $\Gamma_N \cap \Gamma_D = \emptyset$, $\Gamma_N \cup \Gamma_D = \Gamma$ (the last equality holds up to a set of zero two-dimensional Hausdorff measure). The first line in (6) expresses the fact that the solid boundary is impervious and tangential components of the velocity fulfil Navier’s slip boundary condition, the second line in (6) says that on $\Gamma_N$ there is no flux of
concentration through the boundary and a nonhomogeneous Dirichlet boundary condition for \( c \) holds on \( \Gamma_D \).

We will also require that for some open \( \Omega_0 \subseteq \Omega \)

\[
\int_{\Omega_0} p(t, x) \, dx = h(t) \quad \text{for all } t \in [0, T],
\]

where \( h \) is a given function. The above requirement concerning the mean value of the pressure over a domain of non-zero area measure is completely consistent with how pressure measurements are made, namely a pressure gauge sensing the total force due to the normal stress on a small portion of the surface area on which the pressure gauge is mounted.

Regarding the admissible structure for the constitutive quantities \( S \) and \( q_c \) characterizing the specific fluid, we assume that \( S : [0, 1] \times \mathbb{R}^{3 \times 3}_{sym} \rightarrow \mathbb{R}^{3 \times 3}_{sym} \)

that appears in (2) is a continuous mapping such that for some \( r > 1 \)

there are \( C_1, C_2, C_3 \in (0, \infty) \) and a function \( \gamma_1 \in L^\infty([0, 1]) \),

such that for all \( c \in [0, 1] \) and \( D \in \mathbb{R}^{3 \times 3}_{sym} \),

\[
C_1|D|^r - C_3 \leq S(c, D) \cdot D \quad \text{and} \quad |S(c, D)| \leq C_2 \gamma_1(c)|D|^{r-1} + C_3
\]

and

for all \( D_1, D_2 \in \mathbb{R}^{3 \times 3}_{sym}, D_1 \neq D_2 \) and all \( c \in [0, 1] \) there holds

\[
(S(c, D_1) - S(c, D_2)) \cdot (D_1 - D_2) > 0.
\]

We also assume that \( K \) that appears in (3) is a continuous mapping of \([0, 1] \times \mathbb{R}_{>0} \) into \( \mathbb{R}^{3 \times 3} \) such that for some \( \beta \in \mathbb{R} \) the flux vector \( q_c = q_c(c, z, s) = -K(c, s)z \) fulfills the conditions

there are \( C_4, C_5 \in (0, \infty) \) such that for all \( c \in [0, 1], z \in \mathbb{R}^3, s \in \mathbb{R}^+_{0} \):

\[
C_4(1 + s)^\beta|z|^2 \leq -q_c(c, z, s) \cdot z \quad \text{and} \quad |q_c(c, z, s)| \leq C_5 \gamma_1(c)(1 + s)^\beta|z|.
\]

We are unaware of any experimental evidence concerning the structure of the diffusion matrix \( K \). In fact, one has to recognize that in order to have such information one needs to correlate raw experimental data with the model that we are using and as the model is a new model that is being proposed it is not surprising that such data is unavailable. In light of that we shall attempt to develop the theory for as large range of \( \beta \)’s as possible.

### 2.2 Function spaces, definition of solution, main theorem

We first characterize the regularity of the domain by requiring that the boundary of \( \Omega \) is smooth enough so that \( L^q \)-regularity for certain values of \( q \in [1, \infty] \) holds. We shall make this statement more precise in the next paragraph.

Let, for a given \( z \in L^q(\Omega) \) with \( \int_{\Omega} z \, dx = 0 \), the symbol \( \mathcal{N}_{\Omega_0}^{-1}(z) \) denote the unique solution of the Neumann problem
\[ \triangle u = z \text{ in } \Omega, \quad \nabla u \cdot n = 0 \text{ on } \partial \Omega, \quad \int_{\Omega_0} u \, dx = 0. \] (11)

For the special case \( \Omega_0 = \Omega \) we use abbreviation \( \mathcal{N}^{-1}(z) := \mathcal{N}_\Omega^{-1}(z) \). Thus, in particular, denoting \( g^v := \mathcal{N}^{-1}(\text{div} \, u) \) we can define the vector \( v_{\text{div}} \), as

\[ v_{\text{div}} := v - \nabla g^v \quad (\implies v = v_{\text{div}} + \nabla g^v \text{ Helmholz decomposition}). \] (12)

It is known, see Grisvard [1985, Proposition 2.5.2.3, p. 131], that if \( \Omega \in C^{0,1} \), then for any \( s \in (1, \infty) \)

\[ \|g^v\|_{1,s} \leq C(\Omega, s)\|v\|_s \quad \|v_{\text{div}}\|_{s} \leq (C(\Omega, s) + 1)\|v\|_s. \] (13)

We say that a bounded domain \( \Omega \subset \mathbb{R}^3 \) with Lipschitz boundary is of the class \( \mathcal{R} \), and we then write \( \Omega \in \mathcal{R} \), if the \( L^q \)-regularity theory for the Neumann problem (11) holds for \( q = r \) and \( q = \frac{5r}{5r-6} \), and consequently the following estimates are valid for the same \( q' \):

\[ \|g^v\|_{2,q} \leq C_{\text{reg}}(\Omega, q)\|v\|_q \quad \|v_{\text{div}}\|_{1,q} \leq (C_{\text{reg}}(\Omega, q) + 1)\|v\|_{1,q}. \] (14)

It is known, see Grisvard [1985, Proposition 2.5.2.3, p. 131], that if \( \Omega \in C^{1,1} \), then (14) holds for any \( q \in (1, \infty) \), and thus \( \Omega \in \mathcal{R} \). For another conditions that are sufficient for \( \Omega \in \mathcal{R} \) we refer the reader to Bulíček et al. [2007] where this issue is discussed in detail.

