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BOUNDARY TRACE INEQUALITIES AND REARRANGEMENTS

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ABSTRACT. A new approach to boundary trace inequalities for Sobolev functions is presented, which reduces any trace inequality involving general rearrangement-invariant norms to an equivalent, considerably simpler, one-dimensional inequality for a Hardy-type operator. In particular, improvements of classical boundary trace embeddings and new optimal trace embeddings are derived.

1. Introduction and main results

Let \( \Omega \) be an open bounded subset of \( \mathbb{R}^n \), \( n \geq 2 \), having a Lipschitz boundary \( \partial \Omega \). A basic result in the theory of Sobolev spaces asserts that a linear bounded operator
\[
\text{Tr} : W^{1,1}(\Omega) \to L^1(\partial\Omega),
\]
the trace operator, exists such that
\[
\text{Tr} u = u|_{\partial\Omega}
\]
whenever \( u \) is a continuous function on \( \Omega \). Here, \( L^1(\partial\Omega) \) denotes the Lebesgue space of summable functions on \( \partial\Omega \) with respect to the \((n-1)\)-dimensional Hausdorff measure \( H^{n-1} \).

The theory of boundary traces in Sobolev spaces has a number of applications, especially to boundary-value problems for partial differential equations, and has been developed, via different methods and in different settings, by various authors, including Besov [Be], Gagliardo [Ga], Lions and Magenes [LM]. Extensions of this theory to the case where the restriction of \( H^{n-1} \) to \( \partial\Omega \) is replaced by more general measures in \( \Omega \) have also been extensively studied in a vast literature. We limit ourselves to quoting the survey papers [Ma2, Ve] and the monographs [AH, Ma1, OK, Zi] as references on this topic.

Embedding theorems for the trace operator \( \text{Tr} \), in the spirit of the standard Sobolev embeddings in \( \Omega \), are well known. Loosely speaking, they tell us that if \( u \) enjoys either stronger summability properties of derivatives, or higher-order differentiability properties (or both), then \( \text{Tr} u \) turns out to have a higher degree of summability.

To be more specific, given any multindex \( \alpha = (\alpha_1, \ldots, \alpha_k) \), set \( |\alpha| = \alpha_1 + \cdots + \alpha_k \), and, given any positive integer \( m \), denote by \( N = N(m, n, \alpha) \) the number of multiindices \( \alpha \) such that \( |\alpha| \leq m \). For any real-valued \( m \)-times weakly differentiable function \( u \) in \( \Omega \), we denote by \( D^m u \) the \( N \)-vector of all its partial derivatives \( \frac{\partial^m u}{\partial x^\alpha} \) with \( 0 \leq |\alpha| \leq m \), and by \( |D^m u| \) its Euclidean norm. Then the Sobolev space \( W^{m,p}(\Omega) \) is defined, for \( p \in [1, \infty] \), as
\[
W^{m,p}(\Omega) = \{ u : u \text{ is a real-valued } m \text{-times weakly differentiable function in } \Omega \text{ such that } |||D^m u|||_{L^p(\Omega)} < \infty \}.
\]
Now, assume that $1 \leq m < n$. The standard trace embedding theorem (cf. e.g. [Ad]) tells us that if $1 \leq p < \frac{n}{m}$, then a constant $C = C(\Omega, p, m)$ exists such that
\begin{equation}
\| \text{Tr} u \|_{L \frac{p(n-1)}{n-m} (\partial \Omega)} \leq C \| u \|_{W^{m,p}(\Omega)}
\end{equation}
for every $u \in W^{m,p}(\Omega)$, whereas, if $p > \frac{n}{m}$, then a constant $C = C(\Omega, p, m)$ exists such that
\begin{equation}
\| \text{Tr} u \|_{L^\infty (\partial \Omega)} \leq C \| u \|_{W^{m,p}(\Omega)}
\end{equation}
for every $u \in W^{m,p}(\Omega)$.

A classical approach to trace inequalities such as (1.3) and (1.4) makes use of local coordinates on $\partial \Omega$ and of Sobolev inequalities in $\Omega$, and heavily relies upon the fact that Lebesgue norms of $\text{Tr} u$ and of $D^m u$ are involved. On the other hand, such a method does not seem to allow extensions to trace inequalities when more general norms are in play. Alternative techniques can be used to deal with specific instances. For example, a representation formula for Sobolev functions in terms of Riesz potentials enables one to effectively treat the limiting case when $p = \frac{n}{m}$, and to prove that a constant $C = C(\Omega, m)$ exists such that
\begin{equation}
\| \text{Tr} u \|_{\exp L \frac{n}{n-m} (\partial \Omega)} \leq C \| u \|_{W^{m,n/m}(\Omega)}
\end{equation}
for every $u \in W^{m,\frac{n}{m}}(\Omega)$ (see e.g. [AH, 7.6.4]). Here, $\exp L \frac{n}{n-m} (\partial \Omega)$ is the Orlicz space associated with the Young function $\exp(t \frac{n}{n-m}) - 1$. The shortcoming of such an approach is that function spaces defined in terms of potentials are not always equivalent to corresponding spaces defined by derivatives. In fact, none of the available methods seems to cover the whole range of situations of interest in applications.

Thus, the problem can be raised of a unified flexible approach to trace inequalities fitting a fairly general class of function spaces. A powerful tool in the study of (first-order) Sobolev type inequalities in the whole domain $\Omega$ for functions vanishing on $\partial \Omega$ is symmetrization. The strength of this elegant technique, which relies upon the Pólya–Szegő principle on the decrease of gradient norms under radially non-increasing rearrangement ([BZ, Ka, Ta]) and has led to such results as those of Aubin [Au] and Talenti [Ta], and of Moser [Mo], is in that the relevant Sobolev inequalities are reduced to one-dimensional inequalities.

The purpose of the present paper is to prove that a similar reduction can be carried over to trace inequalities (of arbitrary order) as well. Unfortunately, although suitable (weaker) symmetrization results can be proved and applied to derive Sobolev inequalities for functions which need not vanish on $\partial \Omega$ ([Ci1, Ci2, RT]), or, as recently pointed out in [Ci3], even for second-order Sobolev inequalities, they do not seem adaptable to handle trace inequalities. This notwithstanding, we show that a general principle still holds, ensuring that any trace inequality involving general rearrangement-invariant (r.i. for short) norms is equivalent to a corresponding one-dimensional inequality. This can be proved by combining rearrangement methods with interpolation techniques, via an argument in the spirit of the one recently used in [KP] to characterize Sobolev inequalities of arbitrary order in r.i. spaces on open sets $\Omega$.

In order to state our results, let us recall a few facts from the theory of function spaces. An r.i. space $X(\mathcal{R})$ on a positive measure space $(\mathcal{R}, \nu)$ is, roughly speaking, a Banach function space in which the norm of any function depends only on its degree of summability, i.e. on the measure of its level sets. A precise definition of r.i. spaces, as well as other definitions and properties concerning function spaces to be used throughout the paper can be found in Section 2. Customary examples of r.i. spaces are Lebesgue spaces, Orlicz spaces, Lorentz spaces and their generalizations, including Lorentz-Zygmund spaces.

A Sobolev-type space can be associated with any r.i. space $X(\Omega)$ defined on an open subset $\Omega$ of $\mathbb{R}^n$ endowed with the Lebesgue measure. For any integer $m \geq 1$, we define the generalized
Sobolev space built upon \( X(\Omega) \) as
\[
W^m X(\Omega) = \{ u : u is a real-valued \( m \)-times weakly differentiable function in \( \Omega \) such that \( \| D^m u \|_{X(\Omega)} < \infty \}. \]

Our main result asserts that any trace inequality between the norms \( \| f \|_{Y(\partial \Omega)} \) and \( \| f \|_{W^m X(\Omega)} \) is equivalent to a one-dimensional inequality for a suitable Hardy-type operator, between the norms \( \| f \|_{Y(\partial \Omega)} \) and \( \| f \|_{\bar{X}(0,1)} \). Here, \( X(\Omega) \) is an r.i. space on \( \Omega \) equipped with the Lebesgue measure \( | \cdot | \), and \( Y(\partial \Omega) \) is an r.i. space on \( \partial \Omega \) equipped with the \((n-1)\)-dimensional Hausdorff measure. Moreover, \( \| f \|_{Y(\partial \Omega)} \) and \( \| f \|_{\bar{X}(0,1)} \) denote the norms in the representation spaces \( Y(0,1) \) and \( \bar{X}(0,1) \) of \( Y(\partial \Omega) \) and \( X(\Omega) \) on the unit interval \((0,1)\).

