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Abstract. We systematically derive models that would be suitable to describe flows of compressible fluids with the material moduli depending on the symmetric part of the velocity gradient and temperature, within the context of a thermodynamic framework that has been quite successful in developing models to describe the response of bodies that produce entropy while undergoing processes.

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1. Introduction

All real bodies, whether fluid-like or solid-like, are compressible. However in a large class of processes which the body is subject to, which correspond to class of processes that such bodies usually undergo in the real world, it may happen that the volume of every subpart of the body remains nearly constant, that is the body may be approximated in such processes as being incompressible. Under normal operating conditions, most liquids can be approximated as incompressible. However when such liquids are subject to very high pressure one finds that small volume changes can, and do, occur. For instance, Dowson and Higginson [2] provide an explicit formula for the changes in density due to the applied pressure. Gases, on the other hand are compressible though in certain special motions their volume does not change, i.e., compressible gases can undergo isochoric motion. The compressibility of fluids have relevance to the development of shocks and other important phenomena.

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In this short paper, we investigate the development of constitutive relations for compressible fluids within the thermodynamic framework that has been put into place recently by Rajagopal and co-workers (see [11], [12], [13] and [4]).

The thermodynamic framework consists in assuming how the material stores energy, dissipates energy, produces entropy, etc., and then seeking amongst the allowable class of constitutive relations which are appropriate those that maximize the rate of entropy production. We first show how the classical compressible Navier-Stokes fluid model fits in within the context of the above framework. We then proceed to develop the compressible counterparts for the incompressible power-law fluid models, the incompressible Bingham fluid\(^1\), and other non-Newtonian fluid models. The thermodynamic procedure that we adopt has the advantage in that we make constitutive assumptions for scalars such as the stored energy (or Helmholtz potential), and the rate of entropy production and obtain representations for a tensor quantity such as the stress. The efficacy and generality of the procedure is borne out by its success in describing a plethora of disparate response of bodies such as viscoelasticity, twinning, solid to solid phase transition to describe the response of shape memory alloys and shape memory polymers, the response of single crystal super alloys, crystallization of semi-crystalline polymers, inelastic response of solids, to name a few examples (see the review articles [9] and [10] for a discussion of many of these examples).

In the next section, we introduce the basic balance laws and the second law of thermodynamics. We also discuss the procedure of maximization of the rate of entropy production in the section. In section 3 we discuss the development of a variety of compressible fluid models which includes Bingham-like compressible fluids and compressible power-law fluids.

2. Balance equations. Equation for the entropy production

We are interested in the transparent development of a hierarchy of models which are consistent with thermodynamics, in the sense that balance laws for mass, linear and angular momentum and energy, and the second law of thermodynamics are met, that might be capable of describing the nonlinear response of compressible fluids. We recall that the Cauchy stress in the classical compressible Navier-Stokes fluid model bears a linear relationship to the symmetric part of the velocity gradient. Thus, put differently, we intend to develop models to characterize responses that cannot be adequately described either by a compressible Navier-Stokes(-Fourier) fluid (as the body might exhibit one of the following non-Newtonian characteristics, for example the dependence of material moduli on the

\(^1\)A fluid is usually understood as a body that cannot sustain shear. Thus, a fluid body with a threshold for flow that defines a Bingham fluid is a contradiction in terms. What one really means by such a threshold is that the "flow" is so small as to be indiscernible in the time-scale of observation for the specific range of force (stress) that the fluid is subject to, and beyond this threshold the flow of the fluid is easily discernible. However, appropriate time and length scales can be picked within which the fluid will appear to be flowing freely.
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symmetric part of the velocity gradient, stress relaxation, normal stress differences in a simple shear flow, nonlinear creep, etc.), or by incompressible non-Newtonian fluid models because significant volume changes occur. To include both nonlinear response and the fact that the material is compressible seems to be important in applications such as the compaction of asphalt, flows of granular materials, etc. The approach incorporated in this study is based on two concepts: implicit constitutive relationships between the stress and the kinematical variables that remarkably expand our ability to capture disparate material responses without the need for introducing a set of internal variables whose physical underpinnings are not clear (see [5], [6]) and the notion of maximization of the entropy production functional with respect to thermodynamical fluxes in order to select the appropriate constitutive relationships that relate thermodynamical fluxes and thermodynamical affinities (see [11]). This paper extends the methodology that has been developed in [12] within the context of incompressible fluids to fluids that are compressible.

At the continuum level, motions of fluids are usually described in terms of the density \( \rho \), the velocity field \( \mathbf{v} = (v_1, v_2, v_3) \), the internal energy \( e \), the Cauchy stress \( \mathbf{T} = \{T_{ij}\}_{i,j=1}^3 \) and the heat flux \( \mathbf{q} = (q_1, q_2, q_3) \) through the system of equations that are a consequence of the balance of mass, balance of linear and angular momentum, and balance of energy applied to each sub-volume of the material, that lead to local forms for these balance laws in the form

\[
\begin{align*}
\dot{\rho} &= -\rho \text{div} \mathbf{v}, \\
\rho \dot{\mathbf{v}} &= \text{div} \mathbf{T} + \rho \mathbf{f}, \\
\rho \dot{E} &= \text{div}(\mathbf{T} \mathbf{v} - \mathbf{q}) + \rho r + \rho \mathbf{f} \cdot \mathbf{v},
\end{align*}
\]

where \( E = \frac{1}{2} |\mathbf{v}|^2 + e \) (the total energy) is the sum of specific kinetic energy \( \frac{1}{2} |\mathbf{v}|^2 \) and specific internal energy \( e \), \( \mathbf{f} \) represents the specific body forces (such as gravity) and \( r \) stands for the specific sources of energy (radiation, for example).