Before giving the definition of what we mean by a solution to (4)–(7), we need to introduce subspaces (and their duals) of vector-valued Sobolev functions from \( W^{1,q}(\Omega)^3 \) which have zero normal component on the boundary (note that \( q' = q/(q - 1) \)). We define

\[
L_{n,\text{div}}^q(\Omega) := \left\{ v \in W^{1,q}_n(\Omega); \text{div} \, v = 0 \right\},
\]

\[
W_{n,\text{div}}^{1,q}(\Omega) := \left\{ v \in W_{n}^{1,q}; \text{div} \, v = 0 \right\},
\]

\[
W_{n}^{1,q}(\Omega) := \left\{ h \in W_{n}^{1,q}; \text{tr} \, h = 0 \right\},
\]

\[
W_{n}^{1,q'}(\Omega) := \left\{ h \in W_{n}^{1,q'} \right\},
\]

\[
L_{n,\text{div}}^q(\Omega) := \left\{ v \in W_{n,\text{div}}^{1,q} \right\}.
\]

For \( r, q \in [1, +\infty] \), we also introduce relevant spaces of a Bochner-type, namely,

\[
X_{r,q} := \left\{ u \in L^r(0, T; W_{n}^{1,r}) \cap L^q(0, T; L^q(\Omega)^3), \text{tr} \, u \in L^2(0, T; (L^2(\partial \Omega))^3) \right\},
\]

\[
X_{r,q}^{\text{div}} := \left\{ u \in X_{r,q}, \text{div} \, u = 0 \right\},
\]

\[
\mathcal{C}(0, T; L_2^q(\Omega)^3) := \left\{ u \in L^\infty(0, T; L^q(\Omega)^3); (u(t), \varphi) \in \mathcal{C}([0, T])\forall \varphi \in C(\overline{\Omega})^3 \right\}.
\]

In the last definition, we used the notation \( (f, g) \) for \( \int_{\Omega} f(x)g(x) \, dx \) if \( fg \in L^1(\Omega) \). In an analogous manner, we shall use the symbols \( (f, g)_Q \), \( (f, g)_{\Omega} \), \( (f, g)_r \), etc. If \( f \in X \) and \( g \in X^* \) we often use the symbol \( \langle f, g \rangle \) instead of
\[\langle g, f \rangle_{X_\cdot, X} \] The same bracket notation is used for vector functions \( f, h \) and tensor functions \( F, H \) as well.

We assume that \( f \) that appears on the right hand side of Eq. (4), the prescribed function \( h \) for the pressure (see (7)), initial values \( v_0 \) and \( c_0 \) as appear in (5), satisfy

\[
f \in L'^r(0, T; W^{1,r}_n), \quad h \in L'^{r}(0, T),
\]

\[
v_0 \in L^2_{n,\text{div}}, \quad 0 \leq c_0(x) \leq 1 \text{ for a.a. } x \in \Omega, \tag{15}
\]

and we also require that a given concentration \( c_b \) on \( \Gamma_D \) appearing in \((6)_4\) fulfils

For \( n := \max\{2, r/(r - 2\beta)\} \) there is \( \tilde{c}_b \in L^\infty(Q) \cap L^n(0, T; W^{1,n}(\Omega)) \)

with \( \tilde{c}_{b,t} \in L^1(\Omega) \) such that \( 0 \leq \text{tr}\tilde{c}_b = c_b \leq 1 \) on \( \Gamma_D \).

**Definition 2.1.** Let \( \Omega \in \mathbb{R}, \alpha \in [0, \infty) \) and (15)–(16) hold. Assume that \( S \) satisfies (8)-(9) with \( r > \frac{\alpha}{2} \), and \( q_c \) satisfies (10) with \(-r < 2\beta < r\). Setting

\[
m = \min\{\frac{2r}{r - 2\beta}, r'\}, \quad q = \min\{2, \frac{2r}{r - 2\beta}\} \text{ and } s = \min\{2, \frac{2r}{r - 2\beta}\},
\]

we say that \((v, p, c)\) is a weak solution to (4)-(7) if

\[
v \in C([0, T]; L^2_w(\Omega)^3) \cap X_{\text{div}}^{r, n}, \quad v_\cdot t \in L^m(0, T; W^{1,m}_n), \tag{18}
\]

\[
p \in L^m(0, T; W^{1,m}_n) \text{ and } \int_{\Omega_0} p(x,t) \, dx = h(t) \text{ for a.a. } t \in (0, T), \tag{19}
\]

\[
c - \tilde{c}_b \in L^q(0, T; W^{1,q}_{r_D}(\Omega)), \quad c_\cdot t \in L^{s'}(0, T; W^{1,s'}_{r_D}(\Omega)), \tag{20}
\]

\[
0 \leq c \leq 1 \text{ a.e. in } Q, \tag{21}
\]

\[
(1 + |D(v)|^2)^{\frac{\alpha}{2}} \nabla c \in L^2(0, T; L^2(\Omega)^3), \tag{22}
\]

and \((v, p, c)\) fulfil the following weak formulations

\[
\langle v_\cdot t, \varphi \rangle - \langle v \otimes v, \nabla \varphi \rangle_Q + \alpha(v, \varphi)_r + \langle S(c, D(v)), D(\varphi) \rangle_Q = \langle f, \varphi \rangle + \langle p, \text{div} \varphi \rangle_Q \quad \text{for all } \varphi \in L^{m'}(0, T; W^{1,m'}_n), \tag{23}
\]

\[
(c_\cdot t, \varphi)_Q - \langle cv, \nabla \varphi \rangle_Q + \langle K(c, |D(v)|^2) \nabla c, \nabla \varphi \rangle_Q = 0 \quad \text{for all } \varphi \in L^s(0, T; W^{1,s}_{r_D}(\Omega)), \tag{24}
\]

and \((v, c)\) meet the initial conditions in the following sense

\[
\lim_{t \to 0^+} \|v(t) - v_0\|^2_2 + \|c(t) - c_0\|^2_2 = 0. \tag{25}
\]

**Theorem 2.1.** Let \( r > \frac{\alpha}{2} \). Then for any data fulfilling (15)–(16) and for any \( T \in (0, \infty) \) there exists a weak solution to (4)-(7) in the sense of Definition 2.1.
To the best of our knowledge, this is the first result concerning long time existence of solutions to a model such as (4) where the material coefficients depend on $c$ (concentration) and $|\mathbf{D}|^2$ (shear rate). In addition, the result holds for large data fulfilling (15)–(16) and the result concerns flows in general domains (with $C^{1,1}$ boundary, for example) under reasonable Navier's slip boundary conditions. Following Wolf [2007], the result can be however extended to no-slip boundary conditions and to more general domains (we then however lose the integrability of the pressure), and it is also possible to include lower values of the power-law index $r$ (this follows by applying the approach due to Diening et al. [2008]). We discuss these possible extensions in Section 4 in more detail.