**Theorem 1.1.** Let \( \Omega \) be an open bounded subset of \( \mathbb{R}^n \), \( n \in \mathbb{N} \), \( n \geq 2 \), having a Lipschitz boundary. Let \( X(\Omega) \) and \( Y(\partial \Omega) \) be r.i. spaces. Let \( m \in \mathbb{N} \), \( 1 \leq m < n \). Assume that a constant \( C_1 \) exists such that
\[
(1.6) \quad \left\| \int_{t_0}^{t} f(s) s^{m-1} ds \right\|_{Y(0,1)} \leq C_1 \| f \|_{\bar{X}(0,1)}
\]
for every non-increasing function \( f \in \bar{X}(0,1) \). Then there exists a constant \( C_2 = C_2(C_1, \Omega, m) \) such that
\[
(1.7) \quad \| \text{Tr} u \|_{Y(\partial \Omega)} \leq C_2 \| u \|_{W^m X(\Omega)}
\]
for every \( u \in W^m X(\Omega) \).

Conversely, if \( (1.7) \) holds for every \( u \in W^m X(\Omega) \), then there exists a constant \( C_1 = C_1(C_2, \Omega, m) \) such that \( (1.6) \) holds for every \( f \in \bar{X}(0,1) \).

Let us notice the analogy of Theorem 1.1 with a parallel result concerning Sobolev inequalities in \( \Omega \), to which we alluded above, asserting that the inequality \( \| u \|_{Y(\Omega)} \leq C \| u \|_{W^m X(\Omega)} \) is equivalent to an inequality like \( (1.6) \), where \( t^m \) is just replaced by \( t \) as a lower limit of integration in the Hardy-type operator.

Theorem 1.1 can be applied to the proof of optimal trace inequalities in a number of situations where the Hardy-type inequality \( (1.6) \) is known. For instance, since \( W^m X(\Omega) = W^{m,p}(\Omega) \) if \( X(\Omega) = L^p(\Omega) \) for some \( p \in [1, \infty] \), inequalities \((1.3)-(1.5)\) are easily recovered via standard weighted Hardy inequalities ([Ma1, OK]).

A general consequence of Theorem 1.1 is contained in Theorem 1.2 below, which provides a characterization of the optimal r.i. space \( Y(\partial \Omega) \) in \( (1.7) \) once the r.i. space \( X(\Omega) \) is given. By an optimal r.i. space \( Y(\partial \Omega) \) we mean that \( (1.7) \) holds, and that if \( Z(\partial \Omega) \) is another r.i. space such that \( (1.7) \) holds with \( Y(\partial \Omega) \) replaced by \( Z(\partial \Omega) \), then
\[
Y(\partial \Omega) \rightarrow Z(\partial \Omega).
\]
Here, \( \rightarrow \) stands for continuous embedding.

In the statement of Theorem 1.2, \( X'(\Omega) \) and \( Y'(\partial \Omega) \) denote the associate spaces of \( X(\Omega) \) and \( Y(\partial \Omega) \) which, in a sense, play a role of duals in the framework of r.i. spaces. Moreover, given any finite positive measure space \((R, \nu)\) and any \( \nu \)-measurable function \( w \) in \( R \), we denote by \( w^{**} \) the Hardy-Littlewood maximal function associated with the decreasing rearrangement \( w^* \) of \( w \) with respect to \( \nu \); namely,
\[
w^{**}(s) = \frac{1}{s} \int_{0}^{s} w^*(y) dy \quad \text{for } s > 0.
\]

**Theorem 1.2.** Let \( \Omega \), \( n \) and \( m \) be as in Theorem 1.1, and let \( X(\Omega) \) be an r.i. space. Then, the r.i. space \( Y(\partial \Omega) \), whose associate norm is given by
\[
(1.8) \quad \| f \|_{Y'(\partial \Omega)} = \left\| \frac{1}{t^m} W^{**}(t^{\frac{1}{m}}) \right\|_{\bar{X}(0,1)}
\]
for every $\mathcal{H}^{n-1}$-measurable function $w$ on $\partial \Omega$, is optimal in (1.7).

Theorem 1.2 can be used to derive optimal trace embeddings whenever the associate norms appearing in (1.8) can be explicitly computed. For instance, in the next theorem, we present an application of Theorem 1.2 to the proof of trace inequalities in Lorentz spaces, which, in particular, improve (1.3) and (1.4). This can be regarded as a counterpart of results dealing with Sobolev embeddings in $\Omega$ which go back to [O’N, Pe] in the standard case, and to [BW, Ha, Ma1] in the limiting case. In what follows, Lorentz spaces on a measure space $(\mathcal{L}, \mu)$ are denoted by $L^{p;q}(\mathcal{L})$; the notation $L^{p;q}(\log L)^m(\mathcal{L})$ is adopted for the more general Lorentz–Zygmund spaces (see Section 2 for definitions).

**Theorem 1.3.** Let $\Omega$, $n$ and $m$ be as in Theorem 1.1.

(i) If $1 \leq p < \frac{n}{m}$ and $1 \leq q \leq \infty$, then a constant $C = C(\Omega, p, q, m)$ exists such that

$$
||\text{Tr} u||_{L^{\frac{n(n-1)}{nmp}}(\partial \Omega)} \leq C||u||_{W^{m,p,q}(\Omega)}
$$

for every $u \in W^m L^{p,q}(\Omega)$.

(ii) If $p = \frac{n}{m}$ and $1 < q \leq \infty$, then a constant $C = C(\Omega, q, m)$ exists such that

$$
||\text{Tr} u||_{L^{\infty,q}(\log L)^{-1}(\partial \Omega)} \leq C||u||_{W^{m,p,q}(\Omega)}
$$

for every $u \in W^m L^{\frac{n}{p}}(\Omega)$.

(iii) If either $p = \frac{n}{m}$ and $q = 1$ or $p > \frac{n}{m}$ and $1 \leq q \leq \infty$, then a constant $C = C(\Omega, p, q, m)$ exists such that

$$
||\text{Tr} u||_{L^{\infty}(\partial \Omega)} \leq C||u||_{W^{m,p,q}(\Omega)}
$$

for every $u \in W^m L^{p,q}(\Omega)$.

The spaces $L^{\frac{n(n-1)}{nmp}}(\partial \Omega)$, $L^{\infty,q}(\log L)^{-1}(\partial \Omega)$ and $L^{\infty}(\partial \Omega)$ are optimal among all r.i. spaces on $\partial \Omega$ in (1.9), (1.10) and (1.11), respectively.

Notice that inequalities (1.3) and (1.4) are not only generalized, but also improved by (1.9) and (1.10), respectively. Indeed, $L^{p,p}(\Omega) = L^{p}(\Omega)$, whence $W^m L^{p,p}(\Omega) = W^m,p(\Omega)$, and, on the other hand, $L^{\frac{n(n-1)}{nmp}}(\partial \Omega) \subsetneq L^{\frac{n}{mp}}(\partial \Omega)$ and $L^{\infty,\frac{n}{m}}(\log L)^{-1}(\partial \Omega) \subsetneq \exp L^{\frac{n}{mp}}(\partial \Omega)$.

2. BACKGROUND AND PRELIMINARIES

Let $(\mathcal{R}, \nu)$ be a finite positive measure space and let $\mathcal{M}(\mathcal{R})$ be the set of all real-valued $\nu$-measurable functions on $\mathcal{R}$. Given any function $u \in \mathcal{M}(\mathcal{R})$, its decreasing rearrangement $u^*$ is defined as

$$
u^*(s) = \sup \{t \geq 0 : \nu(\{x \in \mathcal{R} : |u(x)| > t\}) > s\}
$$

for $s > 0$.