For any scalar quantity \( z \), the symbol \( \dot{z} \) denotes the material derivative of \( z \) that can be expressed as

\[
\dot{z} = z_t + \nabla z \cdot \mathbf{v} = \frac{\partial z}{\partial t} + \sum_{k=1}^3 \frac{\partial z}{\partial x_k} v_k.
\]

Next, we introduce the second law of thermodynamics and we follow Callen [1] in this regard. We assume the existence of specific entropy \( \eta \) as a function of the state variables (which in our study are \( \rho \) and \( e \) and introduce the specific entropy through \( \eta = \tilde{\eta}(e, \rho) \)) and assume that it satisfies the following properties:

- \( \tilde{\eta} \) is an increasing function with respect to \( e \) and the absolute temperature is given by
  \[
  \theta := \frac{1}{e} \frac{\partial \tilde{\eta}}{\partial e};
  \]

- \( \eta \to 0^+ \) as \( \theta \to 0^+ \).
- \( S(t) := \int q_1(\eta(t)) dt \) tends to its maximum as \( t \to \infty \) provided that the body is thermally and mechanically isolated.
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Note that if one prefers to use \( \eta \) and \( \varrho \) as the state variables the first property implies that
\[
e = \tilde{e}(\eta, \varrho) \quad \text{and} \quad \theta = \frac{\partial e}{\partial \eta}.
\]  
(2.2)

Using (2.2), we can express the time derivative of \( E \) as
\[
\dot{E} = \dot{\varepsilon} \mathbf{v} + \mathbf{v} \cdot \dot{\varepsilon} \mathbf{v} = \theta \dot{\eta} + \frac{\partial \tilde{e}}{\partial \varrho} \dot{\varrho} + \mathbf{v} \cdot \dot{\varepsilon} \mathbf{v},
\]
and thus, using all the equations listed in (2.1), we conclude that
\[
\varrho \dot{\eta} = \text{div} (\mathbf{T} \mathbf{v} - \mathbf{q}) + \varrho \mathbf{f} \cdot \mathbf{v} + \varrho r + \varrho^2 \frac{\partial \tilde{e}}{\partial \varrho} \text{div} \mathbf{v} - \varrho \mathbf{v} \cdot \dot{\varepsilon} \mathbf{v}
\]
(2.3)

where \( \mathbf{D}(\mathbf{v}) \) is the symmetric part of the velocity gradient. Setting
\[
p(\eta, \varrho) = \varrho^2 \frac{\partial \tilde{e}}{\partial \varrho}(\eta, \varrho),
\]
(2.5)

\[
m = \frac{1}{3} \text{tr} \mathbf{T} \quad \text{(mean normal stress)},
\]
(2.6)

\[
\mathbf{S} = \mathbf{T} - \frac{m}{3} \mathbf{I} \quad \text{(deviatoric part of the Cauchy stress)},
\]
(2.7)

dividing the result of (2.3) by \( \theta \) and rewriting \( \text{div} \mathbf{q}/\theta \) in a divergence form we arrive at
\[
\varrho \dot{\eta} + \text{div} (\mathbf{q}/\theta) = \frac{1}{3} \left[ \mathbf{T} \cdot \mathbf{D}(\mathbf{v}) + \varrho r - \frac{2}{3} \cdot \nabla \theta + p(\eta, \varrho) \text{div} \mathbf{v} \right]
\]
\[
= \frac{1}{\theta} \left[ \mathbf{S} \cdot \mathbf{D} + (m + p(\eta, \varrho)) \text{div} \mathbf{v} + \varrho r - \frac{2}{3} \cdot \nabla \theta \right],
\]
(2.8)

where we have used notation \( \mathbf{A}^d := \mathbf{A} - \frac{1}{3} (\text{tr} \mathbf{A}) \mathbf{I} \), and we write \( \mathbf{D}^d \) instead of \( \mathbf{D}^d(\mathbf{v}) \).

\[\text{Note that (2.2) and (2.5) are completely compatible with Gibb’s state equation}
\]
\[
\theta D\eta = De + p D \left( \frac{1}{3} \mathbf{1} \right).
\]
(2.4)

Indeed, considering \( \eta \) and \( \varrho \) as independent variables, and assuming \( e = \tilde{e}(\eta, \varrho) \) and \( p = \tilde{p}(\eta, \varrho) \), it follows from (2.4) that
\[
\theta d\eta = \frac{\partial \tilde{e}}{\partial \eta}(\eta, \varrho) d\eta + \frac{\partial \tilde{e}}{\partial \varrho}(\eta, \varrho) d\varrho - \frac{1}{\varrho^2} p(\eta, \varrho) d\varrho,
\]
or equivalently
\[
[\theta - \frac{\partial \tilde{e}}{\partial \eta}(\eta, \varrho)] d\eta + \left[ \frac{p(\eta, \varrho)}{\varrho^2} - \frac{\partial \tilde{e}}{\partial \varrho}(\eta, \varrho) \right] d\varrho = 0,
\]
which gives trivially the announced equivalence of (2.2), (2.5) and (2.4).
The terms that appear on the right-hand side of (2.8) represent the entropy associated with different physical changes that take place in the body. The second law of thermodynamics says that the production of entropy has to be non-negative\(^3\). More explicitly, on setting
\[
\varrho \xi := \frac{1}{\varrho} [\mathbf{S} \cdot \mathbf{D}^d + (m + p(\eta, \varrho)) \text{div} \mathbf{v} - \frac{q}{\varrho} \cdot \nabla \theta + \varrho r]
\] (2.9)
or
\[
\varrho \theta \xi := \mathbf{S} \cdot \mathbf{D}^d + (m + p(\eta, \varrho)) \text{div} \mathbf{v} - \frac{q}{\varrho} \cdot \nabla \theta + \varrho r ,
\] (2.10)
the second law of thermodynamics implies that
\[
\zeta := \varrho \theta \xi \geq 0. \tag{2.11}
\]