The related studies that we are aware of wherein the existence of weak solution are established for different models of chemically reacting fluids are due to Roubíček [2005, 2007] who in addition considers a system of chemical reactions together with electrical and thermal stimuli. However, on the other hand he only treats the case $r \geq \frac{11}{5}$ and the fluxes related to the chemical reactions are independent of the shear rate.

The problem considered here shares certain similarities with the initial boundary value problems for inhomogeneous incompressible fluids driven by the system of equations

\[
\begin{align*}
\text{div} \, \mathbf{v} &= 0, \quad \rho_t + \text{div}(\rho \mathbf{v}) = 0, \\
(\rho \mathbf{v})_t + \text{div}(\rho \mathbf{v} \otimes \mathbf{v}) - \text{div} \mathbf{S} &= \rho \mathbf{f} - \nabla p, \quad \mathbf{S} = 2\mu(\rho, |\mathbf{D}(\mathbf{v})|^2)\mathbf{D}(\mathbf{v}) .
\end{align*}
\]

(26)

We refer to Antontsev et al. [1990] and Lions [1996] for the analysis of models when $\mathbf{S} = 2\mu(\rho)\mathbf{D}(\mathbf{v})$. The models where the viscosity $\mu$ also depends in a polynomial way on $|\mathbf{D}(\mathbf{v})|^2$ are analyzed in Fernández-Cara et al. [1997] (for $r \geq \frac{12}{5}$), in Guillén-González [2004] (for $r \geq 2$ and the spatially periodic problem), and in Frehse and Růžička [2007] (for $r \geq \frac{11}{5}$ and the problem with no-slip boundary conditions). On contrary, the result presented here holds even for $r < 2$ and one needs to handle the additional nonlinear diffusion term in (4) that provides information on the gradient of the concentration (hardly available for $\rho$ in (26)).

3 A proof of Theorem 2.1

Before establishing the main existence result we would like to make a few comments about the construction of the proofs in this paper. Many of the lemmas and other results such as interpolation inequalities that are necessary to establish the main theorem have been established elsewhere with regard to problems with similar mathematical structures. Thus, in order to avoid repetition we do not document detailed proofs for the same but merely refer the reader to where such proofs can be found. In this paper we shall only focus on the main or new parts of the proofs.
3.1 An \((\varepsilon, \eta)\)-approximate problem and uniform estimates

For positive and fix \(\varepsilon\) and \(\eta\), we consider the following approximation

\[
\begin{align*}
-\varepsilon \Delta p + \text{div } v &= 0 \text{ in } Q, \quad \frac{\partial p}{\partial n} = 0 \text{ on } \Gamma, \quad \int_{\partial\Omega} p \, dx = h(t), \quad (27) \\
v_t + \text{div}(v_t \otimes v) - \text{div} S(c, D(v)) &= -\nabla p + f, \quad (28) \\
c_t + \text{div}(v_t c) &= -\text{div} q(c, \nabla c, D(v)), \quad (29)
\end{align*}
\]

where for a given \(v \in L^r(0, T; W^{1, r}_n)\) we use \(v_{\eta}\) to denote a function defined through

\[
v_{\eta} := ((v \omega_{\eta}) \ast r_{\eta})_{\text{div}},
\]

where \(r_{\eta}(x) := \frac{1}{\eta^n} r\left(\frac{x}{\eta}\right)\) with \(r \in D(\mathbb{R}^3)\) nonnegative, radially symmetric, meeting \(\int_{\mathbb{R}^3} r \, dx = 1\) (i.e., \(r\) is regularization kernel), and \(\omega_{\eta}\) is a smooth function such that \(\text{dist}(\text{supp } \omega_{\eta}, \partial \Omega) \geq \eta\), and \(\omega_{\eta} = 1\) for all \(x\) fulfilling \(\text{dist}(x, \partial \Omega) \geq 2\eta\), and \(z_{\text{div}} := z - \nabla \nabla^{-1}(\text{div } z)\) (see (12)).

Rewriting (27) as \(p - h/\Omega_\eta = \frac{1}{2} N_{\Omega_\eta}^{-1}(\text{div } v)\), and inserting it into (28) we can also equivalently rewrite (27)–(29) as the system for \(v\) and \(c\) only.

We will assume (referring to Bulíček et al. [2007] for details concerning the solvability of similar, yet a more complicated system) that for fix \(\varepsilon, \eta > 0\) there is a solution \((v, p, c) = (v^{\varepsilon, \eta}, p^{\varepsilon, \eta}, c^{\varepsilon, \eta})\) satisfying

\[
\begin{align*}
\varepsilon(\nabla p, \nabla \psi)_\Omega &= (v, \nabla \psi)_\Omega \text{ for a.a. } t \in (0, T) \text{ and all } \psi \in W^{1, 2}(\Omega), \quad (30) \\
\langle v_t, \varphi \rangle - (v_{\eta} \otimes v, \nabla \varphi)_Q + \alpha(v, \varphi)_Q + (S(c, D(v), D(\varphi))_Q &= (p, \text{div } \varphi)_Q + \langle f, \varphi \rangle \quad \text{for all } \varphi \in L^r(0, T; W^{1, r}_n), \quad (31) \\
\langle c_t, \varphi \rangle - ((v_{\eta} c, \nabla \varphi) - (q, c, D(v), \nabla c), \nabla \varphi)_Q &= 0 \quad \text{for all } \varphi \in L^s(0, T; W^{1, s}_n(\Omega)). \quad (32)
\end{align*}
\]

Even more, we will require that we can take \(\varphi = v\) in (31), \(\varphi = c - \bar{c}_0\) in (32) and \(\psi = p\) in (30). These additional assumptions can be easily verified on the level of Galerkin approximations, see Bulíček et al. [2007] for details. Here, this requirement enables us to derive the uniform estimates that are independent of \(\varepsilon, \eta\) in the function spaces that appear in Definition 2.1.