A basic property of rearrangements is the Hardy-Littlewood inequality, which tells us that, if $u, v \in \mathcal{M}(\mathcal{R})$, then

$$
\int_\mathcal{R} |u(x)v(x)|d\nu(x) \leq \int_0^\infty u^*(s)v^*(s)ds.
$$

A Banach space $X(\mathcal{R})$ of functions in $\mathcal{M}(\mathcal{R})$, equipped with the norm $\| \cdot \|_{X(\mathcal{R})}$, is said to be a rearrangement-invariant space if the following five axioms hold:

(P1) $0 \leq v \leq u$ $\nu$-a.e. implies $\|v\|_{X(\mathcal{R})} \leq \|u\|_{X(\mathcal{R})}$;

(P2) $0 \leq u_n / u$ $\nu$-a.e. implies $\|u_n\|_{X(\mathcal{R})} / \|u\|_{X(\mathcal{R})}$;

(P3) $\|1\|_{X(\mathcal{R})} < \infty$;

(P4) a constant $C = C(\nu, \mathcal{R})$ exists such that $\int_\mathcal{R} |u(x)|d\nu(x) \leq C\|u\|_{X(\mathcal{R})}$ for every $u \in X(\mathcal{R})$;

(P5) $\|u\|_{X(\mathcal{R})} = \|v\|_{X(\mathcal{R})}$ whenever $u^* = v^*$.​
Given an r.i. space \( X(\mathcal{R}) \) on \((\mathcal{R}, \nu)\), the set

\[
X'(\mathcal{R}) = \left\{ u \in \mathcal{M}(\mathcal{R}) : \int_{\mathcal{R}} |u(x)v(x)| \, d\nu(x) < \infty \text{ for every } v \in X(\mathcal{R}) \right\},
\]
equipped with the norm

\[
\|u\|_{X'(\mathcal{R})} = \sup_{\|v\|_{X(\mathcal{R})} \leq 1} \int_{\mathcal{R}} |u(x)v(x)| \, d\nu(x),
\]
is called the associate space of \( X(\mathcal{R}) \). It turns out that \( X'(\mathcal{R}) \) is again an r.i. space and that \( X''(\mathcal{R}) = X(\mathcal{R}) \). Furthermore, the Hölder inequality

\[
\int_{\mathcal{R}} |u(x)v(x)| \, d\nu(x) \leq \|u\|_{X(\mathcal{R})} \|v\|_{X'(\mathcal{R})}
\]
holds for every \( u \) and \( v \) in \( \mathcal{M}(\mathcal{R}) \). Note that for any r.i. spaces \( X(\mathcal{R}) \) and \( Y(\mathcal{R}) \), the embeddings \( X(\mathcal{R}) \to Y(\mathcal{R}) \) and \( Y'(\mathcal{R}) \to X'(\mathcal{R}) \) are equivalent.

For every r.i. space \( X(\mathcal{R}) \) on \((\mathcal{R}, \nu)\), there exists a unique r.i. space \( \overline{X}(0, 1) \) on \((0, 1)\), endowed with the one-dimensional Lebesgue measure, satisfying

\[
\|u\|_{X(\mathcal{R})} = \|u^*(\nu(\mathcal{R})s)\|_{\overline{X}(0, 1)}
\]
for every \( u \in X(\mathcal{R}) \). Such a space, equipped with the norm

\[
\|f\|_{\overline{X}(0, 1)} = \sup_{\|u\|_{X(\mathcal{R})} \leq 1} \int_{0}^{1} f^*(s)u^*(\nu(\mathcal{R})s) \, ds,
\]
is called the representation space of \( X(\mathcal{R}) \).

Given any \( s > 0 \), the dilation operator \( E_s \), defined at \( f \in \mathcal{M}(0, 1) \) by

\[
(E_s f)(t) = \begin{cases} 
  f(t/s) & \text{if } 0 < t \leq s \\
  0 & \text{if } s < t < 1,
\end{cases}
\]
is bounded on any r.i. space \( X(0, 1) \), with norm not exceeding \( \max\{1, 1/s\} \).

Hardy’s Lemma tells us that if \( f_1 \) and \( f_2 \) are nonnegative measurable functions in \((0, \infty)\) such that

\[
\int_{0}^{s} f_1(r) \, dr \leq \int_{0}^{s} f_2(r) \, dr \quad \text{for } s > 0,
\]
then

\[
\int_{0}^{\infty} f_1(r)g(r) \, dr \leq \int_{0}^{\infty} f_2(r)g(r) \, dr
\]
for every non-increasing function \( g : (0, \infty) \to [0, \infty) \). A consequence of this result is that if the functions \( u, v \in \mathcal{M}(\mathcal{R}) \) satisfy

\[
\int_{0}^{s} u^*(r) \, dr \leq \int_{0}^{s} v^*(r) \, dr \quad \text{for } s > 0,
\]
then

\[
\|u\|_{X(\mathcal{R})} \leq \|v\|_{X(\mathcal{R})}
\]
for every r.i. space \( X(\mathcal{R}) \).

Basic examples of r.i. spaces are the Lebesgue spaces, defined for \( 1 \leq p \leq \infty \) by

\[
L^p(\mathcal{R}) = \left\{ u \in \mathcal{M}(\mathcal{R}) : \|u\|_{L^p(\mathcal{R})} < \infty \right\},
\]
where

\[
\|u\|_{L^p(\mathcal{R})} = \begin{cases} 
  \left( \int_{\mathcal{R}} |u(x)|^p \, d\nu(x) \right)^{1/p} < \infty, & 1 \leq p < \infty, \\
  \text{ess sup}_{x \in \mathcal{R}} |u(x)| < \infty, & p = \infty.
\end{cases}
\]
We shall also work with the Lorentz spaces, defined either for \( p = q = 1 \) or \( p = q = \infty \) or \( 1 < p < \infty \) and \( 1 \leq q \leq \infty \) as
\[
L^{p,q}(\mathcal{R}) = \{ u \in \mathcal{M}(\mathcal{R}) : \| u \|_{L^{p,q}(\mathcal{R})} < \infty \},
\]
where
\[
\| u \|_{L^{p,q}(\mathcal{R})} = \| u^*(t) t^{\frac{p}{p-1}} \|_{L^q(0,\infty)},
\]
and, more generally, with the Lorentz–Zygmund spaces, defined for \( 1 < p < \infty \), \( 1 \leq q \leq \infty \) and \( \alpha \in \mathbb{R} \) as
\[
L^{p,q}(\log L)^\alpha(\mathcal{R}) = \{ u \in \mathcal{M}(\mathcal{R}) : \| u \|_{L^{p,q}(\log L)^\alpha(\mathcal{R})} < \infty \},
\]
where
\[
\| u \|_{L^{p,q}(\log L)^\alpha(\mathcal{R})} = \| u^*(t) t^{\frac{1}{p} - \frac{1}{q}} (1 + |\log t|)^\alpha \|_{L^q(0,\infty)}.
\]

Let us notice that, in spite of the notation, the quantities \( \cdot \| L^{p,q}(\mathcal{R}) \) and \( \cdot \| L^{p,q}(\log L)^\alpha(\mathcal{R}) \) need not be norms; however, they can be turned into equivalent norms, when \( 1 < p < \infty \), on replacing \( u^* \) by \( u^{**} \) on the right-hand sides of (2.7) and (2.8).

If \( 1 < p < \infty \) and \( 1 \leq q \leq \infty \), then
\[
(L^{p,q}(\mathcal{R}))' = L^{p',q'}(\mathcal{R}),
\]
up to equivalent norms.

Another important class of r.i. spaces is provided by the Orlicz spaces. Given a Young function \( A \), namely, a convex function from \([0, \infty)\) into \([0, \infty)\) such that \( A(0) = 0 \), the Orlicz space \( L^A(\mathcal{R}) \) is defined by
\[
L^A(\mathcal{R}) = \{ u \in \mathcal{M}(\mathcal{R}) : \| u \|_{L^A(\mathcal{R})} < \infty \},
\]
where
\[
\| u \|_{L^A(\mathcal{R})} = \inf \left\{ \lambda > 0 : \int_{\mathcal{R}} A\left( \frac{|u(x)|}{\lambda} \right) \, dv(x) \leq 1 \right\}.
\]

Moreover, \( (L^A(\mathcal{R}))' = \tilde{L}^A(\mathcal{R}) \) (with equivalent norms), where \( \tilde{A} \) is the Young conjugate of \( A \), defined as \( \tilde{A}(t) = \sup\{ st - A(s) : s > 0 \} \) for \( t \geq 0 \).

For a comprehensive treatment of r.i. spaces we refer the reader to [BS].

Let us next record some special results from the theory of interpolation which we shall need in what follows.

Consider a pair \((X_0, X_1)\) of Banach spaces which are compatible in the sense that they are continuously embedded into a common Hausdorff topological vector space \( H \). Their \( K\)-functional is defined for each \( u \) in the vector sum \( X_0 + X_1 \) by
\[
K(t, u; X_0, X_1) = \inf_{u = v + w} \left( \| v \|_{X_0} + t \| w \|_{X_1} \right) \quad \text{for } t > 0.
\]

The \( K\)-functional for pairs of Lorentz spaces \( L^{p,q}(\mathcal{R}) \) is given, up to equivalence, by the following result.