In (2.10), we identify four mechanisms leading to entropy production: the first one associated with isochoric motions such as shear, the second due to volume changes, the third due to conduction of heat and the last one due to radiation. Note that all these terms are in the form of the product of a thermodynamical affinity taken from the set \(\{\mathbf{D}^d, \text{div} \mathbf{v}, \nabla \theta, \varrho\}\) and a corresponding thermodynamical flux taken from the set \(\{\mathbf{S}, (m + p(\eta, \varrho)), q/\varrho, r\}\). It is important to notice that variations of fluxes cause changes in affinities, and not the other way around\(^4\). This point cannot be overemphasized for otherwise we would be turning causality upside down. It is forces (stresses) that cause deformation (and hence velocity gradients) and not vice-versa (see [8] for a discussion of related issues). We refer the reader to [11] for a discussion of these concepts within the thermodynamics of irreversible processes.

In the above setting, the state variables are the density \(\varrho\), the velocity \(\mathbf{v}\) and the entropy \(\eta\) fulfilling the equations (compare with (2.1)\(_1\), (2.1)\(_2\) and (2.8))
\[
\dot{\varrho} = -\varrho \text{div} \mathbf{v}, \quad \varrho \dot{\mathbf{v}} = \nabla m + \text{div} \mathbf{S} + \varrho \mathbf{f}, \quad \mathbf{S} = \mathbf{S}^T, \tag{2.12}
\]
which should be completed by providing relationships between \(\mathbf{S}, m, q\) and \(r\) and the appropriate kinematical and thermal variables.

Interestingly, the same choice of unknowns is used by Feireisl [3] who is not concerned with causality, or affinities and fluxes, when establishing long-time and large date existence theory for the complete thermodynamical system governing

\(^3\)Whether one should require the second law to hold for the body as a whole, or for each subset of the body, thereby leading to a local form of the law, is a matter of contention.

\(^4\)The choice of terminology flux and affinity leaves much to be desired. One might find the terminology "cause" and "effect" to be more appropriate. However, as some scientist might eschew the use of cause and effect on account of their philosophical learnings, the choice "instigator" and "educt" might be more appropriate. However, in view of the current usage we shall continue to use the terms "flux" and "affinity".
the flows of compressible Navier-Stokes-Fourier fluids\footnote{Feireisl deals with the concept of weak solution. Within such a framework Feireisl replaces the equality sign in (2.12) by the inequality $\dot{\rho}_{\frac{1}{2}} \geq \ldots$ and in addition requires, that for flows of a thermally and mechanically isolated fluid that is not slipping at the boundary, the total energy of the weak solution is conserved.}. Neither is the choice for the unknowns conjoined with the philosophy of implicit constitutive relations advocated here nor is the analysis based on the thermodynamic framework that is articulated in this work.

In this paper, we restrict ourselves to processes that take place at uniform temperature (consequently, $\nabla \theta = 0$). We also neglect radiation (i.e., $r = 0$) and consider the special class of materials for which $p = p(\rho)$. Under such circumstances, (2.10) reduces to

$$\zeta = S \cdot D^d + (m + p(\rho)) \text{div} \ v. \tag{2.13}$$

3. Compressible Navier-Stokes fluid

The aim of this section is twofold. In order to explain the usefulness and efficacy of our representation of the classical compressible Navier-Stokes fluid model (these are the equations for div $v$ and $D^d$ below), we use the form (2.13) for expressing the rate of dissipation $\zeta$ in terms of the mean normal stress $m$ and the deviatoric part $S$. Using this form, we show that one can arrive at a more meaningful understanding of the status of the Euler fluid within the larger class. We also see with more clarity the essence and implication of the Stokes hypothesis. This point of view is motivated by the fact that a change in the stress (force per unit area) is the cause for the variations in the velocity gradient, and not other way around. The second goal is then to perform an “inverse” operation: assuming the constitutive relationships for the scalar quantity $\zeta$ (that is of the form derived in the first part), we obtain the constitutive relationship for $T$. This is achieved by maximizing $\zeta$ with respect to $m$ and the components of $S$, and also requiring that the identity (2.13) holds as a constraint. Although the maximization of the entropy production with respect to independent variables (thermodynamical fluxes) in order to identify constitutive relations for tensorial and vectorial quantities has been used already in the late nineties by Rajagopal and Srinivasa, we refer to a more recent work \cite{11} for a detailed exposition.

3.1. Classical compressible Navier-Stokes fluid

In this subsection, we start with the classical definition of a compressible Navier-Stokes fluid and make certain observations concerning the representation for the stress and the restrictions on the material moduli that characterize the fluid.
A compressible Navier-Stokes fluid is characterized, at uniform temperature, by the constitutive equation
\[
\mathbf{T} = -p(\rho)\mathbf{I} + 2\mu(\rho)\mathbf{D} + \lambda(\rho)(\text{div } \mathbf{v})\mathbf{I}
\]
\[
= -p(\rho)\mathbf{I} + 2\mu(\rho)\mathbf{D}^d + \frac{2\mu(\rho) + 3\lambda(\rho)}{3}(\text{div } \mathbf{v})\mathbf{I}.
\]
(3.1)