Taking \(\varphi = v\) in (31) and \(\psi = p\) in (30) and adding the result, we obtain (after using integration by parts)

\[
\sup_{t \in (0, T)} \|v(t)\|_2^2 + \int_0^T (S(c, D(v)), D(v)) + \alpha \|v\|_{2, \partial \Omega}^2 + \varepsilon \|\nabla p\|_2^2 \, dt \\
\leq \|v_0\|_2^2 + \int_0^T \langle f, v \rangle \, dt.
\]

Applying (8) and Korn's inequality (Lemma 5.2 in Appendix) to the second term, the standard duality estimates and Young's inequality to the last term
in (33), and also the interpolation inequality (83) with \( q := \frac{2r}{r} \), we conclude that

\[
\sup_{t \in (0,T)} \|v(t)\|_2^2 + \int_0^T \|v(t)\|_1^2 + \|v(t)\|_2^2 + c\|\nabla p(t)\|_2^2 \, dt \leq C. \quad (34)
\]

Next, we define \( \varphi_1 := \min\{0,c\} \) and \( \varphi_2 := \max\{c,1\} - 1 \). Since \( c = c_b \) on \( \Gamma_D \), using (16) we get that \( \varphi_1, \varphi_2 = 0 \) on \( \Gamma_D \). Thus, setting \( \varphi = \varphi_i \chi_{[0,t]} \) in (32) we get

\[
\|\varphi_i(t)\|_2^2 - 2 \int_0^t (v_{\eta}c, \nabla \varphi_i)_Q - 2 \int_0^t (q_c(c, D(v), c), \nabla \varphi_i)_Q = 2\|\varphi_i(0)\|_2^2.
\]

Hence, using \( \text{div} v = 0 \) and assumption (10) we deduce that

\[
\|\varphi_1(t)\|_2^2 + \|\varphi_2(t)\|_2^2 \leq \|\varphi_1(0)\|_2^2 + \|\varphi_2(0)\|_2^2.
\]

It follows from (16) that \( \varphi_i(0,x) = 0 \) for a.a. \( x \in \Omega \). Combining all these results we finally obtain that

\[
0 \leq c(x,t) \leq 1 \quad \text{for a.a.} \ (x,t) \in Q. \quad (35)
\]

Next, taking \( \varphi = c - \tilde{c}_b \) in (32), we arrive at

\[
\langle c, c - \tilde{c}_b \rangle - (v_{\eta}c, \nabla (c - \tilde{c}_b))_Q - (q_c(c, D(v), c), \nabla (c - \tilde{c}_b))_Q = 0.
\]

Adding and subtracting \( \tilde{c}_b \) into the first term, integrating the result w.r.t. time, using (16), (35) and the facts that \( \text{div} v = 0 \) and \( v_{\eta} \cdot n = 0 \) on \( \partial \Omega \), we find that

\[
-(q_c(c, D(v), c), \nabla c)_Q \leq 1 + (\tilde{c}_b, \tilde{c}_b + c)_Q - (v_{\eta}c, \nabla \tilde{c}_b)_Q
\]

\[
- (q_c(c, D(v), c), \nabla \tilde{c}_b)_Q. \quad (36)
\]

On using the assumptions (10) and (16), the uniform estimates (34) and (35), and the fact that \( 2\beta \leq r \) we deduce from (36) that

\[
\int_Q (1 + |D(v)|^2)^\beta \, |\nabla c|^2 \, dx \, dt \leq C + C \int_Q (1 + |D(v)|^2)^\beta \, |\nabla \tilde{c}_b| \, dx \, dt
\]

\[
\leq C + \frac{1}{2} \int_Q (1 + |D(v)|^2)^\beta \, |\nabla c|^2 \, dx \, dt + C \int_Q (1 + |D(v)|^2)^\beta \, |\nabla \tilde{c}_b|^2 \, dx \, dt
\]

\[
(16), (34) \leq C + \frac{1}{2} \int_Q (1 + |D(v)|^2)^\beta \, |\nabla c|^2 \, dx \, dt.
\]

Thus,

\[
\int_Q (1 + |D(v)|^2)^\beta \, |\nabla c|^2 \, dx \, dt \leq C. \quad (37)
\]

Therefore, for \( \beta \geq 0 \) we simply conclude that
\[ \int_0^T \|c\|_{1,2}^2 \, dt \leq C. \] (38)

For \( \beta < 0 \), recalling that then \( q = \frac{2r}{r-2\beta} > 1 \) (see definition in (17)), we can compute

\[ \int_Q |\nabla c|^q = \int_Q |\nabla c|^q(1 + |D(v)|^2)^{\frac{\beta q}{2}} (1 + |D(v)|^2)^{-\frac{\beta q}{2}} \]
\[ \leq \int_Q |\nabla c|^2(1 + |D(v)|^2)^{\beta} + \int_Q (1 + |D(v)|^2)^{-\frac{\beta q}{2}} \]
\[ = \int_Q |\nabla c|^2(1 + |D(v)|^2)^{\beta} + \int_Q (1 + |D(v)|^2)^{\beta q} \quad \text{(34),(37)} \]
\[ \leq C. \] (39)

It is a direct consequence of (34), (35), (38) and (39), and the assumptions (8) and (10), that

\[ \int_Q |S(c, D(v))|^{r'} + |q_c(c, D(v), \nabla c)|^s \, dx \, dt \leq C \] (40)

with \( s = \min\{2, \frac{2r}{r+2\beta}\} > 1 \) defined in (17).