**Theorem 2.1.** (Holmstedt’s formulas, [Ho, Theorem 4.1]) Assume that either \( p_0 = q_0 = 1 \), or \( 1 < p_0 < p_1 < \infty \) and \( 1 \leq q_0, q_1 < \infty \). Let \( \frac{1}{\alpha} = \frac{1}{p_0} - \frac{1}{p_1} \). Then,
\[
K(t, u; L^{p_0,q_0}(\mathcal{R}), L^{p_1,q_1}(\mathcal{R})) \approx \left( \int_0^t s^{\frac{1}{p_0} - \frac{1}{q_0}} u^*(s) \, ds \right)^{\frac{q_0}{q_1}} + t \left( \int_t^\infty s^{\frac{1}{p_1} - \frac{1}{q_1}} u^*(s) \, ds \right)^{\frac{q_1}{q_1}}
\]
for \( t > 0 \). Furthermore, if either \( p_0 = q_0 = 1 \), or \( 1 < p_0 < \infty \) and \( 1 \leq q_0 < \infty \), then
\[
K(t, u; L^{p_0,q_0}(\mathcal{R}), L^\infty(\mathcal{R})) \approx \left( \int_0^t s^{\frac{1}{p_0} - \frac{1}{q_0}} u^*(s) \, ds \right)^{\frac{q_0}{q_0}} \quad \text{for } t > 0.
\]
Here, and in other similar estimates for $K$–functionals, we write $\approx$ to denote that the two sides are bounded by each other up to constants independent of $u$ and $t$.

As far as the $K$-functional for pairs of Sobolev spaces is concerned, let us recall that

\begin{equation}
K(t, u; W^{m, 1}(\Omega), W^{m, \infty}(\Omega)) \approx \int_0^t |D^m u|^*(s) \, ds \quad \text{for } t > 0 \tag{2.12}
\end{equation}

(see [DS]). Owing to (2.12), the reiteration theorem [BS, p. 311] and Holmstedt’s formulas, if either $p_0 = q_0 = 1$, or $1 < p_0 < p_1 < \infty$ and $1 \leq q_0, q_1 < \infty$, then

\[ K(t, u; W^m L^{p_0, q_0}(\Omega), W^m L^{p_1, q_1}(\Omega)) \approx K(t, |D^m u|; L^{p_0, q_0}(\Omega), L^{p_1, q_1}(\Omega)) \quad \text{for } t > 0. \]

Hence,

\begin{equation}
K(t, u; W^m L^{p_0, q_0}(\Omega), W^m L^{p_1, q_1}(\Omega)) \approx \left( \int_0^t \left[ \frac{1}{s^{\alpha_0}} - \frac{1}{s} \right] |D^m u|^*(s) \, ds \right)^{\frac{1}{\alpha_0}}
+ t \left( \int_0^\infty \left[ \frac{1}{s^{\alpha_1}} - \frac{1}{s} \right] |D^m u|^*(s) \, ds \right)^{\frac{1}{\alpha_1}} \quad \text{for } t > 0, \tag{2.13}
\end{equation}

where $\alpha$ is defined as in Theorem 2.1. Furthermore, if either $p_0 = q_0 = 1$, or $1 < p_0 < \infty$ and $1 \leq q_0 < \infty$, then

\[ K(t, u; W^m L^{p_0, q_0}(\Omega), W^m L^{\infty}(\Omega)) \approx \left( \int_0^t \left[ \frac{1}{s^{\alpha_0}} - \frac{1}{s} \right] |D^m u|^*(s) \, ds \right)^{\frac{1}{\alpha_0}} \quad \text{for } t > 0. \]

The $K$–functional stems its importance from the result contained in the next theorem, an easy consequence of its definition.

**Theorem 2.2.** Let $(X_0, X_1)$ and $(Y_0, Y_1)$ be two compatible pairs of Banach spaces. Let $T$ be a sublinear operator satisfying $T : X_0 \to Y_0$ and $T : X_1 \to Y_1$. Then there exists a positive constant $C$, depending only on the norms of $T$ between $X_0$ and $Y_0$ and between $X_1$ and $Y_1$, such that

\[ K(t, Tw; Y_0, Y_1) \leq CK(t, u; X_0, X_1), \quad \text{for every } u \in X_0 + X_1, \quad \text{and } t > 0. \]

Let us finally recall a classical interpolation theorem by Calderón.

**Theorem 2.3.** ([BS, Chapter 3, Theorem 2.12]) Let $(\mathcal{R}, \nu)$ be a positive measure space and let $T$ be a sublinear operator such that $T : L^1(\mathcal{R}) \to L^1(\mathcal{R})$ and $T : L^\infty(\mathcal{R}) \to L^\infty(\mathcal{R})$. Then

\[ T : X(\mathcal{R}) \to X(\mathcal{R}) \]

for every r.i space $X(\mathcal{R})$, with norm depending only on the norms of $T$ in $L^1(\mathcal{R})$ and in $L^\infty(\mathcal{R})$.

3. Proofs

We begin with a particular case of Theorem 1.1 corresponding to first-order inequalities, namely to the case when $m = 1$. The motivation for enucleating this intermediate step as a separate result is twofold. First, it is needed in the derivation of a first-order trace inequality in Lorentz spaces to be applied in the proof of Theorem 1.1. Second, its proof is quite transparent, since certain technicalities arising in the general case are missing, yet it already contains the main underlying ideas.

**Theorem 3.1.** Let $\Omega$, $X(\Omega)$ and $Y(\partial\Omega)$ be as in Theorem 1.1. If there exists a constant $C_1$ such that

\begin{equation}
\left\| \int_0^1 f(s) s^{-\frac{1}{m}} \, ds \right\|_{Y(0, 1)} \leq C_1 \|f\|_{X(0, 1)} \tag{3.1}
\end{equation}
for every nonnegative non-increasing function $f \in X(0,1)$, then a constant $C_2 = C_2(C_1, \Omega)$ exists such that

$$\|\text{Tr } u\|_{X(\partial \Omega)} \leq C_2\|u\|_{W^{1,X}(\Omega)}$$

for every $u \in W^{1,X}(\Omega)$.

Conversely, if (3.2) holds for every $u \in W^{1,X}(\Omega)$, then there exists a constant $C_1 = C_1(C_2, \Omega)$ such that (3.1) holds for every $f \in X(0,1)$.

Proof. Let us first show that (3.1) implies (3.2). By (1.1), a constant $C = C(\Omega)$ exists such that

$$\|\text{Tr } u\|_{L^1(\partial \Omega)} \leq C\|u\|_{W^{1,1}(\Omega)}$$

for every $u \in W^{1,1}(\Omega)$. On the other hand, the Sobolev embedding theorem in Lorentz spaces entails that

$$W^1L^{n,1}(\Omega) \rightarrow L^\infty(\Omega)$$

([O’N, Pe]). Thus, there exists a constant $C = C(\Omega)$ such that

$$\|\text{Tr } u\|_{L^\infty(\partial \Omega)} \leq C\|u\|_{W^{1,n,1}(\Omega)}.$$ 

Hence, Theorem 2.2 applied to the linear operator $\text{Tr}$ tells us that

$$K(t, \text{Tr } u; L^1(\partial \Omega), L^\infty(\partial \Omega)) \leq CK(t, u; W^{1,1}(\Omega), W^1L^{n,1}(\Omega)) \quad \text{for } t > 0.$$ 

We have, by (2.11),

$$K(t, \text{Tr } u; L^1(\partial \Omega), L^\infty(\partial \Omega)) \approx \int_0^t (\text{Tr } u)^*(s) \, ds \quad \text{for } t > 0$$

and, by (2.13)

$$K(t, u; W^{1,1}(\Omega), W^1L^{n,1}(\Omega)) \approx \int_0^{t_n} |Du|^*(s) \, ds + t \int_{t_n}^\infty |Du|^*(s)s^{-\frac{1}{n}} \, ds$$

$$= \frac{1}{n} \int_0^{t_n} s^{-\frac{1}{n}} \left( \int_s^\infty |Du|^*(r)r^{-\frac{1}{n}} \, dr \right) \, ds$$

$$= \int_0^t \left( \int_{\tau_n}^\infty |Du|^*(r)r^{-\frac{1}{n}} \, dr \right) \, d\tau \quad \text{for } t > 0.$$ 

Thus, a constant $C = C(\Omega)$ exists such that

$$\int_0^t (\text{Tr } u)^*(s) \, ds \leq C \int_0^{t_{n}} |Du|^*(s)s^{-\frac{1}{n}} \, ds \, d\tau$$

$$= C^2 \int_0^t \left( \int_{(C_2)s_n}^\infty |Du|^*(\varrho)\varrho^{-\frac{1}{n}} \, d\varrho \right) \, ds \quad \text{for } t > 0.$$ 

The following chain holds:
for every
the subadditivity of the operation
Proof. for some constant
is finite is an r.i. space on
of brevity. Let us just mention that the triangle inequality for the norm
(3.6)
\[ \| \text{Tr} u \|_{Y(\partial \Omega)} = \| (\text{Tr} u)^* (\mathcal{H}^{n-1}(\partial \Omega) t) \|_{Y(0,1)} \] (by (2.4))
\[ \leq C \left\| \int_0^\infty |Du|^*(s) s^{\frac{1}{m}} \, ds \right\|_{Y(0,1)} \] (by (3.3) and (2.6))
\[ = C' \left\| \int_0^\infty |Du|^*(|\Omega| s) s^{-\frac{1}{m}} \, ds \right\|_{Y(0,1)} \]
\[ \leq C'' \left\| \int_0^1 |Du|^*(|\Omega| s) s^{-\frac{1}{m}} \, ds \right\|_{Y(0,1)} \] (by the boundedness of the dilation operator)
\[ = C'' \| Du \|_{X(0,1)} \]
\[ = C''' \| u \|_{W^1 X(\Omega)} \]
for suitable constants \( C = C(\Omega), C' = C'(\Omega), C'' = C''(\Omega) \) and \( C''' = C'''(C_1, \Omega) \). Hence, (3.2) follows.