This relationship includes the constitutive equations for both \(m\) and \(S\) (see (2.6), (2.7) for their definitions), namely
\[
m + p(\rho) = \frac{2\mu(\rho) + 3\lambda(\rho)}{3}(\text{div } \mathbf{v}) \iff \text{div } \mathbf{v} = \frac{3}{2\mu(\rho) + 3\lambda(\rho)}(m + p(\rho))
\]
and
\[
S = 2\mu(\rho)\mathbf{D}^d \iff \mathbf{D}^d = \frac{1}{2\mu(\rho)}S.
\]
(3.2)

While the equations that are written first in (3.2) and (3.3) represent the usual forms in which the constitutive equations for the Navier-Stokes fluid are formulated, the equations that are written as second in (3.2) and (3.3) reflect better the fact that changes in \(m + p(\rho)\) and \(S\) initiate the changes in \(\text{div } \mathbf{v}\) and \(\mathbf{D}^d\), respectively. Thus, these equations express in a more transparent way the fact that \(m + p(\rho)\) and \(S\) are causes while \(\text{div } \mathbf{v}\) and \(\mathbf{D}^d\) are effects. There are however other features that make (3.2)\(_2\), (3.3)\(_2\) preferable to (3.2)\(_1\), (3.3)\(_1\).

Inserting (3.2)\(_2\) and (3.3)\(_2\) into (2.13) we conclude that
\[
\zeta = \frac{1}{2\mu(\rho)}|S|^2 + \frac{3}{2\mu(\rho) + 3\lambda(\rho)}(m + p(\rho))^2.
\]
(3.4)

If
\[
\mu(\rho) > 0 \quad \text{and} \quad 2\mu(\rho) + 3\lambda(\rho) > 0,
\]
then the second law of thermodynamics written in the form (2.11) is met.

Note that it follows from (3.4), (3.5) that
\[
\zeta = 0 \iff S = 0 \quad \text{and} \quad m = -p(\rho),
\]
(3.6)
implying that
\[
\mathbf{T} = -p(\rho)\mathbf{I},
\]
(3.7)
which is the constitutive relation for the Euler (or elastic\(^6\)) fluid.

At this point, it may be worth recalling the traditional manner in which the second law of thermodynamics is incorporated into the constitutive theory: it consists in inserting (3.2)\(_1\) and (3.3)\(_1\) into (2.13) to arrive at
\[
\zeta = 2\mu(\rho)|\mathbf{D}^d|^2 + \frac{2\mu(\rho) + 3\lambda(\rho)}{3}(\text{div } \mathbf{v})^2.
\]
(3.8)

In order to fulfill (2.11) one then requires that (compare with (3.5))
\[
\mu(\rho) \geq 0 \quad \text{and} \quad 2\mu(\rho) + 3\lambda(\rho) \geq 0.
\]
(3.9)

\(^6\)See [7] for details.
Based on (3.8), (3.9) it is sometimes said that the ideal fluid (meaning the fluid that does not dissipate energy) is inviscid if
\[
\mu(g) = \lambda(g) = 0. 
\] (3.10)
In no real fluid are the viscosities zero. However, it is possible in certain flows that the deviatoric part of the Cauchy stress is absent (i.e., $\mathbf{S} = \mathbf{0}$) and the stress is essentially spherical (i.e., given by (3.7)). This is the situation that seems be relevant in boundary layer theory, wherein outside the boundary layer there is no effect of shear, and $\mathbf{S} = \mathbf{0}$ and stress is given by (3.7).

The above discussion provides several arguments that should support the preference of (3.4) with (3.5).

3.2. Rederivation of the compressible Navier-Stokes constitutive relation by means of the maximization of the rate of entropy production

In this subsection, we reverse the previous process and assuming a form for the rate of dissipation $\zeta$ arrive at a representation for the stress $\mathbf{T}$. That is, assuming the constitutive equations for the scalar quantity $\zeta$ of the form (3.4)-(3.5), i.e.,
\[
\zeta = \tilde{\zeta}(g, \mathbf{S}, m) := \frac{1}{2\mu(g)}|\mathbf{S}|^2 + \frac{3}{2\mu(g) + 3\lambda(g)}(m + p(g))^2 \geq 0, 
\] (3.11)
we determine the constitutive relationship for the tensorial quantity $\mathbf{T}$, requiring that (2.13) holds.

To achieve this end, we follow the approach developed by Rajagopal and Srinivasa [11] and [12], and we maximize $\tilde{\zeta}$ with respect to $\mathbf{S}$ and $m$ so that (2.13) holds as the constraint. The constrained optimization procedure leads then to the optimality conditions for an auxiliary function
\[
\tilde{\Phi}(\mathbf{S}, m) := \tilde{\zeta} + \lambda(\tilde{\zeta} - \mathbf{S} : \mathbf{D}^d - (m + p(g)) \text{div} \mathbf{v}),
\]
and these conditions take the form
\[
\begin{align*}
\frac{\partial \tilde{\Phi}}{\partial \mathbf{S}} &= 0 \quad \iff \quad \left(1 + \frac{\lambda}{\mu(g)}\right) \frac{\partial \tilde{\zeta}}{\partial \mathbf{S}} = \mathbf{D}^d, \\
\frac{\partial \tilde{\Phi}}{\partial m} &= 0 \quad \iff \quad \left(1 + \frac{\lambda}{\mu(g)}\right) \frac{\partial \tilde{\zeta}}{\partial m} = \text{div} \mathbf{v}.
\end{align*}
\] (3.12)

Taking the scalar product of (3.12)\textsubscript{1} with $\mathbf{S}$ and multiplying (3.12)\textsubscript{2} by $m + p(g)$, and comparing their sum with (2.13) we conclude that
\[
\frac{1 + \lambda}{\lambda} = \frac{\tilde{\zeta}}{\frac{\partial \zeta}{\partial \mathbf{S}}} \mathbf{S} + \frac{\partial \zeta}{\partial m}(m + p(g)). 
\] (3.13)

