Existence results for unsteady flows of nonhomogeneous non-Newtonian incompressible fluids - monotonicity methods in generalized Orlicz spaces

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Abstract

Our purpose is to show existence of weak solutions to unsteady flow of non-Newtonian incompressible nonhomogeneous fluids with nonstandard growth conditions of the Cauchy stress tensor. We are motivated by the fluids of strongly inhomogeneous behavior and characterized by rapid shear thickening. Since we are interested in flows with the rheology more general then power-law-type, we describe the growth conditions with help of general $x$–dependent convex function and formulate our problem in generalized Orlicz spaces.

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1 Introduction and formulation of the problem

Our interest is directed to the phenomena of viscosity increase under various stimuli: shear rate, magnetic or electric field. Particularly we want to focus on magnetorheological (MR) and shear thickening fluids (STF). Both types of fluid have an ability of transferring rapidly from liquid to solid state. The magnetorheological fluids are used for modern suspension system, clutches or crash-protection systems in cars. Moreover, they are used in shock absorbers providing seismic protection. Another possible application appears in military armor. The so-called liquid body armor is the solution, which consists in soaking the armor material with fluid, which increases the viscosity under the stimulus within few milliseconds. This allows for improving the flexibility and reducing the weight of the protection. For the liquid body armor, more promising solution for the moment are shear thickening fluids.
It is worth mentioning here that contrary to the motion of classical fluid, which is assumed to be described by the equations invariant w.r.t. full euclidian isometry group, equations below should be invariant only w.r.t. to rotation along the line of field.

We focus on the motion of incompressible, inhomogeneous non-Newtonian fluid, which can be described by the system of equations:

\[
\begin{align*}
\partial_t \varrho + \text{div}_x (\varrho \mathbf{u}) &= 0 \quad \text{in} \quad Q, \\
\partial_t (\varrho \mathbf{u}) + \text{div}_x (\varrho \mathbf{u} \otimes \mathbf{u}) - \text{div}_x S(x, [\mathbf{u}]) + \nabla_x P &= \varrho \mathbf{f} \quad \text{in} \quad Q, \\
\text{div}_x \mathbf{u} &= 0 \quad \text{in} \quad Q, \\
\mathbf{u}(0,x) &= \mathbf{u}_0 \quad \text{in} \quad \Omega, \\
\varrho(0,x) &= \varrho_0 \quad \text{in} \quad \Omega, \\
\mathbf{u}(t,x) &= 0 \quad \text{on} \quad (0,T) \times \partial \Omega,
\end{align*}
\]  

(1.1)

where \( \varrho : Q \to \mathbb{R} \) is the mass density, \( \mathbf{u} : Q \to \mathbb{R}^3 \) denotes the velocity field, \( P : Q \to \mathbb{R} \) the pressure, \( S \) the Cauchy stress tensor, \( \mathbf{f} : Q \to \mathbb{R}^3 \). The set \( \Omega \subset \mathbb{R}^3 \) is a bounded domain with a smooth (\( C^1 \) or \( C^{0,1} \) and sufficiently small Lipschitz constant) boundary \( \partial \Omega \). We denote by \( Q = (0,T) \times \Omega \) the time-space cylinder with some given \( T \in (0,\infty) \). \( \mathbb{D}[\mathbf{u}] = \frac{1}{2} (\nabla_x \mathbf{u} + \nabla_x^T \mathbf{u}) \) is a symmetric part of the velocity gradient.

It is supposed that initial density is bounded, i.e.,

\[
\varrho(0,\cdot) = \varrho_0 \in L^\infty(\Omega)
\]

and

\[
0 < \varrho_* \leq \varrho_0(x) \leq \varrho^* < \infty \quad \text{for a.a.} \ x \in \Omega.
\]

(1.3)

Standard growth conditions of the Cauchy stress tensor, namely polynomial growth, see e.g. [14], [7]

\[
|S(x,\xi)| \leq c(1+|\xi|^2)^{(p-2)/2}|\xi|
\]

\[
S(x,\xi) : \xi \geq c(1+|\xi|^2)^{(p-2)/2}|\xi|^2
\]

(1.4)

can not suffice to describe our model. We want to investigate the processes where growth is faster than polynomial and possibly different in various directions of the shear rate. Therefore we formulate the growth conditions of the Cauchy stress tensor using general convex function \( M \) called the \( N \)-function. Now we are able to describe the effect of rapidly shear thickening fluids.

We also assume that Cauchy stress tensor \( S : \Omega \times \mathbb{R}^{3 \times 3}_{\text{sym}} \to \mathbb{R}^{3 \times 3}_{\text{sym}} \) satisfies (\( \mathbb{R}^{3 \times 3}_{\text{sym}} \) stands for the space of \( 3 \times 3 \) symmetric matrices):

1. \( S(x,K) \) is a Carathéodory function (i.e., measurable function of \( x \) for all \( K \in \mathbb{R}^{3 \times 3}_{\text{sym}} \) and continuous function of \( K \) for a.a. \( x \in \Omega \)) and \( S(x,0) = 0 \).
2. There exist a positive constant $c_S$, an $N$–function $M$ and $M^*$ denotes the complementary function to $M$ (the definitions of an $N$–function and complementary function appear below) such that for all $K \in \mathbb{R}^{3\times 3}$ and a.a. $x \in \Omega$ it holds
\[ S(x, K) : K \geq c_S \{ M(x, K) + M^*(x, S(x, K)) \} \quad (1.5) \]

3. $S$ is monotone, i.e. for all $K_1, K_2 \in \mathbb{R}^{3\times 3}_\text{sym}$, and a.a. $x \in \Omega$
\[ [S(x, K_1) - S(x, K_2)] : [K_1 - K_2] \geq 0. \]

The appropriate spaces to capture such formulated problem are the Orlicz spaces. We also allow the stress tensor to depend on $x$, this provides the possibility of considering the influence of magnetic field and its significant influence on the increase of viscosity. Thus we use the generalized Orlicz spaces, often called Orlicz-Musielak spaces (see [16] for more details). For definitions and preliminaries of $N$–functions and Orlicz spaces see Section 2. Contrary to [16] we consider the $N$–function $M$ not dependent only on $|\xi|$, but on whole vector $\xi$. It results from the fact that the stress tensor $S$ may differ in different directions, it depends on whole tensor $\mathbb{D}[u]$, not only on absolute value. The spaces with $N$–function dependent on vector-valued argument were investigated in [22].

The example of the generalized Orlicz space is generalized Lebesgue space, in this case $M(x, \xi) = |\xi|^{p(x)}$. This kind of spaces were applied in [18] to description of flow of electrorheological fluid. The standard assumption in this work was $1 \leq p_0 \leq p(x) \leq p_\infty < \infty$. For this reason there is satisfied the so-called $\Delta_2$–condition and consequently the space is reflexive and separable. One of the main problems in our model is that the $\Delta_2$–condition is not satisfied and we lose the above properties.

An interesting obstacle here is the lack of the classical integration by parts formula, cf. [9, Section 4.1]. To extend it for the case of generalized Orlicz spaces we would essentially need that $C^\infty$–functions are dense in $L_M(Q)$ and $L_M(Q) = L_M(0,T; L_M(\Omega))$. The first one only holds if $M$ satisfies $\Delta_2$–condition. The second one is not the case in Orlicz and generalized Orlicz spaces. We recall the proposition from [4] (although it is stated for Orlicz spaces with $M = M(|\xi|)$). One can conclude that (1.6) means that $M$ must be equivalent to some power $p$, $1 \leq p \leq \infty$. Hence, if (1.6) should hold, very strong assumptions must be satisfied by $M$. Surely they would provide $L_M(\Omega)$ to be separable and reflexive.

**Proposition 1.1** Let $I$ be the time interval and $\Omega \subset \mathbb{R}^d$, $M = M(|\xi|)$ an $N$–function, and $L_M(I \times \Omega), L_M(I; L_M(\Omega))$ the Orlicz spaces on $I \times \Omega$ and the vector valued Orlicz space on $I$ respectively. Then
\[ L_M(I \times \Omega) = L_M(I; L_M(\Omega)), \]
if and only if there exist constants $k_0, k_1$ such that
\[ k_0 M^{-1}(s) M^{-1}(x) \leq M^{-1}(sx) \leq k_1 M^{-1}(s) M^{-1}(x) \quad (1.6) \]
for every $s \geq 1/|I|$ and $x \geq 1/|\Omega|$. 


Let us recall some existence results focused on similar problems like we are interested in. Most of the available results concerning nonhomogeneous incompressible fluids deal with the polynomial dependence between $\mathbb{S}$ and $\mathbb{D}[\mathbf{u}]$. The analysis of nonhomogeneous Newtonian ($p = 2$ in (1.4)) fluids was investigated by Antontsev, Kazhikhov and Monakhov [2] in the seventies. P.L. Lions in [13] presented concept of renormalized solutions and obtained new convergence and continuity properties of the density. Frehse and Růžička showed in [8] existence of weak solution for generalized Newtonian fluid of power-low type for $p > 11/5$. Authors needed also existence of potential to $\mathbb{S}$. Resent results concerning fluids of where the growth condition is in (1.4) type for $p \geq 11/5$ belong to Frehse, Málek and Růžička [7]. The novelty of this paper is that the authors considered the full thermodynamic model for nonhomogeneous incompressible fluid. Particulary in [8] and [7] the reader can find the concept of integration by parts formula, which we adapted to our case. More details concerning references the reader can find in [7] and [8].

Let us recall some existing analytical results concerning the abstract parabolic problems in non-separable Orlicz spaces

$$\partial_t \mathbf{u} + A(t,x,\nabla_x \mathbf{u}) = \mathbf{f} \quad \text{in} \quad Q,$$
$$\mathbf{u}(x,t) = 0 \quad \text{on} \quad (0,T) \times \partial \Omega,$$  \hspace{1cm} (1.7)
$$\mathbf{u}(x,0) = \mathbf{u}_0(x) \quad \text{in} \quad \Omega.$$  

In [4] the operator $A$ was assumed to be an elliptic second-order operator in divergence form and monotone. The growth and coercivity conditions were more general than the standard growth conditions in $L^p$, namely the $N$–function formulation was stated. Under the assumptions on the $N$–function $M: \xi^2 \ll M(|\xi|)$ (i.e., $\xi^2$ grows essentially less rapidly than $M(|\xi|)$) and $M^*$ satisfies a $\Delta_2$–condition, the existence results to (1.7) was established.