Finally, it follows from (31) and (32) and the above established estimates that (for details see Málek et al. [1996] or Bulíček et al. [2007])

\[ \|v\|^r_{L^r(\tau; H^1)} \quad \text{and} \quad \|c\|^s_{L^s(0, T; W^{-1, s}(\Omega))} \leq C. \] (41)

To obtain uniform estimates on \( p \), we assume without loss of generality that \( h(t) \equiv 0 \). We introduce \( p_0 := p - \frac{1}{|\Omega|} \int_{\Omega} p \, dx \) and observe by contradiction and since \( \int_{\Omega} p(t) \, dx = h(t) = 0 \) that there is \( C \) independent of \( \varepsilon, \eta \) such that

\[ \|p\|_{q} \leq C\|p_0\|_{q}. \] (42)

Consequently, it suffices to find uniform estimates for \( p_0 \). For this purpose, we consider \( \varphi \) in (31) first of the form \( \varphi = \nabla N^{-1}(|p_0|^{r'-2}p_0 - \frac{1}{|\Omega|} \int_{\Omega} |p_0|^{r'-2}p_0 \, dx) \) and then \( \varphi = \nabla N^{-1}(|p_0|^{m-2}p_0 - \frac{1}{|\Omega|} \int_{\Omega} |p_0|^{m-2}p_0 \, dx) \) (with \( m \) defined in (17) as \( m = \min\{r', \frac{2r}{r+2\beta}\} \)). Such a choice of \( \varphi \) clearly leads to

\[ (p, \text{div} \varphi)_Q = \int_0^T \|p_0\|_{\alpha}^\alpha \quad \text{(first with } \alpha = r' \text{ and then with } \alpha = m). \]

Replacing the left hand side by means of (31) and proceeding step by step as in Bulíček et al. [2007], and in particular using the fact that

\[ \int_0^T \langle v, \tau, \varphi \rangle \, dt = -\frac{2\varepsilon}{\alpha} \|p_0(T)\|_{\alpha}^\alpha \leq 0 \quad (\alpha = r' \text{ or } m), \]
we conclude that
\[
\int_0^T \| p_0 \|^r \leq C(\eta) \quad \text{(42)} \quad \int_0^T \| p \|^r \leq C(\eta),
\]
\[
\int_0^T \| p_0 \|^m \leq C \quad \text{(42)} \quad \int_0^T \| p \|^m \leq C.
\]

Using (31) and (34), these estimates imply that (see Bulíček et al. [2007] for details)
\[
\int_0^T \| v_{x,t} \|_{W^{-1, r'}}^r \, dt \leq C(\eta), \quad \int_0^T \| v_{x,t} \|_{W^{-1, m}}^m \, dt \leq C.
\]

### 3.2 Limit $\varepsilon \to 0$

Let $(v^\varepsilon, c^\varepsilon, p^\varepsilon)$ be used in this subsection to denote $(v^{\varepsilon, \eta}, c^{\varepsilon, \eta}, p^{\varepsilon, \eta})$, the solution of the $(\varepsilon, \eta)$-approximation (30)–(32). We introduce the notation
\[
q^\varepsilon_c := q_c(c^\varepsilon, D(v^\varepsilon), \nabla c^\varepsilon), \quad S^\varepsilon := S(c^\varepsilon, D(v^\varepsilon)).
\]

It is then a consequence of the estimates (34)–(44) (that all are independent of $\varepsilon$), Corollary 5.1 and Aubin-Lions Lemma 5.4 that we can find (not relabeled) subsequences of $\{v^\varepsilon, c^\varepsilon, p^\varepsilon, S^\varepsilon, q^\varepsilon_c\}$ such that

- $v^\varepsilon_t \rightharpoonup v_{x,t}$ weakly in $L^r(0, T; W^{-1, r'})$,
- $v^\varepsilon \rightharpoonup v$ weakly in $L^r(0, T; W^{1, r})$,
- $v^\varepsilon \rightharpoonup v$ strongly in $L^h(0, T; L^h(\Omega))$ for all $h < \frac{5r}{3}$,
- $\text{tr } v^\varepsilon \rightharpoonup \text{tr } v$ strongly in $L^2(0, T; L^2(\partial \Omega))$,
- $c^\varepsilon_t \rightharpoonup c_{x,t}$ weakly in $L^r(0, T; W^{-1, r'})$,
- $c^\varepsilon \rightharpoonup c$ weakly in $L^q(0, T; W^{1, q}(\Omega))$,
- $c^\varepsilon \rightharpoonup c$ strongly in $L^s(0, T; L^s(\Omega))$ for all $s < \infty$,
- $p^\varepsilon \rightharpoonup p$ weakly in $L^r(0, T; L^r(\Omega))$,
- $S^\varepsilon \rightharpoonup S$ weakly in $L^r(0, T; L^r(\Omega)^{3 \times 3})$,
- $q^\varepsilon_c \rightharpoonup q_{c}$ weakly in $L^s(0, T; L^s(\Omega))$.

Since $\sqrt{\varepsilon} \nabla p^\varepsilon$ is uniformly bounded in $L^2(0, T; L^2(\Omega))$, applying (46) to (30) we directly conclude that $\text{div } v = 0$ in $Q$.

In order to take the limit in (31) and (32) we first identify the limits of $S^\varepsilon$ and $q^\varepsilon$. To prove that $\mathfrak{S} = S(c, D(v))$, it is enough to establish almost everywhere convergence for $\nabla v^\varepsilon$. To show this, we observe that (53), (51), (8) and Lebesgue Dominated Convergence theorem (used to show that $\| S(c^\varepsilon, D(v)) - S(c, D(v)) \|_{r'} \to 0$ as $\varepsilon \to 0$) imply that $\langle S(c^\varepsilon, D(v)), D(v^\varepsilon - v) \rangle_Q \to 0$ as
Thus, to identify the limit of (56) we know that \( \text{for all } h < r, \)

\[
\int_{\Omega} \phi \cdot (\kappa, |D(v^\varepsilon)|^2)^\kappa \, dx \rightarrow 0
\]

which is sufficient to prove (modulo subsequence) that \( \mathbf{F}_v = S(c, D(v)) \). To show that also \( \mathbf{F}_{c,v} = q_v(c, D(v), \nabla c) \), we first observe (using (51), (55) and (10)) that

\[
K(c^\varepsilon, |D(v^\varepsilon)|^2) \rightarrow K(c, |D(v)|^2)
\]

strongly in \( L^h(0, T; L^h(\Omega)^{3 \times 3}) \) (56) for all \( h \in [1, \frac{1}{2}] \) if \( \beta > 0 \) and all \( h \in (1, \infty) \) if \( \beta \leq 0 \). Thus, for \( \beta \leq 0 \) it is a direct consequence of convergence properties shown above that \( \mathbf{F}_v = q_v(c, D(v), \nabla c) \). For \( \beta \geq 0 \) we proceed more carefully. On using the notation \( \kappa^\varepsilon := |K(c^\varepsilon, D(v^\varepsilon))| \), (56) implies that for all \( h \in (1, \infty) \)