Since
\[
\frac{\partial \zeta}{\partial \mathbf{S}} = \frac{1}{\mu(g)} \mathbf{S} \quad \text{and} \quad \frac{\partial \zeta}{\partial m} = \frac{6}{2\mu(g) + 3\lambda(g)}(m + p(g)), 
\] (3.14)
we conclude from (3.13), (3.11) and (3.14) that \( \frac{1+\lambda}{\lambda} = \frac{1}{2} \). This together with (3.12) and (3.14) implies that

\[
D^d = \frac{1}{2\mu(\rho)} S \quad \text{and} \quad \text{div } v = \frac{3}{2\mu(\rho) + 3\lambda(\rho)} (m + p(\rho)),
\]

or in compact form

\[
D := D(v) = \frac{1}{2\mu(\rho)} S + \frac{1}{2\mu(\rho) + 3\lambda(\rho)} (m + p(\rho)) I.
\]

Clearly, (3.15) is equivalent to (3.1).

Note that if the material is incompressible, i.e., \( \text{div } v = 0 \) then it follows from (3.12) that \( \tilde{\zeta} \) is independent of \( m \). From a physical standpoint, we understand a fluid being incompressible by being reflected in its ability to withstand any spherical state of stress, i.e., \( T = a I \), where \( a \) can take any value. Thus, we say that the stress in the fluid can only be constitutively specified to within this arbitrary spherical state of stress. This result is reflected in a natural way in the above analysis in that the rate of dissipation is independent of \( m \), i.e., the rate of dissipation is independent of the mean normal stress. To put it differently, one cannot determine the value of \( m \) as the rate of dissipation which determines the form of the stress is independent of the mean normal stress \( m \). It is also worth bearing in mind that the mean normal stress \( m \) may or may not have any relationship to the thermodynamic pressure defined through (2.5). In fact for incompressible fluids, it does not.

3.3. Rederivation of the compressible Navier-Stokes constitutive relation by means of the maximization of the rate of entropy production - a modified approach

Equation (2.13) captures two different dissipative mechanisms, one due to the shear stress and another due to volume changes, i.e., compressibility. It may be acceptable under certain conditions to assume that these mechanisms are independent, which suggests the decomposition of \( \zeta \) into the sum of \( \zeta_1 \) and \( \zeta_2 \) and write instead of (2.13)

\[
\zeta_1 = \tilde{\zeta}_1(\rho, S) = \frac{1}{2\mu(\rho)} |S|^2,
\]

\[
\zeta_2 = \tilde{\zeta}_2(\rho, m) = \frac{3}{2\mu(\rho) + 3\lambda(\rho)} (m + p(\rho))^2.
\]

Maximization of \( \tilde{\zeta}_1 \) with respect to \( S \) so that (3.17) holds, leads to (3.15). Similarly, maximization of \( \tilde{\zeta}_2 \) with respect to \( m \) so that (3.18) holds results in (3.15). Thus, the constitutive relation characterizing a compressible Navier-Stokes fluid is rederived once more.
While in this simple framework, the two approaches presented in Subsections 3.2 and 3.3 lead to the same constitutive form for $T$, we will see in Section 4 that in general within the "non-Newtonian" context, making independent assumptions for the rate of dissipation associated with the different mechanisms corresponding to the shear rate and volume change lead to different results.

4. A compressible fluid whose material moduli depend on the shear-rate and volumetric-rate

The aim of this section is to provide various generalizations of incompressible power-law fluids so that we can obtain corresponding compressible fluid-like bodies.

4.1. General cases

We investigate two general cases that are based on the assumption that the rate of entropy production is a non-negative function of $\rho, m + p(\rho), \text{div} v, \mathbf{S}$ and $\mathbf{D}^d$.

In both cases we apply the notion of maximum rate of dissipation, as recalled in Subsection 3.2.

First, we assume that $\zeta = \tilde{\zeta}(\rho, m + p(\rho), \text{div} v, \mathbf{S}, \mathbf{D}^d)$ and $\tilde{\zeta}(\rho, 0, \text{div} v, 0, \mathbf{D}^d) = 0$. (4.1)

The requirement on the invariance of $\tilde{\zeta}$ with respect to any orthogonal transformation leads then to the form (note that $\text{tr} \mathbf{S} = \text{tr} \mathbf{D}^d = 0$ and $\text{tr} \mathbf{A}^2 = |\mathbf{A}|^2$)

$$\zeta = \tilde{\zeta}(\rho, m + p(\rho), \text{div} v, |\mathbf{D}^d|^2, \text{tr}((\mathbf{D}^d)^3),$$

$$\text{tr}((\mathbf{S} \mathbf{D}^d)^2), |\mathbf{S}|^2, \text{tr}(\mathbf{S}^2(\mathbf{D}^d)^2), \text{tr}(\mathbf{S}^2(\mathbf{D}^d)^2), \text{tr}(\mathbf{S}^3))$$

(4.2)

As $\mathbf{S}$ and $m$ are the causes for the microstructural changes in the material which are associated with the production of entropy, we maximize $\tilde{\zeta}$ of the form (4.2) with respect to $m$ and $\mathbf{S}$, requiring that (2.13) holds as a constraint. Necessary conditions for this constrained maximization are

$$\mathbf{D}^d = \frac{1 + \lambda}{\lambda} \frac{\partial \tilde{\zeta}}{\partial \mathbf{S}}$$

and

$$\text{div} v = \frac{1 + \lambda}{\lambda} \frac{\partial \tilde{\zeta}}{\partial m}.$$ (4.3)