The review paper [17] summarizes the monotone-like mappings techniques in Orlicz and Orlicz–Sobolev spaces. The authors need the essential modifications of the notions like monotonicity, pseudomonotonicity, operators of type $(M)$, $(S_+)$, et al. The reason is that Orlicz–Sobolev spaces are not reflexive in general. Moreover, the nonlinear differential operators in divergence form with standard growth conditions are neither bounded nor everywhere defined. First results concerning non-Newtonian fluid with assumption that $\mathbb{S}$ satisfied 1.-2. conditions and strict monotonicity were established by Gwiazda et al. [10]. The stronger assumption on $\mathbb{S}$ was crucial for the applied tools (Young measures). This restriction were abandoned in [12], where authors used generalization of Mintys trick for non-reflexive spaces. Above results were established for $p \geq 11/5$, but without including to the system dependence on density.

We want to extend the existence theory to more general then the class of polynomial growth conditions and to the system where density is not assumed to be constant.

Before we state our main theorem, let us denote some spaces. By $\mathcal{D}(\Omega)$ we mean the set of $C^\infty$-functions with compact support. Let $\mathcal{V}$ be the set of all functions
which belong to $\mathcal{D}(\Omega)$ and are divergence-free. Moreover, by $L^p, W^{1,p}$ we mean the standard Lebesgue and Sobolev spaces respectively, by $H$ the closure of $\mathcal{V}$ w.r.t. the $\|\cdot\|_{L^2}$ norm and by $W^{1,p}_{0,\text{div}}$ the closure of $\mathcal{V}$ w.r.t. the $\|\nabla(\cdot)\|_{L^p}$ norm. Let $W^{-1,p'} = (W^{1,p}_0)^*$, $W_{\text{div}}^{-1,p'} = (W^{1,p}_{0,\text{div}})^*$. By $p'$ we mean the conjugate exponent to $p$, namely $\frac{1}{p} + \frac{1}{p'} = 1$.

If $X$ is a Banach space of scalar functions, then $X^3$ or $X^{3\times3}$ denotes the space of vector- or tensor-valued functions where each their component belongs to $X$. The symbols $L^p(0,T;X)$ and $C([0,T];X)$ mean the standard Bochner spaces.

We will use $C([0,T];H_{\text{weak}})$ to denote the space of functions $u \in L^\infty(0,T;H)$ which satisfy $(u(t),\varphi) \in C([0,T])$ for all $\varphi \in H$.

By $(a,b)$ we mean $\int_\Omega a(x) \cdot b(x)dx$ and $(a,b)$ denotes a duality parting.

**Definition 1.1** We call the pair $\varrho, u$ a weak solution to (1.1) if

\[
0 < \varrho_s \leq \varrho(t,x) \leq \varrho^* \quad \text{for a.a.} \quad (t,x) \in Q,
\]
\[
\varrho \in C([0,T];L^q(\Omega)), \quad q \in [1,\infty),
\]
\[
\partial_t \varrho \in L^{5p/3}(0,T;(W^{1,5p/(5p-3)}_0)^*),
\]
\[
u \in C([0,T];H_{\text{weak}}) \cap L^p(0,T;W^{1,p}_{0,\text{div}}(\Omega)),
\]
\[
\mathbb{D}[u] \in L_M(Q),
\]
\[
\partial_t (\varrho u) \in L^p(0,T;(W^{1,p}_{0,\text{div}}(\Omega))^*)
\]

and following identities are satisfied

\[
\int_0^T \langle \partial_t \varrho, z \rangle - (\varrho u, \nabla_x z)dt = 0 \quad \text{for all} \quad z \in L^r(0,T;W^{1,r}) \quad \text{with} \quad r = \frac{5p}{5p-3},
\]

i.e.

\[
\int_{s_1}^{s_2} \int_\Omega \varrho \partial_t z + (\varrho u) \cdot \nabla_x z dxdt = \int_\Omega \varrho z(s_2) - \varrho z(s_1)dx
\]

for all $z$ smooth and $s_1, s_2 \in [0,T]$, $s_1 < s_2$

and

\[
- \int_0^T \int_\Omega \varrho u \cdot \partial_t \varphi - \varrho u \otimes u : \nabla_x \varphi + \mathbb{S}(x,\mathbb{D}[u]) : \mathbb{D}[\varphi] dxdt
\]

\[
= \int_0^T \int_\Omega \varrho f \cdot \varphi dxdt + \int_\Omega \varrho_0 u_0 \cdot \varphi(0)dx \quad \text{for all} \quad \varphi \in \mathcal{D}(-\infty,T);\mathcal{V},
\]

and initial conditions are achieved in the following way

\[
\lim_{t \to 0^+} \|\varrho(t) - \varrho_0\|_{L^q(\Omega)} + \|u(t) - u_0\|^2_{L^2(\Omega)} = 0 \quad \text{for any} \quad q \in [1,\infty).
\]
Theorem 1.1 Let \( M \) be an \( N \)-function satisfying for some \( c > 0 \) and
\[
p \geq \frac{11}{5}
\] (1.12)
the condition
\[
M(x, \xi) \geq c|\xi|^p.
\] (1.13)
Moreover, let \( S \) satisfy conditions 1.-3. and \( u_0 \in H(\Omega)^3, \varrho_0 \in L^\infty(\Omega) \) with \( 0 < \varrho_* \leq \varrho_0(x) \leq \varrho^* < +\infty \) for a.a. \( x \in \Omega \) and \( f \in L^2(0,T;L^2(\Omega)^3) \) Then there exists a weak solution to (1.1).

Our paper is organized as follows: Section 2 presents the notation and some properties of generalized Orlicz spaces. In Section 3 our main result of existence of weak solutions to the system (1.1) is proved. Section 4 (Appendix) contains some technical facts used in Section 3.

2 Notation and properties of Orlicz spaces

Definition 2.1 Let \( \Omega \) be a bounded domain in \( \mathbb{R}^3 \), a function \( M : \Omega \times \mathbb{R}^{3 \times 3}_{\text{sym}} \to \mathbb{R}_+ \) is said to be an \( N \)-function if it satisfies the following conditions

1. \( M \) is a Carathéodory function such that \( M(x, \mathbb{K}) = 0 \) if and only if \( \mathbb{K} = 0 \), \( M(x, \mathbb{K}) = M(x, -\mathbb{K}) \) a.e. in \( \Omega \),
2. \( M(x, \mathbb{K}) \) is a convex function w.r.t. \( \mathbb{K} \),
3. \[
\lim_{|\mathbb{K}| \to 0} \sup_{x \in \Omega} \frac{M(x, \mathbb{K})}{|\mathbb{K}|} = 0,
\] (2.14)
4. \[
\lim_{|\mathbb{K}| \to \infty} \inf_{x \in \Omega} \frac{M(x, \mathbb{K})}{|\mathbb{K}|} = \infty.
\] (2.15)

Definition 2.2 The complementary function \( M^* \) to a function \( M \) is defined by
\[
M^*(x, \mathbb{L}) = \sup_{\mathbb{K} \in \mathbb{R}^{3 \times 3}_{\text{sym}}} (\mathbb{K} : \mathbb{L} - M(x, \mathbb{K}))
\]
for \( \mathbb{L} \in \mathbb{R}^{3 \times 3}_{\text{sym}}, x \in \Omega \).

The complementary function \( M^* \) is also an \( N \)-function.

The generalized Orlicz class \( L_M(Q)^{3 \times 3}_{\text{sym}} \) is the set of all measurable functions \( \mathbb{K} : \Omega \to \mathbb{R}^{3 \times 3}_{\text{sym}} \) such that
\[
\int_Q M(x, \mathbb{K}(t,x))dxdt < \infty.
\]
The generalized Orlicz space $L_M(Q)^{3\times 3}_\text{sym}$ is defined as the set of all measurable functions $K : \Omega \to \mathbb{R}^{3\times 3}_\text{sym}$ which satisfy
\[\int_Q M(x, \lambda K(t, x)) \, dx \, dt \to 0 \quad \text{as } \lambda \to 0.\]

The generalized Orlicz space is a Banach space with respect to the Luxemburg norm
\[\|K\|_M = \inf \left\{ \lambda > 0 : \int_\Omega M\left(x, \frac{K(t, x)}{\lambda}\right) \, dx \, dt \leq 1 \right\}.\]

Let us denote by $E_M(Q)^{3\times 3}_\text{sym}$ the closure of all measurable, simple functions on $Q$ in $L_M(Q)^{3\times 3}_\text{sym}$. The space $L_M^*(Q)^{3\times 3}_\text{sym}$ is the dual space of $E_M(Q)^{3\times 3}_\text{sym}$.

The functional
\[\varrho(K) = \int_Q M(x, K(x)) \, dx \, dt\]

is a modular in the space of measurable functions $K : \Omega \to \mathbb{R}^{3\times 3}_\text{sym}$. A sequence $\{z^j\}_{j=1}^\infty$ converges modularly to $z$ in $L_M(Q)^{3\times 3}_\text{sym}$ if there exists $\lambda > 0$ such that
\[\int_Q M\left(x, \frac{z^j - z}{\lambda}\right) \, dx \, dt \to 0 \quad \text{as } j \to \infty.\]

We will write $z^j \xrightarrow{M} z$ for the modular convergence in $L_M(Q)^{3\times 3}_\text{sym}$.

We say that an N-funtion $M$ satisfies $\Delta_2$-condition if for some nonnegative, integrable in $\Omega$ function $h_M$ and a constant $C_M > 0$
\[M(x, 2K) \leq C_M M(x, K) + h_M(x) \quad \text{for all } K \in \mathbb{R}^{d\times d}_\text{sym} \text{ a.a. } x \in \Omega. \quad (2.16)\]

If this condition fails we lose numerous properties of the space $L_M(Q)^{3\times 3}_\text{sym}$ like separability, reflexivity, cf. [16] and many others. In particular, if (2.16) holds, then $L_M(Q)^{3\times 3}_\text{sym} = L_M(Q)^{3\times 3}_\text{sym}$.