The fact that (3.2) implies (3.1) follows on specializing the general construction which will be given in the proof of Theorem 1.1. \( \square \)

In the proof of Theorems 1.1 and 1.2 we shall need the following auxiliary result in the spirit of [EKP, Theorem 4.5] and [KP, Theorem 3.2].

**Proposition 3.2.** Let \( X(0,1) \) be an r.i. space. Then the space \( X_\sigma(0,1) \) of all measurable functions \( f \) on \( (0,1) \) such that the quantity

\[ \| f \|_{X_\sigma(0,1)} = \left\| \frac{1}{t^{\frac{m}{2}}} \int_0^t f^*(s) \, ds \right\|_{X'(0,1)} \]

is finite is an r.i. space on \( (0,1) \), and \( \| \cdot \|_{X_\sigma(0,1)} \) is a norm. Its associate space \( X'_\sigma(0,1) \) satisfies

\[ \left\| \int_0^1 f(s) s^{\frac{m}{2}} \, ds \right\|_{X'_\sigma(0,1)} \leq \| f \|_{X(0,1)} \]

for every \( f \in X(0,1) \). Moreover, \( X'_\sigma(0,1) \) is the optimal r.i. space in (3.5), in the sense that if \( Z(0,1) \) is an r.i. space such that

\[ \left\| \int_0^1 f(s) s^{\frac{m}{2}} \, ds \right\|_{Z(0,1)} \leq C \| f \|_{X(0,1)} \]

for some constant \( C \) and for every \( f \in X(0,1) \), then

\[ \| g \|_{Z(0,1)} \leq C \| g \|_{X'_\sigma(0,1)} \]

for every \( g \in X'_\sigma(0,1) \).

**Proof.** The fact that \( X_\sigma(0,1) \) is an r.i. space is easily verified. We omit the details for the sake of brevity. Let us just mention that the triangle inequality for the norm \( \| \cdot \|_{X_\sigma(0,1)} \) follows from the subadditivity of the operation \( f(t) \mapsto t^{\frac{m}{2}} \int_0^t f^*(s) \, ds \).
Next, we establish (3.5). Fubini’s theorem ensures that for every pair of nonnegative functions \( f, g \in M(0,1) \) we have

\[
\int_0^1 g(t) \int_0^1 f(s) s^{\frac{n}{n-1}} ds \, dt = \int_0^1 t^{\frac{n}{n-1}} f(t) \int_0^t \frac{1}{s^{\frac{n}{n-1}}} g(s) \, ds \, dt.
\]

Thus, by the very definition of the associate norm, we have

\[
\sup_{\|f\|_{X'(0,1)} \leq 1} \left\| \int_0^1 f(s) s^{\frac{n}{n-1}} ds \right\|_{X'(0,1)} = \sup_{\|f\|_{X(0,1)} \leq 1} \sup_{\|g\|_{X'(0,1)} \leq 1} \left\| \int_0^1 g(t) \int_0^1 f(s) s^{\frac{n}{n-1}} ds \, dt \right\|
= \sup_{\|g\|_{X'(0,1)} \leq 1} \left\| g \right\|_{X'(0,1)} \int_0^1 t^{\frac{n}{n-1}} f(t) \int_0^t \frac{1}{s^{\frac{n}{n-1}}} g(s) \, ds \, dt
= \sup_{\|g\|_{X'(0,1)} \leq 1} \left\| t^{\frac{n}{n-1}} \int_0^t \frac{1}{s^{\frac{n}{n-1}}} g(s) \, ds \right\|_{X'(0,1)}.
\]

Therefore, (3.5) is equivalent to

\[
(3.8) \quad \left\| t^{\frac{n}{n-1}} \int_0^t \frac{1}{s^{\frac{n}{n-1}}} g(s) \, ds \right\|_{X'(0,1)} \leq \|g\|_{X'(0,1)}
\]

for every \( g \in X'(0,1) \). We claim that (3.8) is in turn equivalent to

\[
(3.9) \quad \left\| t^{\frac{n}{n-1}} \int_0^t \frac{1}{s^{\frac{n}{n-1}}} g^*(s) \, ds \right\|_{X'(0,1)} \leq \|g\|_{X'(0,1)}
\]

for every \( g \in X'(0,1) \). Indeed, while (3.9) obviously follows from (3.8) by restricting the inequality to non-increasing functions, the converse implication is a consequence of the fact that, owing to (2.1), \( \int_0^t \frac{1}{s^{\frac{n}{n-1}}} g(s) \, ds \leq \int_0^t \frac{1}{s^{\frac{n}{n-1}}} g^*(s) \, ds \) for \( t > 0 \). Inequality (3.9) is obviously satisfied owing to the definition of \( X'_\sigma \). This proves (3.5).

It remains to show the optimality of \( X'_\sigma(0,1) \). By the same duality argument as above, inequality (3.6) implies that

\[
\left\| t^{\frac{n}{n-1}} \int_0^t \frac{1}{s^{\frac{n}{n-1}}} g^*(s) \, ds \right\|_{X'(0,1)} \leq C \|g\|_{Z'(0,1)}
\]

for every \( g \in Z'(0,1) \). The left-hand side of the last inequality agrees with \( \|g\|_{X'(0,1)} \). Hence, (3.7) follows, owing to the definition of the associate norm.

We are now in a position to prove Theorem 1.1.

Proof of Theorem 1.1. We begin by showing that (1.6) implies (1.7). The Sobolev embedding theorem in Lorentz spaces tells us that

\[
W^m L^{\frac{n}{m-1}}(\Omega) \to L^\infty(\overline{\Omega}).
\]

Thus, a constant \( C = C(\Omega, m) \) exists such that

\[
(3.10) \quad \| \text{Tr} u \|_{L^\infty(\partial \Omega)} \leq C \|u\|_{W^m L^{\frac{n}{m-1}}(\Omega)}
\]
for every $u \in W^mL_\frac{p}{m-n+1}(\Omega)$. On the other hand, again the Sobolev embedding theorem in Lorentz spaces yields

\[(3.11) \quad W^{m,1}(\Omega) \to W^1L_\frac{p}{m-n+1}(\Omega).\]

Moreover, we claim that

\[(3.12) \quad \text{Tr} : W^1L_\frac{p}{m-n+1}(\Omega) \to L_\frac{p}{n-m+1}(\partial\Omega).\]

Indeed, when $m = 1$, embedding (3.12) is nothing but (1.1). Assume now that $m > 1$. Given any $p \in (1, n)$ and any nonnegative function $f \in L^{p,1}(0,1)$, we have that

\[
\left\| \int_{s_n}^1 f(r)r^{-\frac{1}{n}} \, dr \right\|_{L_\frac{p(n-1)}{m+n-p-1}(1,0)} = \int_0^1 \int_{s_n}^1 f(r)r^{-\frac{1}{n}} \, dr \, ds
\]

\[
= \int_0^1 f(r)r^{-\frac{1}{n}} \int_0^{\frac{1}{n}} s^{\frac{n-p}{m+n-p-1}} \, ds \, dr ds = \frac{p(n-1)}{n-p} \int_0^1 f(r)r^{\frac{1}{p}-1} \, dr
\]

\[
\leq \frac{p(n-1)}{n-p} \int_0^1 f(r)r^{\frac{1}{p}-1} \, dr = \frac{p(n-1)}{n-p} \|f\|_{L^{p,1}(1,0)}.
\]

On applying this chain with $p = \frac{n}{n-m+1}$ and making use of Theorem 3.1, we get (3.12).