Using the constrained equation (2.13), we conclude from (4.3) that

$$\frac{1 + \lambda}{\lambda} \frac{\zeta(y, z)}{\partial m(y, z)(m + p(\rho)) + \frac{\partial \zeta}{\partial m}(y, z) \cdot \mathbf{S}} = \beta(y, z).$$ (4.4)

Since

$$\frac{\partial \tilde{\zeta}}{\partial \mathbf{S}} = \sum_{i=1}^{6} \frac{\partial \tilde{\zeta}}{\partial z_i} \frac{\partial z_i}{\partial \mathbf{S}},$$

$$\frac{\partial \tilde{\zeta}}{\partial m} = \sum_{i=1}^{6} \frac{\partial \tilde{\zeta}}{\partial z_i} \frac{\partial z_i}{\partial m},$$

$$\frac{\partial \tilde{\zeta}}{\partial p} = \sum_{i=1}^{6} \frac{\partial \tilde{\zeta}}{\partial z_i} \frac{\partial z_i}{\partial p},$$

$$\frac{\partial \tilde{\zeta}}{\partial \rho} = \sum_{i=1}^{6} \frac{\partial \tilde{\zeta}}{\partial z_i} \frac{\partial z_i}{\partial \rho},$$

$$\frac{\partial \tilde{\zeta}}{\partial v} = \sum_{i=1}^{6} \frac{\partial \tilde{\zeta}}{\partial z_i} \frac{\partial z_i}{\partial v},$$

$$\frac{\partial \tilde{\zeta}}{\partial \mathbf{S}} = \sum_{i=1}^{6} \frac{\partial \tilde{\zeta}}{\partial z_i} \frac{\partial z_i}{\partial \mathbf{S}},$$

$$\frac{\partial \tilde{\zeta}}{\partial \mathbf{D}^d} = \sum_{i=1}^{6} \frac{\partial \tilde{\zeta}}{\partial z_i} \frac{\partial z_i}{\partial \mathbf{D}^d},$$

$$\frac{\partial \tilde{\zeta}}{\partial (m + p(\rho))} = \sum_{i=1}^{6} \frac{\partial \tilde{\zeta}}{\partial z_i} \frac{\partial z_i}{\partial (m + p(\rho))},$$

$$\frac{\partial \tilde{\zeta}}{\partial \text{div} v} = \sum_{i=1}^{6} \frac{\partial \tilde{\zeta}}{\partial z_i} \frac{\partial z_i}{\partial \text{div} v},$$

$$\frac{\partial \tilde{\zeta}}{\partial |\mathbf{D}^d|^2} = \sum_{i=1}^{6} \frac{\partial \tilde{\zeta}}{\partial z_i} \frac{\partial z_i}{\partial |\mathbf{D}^d|^2},$$

$$\frac{\partial \tilde{\zeta}}{\partial ((\mathbf{D}^d)^3)} = \sum_{i=1}^{6} \frac{\partial \tilde{\zeta}}{\partial z_i} \frac{\partial z_i}{\partial ((\mathbf{D}^d)^3)},$$

$$\frac{\partial \tilde{\zeta}}{\partial \text{tr}((\mathbf{S} \mathbf{D}^d)^2)} = \sum_{i=1}^{6} \frac{\partial \tilde{\zeta}}{\partial z_i} \frac{\partial z_i}{\partial \text{tr}((\mathbf{S} \mathbf{D}^d)^2)},$$

$$\frac{\partial \tilde{\zeta}}{\partial |\mathbf{S}|^2} = \sum_{i=1}^{6} \frac{\partial \tilde{\zeta}}{\partial z_i} \frac{\partial z_i}{\partial |\mathbf{S}|^2},$$

$$\frac{\partial \tilde{\zeta}}{\partial \text{tr}(\mathbf{S}^2(\mathbf{D}^d)^2)} = \sum_{i=1}^{6} \frac{\partial \tilde{\zeta}}{\partial z_i} \frac{\partial z_i}{\partial \text{tr}(\mathbf{S}^2(\mathbf{D}^d)^2)},$$

$$\frac{\partial \tilde{\zeta}}{\partial \text{tr}(\mathbf{S}^2(\mathbf{D}^d)^2)} = \sum_{i=1}^{6} \frac{\partial \tilde{\zeta}}{\partial z_i} \frac{\partial z_i}{\partial \text{tr}(\mathbf{S}^2(\mathbf{D}^d)^2)},$$

$$\frac{\partial \tilde{\zeta}}{\partial \text{tr}(\mathbf{S}^3)} = \sum_{i=1}^{6} \frac{\partial \tilde{\zeta}}{\partial z_i} \frac{\partial z_i}{\partial \text{tr}(\mathbf{S}^3)}.$$
we conclude from (4.3) that
\[
\text{div } v = \beta(y, z) \frac{\partial \lambda_i}{\partial m_i}(y, z),
\]
\[
D^i = \alpha_1 D^i + \alpha_2 (D^i)^2 + \alpha_3 S + \alpha_4 SD^i + \alpha_5 S(D^i)^2 + \alpha_6 S^2,
\] (4.6)
where \((i = 1, \ldots, 6)\)
\[
\alpha_i = \alpha_i(y, z) =: \beta(y, z) \frac{\partial \lambda_i}{\partial m_i}(y, z).
\]
We notice that (4.6) is an implicit relation for \(S\) and \(D^i\).