3 Proof of the Theorem 1.1

3.1 Uniform estimates and the existence of approximate solution

Let $\{\omega^n\}_{n=1}^\infty$ be a basis of $W_{0,\text{div}}^{1,p}(\Omega)$ constructed as eigenfunctions to the problem
\[((\omega_i, \varphi))_s = \lambda_i(\omega_i, \varphi) \quad \text{for all } \varphi \in V_s,\]

where
\[V_s \equiv \text{the closure of } \mathcal{V} \text{ w.r.t. the } W^{s,2}(\Omega)-\text{norm}\]
and \((\cdot, \cdot)_s\) denotes the scalar product in \(V_s\). We assume that \(s > \frac{5}{2}\) and then the Sobolev embedding theorem provides

\[ W^{s-1,2}(\Omega) \hookrightarrow L^\infty(\Omega). \]  

We denote \(H^n := \text{span}\{\omega^1, \ldots, \omega^n\}\) and define orthonormal projection \(P^n : H \rightarrow H^n\) by \(P^n u^n := \sum_{i=1}^n (u, \omega^i) \omega^i\) for every \(n \in \mathbb{N}\). We are looking for an approximate solution \(\varrho^n, u^n\) of the system (1.1) in the following way: let \(\alpha^n_j \in C^1([0,T])\) and

\[ u^n(t, x) := \sum_{j=1}^n \alpha^n_j(t) \omega^j(x) \]  

solve the Galerkin system

\[ (\varrho^n \partial_t u^n, \omega^j) + (\varrho^n [\nabla_x u^n] u^n, \omega^j) + (S(x, \mathbb{D}[u^n]), \mathbb{D}\omega^j) = (\varrho^n f^n, \omega^j) \]  

\[ u^n(0) = P^n(u_0) \]  

for all \(1 \leq j \leq n\) and a.a. \(t \in [0,T]\). Moreover \(\varrho^n\) is the solution of the following equation

\[ \partial_t \varrho^n + \text{div}_x(\varrho^n u^n) = 0, \]  

\[ \varrho^n(0) = \varrho^n_0. \]  

We assume additionally that

\[ u^n_0 \rightarrow u_0 \text{ strongly in } H, \]  

\[ \varrho^n_0 \rightarrow \varrho_0 \text{ strongly in } L^\infty(\Omega), \]  

\[ \varrho^n_0 \in C^1(\Omega) \text{ and } \varrho_* \leq \varrho^n_0 \leq \varrho^* \]  

and

\[ f^n \rightarrow f \text{ strongly in } L^{p'}(0,T; W^{-1,p'}). \]  

Let us note that since our approximate solution satisfies (3.20), (3.19) is equivalent to

\[ \int_0^T \langle \partial_t (\varrho^n u^n), \omega^j \rangle - (\varrho^n u^n \otimes u^n, \nabla_x \omega^j) + (S(x, \mathbb{D}[u^n]), \mathbb{D}\omega^j) dt = \int_0^T (\varrho^n f^n, \omega^j) dt \]  

and also (3.20) satisfies

\[ \int_0^T \langle \partial_t \varrho^n, \omega^j \rangle + (\varrho^n u^n, \nabla_x \omega^j) dt = 0. \]  

Before we will prove existence of the approximate solution we want to show that some uniform w.r.t. \(n\) a priori estimates are valid and to present some of their consequences which we will use later.
In first step we will concentrate now on equations (3.20). We will use standard techniques for transport equation and apply the method of characteristics. For this reason we solve with help of Caratheodory’s theory the system of ordinary differential equations
\[
\frac{dy^n(t, x)}{dt} = u^n(t, y^n(t, x)) \quad \text{for } t \in [0, T],
\]
which defines so-called characteristics associated with (3.20). Note that for every \(t \in [0, T]\) the map \(x \rightarrow y^n(t, x)\) is a diffeomorphism of \(\overline{\Omega}\) onto \(\overline{\Omega}\). Using this fact and \(\text{div}_x u^n = 0\) we can see that the solution of (3.20) is given by
\[
\varrho^n(t, y^n(t, x)) = \varrho^n_0(x).
\]
(3.26)
Since (3.26) is satisfied and according to assumptions on \(\varrho^n_0\) we obtain that
\[
0 < \varrho_* \leq \varrho^n(t, x) \leq \varrho^* < +\infty \quad \text{for all } (t, x) \in Q.
\]
(3.27)
For later consideration let us note that Alaoglu-Burbaki Theorem provide existence of subsequence such that
\[
\varrho^n \rightharpoonup \varrho \quad \text{weakly in } L^q(Q) \text{ for any } q \in [1, \infty),
\]
\[
\varrho^n \rightharpoonup^* \varrho \quad \text{weakly* in } L^\infty(Q) \text{ and weakly* in } L^\infty(0, T; L^\infty(\Omega)).
\]
(3.28)
Multiplying (3.19) by \(\alpha^n_j\), summing up over \(j\) and using (3.20) we get
\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} \varrho^n |u^n|^2 dx + (S(x, D[u^n]), D[u^n]) = (\varrho^n f^n, u^n)
\]
(3.29)
Using Korn’s and Young inequalities, assumption (1.13) and (3.27) we are able to estimate right-hand side of (3.29) in following way
\[
|\langle \varrho^n f^n, u^n \rangle| \leq C_1(\Omega, c_c, c_s, \varrho^*, p) \|f^n\|_{L^{p'}(\Omega)} + \frac{c_c}{2} \int_{\Omega} M(x, D[u^n]) dx.
\]
(3.30)
Integrating (3.29) over the time interval \((0, s_0)\), using estimate (3.30), (3.27), the coercivity conditions (1.5) on \(S\), continuity of \(P^n\) uniformly w.r.t. \(n\) and strong convergence \(f^n \rightarrow f\) in \(L^{p'}(0, T; W^{-1,p'})\) we obtain
\[
\int_{\Omega} \frac{1}{2} \varrho^n |u^n(s_0)|^2 + \frac{c_c}{2} \int_{0}^{s_0} \int_{\Omega} M(x, D[u^n]) dx dt + c_c \int_{0}^{s_0} \int_{\Omega} M^*(x, S(x, D[u^n])) dx dt
\]
\[
\leq C_2(\Omega, c_c, c_s, \varrho^*, p, \|f\|_{L^{p'}(0, T; W^{-1,p'})} + \frac{1}{2} \varrho^* \|u_0\|^2_{L^2(\Omega)}),
\]
(3.31)
where $C_2 > 0$. The condition (1.13) provides that \( \{ D[u^n] \}_{n=1}^{\infty} \) is uniformly bounded in the space \( L^p(Q)^{3 \times 3} \) for \( p \geq \frac{11}{5} \)

\[
\int_0^T \| D[u^n] \|^p_{L^p(\Omega)} dt \leq C \tag{3.32}
\]

and hence there exists a subsequence such that

\( D[u^n] \rightharpoonup D[u] \) weakly in \( L^p(Q)^{3 \times 3} \).

According to Korn’s inequality we also obtain

\[
\int_0^T \| \nabla x u^n \|^p_{L^p(\Omega)} dt \leq C \tag{3.33}
\]

and

\[
u^n \rightharpoonup \nu \quad \text{weakly in} \quad L^p(0, T; W^{1,p}_0(\Omega)). \tag{3.34}\]

Using (3.31) we get that the sequence \( \{ S(x, D[u^n]) \}_{n=1}^{\infty} \) is uniformly bounded in Orlicz class \( L_{M^*}(Q)^{3 \times 3} \). Applying Lemma 4.3 we conclude the uniform integrability and hence weak precompactness in \( L^1(Q)^{3 \times 3} \). Therefor we can deduce

\[
\| S(x, D[u^n]) \|_{L^1(Q)} \leq C, \tag{3.35}\]

Namely there exists a \( \overline{S} \in L^1(Q)^{3 \times 3} \) such that

\( S(\cdot, D[u^n]) \rightharpoonup \overline{S} \) weakly in \( L^1(Q)^{3 \times 3} \). \tag{3.36}\]

Moreover from (3.31) using (3.27) we conclude

\[
\sup_{t \in [0,T]} \| u^n(t) \|^2_{L^2(\Omega)} \leq C, \tag{3.37}\]

\[
\sup_{t \in [0,T]} \| \varrho^n(t) |u(t)|^2 \|_{L^1(\Omega)} \leq C,
\]

where \( C \) is positive constant depended on the size of data, but independent of \( n \) and it can change from line to line. It follows immediately that

\[
u^n \rightharpoonup \nu \quad \text{in} \quad L^\infty(0, T; H(\Omega)). \tag{3.38}\]

Since the sequence \( \{ u^n \} \) is uniformly bounded in \( L^p(0, T; W^{1,p}_{0,\text{div}}(\Omega)) \) standard imbedding provide also uniform boundedness in the space \( L^p(0, T; L^{3p/(3-p)}(\Omega)) \). The standard interpolation (see e.g. [19]) of \( L^\infty(0, T; L^2) \) and \( L^p(0, T; L^{3p/(3-p)}(\Omega)) \) (this particular argument deal with the case \( p < 3 \), the case \( p \geq 3 \) can be treated easier with slightly different inequality) gives us, that for \( 1 \leq r \leq 5p/3 \)

\[
\int_0^T \| u^n \|^r_{L^r(\Omega)} dt \leq C_B \tag{3.39}\]
and applying the Hölder inequality to (3.37) we get also
\[ \int_0^T \| \varrho^n u^n \|_{L^{5p/3}(\Omega)}^{5p/3} \, dt \leq C. \]  
(3.40)

Consequently we can subtract subsequences satisfying
\[ u^n \rightharpoonup u \text{ weakly in } L^{5p/3}(0, T; L^{5p/3}(\Omega)) \]  
(3.41)

and there exists \( \overline{\varrho u} \in L^{5p/3}(Q)^3 \) such that
\[ \varrho^n u^n \rightharpoonup \overline{\varrho u} \text{ weakly in } L^{5p/3}(0, T; L^{5p/3}(\Omega)). \]  
(3.42)

Using (3.27), (3.33) and (3.39) and applying the Hölder inequality, we obtain
\[ \int_0^T |(\varrho^n u^n \otimes u^n, \nabla x u^n)| \, dt \leq C \iff p \geq \frac{11}{5} \]  
(3.43)

(here is the restriction for \( p \)).

Using above estimates it follows from weak formulation (3.23) and (3.24) of approximate problem that
\[ \int_0^T \| \partial_t (\varrho^n u^n) \|_{W^{-1, 5p/3}} \, dt \leq C \quad \text{and} \quad \int_0^T \| \partial_t \varrho^n \|_{(W^{1,5p/(5p-3)})^*} \, dt \leq C. \]  
(3.44)

Hence Alaoglu-Burbaki Theorem provide existence of a subsequence such that
\[ \partial_t \varrho^n \rightharpoonup \partial_t \varrho \text{ weakly in } L^{5p/3}(0, T; (W^{1,5p/(5p-3)})^*). \]  
(3.45)

On the basis of above estimates we will show the existence of solutions of (3.19) and (3.20) using Schauder’s fixed point theorem for the operator
\[ \Lambda : B \subset Y \rightarrow B : \tilde{u} \rightarrow u^n \]

where \( Y := L^q(0, T; L^q(\Omega)) \cap L^q(0, T; H^n) \), \( q = 2p' \) is equipped with the norm of the \( L^q(0, T; L^q(\Omega)) \) and \( B \) is the closed ball which will be defined later. For given \( \tilde{u}^n \in B \) the element \( \Lambda \tilde{u}^n \) is a solution of the problem
\[ \begin{align*}
\partial_t \varrho^n + \text{div}_x (\varrho^n \tilde{u}^n) &= 0, \\
\varrho^n(0) &= \varrho^n_0.
\end{align*} \]  
(3.46)

\[ \begin{align*}
(\varrho^n \partial_t \tilde{u}^n, \omega^j) + (\varrho^n [\nabla_x \tilde{u}^n] \tilde{u}^n, \omega^j) + (\mathbb{S}(x, \mathbb{D}[\tilde{u}^n]), \mathbb{D}\omega^j) &= (\varrho^n \tilde{f}^n, \omega^j), \\
\tilde{u}^n(0) &= P^n(u_0).
\end{align*} \]  
(3.47)

It means that in the first step we find solution \( \varrho^n \) of the linear problem (3.46) and next we look for the vector \( u^n \), solution of the linearization (3.47) of the system (3.19).
The system (3.47) can be rewritten as a system of ordinary differential equations (details the reader can find in [2], [15]). We obtain local in time solvability according to Peano’s existence theorem for the system of ordinary differential equations. The global solvability is provided by the a’priori estimates (3.31).