Owing to (3.11) and (3.12), there exists a constant $C = C(\Omega, m)$ such that

\[(3.13) \quad \|\text{Tr} u\|_{L_\frac{p}{n-m+1}(0,1)} \leq C\|u\|_{W^{m,1}(\Omega)}.
\]

Thanks to Theorem 2.2, (3.10) and (3.13), there exists a constant $C = C(\Omega, m)$ such that

\[(3.14) \quad K(t, \text{Tr} u; L_\frac{p}{n-1}(\partial\Omega), L_\infty(\partial\Omega)) \leq CK \left( Ct, u; W^{m,1}(\Omega), W^mL_\frac{p}{m}(\Omega) \right) \quad \text{for} \ t > 0.
\]

By (2.11), we have

\[(3.15) \quad K \left( t, \text{Tr} u; L_\frac{p}{n-m}(\partial\Omega), L_\infty(\partial\Omega) \right) \approx \int_0^t \left[ \int_{s_n}^1 |D^m u|^s(s) \, ds \right] \left[ \int_0^s |D^m u|^s r^{\frac{m}{n}} \, ds \right] ds
\]

\[
\approx \int_0^t \frac{n-m}{n} \left[ \int_{s_n}^1 |D^m u|^s(s) \, ds \right] \left[ \int_0^s r^{\frac{m}{n}} \, dr \right] ds
\]

\[
= \frac{n-m}{n} \int_0^t \frac{1-m}{n} \left[ \int_{s_n}^1 |D^m u|^s r^{\frac{m}{n}} \, dr \right] ds
\]

\[
\quad \text{for} \ t > 0.
\]

Note that the constants of equivalence in (3.15) and in (3.16) depend only on $\Omega$ and $m$.

From (3.14)–(3.16) we get that positive constants $C = C(\Omega, m)$ and $C' = C'(\Omega, m) > 1$ exist such that

\[(3.17) \quad \int_0^t \frac{1-m}{n} \left[ \int_{s_n}^1 (\text{Tr} u)^s(s) \, ds \right] ds \leq C \int_0^t \frac{1-m}{n} \left[ \int_{s_n}^1 |D^m u|^s r^{\frac{m}{n}} \, dr \right] ds
\]

\[
\leq C' \int_0^t \frac{1-m}{n} \left[ \int_{s_n}^1 |D^m u|^s(|\Omega r|^r)^{\frac{m}{n}} \, dr \right] ds \quad \text{for} \ t > 0.
\]
By Proposition 3.2, inequality (1.6) implies that
\begin{equation}
\|f\|_{\mathcal{V}(0,1)} \leq C_1 \|f\|_{X_\sigma(0,1)}
\end{equation}
for \(f \in X'_\sigma(0,1)\), where the norm in the space \(X'_\sigma(0,1)\) is defined by (3.4).

Assume for a moment that we already know that
\begin{equation}
\|(\text{Tr } u)^*(\mathcal{H}^{n-1}(\partial \Omega))\|_{X'_\sigma(0,1)} \leq C \left\| \int_{t_\sigma}^{1} |D^m u|^* (|\Omega| s) \frac{m}{\pi} \, ds \right\|_{X'_\sigma(0,1)}
\end{equation}
for some positive constant \(C = C(m, n, |\Omega|)\) and for every \(u \in W^m X(\Omega)\). Then, from (3.18) and (3.19) we get
\begin{equation}
\| \text{Tr } u \|_{Y(\partial \Omega)} = \|(\text{Tr } u)^*(\mathcal{H}^{n-1}(\partial \Omega))\|_{\mathcal{V}(0,1)} \leq C_1 \|(\text{Tr } u)^*(\mathcal{H}^{n-1}(\partial \Omega))\|_{X'_\sigma(0,1)}
\end{equation}
for some constant \(C = C(1, m, n, |\Omega|)\). Inequality (1.7) follows from (3.20) and (3.5).

It remains to prove (3.19). By (3.17) and (2.4),
\begin{equation}
\int_{0}^{1} t^{\frac{m-n}{\pi}} \|(\text{Tr } u)^*(\mathcal{H}^{n-1}(\partial \Omega))t\|_{L^1} \, dt \leq C \int_{0}^{1} t^{\frac{1-m}{\pi}} \left( \int_{t_\sigma}^{1} |D^m u|^* (|\Omega| s) \frac{m}{\pi} \, ds \right) \, dt
\end{equation}
for some constant \(C = C(\Omega, m) > 1\) and for every non-increasing function \(h : (0, 1) \to [0, \infty)\).

Define the operator \(S\) at a function \(f \in \mathcal{M}(0,1)\) by
\begin{equation}
(Sf)(t) = t^{\frac{1-m}{\pi}} \sup_{t < s < 1} s^{\frac{m-1}{\pi}} f^*(s) \quad \text{for } t > 0.
\end{equation}

On choosing
\[
\begin{align*}
\text{for } t > 0
\end{align*}
\]
in (3.21), we get
\[
\int_{0}^{1} (\text{Tr } u)^*(\mathcal{H}^{n-1}(\partial \Omega))tSf(t) \, dt \leq C \int_{0}^{1} \left( \int_{t_\sigma}^{1} |D^m u|^* (|\Omega| s) \frac{m}{\pi} \, ds \right) Sf(t) \, dt.
\]
Thus,
\[
\|(\text{Tr } u)^*(\mathcal{H}^{n-1}(\partial \Omega))\|_{X'_\sigma(0,1)} = \sup_{\|f\|_{X'_\sigma(0,1)} \leq 1} \int_{0}^{1} (\text{Tr } u)^*(\mathcal{H}^{n-1}(\partial \Omega)t) f^*(t) \, dt
\]
\[
\leq \sup_{\|f\|_{X'_\sigma(0,1)} \leq 1} \int_{0}^{1} (\text{Tr } u)^*(\mathcal{H}^{n-1}(\partial \Omega)t) Sf(t) \, dt
\]
\[
\leq C \sup_{\|f\|_{X'_\sigma(0,1)} \leq 1} \int_{0}^{1} \left( \int_{t_\sigma}^{1} |D^m u|^* (|\Omega| s) \frac{m}{\pi} \, ds \right) Sf(t) \, dt
\]
\[
\leq C' \left\| \int_{t_\sigma}^{1} |D^m u|^* (|\Omega| s) \frac{m}{\pi} \, ds \right\|_{X'_\sigma(0,1)} \|Sf\|_{X'_\sigma(0,1)}
\]
for some constant \(C' = C'(\Omega, m)\), where the last estimate follows from inequality (2.3) combined with the boundedness of the dilation operator in r.i. spaces. We will be done if we show that the operator \(S\) is bounded on the space \(X_\sigma(0,1)\), namely that there exists \(C = C(m, n)\) such that
\begin{equation}
\|Sf\|_{X_\sigma(0,1)} \leq C \|f\|_{X_\sigma(0,1)}
\end{equation}
for \(f \in X_\sigma(0,1)\).
By [KP, Theorem 3.8], there exists a constant $C = C(m, n)$ such that

$$(Sf)^{**}(t) \leq C(Sf^{**})(t) \quad \text{for } t > 0,$$

for every integrable function $f$ in $(0, 1)$. Owing to (3.23), we have

$$\|Sf\|_{X^0(0,1)} = \|t^{\frac{m-1}{n}}(Sf)^{**}(t')\|_{X^0(0,1)} \leq C\|t^{\frac{m-1}{n}}(Sf^{**})(t')\|_{X^0(0,1)}$$

and

$$= C\sup_{t < s < 1} \left( s^{\frac{1}{n}} f^{**}(s) \right) = C\sup_{t < s < 1} \left( s^{\frac{1}{n}} f^{**}(s) \right) \in X^0(0,1).$$

By a straightforward extension of [KP, Theorem 3.9], for any $\alpha \in (0, 1)$ there exists a constant $C = C(\alpha)$ such that

$$\| \sup_{t < s < 1} s^\alpha f^{**}(s) \|_{Z(0,1)} \leq C \| t^\alpha f^{**}(t) \|_{Z(0,1)}$$

for every r.i. space $Z(0,1)$ and for every $f$ as above. On applying (3.24) with $Z(0,1) = X^0(0,1)$ and $\alpha = \frac{m}{n}$, and with $f$ replaced by $f^*(t^\frac{1}{n}) t^{-\frac{1}{n}}$, we conclude that a positive constant $C = C(m, n)$ exists such that

$$\|Sf\|_{X^a(0,1)} \leq C \left| s^{\frac{m-1}{n}} \int_0^s f^*(r^\frac{1}{n}) r^{-\frac{1}{n}} dr \right| \|X^0(0,1)\| = C\|f\|_{X^a(0,1)}.$$
and set
\begin{equation}
    u = v|\Omega.
\end{equation}

We begin by estimating \( \|D^m u\|_{X(\Omega)} \) from above in terms of \( f \). Owing to (3.27) and (3.28),
\begin{equation}
    \left| \left\{ |D^m u| > t \right\} \right| \leq \lambda \left| \left\{ |D^m v| > t \right\} \right| \quad \text{for } t > 0,
\end{equation}
whence
\begin{equation}
    |D^m u|^*(s) \leq |D^m v|^*(s \lambda) \quad \text{for } s > 0.
\end{equation}

One can verify (see e.g. [KP, Proof of Theorem A]) that a constant \( C = C(m, n) \) exists such that
\begin{equation}
    |D^m v(x)| \leq C \left( f(\omega_1|x|^n) + \sum_{i=1}^{m-1} |x|^m \int_0^\infty f(r) r^{-i+m-1} \, dr \right)
    \quad \text{for a.e. } x \in B_R(0)
\end{equation}
if \( x \notin B_R(0) \).