\[
\int_{\Omega} \phi \cdot (\kappa^\varepsilon, |D(v^\varepsilon)|^2)^\kappa \, dx \rightarrow 0
\]

strongly in \( L^h(0, T; L^h(\Omega)^{3 \times 3}) \). Thus, to identify \( \mathbf{F}_{c,v} \), it is enough to identify \( \kappa \nabla c \) (here the symbol \( \kappa \nabla c \) denotes the weak limit of \( \kappa^\varepsilon \nabla c^\varepsilon \)). Then (10), (37) and (35) imply (up to subsequence) that \( \sqrt{\kappa^\varepsilon \nabla c^\varepsilon} \rightarrow \sqrt{\kappa} \nabla c \) weakly in \( L^2(Q) \). Thus, having \( 2\beta < r \) and strong convergence (56), we know that \( \sqrt{\kappa^\varepsilon} \rightarrow \sqrt{\kappa} \) strongly in \( L^2(Q) \). Hence, to identify the limit \( \mathbf{F}_{c,v} \) it is enough to identify the limit \( \sqrt{\kappa} \nabla c \). Using this procedure inductively, we see that it is enough that for some \( k \in \mathbb{N} \) there holds

\[
\left( \frac{\kappa^\varepsilon}{\kappa} \right)^{2-k} \nabla c^\varepsilon \rightarrow \kappa^{2-k} \nabla c \text{ weakly in } L^1(Q).
\]

Hence, we find \( k \) such that \( \frac{2^{k-1}}{\beta} > q' \). Then it is a consequence of (56) and (50) that the convergence (57) is valid that finishes the proof of convergence of \( q_v^\gamma \).

Finally, having all these convergence results (45)-(54) in hands together with the identification of the limit of nonlinear terms, we can take the limit \( \varepsilon \rightarrow 0 \) in (31) and in (32) in a standard way and conclude that for any \( \eta > 0 \) the triplet \( (v, c, p) := (v^n, c^n, p^n) \) fulfills

\[
\langle v, \alpha \rangle - \langle v_0 \otimes v, \nabla \varphi \rangle_Q + \langle S(c, D(v)), \nabla \varphi \rangle_Q + \alpha(v, \varphi)_T - \langle p, \text{div} \varphi \rangle_Q = \langle f, \varphi \rangle \text{ for all } \varphi \in L'(0, T; W^1_n)^c, \quad (58)
\]

\[
\langle c, \alpha \rangle - \langle v_0 c, \nabla \varphi \rangle_Q = \langle q_v(c, D(v), \nabla c), \nabla \varphi \rangle_Q \text{ for all } \varphi \in L^m(0, T; W^{1,m'}(\Omega)) \text{ such that } \varphi|_{\partial \Omega} = 0. \quad (59)
\]
The attainment of initial conditions is again standard and can be proved by using the same methods as those described in Málek et al. [1996].

3.3 Limit $\eta \to 0$

Let $(v^\eta, c^\eta, p^\eta, S^\eta, q_c^\eta)$, the solution of the $\eta$-approximation, fulfill (58) and (59). Our final goal is to take the limit $\eta \to 0$ in (58) and in (59), and to establish the existence of a suitable weak solution to our original problem.

Using weak lower semicontinuity of appropriate norms, we find that (34)–(41), (43) and (44) hold. These estimates together with Aubin-Lions Lemma 5.4 and Corollary 5.1 are sufficient to find a (not relabeled) subsequence of $(v^\eta, c^\eta, p^\eta)$ such that

\begin{align*}
v^\eta_h &\to v, \quad $\text{weakly in } L^m(0,T; W^{-1,m}_n) \cap (X^{T-\frac{5}{3}})^*$, \\
v^\eta &\to v, \quad $\text{weakly in } L^r(0,T; W^{1,r}_n)$, \\
v^\eta &\to v, \quad $\text{strongly in } L^k(0,T; L^k(\Omega)^3)$ for all $k < \frac{5r}{3}$, \\
tr v^\eta &\to tr v, \quad $\text{strongly in } L^2(0,T; L^2(\partial\Omega)^3)$, \\
c^\eta_t &\to c, \quad $\text{weakly in } L^{s'}(0,T; W^{-1,s'}(\Omega))$, \\
c^\eta &\to c, \quad $\text{weakly in } L^q(0,T; W^{1,q}(\Omega))$, \\
c^\eta &\to c, \quad $\text{strongly in } L^h(0,T; L^h(\Omega))$ for all $h < \infty$, \\
p^\eta &\to p, \quad $\text{weakly in } L^m(0,T; L^m(\Omega))$, \\
S^\eta &\to S, \quad $\text{weakly in } L^r(0,T; L^r(\Omega)^{3\times3})$, \\
q_c^\eta &\to q_c, \quad $\text{weakly in } L^s(0,T; L^s(\Omega)^3)$.
\end{align*}

Assume for a moment that $D(v^\eta)$ converges to $D(v)$ almost everywhere in $Q$. Then using the same procedure as in Subsection 3.2 we get $q_c = q_c(c, D(v)\nabla c)$. Then we are able to take the limit in (59) and to achieve (24).

Thus it remains to show that $\nabla v^\eta$ converges a.e. in $Q$. To do it, we follow the approach described in Boccardo and Murat [1992], Frehse et al. [2000] or Bulíček et al. [2007]. We define

\begin{equation}
g^\eta := |\nabla v^\eta|^r + |\nabla v|^r + (|S^\eta| + |S|)(|D(v)| + |D(v)|). \tag{70}
\end{equation}

It follows from (61) and (68) that there is $K \in [1, \infty)$ so that for all $\eta$

\begin{equation}
0 \leq \int_0^T \int_\Omega g^\eta \, dx \, dt \leq K.
\end{equation}

Let $\varepsilon^* > 0$ be arbitrary. Then the following statement is proved in Bulíček et al. [2007]:

\begin{equation}
\text{...}
\end{equation}
Using the assumption (10) we have

and sets \( E_j := \{(t,x) \in Q; L^2 \leq |u^j(t,x) - v(t,x)| < L\} \)  

such that \( \int_{E_j} g^j \, dx \, dt \leq \varepsilon^* \).