Let us set \( B := B_{CB}(0) \), where \( B_{CB}(0) \) is a ball and \( C_1 \) is a constant from (3.39). Previous estimates (3.31) and (3.39) showed that \( \Lambda \) maps \( B \) into \( B \), since \( 2p' < 5p/3 \) for \( p \geq 11/5 \). Using (3.37) and (3.33) we deduce that \( u^n \in L^\infty(0,T;L^2(\Omega)) \cap L^p(0,T;W^{1,p}_{0,div}(\Omega)) \). Now the main difficulty is to show compactness of the operator \( \Lambda \). Similarly like in [18], [2] our plan is to prove that

\[
\int_0^{T-\delta} \|u^n(s+\delta) - u^n(s)\|_{L^2(\Omega)}^2 ds \to 0 \quad \text{as} \quad \delta \to 0
\]

is satisfied. According to [20, Theorem 5] and parabolic embedding theorem \( \Lambda(B) \) is a compact subset of \( Y \). Applying Schauder’s fixed point theorem we can deduce that the fixed point and the corresponding density solve thy system (3.19), (3.20).

To show (3.48) we will follow [2, Chap.3. Lemma 1.2] with some modification introduced in [18] concerning change from \( L^2 \)-structure for \( L^2 \)-structure and additional one concerning the nonlinear term.

Let us fix \( \delta \) and \( s, 0 < \delta < T, 0 \leq s \leq T - \delta \). Next we test (3.47) at time \( t \) by \( u^n(s+\delta) - u^n(s) \). Using integrating by parts formula w.r.t. time, equality

\[
\partial_t \tilde{\varrho}^n = -\text{div}_x(\tilde{\varrho}^n \tilde{u}^n)
\]

and obvious identity

\[
\tilde{\varrho}^n(s+\delta)u^n(s+\delta) - \tilde{\varrho}(s)u^n(s) = \tilde{\varrho}^n(s+\delta)[u^n(s+\delta) - u^n(s)] + [\tilde{\varrho}^n(s+\delta) - \tilde{\varrho}(s)]u^n(s)
\]

we get

\[
\int_\Omega \tilde{\varrho}^n(s+\delta)|u^n(s+\delta) - u^n(s)|^2 + [\tilde{\varrho}^n(s+\delta) - \tilde{\varrho}(s)] |u^n(s+\delta) - u^n(s)| dx

+ \int_s^{s+\delta} \int_\Omega \text{div}_x(\tilde{\varrho}^n \tilde{u}^n)(t) \cdot [u^n(s+\delta) - u^n(s)] + \tilde{\varrho}^n(t)[\nabla_x u^n(t)] \tilde{u}^n(t) \cdot [u^n(s+\delta) - u^n(s)]

+ S(x, D[u^n(t)]) : D[u^n(s+\delta) - u^n(s)]dx dt = \int_s^{s+\delta} \int_\Omega \tilde{\varrho}^n(t)f^n(t) \cdot [u^n(s+\delta) - u^n(s)] dx dt.
\]

Now, let us test (3.46) at time \( t \) by \( u^n(s) \cdot (u^n(s+\delta) - u^n(s)) \) and integrate w.r.t. time \( t \) on \( (s,s+\delta) \) then we obtain.

\[
\int_\Omega [\tilde{\varrho}^n(s+\delta) - \tilde{\varrho}(s)] u^n(s) \cdot [u^n(s+\delta) - u^n(s)] dx

= - \int_s^{s+\delta} \int_\Omega \text{div}_x(\tilde{\varrho}^n(t) \tilde{u}^n(t))(t) \cdot [u^n(s+\delta) - u^n(s)] dx dt.
\]

Substituting above relation into (3.49) some obvious manipulations ((\( \text{div}_x(\tilde{\varrho}^n \tilde{u}^n)u^n(s), [u^n(s+\delta) - u^n(s)] \)) = - (\( \tilde{\varrho}^n(t)[\nabla_x u^n(s)] \tilde{u}^n(t), [u^n(s+\delta) - u^n(s)] \)) - (\( \tilde{\varrho}^n(t)u^n(s) \otimes \tilde{u}^n(t), \nabla_x [u^n(s+\delta) - u^n(s)] \)) = - (\( \tilde{\varrho}^n(t)[\nabla_x u^n(s)] \tilde{u}^n(t), [u^n(s+\delta) - u^n(s)] \)) - (\( \tilde{\varrho}^n(t)u^n(s) \otimes \tilde{u}^n(t), \nabla_x [u^n(s+\delta) - u^n(s)] \)).
\( \delta - u^\alpha(s) \rangle \rangle \) and \( (3.27) \) provide us
\[
\int_\Omega \| u^n(s + \delta) - u^n(s) \|_{L^2(\Omega)}^2 dx \leq \\
\frac{1}{\theta_\varepsilon} \left\{ \int_s^{s+\delta} \int_\Omega \tilde{g}^n(t) u^n(s) \otimes \tilde{u}^n(t) \cdot \nabla_x [u^n(s + \delta) - u^n(s)] dx dt \\
+ \int_s^{s+\delta} \int_\Omega \tilde{g}^n(t) u^n(t) \otimes \tilde{u}^n(t) \cdot \nabla_x [u^n(s + \delta) - u^n(s)] dx dt \\
- \int_s^{s+\delta} \int_\Omega \tilde{g}^n(t)[\nabla_x u^n(s)] \tilde{u}^n(t) \cdot [u^n(s + \delta) - u^n(s)] dx dt \\
- \int_s^{s+\delta} \int_\Omega \mathbb{S}(x, \mathbb{D}[u^n(t)]) : \mathbb{D}[u^n(s + \delta) - u^n(s)] dx dt \\
+ \int_s^{s+\delta} \int_\Omega \tilde{g}^n(t) f^n(t) \cdot [u^n(s + \delta) - u^n(s)] dx dt \right\}.
\]

Integrating over \((0, T - \delta)\) w.r.t. time \(s\) we can show that for any of the ten addends \(I_k(s), k = 1, 2, \ldots, 10\) in the right-hand side of \(3.50\), the following estimates are valid
\[
\int_0^{T-\delta} I_k(s) ds \leq \kappa_k \delta^{1/q} \quad k = 1, 2, \ldots, 10,
\]
with constant \(\kappa_k\) independent of \(\delta\). To estimate first six integrals let us employ \(3.27\), Fubini's theorem, Hölder inequality, the fact that \(q = 2p'\), \((3.39)\). Since \(\tilde{u}^n \in B\) and \((3.33)\) is satisfied, we obtain for one representative term
\[
| \int_0^{T-\delta} \int_s^{s+\delta} \int_\Omega \tilde{g}^n(t) u^n(s) \otimes \tilde{u}^n(t) \cdot \nabla_x u^n(s + \delta) dx dt ds |
\leq \tilde{g}^* \int_0^{T-\delta} \int_s^{s+\delta} \| u^n(s) \|_{L^q(\Omega)} \| \tilde{u}^n(t) \|_{L^q(\Omega)} \| \nabla_x u^n(s + \delta) \|_{L^p(\Omega)} dt ds \\
\leq \tilde{g}^* \int_0^{T} \| \tilde{u}^n(t) \|_{L^q(\Omega)} \int_{\max(0, t-\delta)}^{\min(T-\delta, t)} \| u^n(s) \|_{L^q(\Omega)} \| \nabla_x u^n(s + \delta) \|_{L^p(\Omega)} ds dt \\
\leq \tilde{g}^* \delta^{1/q} \int_0^{T} \| \tilde{u}^n(t) \|_{L^q(\Omega)} dt \| u^n \|_{L^q(0, T; L^p(\Omega))} \| \nabla_x u^n \|_{L^p(0, T; L^p(\Omega))} \\
\leq \kappa_4 \delta^{1/q}.
\]

Since approximate solution is in the form \((3.18)\), \(\alpha^n \in C^1([0, T])\) and \((3.17), (3.35)\) are satisfied, using Hölder inequality, we get the following estimate
\[
| \int_0^{T-\delta} \int_s^{s+\delta} \mathbb{S}(x, \mathbb{D}[u^n(t)]) : \mathbb{D}[u^n(s + \delta)] dx dt ds |
\leq \int_0^{T} \| \mathbb{S}(x, \mathbb{D}[u^n(t)]) \|_{L^1(\Omega)} \int_{\max(0, t-\delta)}^{\min(T-\delta, t)} \| \mathbb{D} u^n(s + \delta) \|_{L^\infty(\Omega)} ds dt \quad (3.53)
\leq \kappa_2(n) \delta,
\]

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where $\kappa_2$ is not uniform w.r.t. $n$. Using assumptions on $f^n$ and (3.33) we can deduce

$$\left| \int_0^T \int_s^{s+\delta} \int_{\Omega} \bar{\rho}^n(t) f^n(t) \cdot u^n(s+\delta) dxdt ds \right| \leq \bar{\rho}^n \int_0^T \|f^n(t)\|_{W^{-1,p}} \int_{\max(0,t-\delta)}^{\min(T-\delta,t)} \|u^n(s+\delta)\|_{W^{1,p}} ds dt \leq \kappa_3 \delta^{1/p'}.$$  \tag{3.54}

We proceed with the second source term in similar way. Summarizing all above estimates for integrals on the right-hand side of (3.50) we prove (3.48) and existence of approximate solution.

### 3.2 Strong convergence of $\rho^n$ and $u^n$

Using (3.27), (3.28) and (3.45) Aubin Lions Lemma provides that

$$\rho^n \rightarrow \rho \quad \text{strongly in } C([0,T]; W^{-1,5p/3}). \tag{3.55}$$

To show

$$\rho^n u^n \rightharpoonup \rho u \quad \text{weakly in } L^q(Q) \text{ for all } q \in [1, 5p/6] \tag{3.56}$$

we use Div-Curl Lemma.