Define \( g : [0, \infty) \to [0, \infty) \) as
\begin{equation}
    g(s) = \sum_{i=1}^{m-1} \frac{s^{-m/n}}{\omega_n} \int_s^{\omega_n R^n} f(r) r^{-i+m-1} \, dr \quad \text{for } s \geq 0.
\end{equation}

From (3.32)–(3.34) we deduce that
\begin{equation}
    |D^m u|^*(s) \leq |D^m v|^*(s) \left( \frac{s}{\lambda} \right) \leq C \left( \chi_{[0,\omega_1 R^n]} f + \chi_{[0,\omega_1 R^n]} g \right)^* \left( \frac{s}{\lambda} \right)
\end{equation}
\begin{equation}
    \leq C \left( \chi_{[0,\omega_1 R^n]} f \right)^* \left( \frac{s}{2\lambda} \right) + C \left( \chi_{[0,\omega_1 R^n]} g \right)^* \left( \frac{s}{2\lambda} \right) \quad \text{for } s > 0.
\end{equation}

Hence,
\begin{equation}
    \|D^m u\|_{X(\Omega)} = \|D^m u|^*(\Omega\|X(0,1)
\begin{align*}
    &\leq C \left( \chi_{[0,\omega_1 R^n]} f \right)^* \left( \frac{s|\Omega|}{2\lambda} \right) \|X(0,1) + C \left( \chi_{[0,\omega_1 R^n]} g \right)^* \left( \frac{s|\Omega|}{2\lambda} \right) \|X(0,1)
    &\leq C' \left\| \chi_{[0,\omega_1 R^n]} f \right\|_{X(0,1)} + C' \left\| \chi_{[0,\omega_1 R^n]} g \right\|_{X(0,1)}
    &= C' \left\| \chi_{[0,\omega_1 R^n]} (\omega_n R^n s) f(\omega_n R^n s) \right\|_{X(0,1)} + C' \left\| \chi_{[0,\omega_1 R^n]} (\omega_n R^n s) g(\omega_n R^n s) \right\|_{X(0,1)}
    &= C' \left\| f(\omega_n R^n s) \right\|_{X(0,1)} + C' \left\| g(\omega_n R^n s) \right\|_{X(0,1)}
    &\leq C' \left\| f(\omega_n R^n s) \right\|_{X(0,1)} + C' \sum_{i=1}^{m-1} s^{-i+m} \int_s^{\omega_n R^n} f(r) r^{-i+m-1} \, dr \right\|_{X(0,1)}.
\end{align*}

where \( C \) is the constant appearing in (3.35), and \( C' = C'(m, n, R, \lambda, |\Omega|) \) and \( C'' = C''(m, n, R, \lambda, |\Omega|) \) are suitable positive constants. Notice that the second inequality holds owing to the boundedness of the dilation operator in r.i. spaces, and that the second equality holds thanks to the rearrangement invariance of \( X(0,1) \).

It is easily verified that, for each \( i = 1, \ldots, m-1 \), the operator
\begin{equation}
    f(s) \to s^{-i+m} \int_s^{\omega_n R^n} f(r) r^{-i+m-1} \, dr
\end{equation}
is bounded both in \( L^1(0,1) \) and in \( L^\infty(0,1) \), with norms depending only on \( n \). Hence, by Theorem 2.3, it is bounded in any r.i. space, and, in particular, in \( X(0,1) \), with norm depending
only on its norms in $L^1(0,1)$ and in $L^\infty(0,1)$. Consequently, a positive constant $C = C(n)$ exists such that
\begin{equation}
\left\| \sum_{i=1}^{m-1} a_i \int_s^1 f(\omega_n R^r r) r^{-i+\frac{n}{m}} \frac{dr}{X(0,1)} \right\|_{X(0,1)} \leq C \|f(\omega_n R^r r)\|_{X(0,1)}.
\end{equation}

Combining (3.36) and (3.37) yields
\begin{equation}
\|D^m u\|_{X(\Omega)} \leq C \|f(\omega_n R^r r)\|_{X(0,1)}
\end{equation}
for some positive constant $C = C(m, n, R, \lambda, |\Omega|)$.

We now need a lower bound for $\|\mathrm{Tr} u\|_{Y(\partial \Omega)}$. Set $B_R^{-1}(0) = B_R(0) \cap \{x_n = 0\}$. One has, for $t > 0$,
\begin{equation}
H^{n-1}(\{\mathrm{Tr} u > t\}) = \int_{\partial \Omega} \chi_{\{u > t\}}(x) dH^{n-1}(x)
\end{equation}
\begin{align}
= & \int_{B_R^{-1}(0)} \chi_{\{u > t\}}(x', \xi(x')) \sqrt{1 + |\nabla \xi|^2} dx' \geq \int_{B_R^{-1}(0)} \chi_{\{v > t\}}(x', \xi(x')) dx'
\end{align}
\begin{align}
= & H^{n-1}\left( \{x' \in B_R^{-1}(0) : v(x', \xi(x')) > t \} \right)
\end{align}
\begin{align}
\geq & H^{n-1}\left( \{x' \in B_R^{-1}(0) : \int_{\omega_n(1+L^2)^{\frac{m}{2}}}^{\omega_n R^n} f(r_m) \int_{r_1}^{\omega_n R^n} \cdots \int_{r_{m-1}}^{\omega_n R^n} dr_m \cdots dr_1 > t \} \right)
\end{align}
\begin{align}
= & H^{n-1}\left( \{x' \in B_R^{-1}(0) : \int_{K(\omega_n^{-1} x' \xi(x'))}^{\omega_n R^n} f(r_m) \int_{r_1}^{\omega_n R^n} \cdots \int_{r_{m-1}}^{\omega_n R^n} dr_m \cdots dr_1 > t \} \right),
\end{align}
where
\begin{align}
K = \frac{\omega_n (1 + L^2)^{\frac{m}{2}}}{(\omega_n^{-1})^{n'}}.
\end{align}

Thus,
\begin{equation}
(\mathrm{Tr} u)^*(s) \geq \chi_{[0, \omega_n R^n]}(K s') \int_{K s'}^{\omega_n R^n} f(r_m) \int_{r_1}^{\omega_n R^n} \cdots \int_{r_{m-1}}^{\omega_n R^n} dr_m \cdots dr_1
\end{equation}
\begin{align}
= & \chi_{[0, \omega_n R^n]}(K s') \int_{K s'}^{\omega_n R^n} f(r_m) \int_{r_1}^{\omega_n R^n} \cdots \int_{r_{m-1}}^{\omega_n R^n} \left( r - K s' \right)^{m-1} dr
\end{align}
\begin{align}
= & \left( \omega_n R^n \right)^{m+\frac{m}{n}+1} \chi_{[0, \omega_n R^n]}(K s') \int_{K s'}^{\omega_n R^n} f(\omega_n R^n q) q^{-m+\frac{m}{n}} \left( \omega_n R^n q - K s' \right)^{m-1} dq
\end{align}
\begin{align}
\geq & \frac{(\omega_n R^n)^{\frac{n}{m}}}{2^{m-1}(m-1)!} \chi_{[0, \omega_n R^n]}(K s') \int_{\omega_n R^n} f(\omega_n R^n q) q^{-\frac{m}{n}-1} dq \quad \text{for } s > 0.
\end{align}
Consequently, constants $C = C(n, R)$ and $C' = C'(\Omega)$ exist such that

$$\| \text{Tr} u \|_{Y(\partial \Omega)} = \| \text{Tr}(H^{n-1}(\partial \Omega)s) \|_{Y(0, 1)} \geq C \left\| \chi_{[0, \frac{nR}{2}]}(K(H^{n-1}(\partial \Omega)s)^n) \int_{2K(\frac{nR}{2})(\partial \Omega)} f(\omega_n R^n r)^{\frac{n}{m} - 1} \, dr \right\|_{Y(0, 1)}$$

$$\geq C' \left\| \int_{\frac{nR}{2}}^1 f(\omega_n R^n r)^{\frac{n}{m} - 1} \, dr \right\|_{Y(0, 1)},$$

where the first inequality is due to (3.40), and the second one to the boundedness of the dilation operator. Inequality (1.6) follows from (1.7), (3.38) and (3.41), owing to the arbitrariness of $f$.