For such an obtained sequence \( \{u^j\}_{j=1}^{\infty} \) and \( L \) we define \( u^j \) and the sets \( Q_j \) as

\[
  u^j := (u^j - v) \left( 1 - \min \left\{ \frac{|u^j - v|}{L}, 1 \right\} \right); 
  Q^j := \{(t,x) \in Q; |v - v^j| < L\}.
\]

(71)

By using (61)-(62) and the fact that \(|u^j| \leq L \) in \( Q \) we have (as \( j \to \infty \))

\[
  u^j \to 0 \quad \text{weakly in } L'(0,T; W_n^{1,r}),
\]

(72)

\[
  u^j \to 0 \quad \text{strongly in } L^s(0,T; L^s(\Omega)^3) \quad \forall s < \infty.
\]

(73)

\[
  v^j \to 0 \quad \text{strongly in } L^2(0,T; L^2(\partial\Omega)^3).
\]

(74)

Since (see Bulíček et al. [2007, (2.60)] for details)

\[
  \int_0^T \| \text{div } u^j \|^s_r \, dt \leq C\varepsilon^*,
\]

(75)

the Helmholtz decomposition \( u^j = u_{\text{div}}^j + \nabla g^j \) then implies that

\[
  \int_0^T \| g^j u^j \|_{2,s} \, dt \leq C\varepsilon^*,
\]

(66)

\[
  u_{\text{div}}^j \to 0 \quad \text{strongly in } L^s(0,T; L^s(\Omega)^3) \quad \text{for all } s < \infty.
\]

(77)

Using the assumption (10) we have

\[
  0 \leq (S(c^j, D(v^j)) - S(c^j, D(v)), D(v^j - v))_Q \\
  = -(S(c^j, D(v)), D(v^j - v))_Q + (S(c^j, D(v^j)), D(v^j - v))_Q \\
  \leq: Y_1 + Y_2.
\]

(78)

By virtue of (66), Lebesgue Dominated Convergence theorem and (8), we observe that

\[
  S(c^j, D(v)) \to S(c, D(v)) \quad \text{strongly in } L'(0,T; L'(\Omega)^{3 \times 3}).
\]

(79)

Therefore (as \( j \to \infty \))

\[
  Y_1 := (S(c^j, D(v)), D(v - v^j))_Q \\
  = (S(c^j, D(v)), D(u^j))_Q + (S(c^j, D(v)), D((v - v^j) \frac{|v - v^j|}{L}))_Q \\
  \leq o(1) + (S(c^j, D(v)), D((v - v^j) \frac{|v - v^j|}{L}))_Q |_{E_j} \\
  \leq o(1) + CL + C\varepsilon^* \leq o(1) + C\varepsilon^*,
\]

(79)
where \( o(1) \to 0 \) as \( j \to \infty \). To estimate \( Y_2 \) we set \( \varphi = u_{\text{div}}^j \) in (58) and denoting \( S^j := S(c^j, D(v^j)) \) obtain

\[
Y_2 := (S^j, D(v^j - v))_{Q^j} = (S^j, D(u^j))_{Q^j} + (S^j, D((v - v^j)|_{\frac{v - v^j}{L}})_{Q^j} = (S^j, D(u^j)_{Q^j}) + (S^j, D((v - v^j)_{Q^j})_{Q^j} + (S^j, D((v - v^j)|_{\frac{v - v^j}{L}})_{Q^j} \tag{80}
\]

where

\[
I_1 = -\langle v_t, u_{\text{div}}^j \rangle = -\langle v_t, u_{\text{div}}^j \rangle + \langle v_t - v^j, u_{\text{div}}^j \rangle \leq o(1) + \langle v_t - v^j, u^j \rangle \leq o(1),

I_2 = -\left( \langle |\nabla v^j| v_{\eta(j)}, u_{\text{div}}^j \rangle \right)_{Q^j} \leq C\|u^j\|_{\frac{2}{r-\epsilon} Q} \|v^j\|_{L_\infty Q} \|\nabla v^j\|_{r, Q} \tag{73} \equiv o(1),

I_3 = -\alpha(v^j, u_{\text{div}}^j) \leq C\|u^j\|_{L^2(\Omega)} \tag{74} \equiv o(1),

I_4 = \langle f, u_{\text{div}}^j \rangle = o(1).

Thus, we can conclude that for \( \theta < 1 \)

\[
\int_Q |S^j - S(c^j, D(v)) \cdot D(v^j - v)|^\theta = \int_{Q_j} |\cdots|^\theta + \int_{Q \setminus Q_j} |\cdots|^\theta \leq C (Y_1 + Y_2)^\theta + C|Q| \tag{58} \equiv o(1)
\]

and using strict monotonocity of \( S \), i.e., the assumption (10) we can derive that \( D(v^j) \) converges to \( D(v) \) a.e. in \( Q \). Thus with help of Vitali’s theorem, it is standard to take limit in (58) to obtain (23). The proof is complete.

### 4 Extensions of Theorem 2.1

This section is devoted to the formulation of the extensions that are not proved in detail anywhere but we think they are direct consequences of the approaches described here and those presented in recent studies Wolf [2007] and Diening et al. [2008]. These extensions are focused on including the general domains with no smoothness of the boundary, incorporating the no-slip boundary conditions and lowering the power-law index \( r \). We wish to underline that to our best knowledge we can deal with globally integrable pressure for the low exponents of \( r \) only for domains of the class \( \mathcal{C} \) and for Navier’s slip boundary conditions - the case that is studied in previous sections.

We first formulate the result for the same range of \( r \) as in Theorem 2.1 and for the no-slip boundary conditions (corresponding to \( \alpha = \infty \) in (6)1) which

\footnote{For details concerning the estimate of \( I_1 \) see Bulíček et al. [2007].}
also enables us to consider any bounded domain. The pressure is omitted from the weak formulation. As the pressure is not in general an integrable function it is also not anymore clear how to fix its mean value over any spatial subdomain.