Let take $a^n = (\rho^n, \rho^n u^n_1, \rho^n u^n_2, \rho^n u^n_3)$ and $b^n = (u^n_i, 0, 0, 0)$, $i \in \{1, 2, 3\}$ fixed. Since (3.28), (3.42) and (3.41) we obtain respectively

$$a^n \rightharpoonup (\rho, \overline{\rho u}_1, \overline{\rho u}_2, \overline{\rho u}_3) \quad \text{weakly in } L^q(Q) \text{ for all } q \in [1, 5p/3], \tag{3.57}$$

$$b^n \rightharpoonup (u_i, 0, 0, 0) \quad \text{weakly in } L^{5p/3}(Q).$$

As

$$\text{Div}_{t,x} a^n = \partial_t \rho^n + \text{div}_x (\rho^n u^n) = 0$$

$$\text{Curl}_{t,x} b^n = \begin{pmatrix} 0 \\ - (\nabla_x u^n_i)^T \\ \nabla_x u^n_i \\ 0 \end{pmatrix}$$

and $\nabla_x u^n$ is bounded in $L^p(Q)$ and since $L^p(Q) \hookrightarrow W^{-1,p}(Q)$ we obtain by Curl-Div Lemma that (3.56) holds. Hence by (3.42), (3.56 holds for all $q \in [1, 5p/3]$).

From (3.45) and (3.56) we conclude that $\rho$ and $u$ satisfy (1.9). This in turn implies, by using test function of the form $1_{[t_0, t]} h$, $h \in W^{1,5p/(5p-3)}$ in (1.9), partial integration w.r.t. time and the density $W^{1,5p/(5p-3)}$ in $L^1$ that $\rho \in C([0,T]; L^\infty_{\text{weak}})$, i.e. for all $h \in L^1$ and all $0 \leq t_0 \leq T$ we have

$$\lim_{t \to t_0} (\rho(t), h) = (\rho(t_0), h). \tag{3.58}$$

Using the fact that $\text{div}_x u^n = 0$ and concept of renormalized solution to the equation (3.24), it is possible to strengthen (3.55) and (3.58). If we employ the same steps like Lions in [13], we will be able to deduce that

$$\rho^n \rightarrow \rho \quad \text{strongly in } C([0,T]; L^q(\Omega)) \text{ for all } q \in [1, \infty) \text{ and a.e. in } Q. \tag{3.59}$$
and also
\[ \lim_{t \to 0^+} \| \varrho(t) - \varrho_0 \|_{L^q(\Omega)} = 0 \quad \text{for all } q \in [1, \infty), \] (3.60)
which is the first part of initial condition (1.11).

Let us consider now convergence of \( u^n \) and related sequences. According to (3.44) and (3.56) we obtain
\[ \partial_t (\varrho^n u^n) \rightharpoonup \partial_t (\varrho u) \quad \text{weakly in } L^{p'}(0, T; W^{-1,p'}_{\text{div}}). \] (3.61)
Since (3.37), (3.38) and (3.59) hold we can see
\[ \sqrt{\varrho^n u^n} \rightharpoonup \sqrt{\varrho u} \quad \text{weakly in } L^2(\Omega). \] (3.62)
Let us note that Aubin-Lions Lemma (see [20]), (3.37), (3.27), (3.44) and (3.56) imply that
\[ \varrho^n u^n \rightharpoonup \varrho u \quad \text{strongly in } C(0, T; W^{-1,p'}_{\text{div}}). \] (3.63)
From this and (3.34) it follows that (as \( n \to \infty \))
\[ \int_0^T (\varrho^n u^n, u^n) dt = \int_0^T \langle \varrho^n u^n, u^n \rangle_{(W^{-1,p'}_{\text{div}}, W^{1,p}_{\text{div}})} dt \to \int_0^T \langle \varrho u, u \rangle_{(W^{-1,p'}_{\text{div}}, W^{1,p}_{\text{div}})} dt \]
\[ = \int_0^T (\varrho u, u) dt, \] (3.64)
i.e. \( \| \sqrt{\varrho^n u^n} \|_{L^2(\Omega)} \to \| \sqrt{\varrho u} \|_{L^2(Q)} \). This together with (3.62) imply
\[ \sqrt{\varrho^n u^n} \to \sqrt{\varrho u} \quad \text{strongly in } L^2(\Omega) \] (3.65)
and
\[ \sqrt{\varrho^n u^n}(t) \to \sqrt{\varrho u}(t) \quad \text{strongly in } L^2(\Omega) \text{ for a.a. } t \in [0, T]. \] (3.66)
Using (3.39), (3.59) and (3.65), we come to conclusion that
\[ u^n \to u \quad \text{strongly in } L^q(Q) \text{ for all } q \in [1, 5p/3) \text{ and a.e. in } Q. \] (3.67)
Using this and (3.28) we easily check that for \( p > 6/5 \)
\[ \varrho^n u^n \otimes u^n \to \varrho u \otimes u \quad \text{weakly in } L^q(0, T; W^{-1,q'}_{\text{div}}) \text{ for } q \text{ sufficiently large.} \]
Employing density argument the above property, (3.28) and (3.34) imply that
\[ \varrho^n u^n \otimes u^n \to \varrho u \otimes u \quad \text{weakly in } L^{p'}(0, T; W^{-1,p'}_{\text{div}}) \text{ for } p \geq 11/5. \] (3.68)
3.3 Continuity w.r.t. time in weak topology and initial condition

Using convergence results established in previous subsections, particularly (3.61), (3.68), (3.36) and (3.22) we can take the limit \( n \to \infty \) in (3.23). Hence taking as a test function \( \varphi \) of the form \( \varphi(t, x) = \mathbb{1}_{[s_1, s_2]}(t) \tilde{\varphi}(x) \), \( \tilde{\varphi} \in V_s \), \( s_1, s_2 \in [0, T] \) we obtain

\[
(q(s_2)u(s_2), \tilde{\varphi}) - (q(s_1)u(s_1), \tilde{\varphi}) = \int_{s_1}^{s_2} (qu \otimes u, \nabla \tilde{\varphi}) - (\mathbb{S}, \mathbb{D} \tilde{\varphi}) + (qf, \tilde{\varphi}) dt.
\]

Since (3.27), (3.37), (3.35), we have

\[
|\langle q(s_2)u(s_2), \tilde{\varphi} \rangle - \langle q(s_1)u(s_1), \tilde{\varphi} \rangle| \\
\leq \left( \int_{s_1}^{s_2} q^* \|u\|_{L^2(\Omega)}^2 + \|\mathbb{S}\|_{L^1(\Omega)} + q^* \|f\|_{L^1(\Omega)} dt \right) \|\tilde{\varphi}\|_{L^\infty(\Omega)} \\
\leq \left( q^* |s_2 - s_1| \sup_{t \in [0, T]} \|u(t)\|_{L^2(\Omega)}^2 + \int_{s_1}^{s_2} \|\mathbb{S}\|_{L^1(\Omega)} dt \\
+ q^* |s_2 - s_1|^{1/p} \left( \int_{s_1}^{s_2} \|f\|_{W^{-1, p'}(\Omega)} dt \right)^{1/p'} \right) \|\tilde{\varphi}\|_{V_s}
\]

(3.69)

All integrals and norms are finite, therefore (3.69) leads to conclusion that \( \langle q(\cdot)u(\cdot), \tilde{\varphi} \rangle \) is continuous at \( s_1 \) for all \( \tilde{\varphi} \in V_s \) (in other words, \( qu \in C([0, T]; (V_s)^{\ast\text{weak}}) \) or \( \lim_{s_2 \to s_1} (q(s_2)u(s_2) - q(s_1)u(s_1), \tilde{\varphi}) = 0 \).

Since \( u \in L^\infty(0, T; H) \), \( q \in C([0, T]; L^q(\Omega)) \) for \( q \in [1, \infty) \) and \( V_s \) is dense in \( H \), we observe that \( qu \in C([0, T]; H^{\text{weak}}) \). As a consequence we have

\[
\lim_{s_1 \to 0} (q(s_1)u(s_1) - \varrho_0 u_0, \tilde{\varphi}) = 0 \quad \text{for all} \quad \tilde{\varphi} \in H.
\]

(3.70)

Taking the limit \( n \to \infty \) in (3.29) and integrating it over time interval \((0, s_1)\) we obtain

\[
(q(s_1), |u(s_1)|^2) - (q(s_1), |u(s_1)|^2) = 2 \int_0^{s_1} -(\mathbb{S}, \mathbb{D}u) + (qf, u) dt.
\]

(3.71)

If we employ obvious identity

\[
\|\sqrt{q(s_1)}(u(s_1) - u_0)\|_{L^2(\Omega)}^2 = (q(s_1), |u(s_1)|^2) - 2(q(s_1)u(s_1), u_0) + (q(s_1), |u_0|^2),
\]

(3.72)

then the second part of property (1.11) is an easy consequence of (3.71) and the following operations

\[
\|\sqrt{q(s_1)}(u(s_1) - u_0)\|_{L^2(\Omega)}^2 = (q(s_1), |u(s_1)|^2) - 2(q(s_1)u(s_1), u_0) + (q(s_1), |u_0|^2) \\
= (q(s_1), |u(s_1)|^2) - (q_0, |u_0|^2) - 2(q(s_1)u(s_1) - q_0 u_0, u_0) + (q(s_1) - q_0, |u_0|^2) \\
= 2 \int_0^{s_1} -(\mathbb{S}, \mathbb{D}u) + (qf, u) dt - 2(q(s_1)u(s_1) - q_0 u_0, u_0) + (q(s_1) - q_0, |u_0|^2).
\]

(3.73)
Letting \( s_1 \to 0^+ \) in (3.73) using (3.70), (3.58) and the fact that \((\mathcal{F},\mathbb{D}[\mathbf{u}])\) is nonnegative (because of monotonicity) and \((\varrho \mathbf{f}, \mathbf{u}) \in L^1(0,T;L^1(\Omega))\) we can conclude
\[
\lim_{s_1 \to 0} \| \sqrt{\varrho(s_1)}(\mathbf{u}(s_1) - \mathbf{u}_0) \|_{L^2(\Omega)}^2 = 0. \tag{3.74}
\]
Hence this implies together with (3.27) the second part of (1.11). Above arguments and (3.73), (3.74) provide also the fact which we will use later
\[
\lim_{s_1 \to 0} \left( \varrho(s_1), |\mathbf{u}(s_1)|^2 \right) = \left( \varrho_0, |\mathbf{u}_0|^2 \right). \tag{3.75}
\]