Next, we assume that \(\lambda = \zeta_1 + \zeta_2\) and \(\zeta_i \geq 1\) are given by
\[
\zeta_1 = \zeta_1(\varrho, m + p(\varrho), \text{div } v) \quad (\zeta_1(\varrho, 0, \text{div } v) = 0),
\]
\[
\zeta_2 = \zeta_2(\varrho, S, D^i) \quad (\zeta_2(\varrho, 0, D^i) = 0).
\] (4.8)
Similarly as above, invariance with respect to orthogonal transformations implies that
\[
\zeta_2(\varrho, S, D^i) = \zeta_2(\varrho, |D^i|^2, tr((D^i)^3), tr(SD^i), tr(S(D^i)^2)),
\]
\[
|S|^2, tr(S^2(D^i)), tr(S^2(D^i)^2), tr(S^3))
\] (4.9)
\[
=: \zeta_2(u_1, u_2, u_3, z_1, z_2, z_3, z_4, z_5, z_6)
\]
\[
=: \zeta_2(u, z).
\]
Requiring the maximum rate of dissipation first of (4.7) with (3.18), and then of (4.8) with (3.17) yields
\[
\text{div } v = \frac{1 + \lambda_1}{\lambda_1} \frac{\partial \zeta_1}{\partial m}(\varrho, m + p(\varrho), \text{div } v) \quad \text{and} \quad D^i = \frac{1 + \lambda_2}{\lambda_2} \frac{\partial \zeta_2}{\partial S}(u, z).
\] (4.10)
Since
\[
\frac{1 + \lambda_1}{\lambda_1} = \frac{\partial \zeta_1}{\partial m}(\varrho, m + p(\varrho)) \quad \text{and} \quad \frac{1 + \lambda_2}{\lambda_2} = \frac{\partial \zeta_2}{\partial S}(u, z) =: \gamma(u, z).
\] (4.11)
and
\[
\frac{\partial \zeta_2}{\partial S} = \sum_{i=1}^{6} \frac{\partial \zeta_2}{\partial z_i} \frac{\partial z_i}{\partial S},
\]
we conclude from (4.10) that
\[
\text{div } v = \frac{\zeta_1(\varrho, m + p(\varrho), \text{div } v)}{m + p(\varrho)},
\]
\[
D^i = a_1 D^i + a_2 (D^i)^2 + a_3 S + a_4 SD^i + a_5 S(D^i)^2 + a_6 S^2,
\] (4.12)
where \((i = 1, \ldots, 6)\)
\[
a_i = a_i(u, z) =: \gamma(u, z) \frac{\partial \zeta_2}{\partial z_i}(u, z).
\]
We see from (4.12) and (4.13) that once again we have an implicit relationship between \( m \) and \( \text{div } v \), and between \( \mathbf{S} \) and \( \mathbf{D}^d \). However, the expression for \( \text{div } v \) given in (4.12) is in general different from that given by (4.5). The same comment concern the difference between (4.6) and (4.13). Thus, the two procedures outlined in this section do not, in general, lead to the same constitutive theory. We feel that the first approach that does not resort to the additive decomposition of the rate of entropy production is the more appropriate way to approach the problem.

### 4.2. Compressible power-law fluids: examples

In this part, we illustrate the potential of the above approach by deriving an hierarchy of models of the power-law type; while considering the flow of incompressible fluids, one usually deals with only one mechanism associated with the dissipation of the energy (that due to the shear stress), here we have two mechanisms. It leads, of course, to more flexibility with regard to the development of constitutive equations. The aim of the examples is to characterize several typical cases. As a detailed description of the maximum rate of dissipation criterion was given above when discussing general cases, here we skip these details. In what follows, a quantity with subscript \( * \) is a constant.

**Example 4.1.** Considering a rate of dissipation of the form

\[
\tilde{\zeta} = \frac{1}{2\tilde{\mu}(\varrho, \text{div } \mathbf{v}, |\mathbf{D}^d|^2)}|\mathbf{S}|^2 + \frac{3}{2\tilde{\mu}(\varrho, \text{div } \mathbf{v}, |\mathbf{D}^d|^2)} + 3\lambda_* (m + p(\varrho))^2, \tag{4.14}
\]

where \( \tilde{\mu}(\ldots) \) and \( 2\tilde{\mu}(\ldots) + 3\lambda_* \) are positive, we obtain

\[
\mathbf{D}^d = \frac{1}{2\tilde{\mu}(\varrho, \text{div } \mathbf{v}, |\mathbf{D}^d|^2)}\mathbf{S}, \tag{4.15}
\]

\[
\text{div } \mathbf{v} = \frac{3}{2\tilde{\mu}(\varrho, \text{div } \mathbf{v}, |\mathbf{D}^d|^2)} + 3\lambda_* (m + p(\varrho)), \tag{4.16}
\]

and consequently

\[
\mathbf{T} = m\mathbf{I} + \mathbf{S}
\]

\[
= -p(\varrho)\mathbf{I} + 2\tilde{\mu}(\varrho, \text{div } \mathbf{v}, |\mathbf{D}^d|^2)\mathbf{D}^d + (2\tilde{\mu}(\varrho, \text{div } \mathbf{v}, |\mathbf{D}^d|^2) + 3\lambda_* \text{div } \mathbf{v}) \frac{\text{div } \mathbf{v}}{3} \mathbf{I}. \tag{4.17}
\]

Thus, both shear and bulk viscosity depend on \( \text{div } \mathbf{v} \) and the generalized shear rate \( |\mathbf{D}^d|^2 \).

**Example 4.2.** Choosing the rate of dissipation to be given by

\[
\tilde{\zeta} = \frac{1}{2\mu_*}|\mathbf{S}|^2 + \frac{3}{2\mu_* + 3\lambda(\text{div } \mathbf{v})} (m + p(\varrho))^2, \tag{4.18}
\]

with \( \mu_* \) and \( 2\mu_* + 3\lambda(\text{div } \mathbf{v}) \) positive, leads to the relations

\[
\mathbf{D}^d = \frac{1}{2\mu_*} \mathbf{S}, \tag{4.19}
\]

\[
\text{div } \mathbf{v} = \frac{3}{2\mu_* + 3\lambda(\text{div } \mathbf{v})} (m + p(\varrho)), \tag{4.20}
\]
and consequently
\[ T = -p(\rho)I + 2\mu_\ast \mathbf{D}^d + (2\mu_\ast + 3\lambda(\text{div } \mathbf{v}))\frac{\text{div } \mathbf{v}}{3} I. \] (4.21)

In this case, shear viscosity is constant and the bulk viscosity depends on \( \text{div } \mathbf{v} \).