**Theorem 4.1.** Let $\Omega$ be a bounded domain and (15)–(16) hold (with the exception of the condition fixing the pressure). Assume that $\mathbf{S}$ satisfies (8)-(9) with $r > \frac{8}{5}$, and $q_c$ satisfies (10) with $-r < 2\beta < r$. Let $m$, $q$ and $s$ be defined as in (17). Then there is a weak solution $(\mathbf{v}, c)$ to (4)-(7) such that

\[
\mathbf{v} \in C([0, T]; L^2_w(\Omega)^3) \cap L^r(0, T; W^{1,r}_{0,\text{div}}), \quad \mathbf{v}_t \in L^m(0, T; W^{-1,m}_{0,\text{div}}),
\]
\[
c - c_b \in L^q(0, T; W^{1,q}_{1,\text{div}}(\Omega)), \quad c_t \in L^s(0, T; W^{-1,s'}_{1,\text{div}}(\Omega)),
\]
\[
0 \leq c \leq 1 \text{ a.e. in } Q,
\]
\[
(1 + |\mathbf{D}(\mathbf{v})|^2)^{\frac{\beta}{2}} \nabla c \in L^2(0, T; L^2(\Omega)^3),
\]

and $(\mathbf{v}, c)$ fulfil the following weak formulations

\[
\langle \mathbf{v}_t, \varphi \rangle - \langle \mathbf{v} \otimes \mathbf{v}, \nabla \varphi \rangle_Q - \alpha(\mathbf{v}, \varphi)_T + \langle \mathbf{S}(c, \mathbf{D}(\mathbf{v})), \mathbf{D}(\varphi) \rangle_Q = \langle \mathbf{f}, \varphi \rangle \quad \text{ for all } \varphi \in L^m'(0, T; W^{1,m'}_{0,\text{div}}),
\]
\[
(c_t, \varphi)_Q - \langle \mathbf{c} \mathbf{v}, \nabla \varphi \rangle_Q + \langle \mathbf{K}(c, |\mathbf{D}(\mathbf{v})|^2) \nabla c, \nabla \varphi \rangle_Q = 0 \quad \text{ for all } \varphi \in L^s(0, T; W^{1,s}_{1,\text{div}}(\Omega)),
\]

and $(\mathbf{v}, c)$ attain the initial conditions as in (25).

The next theorem extends the result stated in Theorem 2.1 also to the case $r \in (6/5, 8/5]$.

**Theorem 4.2.** Let $\frac{6}{5} < r \leq \frac{8}{5}$. Then for any data fulfilling (15)–(16) and for any $T \in (0, \infty)$ there exists a weak solution to (4)-(7) in the sense of Definition 2.1.

The extension of the existence result in the sense of Theorem 4.1 to no-slip boundary condition for $r \in (6/5, 8/5]$ can be also proved. We skip the formulation.

The final remark concern the possible extension of the result established here and formulated in the above theorem to model describing unsteady flows of incompressible chemically reacting and heat conducting fluids. Such a problem requires to consider the full thermodynamical system including the balance of energy and possibly also the second law of thermodynamics in combination with the convection-diffusion equation for $c$. We refer an interested reader to combine the results established here with those proved in Bulíček et al. [2008] and Bulíček et al. [2007] where the incompressible Navier-Stokes-Fourier-like systems are analyzed.
5 Appendix

The following lemmas summarize helpful inequalities related to functions belonging to the Sobolev spaces.

Lemma 5.1. Let $1 < q_1, q_2 < \infty$. Set

$$S := \{ v; v \in L^\infty(0,T; L^2(\Omega)^3) \cap L^r(0,T; W^{1,r}_n), v, t \in L^{q_1}(0,T; W^{-1,q_2}_n, \text{div}) \}.$$

If $r > \frac{3}{2}$ and $\{ v^i \}_{i=1}^\infty$ is bounded in $S$, then $\{ \text{tr} v^i \}_{i=1}^\infty$ is precompact in $L^p(0,T; L^s(\partial\Omega)^3)$ for all $p, s \in (1, \infty)$ satisfying

$$p < \left( \frac{5r - 6}{3s - 4} \right), \quad \max(2, r) \leq \frac{2r}{3}. \quad (81)$$

Proof. See Bulíček et al. [2007, Lemma 1.4] where an even more general result is proved.

Corollary 5.1. Let $r > \frac{8}{5}$. Let $\{ v^i \}_{i=1}^\infty$ be bounded in $S$. Then $\{ \text{tr} v^i \}_{i=1}^\infty$ is precompact in $L^2(0,T; L^2(\partial\Omega)^3)$.

Lemma 5.2. (Korn’s inequality) Let $q \in (1, \infty)$. Then there exists a positive constant $C$ depending only on $\Omega$ and $q$ such that for all $v \in W^{1,q}(\Omega)^3$ which has the trace $\text{tr} v \in L^2(\partial\Omega)^3$ and the following inequality holds

$$C \| v \|_{1,q} \leq \| D(v) \|_q + \| v \|_{L^2(\partial\Omega)}. \quad (82)$$

Proof. For the proof we refer the reader to a modification of the result by Nečas [1966] in Bulíček et al. [2007].

Lemma 5.3. (Interpolation inequalities) For $2 \leq q \leq \frac{3r}{r-2}$ (if $6/5 < r < 3$) and for any $q \in [1, \infty)$ if $r \geq 3$ the following inequality hold

$$\| z \|_q \leq \| z \|_2^{\frac{6r - 6q + 2qr}{4r - 6q + 2qr}} \| z \|_{1,r}^{\frac{3r(q - 2)}{4r - 6q + 2qr}}. \quad (83)$$

Proof. See Nirenberg [1966].

Lemma 5.4. (Aubin-Lions) Let $V_1, V_2, V_3$ be Banach reflexive separable spaces such that

$$V_1 \hookrightarrow V_2 \hookrightarrow V_3.$$

Let $1 < p < \infty$, $1 \leq q \leq +\infty$ and $0 < T < \infty$. Then

$$\{ v; v \in L^p(0,T; V_1), v, t \in L^q(0,T; V_3) \}$$

is compactly embedded into $L^p(0,T; V_2)$.

Proof. See for example Simon [1987] or Feireisl [2004, Lemma 6.3].
References