### 3.4 Integration by parts

We multiply each equation in system (3.23) by \( \tilde{\sigma}_h^+ * ((\tilde{\sigma}_h^- * \alpha(t)) \mathbf{1}_{(s_0,s)}) \) and sum over \( i = 1, \ldots, j \). Let
\[
\mathbf{u}^{h,j} \overset{\text{def}}{=} \tilde{\sigma}_h^+ * ((\tilde{\sigma}_h^- * \mathbf{u}^j) \mathbf{1}_{(s_0,s)})
\]
with \( h \leq \min\{s_0,T-s\} \). Since
\[
\int_0^T \left\langle \partial_t(\varrho \mathbf{u}^n), \mathbf{u}^{h,j} \right\rangle dt = \int_0^T \int_\Omega \partial_t(\tilde{\sigma}_h^- * (\varrho \mathbf{u}^n)) \cdot ((\tilde{\sigma}_h^- * \mathbf{u}^j) \mathbf{1}_{(s_0,s)}) dx dt,
\]
after integrating over \((0,T)\) we get in the limit as \( n \to \infty \)
\[
\int_{s_0}^s \int_\Omega (\partial_t(\tilde{\sigma}_h^- \varrho \mathbf{u})) \cdot (\tilde{\sigma}_h^- * \mathbf{u}^j) dx dt - \int_0^T \int_\Omega (\varrho \mathbf{u} \otimes \mathbf{u}) : \nabla \mathbf{u}^{h,j} dx dt
\]
\[
+ \int_0^T \int_\Omega \mathbb{S} : \mathbb{D} \mathbf{u}^{h,j} dx dt = - \int_0^T \int_\Omega \varrho \mathbf{f} \cdot \mathbf{u}^{h,j} dx dt. \tag{3.76}
\]

For any function \( z \) and for \( h > 0 \) from above we denote
\[
(\tilde{\sigma}_h^+ * z)(t,x) := \frac{1}{h} \int_0^h z(t + \tau, x) d\tau,
\]
\[
(\tilde{\sigma}_h^- * z)(t,x) := \frac{1}{h} \int_{-h}^0 z(t + \tau, x) d\tau.
\]
and let us define
\[
D^+ h z := \frac{z(t + h, x) - z(t, x)}{h},
\]
\[
D^- h z := \frac{z(t, x) - z(t - h, x)}{h}.
\]
It is easy to observe that
\[ \partial_t(\tilde{\sigma}_h^+ * z) = D^+ h z \quad \text{and} \quad \partial_t(\tilde{\sigma}_h^- * z) = D^- h z. \] (3.77)

Our aim is to use a test function in (3.76)
\[ u^h \overset{\text{def}}{=} \tilde{\sigma}_h^+ ((\tilde{\sigma}_h^- * u) 1\mathbb{I}_{(s_0,s)}) \]
with \( 0 < h < \min\{s_0, T - s\} \). For this purpose define the truncation operator \( \bar{T}_m : \mathbb{R}^{3 \times 3} \to \mathbb{R}^{3 \times 3} \) such that
\[ \bar{T}_m(\mathbb{K}) = \begin{cases} \mathbb{K} & |\mathbb{K}| \leq m, \\ m \frac{\mathbb{K}}{|\mathbb{K}|} & |\mathbb{K}| > m. \end{cases} \]

The sequence of functions \( \{u^{h,j}\} \) is weakly convergent to \( u^h \) in \( L^p(0, T; W^1_{0,\text{div}}(\Omega)) \) as \( j \to \infty \). Observe the following identity
\[
\int_{s_0}^s \int_{\Omega} (\partial_t(\tilde{\sigma}_h^+ * (\varrho u)) \cdot (\tilde{\sigma}_h^- * u^j)) dx dt = \int_0^T \int_{\Omega} (\varrho u \otimes u) : \nabla_x u^{h,j} dx dt \\
+ \int_0^T \int_{\Omega} (\bar{T}_m(\mathbb{S}) - \mathbb{S}) : \mathbb{D} u^{h,j} dx dt \\
- \int_0^T \int_{\Omega} \bar{T}_m(\mathbb{S}) : \mathbb{D} u^{h,j} dx dt \\
+ \int_0^T \int_{\Omega} \varrho f \cdot u^{h,j} dx dt.
\] (3.78)

Note that if \( p \geq \frac{11}{5} \), then the term \( \int_0^T \int_{\Omega} (\varrho u \otimes u) \cdot \nabla_x u^{h,j} dx dt \to \int_0^T \int_{\Omega} (\varrho u \otimes u) \cdot \nabla_x u^h dx dt \) as \( j \to \infty \). We fix \( k \in \mathbb{N} \) and using Fenchel-Young inequality, the convexity of \( M \) and that \( M^* \) satisfies \( \Delta_2 \)-condition we estimate the integral
\[
\int_0^T \int_{\Omega} |(\bar{T}_m(\mathbb{S}) - \mathbb{S}) : \mathbb{D} u^{h,j}| dx dt \leq \int_0^T \int_{\Omega} M^*(x, 2^k(\bar{T}_m(\mathbb{S}) - \mathbb{S})) dx dt \\
+ \int_0^T \int_{\Omega} M(x, \frac{1}{2^k} \mathbb{D} u^{h,j}) dx dt \\
\leq C_k M_* \int_0^T \int_{\Omega} M^*(x, \bar{T}_m(\mathbb{S}) - \mathbb{S}) dx dt \\
+ k \int_0^T \int_{\Omega} h_{M_*}(x) \mathbb{I}_{(|\mathbb{S}(t,x)| > m)} dx dt \\
+ \frac{1}{2^k} \int_0^T \int_{\Omega} M(x, \mathbb{D} u^{h,j}) dx dt.
\] (3.79)

Note that for each \( h > 0 \)
\[
\lim_{k \to \infty} \frac{1}{2^k} \sup_{j \in \mathbb{N}} \int_0^T \int_{\Omega} M(x, \mathbb{D} u^{h,j}) dx dt = 0. \] (3.80)
Due to the convexity of $M^*$, symmetry and that $M^*(x,0) = 0$ a.e. it holds

$$M^*(x, \mathcal{T}_m(\overline{\Omega}) - \overline{\Omega}) \leq M^*(x, \overline{\Omega}).$$

Since (1.13) holds it is easy to see that there exists some $\tilde{c} > 0$ such that

$$M^*(x, \xi) \leq \tilde{c} |\xi|'.$$

Hence $M^*$ satisfies $\Delta_2$-condition, therefore $\overline{\Omega}$ is an element of $L^*_{M^*}(Q)$ and above inequality yields by the Lebesgue convergence theorem that $\int_{Q} M^*(x, \mathcal{T}_m(\overline{\Omega}) - \overline{\Omega}) dx dt$ converges to zero as $m \to \infty$. Hence

$$\lim_{k \to \infty} \lim_{m \to \infty} \int_{s_0}^{s} \int_{\Omega} C^k_{M^*} M^*(x, \mathcal{T}_m(\overline{\Omega}) - \overline{\Omega}) + k h_{M^*}(x) \mathbb{1}_{(\overline{\Omega}(t,x)) > m} dx dt = 0. \quad (3.81)$$

Since $f \in L^2$ and $\varrho$ is bounded then $\int_0^T \int_{\Omega} \varrho f \cdot u^h dx dt \to \int_0^T \int_{\Omega} \varrho f \cdot u^h dx dt$.

Now we will concentrate on the left hand-side term of (3.76). Letting $j \to \infty$ we obtain

$$L_h := \int_{s_0}^{s} \int_{\Omega} (\partial_t (\tilde{\sigma}_h^* (\varrho u)) \cdot (\tilde{\sigma}_h^* u)) dx dt.$$

Using (3.77) we get

$$L_h = \int_{s_0}^{s} \int_{\Omega} (D^{-h}_x (\varrho u) \cdot (\tilde{\sigma}_h^* u)) dx dt. \quad (3.82)$$

Moreover notice that

$$L_h = \int_{s_0}^{s} \int_{\Omega} (\varrho D^{-h}_x u) \cdot (\tilde{\sigma}_h^* u) dx dt + \int_{s_0}^{s} \int_{\Omega} ((D^{-h}_x \varrho) u(t-h)) \cdot (\tilde{\sigma}_h^* u) dx dt

= \int_{s_0}^{s} \int_{\Omega} \varrho \cdot \frac{1}{2} \partial_t (\tilde{\sigma}_h^* u)^2 dx dt + \int_{s_0}^{s} \int_{\Omega} (\tilde{\sigma}_h^* (\varrho u)) \cdot (\nabla_x (u(t-h) \cdot (\tilde{\sigma}_h^* u))) dx dt,$$

where we used (3.77) and relation $D^{-h}_x \varrho = -\text{div}_x (\tilde{\sigma}_h^* (\varrho u))$, which is provided by the fact that the couple $(\varrho, u)$ solves continuity equation $\partial_t \varrho + \text{div}_x (\varrho u) = 0$ in a weak sense. Inserting $z = \frac{1}{2} |\tilde{\sigma}_h^* u|^2$ into weak formulation of the continuity equation, which means that for all $s_0, s \in [0, T]$

$$\int_{s_0}^{s} \int_{\Omega} (\varrho(z) \cdot \partial_t z + \varrho(z) u(t) \cdot \nabla_x z(t)) dx dt = \int_{\Omega} \varrho(s) \cdot z(s) dx - \int_{\Omega} \varrho(s_0) \cdot z(s_0) dx$$

(for all $z \in L^r(0, T; W^{1,r})$ with $r = 5p/(5p-3)$ and $\partial_t z \in L^{1+\delta}(0, T; L^{1+\delta})$) we obtain

$$L_h = \int_{\Omega} \varrho(s) \cdot (\frac{1}{2} |\tilde{\sigma}_h^* u|^2) dx - \int_{\Omega} \varrho(s_0) \cdot (\frac{1}{2} |\tilde{\sigma}_h^* u|^2) dx$$

$$- \int_{s_0}^{s} \int_{\Omega} (\varrho u) \cdot (\frac{1}{2} \nabla_x |\tilde{\sigma}_h^* u|^2) dx dt + \int_{s_0}^{s} \int_{\Omega} (\tilde{\sigma}_h^* (\varrho u)) \cdot (\nabla_x [u(t-h) \cdot (\tilde{\sigma}_h^* u)]) dx dt$$

$$\quad (3.84)$$
Applying Proposition 4.2 allows to conclude

\[ \lim_{h \to 0^+} L_h = \int_{s_0}^s \int_\Omega (\rho u) \cdot (\frac{1}{2} \nabla_x |u|^2) dx dt \]

\[ + \frac{1}{2} \int_\Omega \rho(s, x)|u(s, x)|^2 dx - \frac{1}{2} \int_\Omega \rho(s_0, x)|u(s_0, x)|^2 dx \]

\[ = \int_{s_0}^s \int_\Omega \rho u \otimes u \cdot \nabla_x u dx dt \]

\[ + \frac{1}{2} \int_\Omega \rho(s, x)|u(s, x)|^2 dx - \frac{1}{2} \int_\Omega \rho(s_0, x)|u(s_0, x)|^2 dx. \]  