**Proof of Theorem 1.2.** The assertion is a direct consequence of Proposition 3.2 and Theorem 1.1.

We conclude with the proof of Theorem 1.3.

**Proof of Theorem 1.3.** (i) Let $X(\Omega) = L^{p,q}(\Omega)$ and $Y(\partial \Omega) = L^{\frac{2(n-1)}{n-m}}(\partial \Omega)$. Owing to Theorem 1.2, it suffices to prove that constants $C = C(\Omega, p, q, m)$ and $C' = C'(\Omega, p, q, m)$ exist such that

$$C\| f \|_{X(0, 1)} \leq \left\| t^{\frac{m-1}{n}} f^{**}(\frac{1}{t^{\frac{1}{n}}}) \right\|_{Y(0, 1)} \leq C'\| f \|_{Y(0, 1)}$$

for every $f \in M(0, 1)$.

We start with the upper bound. First, by (2.9) we have that

$$\| t^{\frac{m-1}{n}} f^{**}(\frac{1}{t^{\frac{1}{n}}}) \|_{X(0, 1)} \leq \sup_{t < s < 1} s^{\frac{m-1}{n}} f^{**}(\frac{1}{s^{\frac{1}{n}}}) \|_{X(0, 1)} \leq C\| t^{\frac{1}{p} - \frac{1}{q}} \sup_{t < s < 1} s^{\frac{m-1}{n}} f^{**}(\frac{1}{s^{\frac{1}{n}}}) \|_{L^{p'}(0, 1)}$$

for some constant $C = C(p, q)$. By [CKOP, Lemma 3.1], there exists a positive constant $C = C(p, q, m, n)$ such that

$$\| t^{\frac{1}{p} - \frac{1}{q}} \sup_{t < s < 1} s^{\frac{m-1}{n}} f^{**}(\frac{1}{s^{\frac{1}{n}}}) \|_{L^{p'}(0, 1)} \leq C\| t^{\frac{1}{p} - \frac{1}{q}} + \frac{m-1}{n} f^{**}(\frac{1}{t^{\frac{1}{n}}}) \|_{L^{p'}(0, 1)}.$$

Thus,

$$\| t^{\frac{m-1}{n}} f^{**}(\frac{1}{t^{\frac{1}{n}}}) \|_{X(0, 1)} \leq C\| t^{\frac{1}{p} - \frac{1}{q} + \frac{m-1}{n}} f^{**}(\frac{1}{t^{\frac{1}{n}}}) \|_{L^{p'}(0, 1)} = C\left\| t^{-\frac{n-mp}{n(q-1)p} - \frac{1}{q}} \int_0^t f^*(s) \, ds \right\|_{L^{p'}(0, 1)} \leq C''\| t^{-\frac{n-mp}{n(q-1)p} - \frac{1}{q}} f^*(t) \|_{L^{p'}(0, 1)} \leq C'''\| f \|_{Y(0, 1)}$$

for some constants $C = C(p, q, m, n)$, $C' = C'(p, q, m, n)$, $C'' = C''(p, q, m, n)$ and $C''' = C'''(p, q, m, n)$. Notice that the last but one inequality rests upon a weighted Hardy inequality (e.g. [OK]).

In order to prove the lower bound in (3.42), we make use of (3.24) and of the estimate $f^{**}(t) \geq f^*(t)$ for $t > 0$, and we get

$$\| t^{\frac{m-1}{n}} f^{**}(\frac{1}{t^{\frac{1}{n}}}) \|_{X(0, 1)} \geq C\| \sup_{t < s < 1} s^{\frac{m-1}{n}} f^{**}(\frac{1}{s^{\frac{1}{n}}}) \|_{X(0, 1)} \geq C'\| \sup_{t < s < 1} s^{\frac{m-1}{n}} f^{**}(\frac{1}{s^{\frac{1}{n}}}) \|_{L^{p'}(0, 1)} \geq C'\| \frac{1}{p} - \frac{1}{q} + \frac{m-1}{n} f^{**}(\frac{1}{t^{\frac{1}{n}}}) \|_{L^{p'}(0, 1)} \geq C''\| t^{\frac{1}{p} - \frac{1}{q} + \frac{m-1}{n}} f^*(\frac{1}{t^{\frac{1}{n}}}) \|_{L^{p'}(0, 1)} = C''\| t^{-\frac{n-mp}{n(q-1)p} - \frac{1}{q}} f^*(t) \|_{L^{p'}(0, 1)} \geq C'''\| f \|_{Y(0, 1)},$$

for some positive constants $C = C(p, q, m, n)$, $C' = C'(p, q, m, n)$, $C'' = C''(p, q, m, n)$ and $C''' = C'''(p, q, m, n)$. Equation (3.42) follows.
(ii) Let \( X(\Omega) = L^{\infty}(\log L)^{-1}(\partial\Omega) \) and \( Y(\partial\Omega) = L^{\infty}(\log L)^{-1}(\partial\Omega) \). Again, it suffices to show that (3.42) holds. By [Sa, Remark after Theorem 1, p. 147], constants \( C = C(q) \) and \( C' = C'(q) \) exist such that

\[
C\|f\|_{Y'(0,1)} \leq \|t^{1-\frac{q}{q}} f^{**}(t)\|_{L^{q'}(0,1)} \leq C'\|f\|_{Y'(0,1)}
\]

for \( f \in Y'(0,1) \). Thus, by (3.43) and (3.44), we obtain

\[
\|t^{\frac{m-1}{n}} f^{**}(t^{\frac{1}{n}})\|_{X'(0,1)} \leq C\sup_{t<s<1} \|s^{\frac{m-1}{n}} f^{**}(s^{\frac{1}{n}})\|_{L^{\frac{m}{m-n}}(0,1)} = C\|t^{1-\frac{m}{n}} \cdot s^{1-\frac{m}{n}} f^{**}(s^{\frac{1}{n}})\|_{L^{q'}(0,1)} 
\]

\[
\leq C'\|t^{1-\frac{m}{n}} \cdot s^{1-\frac{m}{n}} f^{**}(s^{\frac{1}{n}})\|_{L^{q'}(0,1)} = C''\|t^{1-\frac{q}{q}} f^{**}(t)\|_{L^{q'}(0,1)} \leq C''\|f\|_{Y'(0,1)}
\]

for some constants \( C = C(q, m, n) \), \( C' = C'(q, m, n) \), \( C'' = C''(q, m, n) \) and \( C''' = C'''(q, m, n) \).

Conversely, on exploiting (3.24) again, we have that

\[
\|t^{\frac{m-1}{n}} f^{**}(t^{\frac{1}{n}})\|_{X'(0,1)} \geq C\sup_{t<s<1} \|s^{\frac{m-1}{n}} f^{**}(s^{\frac{1}{n}})\|_{L^{\frac{m}{m-n}}(0,1)} = C\|t^{1-\frac{m}{n}} \cdot s^{1-\frac{m}{n}} f^{**}(s^{\frac{1}{n}})\|_{L^{q'}(0,1)} 
\]

\[
\geq C\|t^{1-\frac{m}{n}} \cdot s^{1-\frac{m}{n}} f^{**}(s^{\frac{1}{n}})\|_{L^{q'}(0,1)} = C'\|t^{1-\frac{q}{q}} f^{**}(t)\|_{L^{q'}(0,1)} \geq C'\|f\|_{Y'(0,1)}
\]

for some constants \( C = C(q, m, n) \), \( C' = C'(q, m, n) \), \( C'' = C''(q, m, n) \).

(iii) When \( p = \frac{n}{m} \) and \( q = 1 \), inequality (1.11) agrees with (3.10). If \( p > \frac{n}{m} \) and \( 1 \leq q < \infty \), we have

\[
L^{p,q}(\Omega) \rightarrow L^{\frac{m}{m-1}}(\Omega),
\]

and hence (1.11) follows from (3.45) and (3.10). \( \square \)

References


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