(3.85)

Next, we will investigate the convergence of the term \( \int_{s_0}^s \int_\Omega (\rho u \otimes u \cdot \nabla_x u^h) dx dt. \) Since condition (1.13) provides that \( D u \in L^p(0, T; L^p(\Omega)) \) and due to Korn’s inequality \( \nabla u \in L^p(0, T; L^p(\Omega)) \) also the sequence \( \nabla_x u^h = \nabla_x (\tilde{\sigma}_h^* \otimes (\tilde{\sigma}_h^- * u) \mathbb{1}_{(s_0, s)}) \) is uniformly bounded in \( L^p(0, T; L^p(\Omega)) \). Hence we obtain, for subsequence if needed,

\[ \lim_{h \to 0^+} \int_{s_0}^s \int_\Omega (\rho u \otimes u \cdot \nabla_x u^h) dx dt = \int_{s_0}^s \int_\Omega (\rho u \otimes u \cdot \nabla_x u) dx dt. \]  

(3.86)

Let us concentrate now on the term

\[ \int_0^T \int_\Omega \tilde{\sigma}_h^+ \cdot ((\tilde{\sigma}_h^- \cdot D[u]) \mathbb{1}_{(s_0, s)}) dx dt = \int_{s_0}^s \int_\Omega (\tilde{\sigma}_h^- \cdot \tilde{\sigma}_h^+ \cdot D[u]) dx dt. \]

Sequences \( \{\tilde{\sigma}_h^- \cdot \tilde{\sigma}_h^+\} \) and \( \{\tilde{\sigma}_h^- \cdot D[u]\} \) converge in measure in \( Q \) due to Proposition 4.3. Moreover, since \( M \) and \( M^* \) are convex nonnegative functions, then the weak lower semicontinuity and estimate (3.31) provide that the integrals

\[ \int_0^T \int_\Omega M(x, D[u]) dx dt \quad \text{and} \quad \int_0^T \int_\Omega M^*(x, \tilde{\sigma}_h^- \cdot \tilde{\sigma}_h^+ \cdot D[u]) dx dt \]

are finite. Hence Proposition 4.4 implies that the sequences \( \{\tilde{\sigma}_h^- \cdot \tilde{\sigma}_h^+\} \) and \( \{\tilde{\sigma}_h^- \cdot D[u]\} \) are uniformly integrable and hence according to Lemma 4.2 we have

\[ \tilde{\sigma}_h^- \cdot\cdot D[u] \xrightarrow{M} D[u] \quad \text{in} \quad L_M(Q), \]

\[ \tilde{\sigma}_h^- \cdot\cdot \tilde{\sigma}_h^+ \xrightarrow{M^*} \tilde{\sigma}_h^- \quad \text{in} \quad L_{M^*}(Q). \]  

(3.87)

Applying Proposition 4.2 allows to conclude

\[ \lim_{h \to 0^+} \int_{s_0}^s \int_\Omega (\tilde{\sigma}_h^- \cdot \tilde{\sigma}_h^+ \cdot D[u]) dx dt = \int_{s_0}^s \int_\Omega \tilde{\sigma}_h^- \cdot D[u] dx dt. \]  

(3.88)

We treat the source term in similar way like convective one. Note that estimate (3.37)1 provides that \( u \in L^2(0, T; L^2(\Omega)) \) and consequently the sequence \( \{u^h\} \) is
uniformly bounded in $L^2(0,T; L^2(\Omega))$. Since $f \in L^2(0,T; L^2(\Omega))$ and $\varrho$ satisfies (3.27) we can conclude

$$\lim_{h \to 0^+} \int_{s_0}^s \int_\Omega (\varrho f) \cdot u^h dx dt = \int_{s_0}^s \int_\Omega \varrho f \cdot u dx dt.$$  

(3.89)

Summarizing arguments (3.85), (3.88) and (3.89) we are able to pass to the limit in (3.76) and we obtain

$$\frac{1}{2} \int_\Omega \varrho(s,x)|u(s,x)|^2 dx + \int_{s_0}^s \int_\Omega \mathbb{S} : \mathbb{D}[u] dx dt = \int_{s_0}^s \int_\Omega \varrho f \cdot u dx dt + \frac{1}{2} \int_\Omega \varrho(s_0, x)|u(s_0, x)|^2 dx.$$ 

(3.90)

Using the property (3.75) and letting $s_0 \to 0$ we obtain

$$\frac{1}{2} \int_\Omega \varrho(s,x)|u(s,x)|^2 dx + \int_{s_0}^s \int_\Omega \mathbb{S} : \mathbb{D}[u] dx dt = \int_{s_0}^s \int_\Omega \varrho f \cdot u dx dt + \frac{1}{2} \int_\Omega \varrho(s_0, x)|u_0(x)|^2 dx.$$ 

(3.91)

Hence integrating (3.29) over the interval $(0,s)$ allows to conclude that

$$\limsup_{n \to \infty} \int_{Q^s} \mathbb{S}(x, \mathbb{D}[u^n]) : \mathbb{D}[u^n] dx dt = \int_{Q^s} \mathbb{S} : \mathbb{D}[u] dx dt.$$ 

(3.92)

### 3.5 Monotonicity method

By $Q^s$ we will mean the set $(0,s) \times \Omega$. Since $\mathbb{S}$ is monotone, then we have

$$\int_{Q^s} (\mathbb{S}(x, w) - \mathbb{S}(x, \mathbb{D}[u^n])) : (w - \mathbb{D}[u^n]) dx dt \geq 0$$ 

(3.93)

for all $w \in L^\infty(Q)^{3 \times 3}$. Observe that also $\mathbb{S}(x, w) \in L^\infty(Q)^{3 \times 3}$. We prove it by contradiction, i.e. let us assume that $\mathbb{S}(x, w)$ is unbounded. Then, since $M$ is nonnegative, by (1.5), it holds

$$|w| \geq \frac{M^*(x, \mathbb{S}(x, w))}{|\mathbb{S}(x, w)|}.$$ 

The right-hand side tends to infinity as $|\mathbb{S}(x, w)| \to \infty$ by (2.15), which contradicts that $w \in L^\infty(Q)^{3 \times 3}$. Before passing to the limit with $n \to \infty$, we rewrite (3.93)

$$\int_{Q^s} \mathbb{S}(x, \mathbb{D}[u^n]) : \mathbb{D}[u^n] dx dt \geq \int_{Q^s} \mathbb{S}(x, \mathbb{D}[u^n]) : w dx dt + \int_{Q^s} \mathbb{S}(x, w) : (\mathbb{D}[u^n] - w) dx dt$$ 

(3.94)

hence

$$\int_{Q^s} \mathbb{S} : \mathbb{D}[u] dx dt \geq \int_{Q^s} \mathbb{S} : w dx dt + \int_{Q^s} \mathbb{S}(x, w) : (\mathbb{D}[u] - w) dx dt$$ 

(3.95)
and consequently
\[ \int_{Q^s} (S(x, w) - \overline{S}) : (w - D[u]) dx dt \geq 0. \] (3.96)

Let \( k > 0 \) and denote by
\[ Q_k = \{(t, x) \in Q^s : |D[u](t, x)| \leq k \text{ a.e. in } Q^s\} \]
and let \( 0 < j < i \) be arbitrary and \( h > 0 \)

\[ w = (D[u]) 1_{Q_i} + h v 1_{Q_j}, \]

where \( v \in L^\infty(Q)^{3 \times 3} \) is arbitrary. By (3.96)
\[ -\int_{Q^s \setminus Q_i} (S(x, 0) - \overline{S}) : D[u] dx dt + h \int_{Q_j} (S(x, D[u] + h v) - \overline{S}) : v dx dt \geq 0. \] (3.97)

Note that \( S(x, 0) = 0 \). Obviously
\[ \int_{Q^s \setminus Q_i} \overline{S} : D[u] dx dt = \int_Q (\overline{S} : D[u]) 1_{Q^s \setminus Q_i} dx dt \]
and since
\[ \int_Q \overline{S} : D[u] dx dt < \infty \]
then as \( i \to \infty \) we have
\[ (\overline{S} : D[u]) 1_{Q^s \setminus Q_i} \to 0 \text{ a.e. in } Q. \]

Hence by the Lebesgue dominated convergence theorem
\[ \lim_{i \to \infty} \int_{Q^s \setminus Q_i} \overline{S} : D[u] dx dt = 0. \]

Letting \( i \to \infty \) in (3.97) and dividing by \( h \), we get
\[ \int_{Q_j} (S(x, D[u] + h v) - \overline{S}) : v dx dt \geq 0. \]

Since \( D[u] + h v \to D[u] \) a.e. in \( Q_j \) when \( h \to 0^+ \) and \( S(x, D[u] + h v) \) is uniformly bounded in \( L^\infty(Q_j)^{3 \times 3}, |Q_j| < \infty \), hence by Vitali’s lemma we conclude
\[ S(x, D[u] + h v) \to S(x, D[u]) \text{ in } L^1(Q_j) \]
and
\[ \int_{Q_j} (S(x, D[u] + h v) - \overline{S}) : v dx dt \to \int_{Q_j} (S(x, D[u]) - \overline{S}) : v dx dt \]
when $h \to 0^+$. Consequently,

$$\int_{Q_j} (S(x, D[u]) - S) : \mathbf{v} \, dx \, dt \geq 0$$

for all $\mathbf{v} \in L^\infty(Q)^{3 \times 3}$. The choice

$$\mathbf{v} = \begin{cases} 
- \frac{S(x, D[u]) - S}{|S(x, D[u]) - S|} & \text{for } S(x, D[u]) \neq \overline{S}, \\
0 & \text{for } S(x, D[u]) = \overline{S}, 
\end{cases}$$

yields

$$\int_{Q_j} |S(x, D[u]) - \overline{S}| \, dx \, dt \leq 0.$$

Hence

$$S(x, D[u]) = \overline{S} \text{ a.e. in } Q_j.$$  \hfill (3.98)

Since $j$ was arbitrary, (3.98) holds a.e. in $Q$. Since it holds for almost all $s$ such that $0 < s < T$, hence $\overline{S} = S(x, D[u])$ a.e. in $Q$.

### 4 Appendix

#### 4.1 Curl-Div Lemma

Here we recall Curl-Div Lemma which is formulated and reproved in [6], see also [21] for original Tartar’s work. Let $\mathbf{a} = (a_0, a_1, a_2, a_3)$, $\text{Div}_{t,x} \mathbf{a} = \partial_t (a_0) + \sum_{i=1}^3 \partial_x (a_i)$ and $\text{Curl}_{t,x} \mathbf{a} = \nabla_{t,x} \mathbf{a} - (\nabla_{t,x} \mathbf{a})^T$, $\nabla_{t,x} \mathbf{a} = (\partial_t \mathbf{a}^T, \partial_x \mathbf{a}^T, \partial_{x_2} \mathbf{a}^T, \partial_{x_3} \mathbf{a}^T)$.

**Lemma 4.1 (Curl-Div)** Let $Q \subset \mathbb{R}^4$ be a bounded set, $p, q, l, s \in (1, \infty)$ be such that

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{l}.$$

Assume that $\{a^n\}$ and $\{b^n\}$ satisfy

$$a^n \rightharpoonup a \text{ weakly in } L^p(Q)^4$$

$$b^n \rightharpoonup b \text{ weakly in } L^q(Q)^4$$

and $\text{Div}_{t,x} a^n$, $\text{Curl}_{t,x} b^n$ are precompact in $W^{-1,s}(Q)$, $W^{-1,s}(Q)^{4 \times 4}$ respectively. Then

$$a^n \cdot b^n \rightharpoonup a \cdot b \text{ weakly in } L^l(Q).$$
4.2 Properties of Orlicz spaces

For completeness of the paper let us recall some general facts on Orlicz spaces, see e.g. [16] and technical facts which are rewritten from [10] and [12].

**Proposition 4.1 (Fenchel-Young Inequality)** Let $M$ be an $N$–function and $M^*$ a complementary to $M$. Then the following inequality is satisfied

$$|\xi \cdot \eta| \leq M(x, \xi) + M^*(x, \eta)$$

for all $\xi, \eta \in \mathbb{R}^{d \times d}$ and a.a. $x \in \Omega$.

We recall an analogue to the Vitali’s lemma, however for the modular convergence instead of the strong convergence in $L^p$.

**Lemma 4.2** Let $z_j : Q \to \mathbb{R}^d$ be a measurable sequence. Then $z_j \overset{M}{\rightarrow} z$ in $L_M(Q)$ modularly if and only if $z_j \overset{\lambda}{\rightarrow} z$ in measure and there exist some $\lambda > 0$ such that the sequence $\{M(\cdot, \lambda z_j)\}$ is uniformly integrable, i.e.,

$$\lim_{R \to \infty} \left( \sup_{j \in \mathbb{N}} \int_{\{(t,x) : |M(x, \lambda z_j)| \geq R\}} M(x, \lambda z_j) \, dx \, dt \right) = 0.$$ 

**Proof:** Note that $z_j \to z$ in measure if and only if $M(\cdot, \frac{z_j - z}{\lambda}) \to 0$ in measure for all $\lambda > 0$. Moreover the convergence $z_j \to z$ in measure implies that for all measurable sets $A \subset Q$ it holds

$$\lim_{j \to \infty} \int_{A} M(x, z_j) \, dx \, dt \geq \int_{A} M(x, z) \, dx \, dt.$$ 

Note also that the convexity of $M$ implies

$$\int_{A} M \left( x, \frac{z_j - z}{\lambda} \right) \, dx \, dt \leq \int_{A} M \left( x, \frac{z_j}{2\lambda} \right) \, dx \, dt + \int_{A} M \left( x, \frac{z}{2\lambda} \right) \, dx \, dt.$$ 

Hence by the classical Vitali’s lemma for $f^j(x) = M \left( x, \frac{z_j - z}{\lambda} \right)$ we obtain that $f^j \to 0$ strongly in $L^1(Q)$. \hfill \Box

**Lemma 4.3** Let $M$ be an $N$–function and for all $j \in \mathbb{N}$ let $\int_Q M(x, z_j) \, dx \, dt \leq c$. Then the sequence $\{z_j\}$ is uniformly integrable.

**Proof:** Let us define $\delta(R) = \min_{|\xi| = R} \frac{M(x, \xi)}{|\xi|}$. Then for all $j \in \mathbb{N}$ it holds

$$\int_{\{(t,x) : |z_j(t,x)| \geq R\}} M(x, z_j(t,x)) \, dx \, dt \geq \delta(R) \int_{\{(t,x) : |z_j(t,x)| \geq R\}} |z_j(t,x)| \, dx \, dt.$$
Since the left-hand side is bounded, then we obtain
\[
\sup_{j \in \mathbb{N}} \int_{\{(t,x):|z_j(t,x)| \geq R\}} |z_j(t,x)| \, dx \, dt \leq \frac{C}{\delta(R)}.
\]
Using condition (2.15) we obtain uniform integrability. □

**Proposition 4.2** Let \( M \) be an \( N \)-function and \( M^* \) its complementary function. Suppose that the sequences \( \psi^j : Q \rightarrow \mathbb{R}^d \) and \( \phi^j : Q \rightarrow \mathbb{R}^d \) are uniformly bounded in \( L_M(Q) \) and \( L_{M^*}(Q) \) respectively. Moreover \( \psi^j \xrightarrow{M} \psi \) modularly in \( L_M(Q) \) and \( \phi^j \xrightarrow{M} \phi \) modularly in \( L_{M^*}(Q) \). Then \( \psi^j \cdot \phi^j \rightarrow \psi \cdot \phi \) strongly in \( L^1(Q) \).

**Proof:** Due to Lemma 4.2 the modular convergence of \( \{\psi^j\} \) and \( \{\phi^j\} \) implies the convergence in measure of these sequences and consequently also the convergence in measure of the product. Hence it is sufficient to show the uniform integrability of \( \{\psi^j \cdot \phi^j\} \). Notice that it is equivalent with the uniform integrability of the term \( \left\{ \frac{\psi^j}{\lambda_1} \cdot \frac{\phi^j}{\lambda_2} \right\} \) for any \( \lambda_1, \lambda_2 > 0 \). The assumptions of the proposition provide there exist some \( \lambda_1, \lambda_2 > 0 \) such that the sequences
\[
\left\{ M \left( x, \frac{\psi^j}{\lambda_1} \right) \right\} \quad \text{and} \quad \left\{ M^* \left( x, \frac{\phi^j}{\lambda_2} \right) \right\}
\]
are uniformly integrable. Hence let us use the same constants and estimate with the help of Fenchel-Young inequality
\[
\left| \frac{\psi^j}{\lambda_1} \cdot \frac{\phi^j}{\lambda_2} \right| \leq M \left( x, \frac{\psi^j}{\lambda_1} \right) + M^* \left( x, \frac{\phi^j}{\lambda_2} \right).
\]
Obviously the uniform integrability of the right-hand side provides the uniform integrability of the left-hand side and this yields the assertion. □

**Remark:** The same proofs for Proposition 4.3 and 4.4 works if instead of a standard mollifier \( \varrho^j \) we will take
\[
\tilde{\sigma}^+_h = \frac{1}{h} \mathbb{1}(\tau)_{[0,h]} \quad \text{or} \quad \tilde{\sigma}^-_h = \frac{1}{h} \mathbb{1}(\tau)_{[-h,0]}
\]
with \( h > 0 \).

**Proposition 4.3** Let \( \varrho^j \) be a standard mollifier, i.e., \( \varrho \in C^\infty(\mathbb{R}) \), \( \varrho \) has a compact support and \( \int_{\mathbb{R}} \varrho(\tau) \, d\tau = 1, \varrho(t) = \varrho(-t) \). We define \( \varrho^j(t) = j \varrho(jt) \). Moreover let \( * \) denote a convolution in the variable \( t \). Then for any function \( \psi : Q \rightarrow \mathbb{R}^d \) such that \( \psi \in L^1(Q) \) it holds
\[
(\varrho^j \ast \psi)(t,x) \rightarrow \psi(t,x) \quad \text{in measure.}
\]
Proof: For a.a. \( x \in \Omega \) the function \( \psi(\cdot, x) \in L^1(0,T) \) and \( \varrho^j \ast \psi(\cdot, x) \to \psi(\cdot, x) \) in \( L^1(0,T) \) and hence \( \varrho^j \ast \psi \to \psi \) in measure on the set \([0,T] \times \Omega\).

Proposition 4.4 Let \( \varrho^j \) be defined as in Proposition 4.3. Given an N–function \( M \) and a function \( \psi : Q \to \mathbb{R}^d \) such that \( \psi \in L_M(Q) \). Then the sequence \( \{M(x, \varrho^j \ast \psi)\} \) is uniformly integrable.

Proof: We start with an abstract fact concerning the uniform integrability. Namely, the following two conditions are equivalent for any measurable sequence \( \{z^j\} \)

(a) \( \forall \varepsilon > 0 \ \exists \delta > 0 : \sup_{j \in \mathbb{N}} \sup_{|A| \leq \delta} \int_A |z^j(x)| \, dx \, dt \leq \varepsilon \),

(b) \( \forall \varepsilon > 0 \ \exists \delta > 0 : \sup_{j \in \mathbb{N}} \int_Q \left| |z^j(x)| - \frac{1}{\sqrt{\delta}} \right|_+ \, dx \, dt \leq \varepsilon \),

where we use the notation \( |\xi|_+ = \max\{0, \xi\} \).

The implication (a) \( \Rightarrow \) (b) is obvious. To show that also (b) \( \Rightarrow \) (a) holds let us estimate

\[
\sup_{j \in \mathbb{N}} \sup_{|A| \leq \delta} \int_A |z^j| \, dx \, dt \leq \sup_{|A| \leq \delta} |A| \cdot \frac{1}{\sqrt{\delta}} + \sup_{j \in \mathbb{N}} \int_Q \left| |z^j| - \frac{1}{\sqrt{\delta}} \right|_+ \, dx \, dt 
\leq \sqrt{\delta} + \sup_{j \in \mathbb{N}} \int_Q \left| |z^j| - \frac{1}{\sqrt{\delta}} \right|_+ \, dx \, dt.
\]

Notice that since \( M \) is a convex function, then the following inequality holds for all \( \delta > 0 \)

\[
\int_Q \left| M(x, \psi) - \frac{1}{\sqrt{\delta}} \right|_+ \, dx \, dt \geq \int_Q \left| M(x, \varrho^j \ast \psi) - \frac{1}{\sqrt{\delta}} \right|_+ \, dx \, dt. \tag{4.99}
\]

Finally, since \( \psi \in L_M(Q) \), then also \( \int_Q \left| M(x, \psi) - \frac{1}{\sqrt{\delta}} \right|_+ \, dx \, dt \) is finite and hence taking supremum over \( j \in \mathbb{N} \) in (4.99) we prove the assertion.

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