**Example 4.3.** Taking the rate of dissipation to be
\[ \dot{\zeta} = \frac{1}{2\mu_{\text{shear}}(\text{div } \mathbf{v}, |\mathbf{D}^d|^2)}|\mathbf{S}|^2 + \frac{1}{\mu_{\text{bulk}}(\text{div } \mathbf{v}, |\mathbf{D}^d|^2)}(m + p(\rho))^2 \] (4.22)
with \( \mu_{\text{shear}}(\ldots) \) and \( \mu_{\text{bulk}}(\ldots) \) positive, we obtain the relations
\[ \mathbf{D}^d = \frac{1}{2\mu_{\text{shear}}(\ldots)}\mathbf{S}, \] (4.23)
\[ \text{div } \mathbf{v} = \frac{1}{\mu_{\text{bulk}}(\ldots)}(m + p(\rho)), \] (4.24)
and consequently
\[ T = -p(\rho)I + 2\mu_{\text{shear}}(\ldots)\mathbf{D}^d + \mu_{\text{bulk}}(\ldots)(\text{div } \mathbf{v})I. \] (4.25)
Here, both viscosities depend on \( \text{div } \mathbf{v} \) and \( |\mathbf{D}^d|^2 \).

**Example 4.4.** Let \( \zeta = \zeta_1 + \zeta_2 \) and let
\[ \zeta_1 = \zeta_1(\rho, m) = \frac{1}{K(\rho)}(\delta_\ast + (m + p(\rho))^2)^\frac{\beta-2}{\beta} (m + p(\rho))^2 \] (4.26)
\[ \zeta_2 = \zeta_2(\rho, |\mathbf{S}|^2) = \frac{1}{L(\rho)}(\epsilon_\ast + |\mathbf{S}|^2)^\frac{\sigma-2}{\sigma}|\mathbf{S}|^2 \] (4.27)
where \( \epsilon_\ast, \delta_\ast \) are positive constants, and \( K(\rho), L(\rho) \) are positive functions, and \( \sigma, \beta \in (1, \infty) \) are the so-called power-law indices. As with (4.12) and (4.13) we observe that
\[ \text{div } \mathbf{v} = \frac{1}{K(\rho)}(\delta_\ast + (m + p(\rho))^2)^\frac{\beta-2}{\beta} (m + p(\rho)), \] (4.28)
\[ \mathbf{D}^d = \frac{1}{L(\rho)}(\epsilon_\ast + |\mathbf{S}|^2)^\frac{\sigma-2}{\sigma}\mathbf{S}. \] (4.29)
For simplicity, suppose that \( \delta_\ast = \epsilon_\ast = 0 \). Then it follows from (4.28), (4.29) that
\[ m + p(\rho) = [K(\rho)]\frac{1}{\beta-1}|\text{div } \mathbf{v}|^{\frac{\beta-2}{\beta}}|\text{div } \mathbf{v}| =: k(\rho)|\text{div } \mathbf{v}|^{\frac{\beta-2}{\beta}} \text{div } \mathbf{v}, \]
\[ \mathbf{S} = [L(\rho)]^{\frac{1}{\sigma-1}}|\mathbf{D}^d|^{\frac{\sigma-2}{\sigma}}|\mathbf{D}^d| =: \ell(\rho)|\mathbf{D}^d|^{\frac{\sigma-2}{\sigma}}|\mathbf{D}^d|. \]
Thus
\[ T = -p(\rho)I + \ell(\rho)|\mathbf{D}^d|^{\frac{\sigma-2}{\sigma}}|\mathbf{D}^d| + k(\rho)|\text{div } \mathbf{v}|^{\frac{\beta-2}{\beta}}(\text{div } \mathbf{v})I, \] (4.30)
and the generalized shear viscosity \( \ell(\rho)|\mathbf{D}^d|^{\frac{\sigma-2}{\sigma}} \) depends on \( \rho \) and the shear rate, while the generalized bulk viscosity \( k(\rho)|\text{div } \mathbf{v}|^{\frac{\beta-2}{\beta}} \) depends on \( \rho \) and \( \text{div } \mathbf{v} \).
In order to show the versatility of the procedure to deliver constitutive relations for compressible fluids, we consider some additional examples. All these examples have a feature in common, namely in the constitutive equation for the rate of dissipation we set \( m = -p(\rho) \) and the same concerns the equation (3.13) that then simplifies to
\[
\zeta = \mathbf{S} \cdot \mathbf{D}^d. 
\]
(4.31)
It means that the fluid does not dissipate the energy due to volumetric changes. On the other hand the material under consideration dissipates energy due to, for example, shear effects. The governing equations for such fluid motions are
\[
\dot{\rho} = -\rho \text{div } \mathbf{v}, \\
\rho \dot{\mathbf{v}} = \text{div } \mathbf{T} + \rho \mathbf{f} \quad \text{with } \mathbf{T} = -p(\rho)\mathbf{I} + \mathbf{S},
\]
where \( \mathbf{S} \) and \( \mathbf{D}^d \) are related by an (explicit/implicit) constitutive relationship.

**Example 4.5.** Choosing the rate of dissipation to be
\[
\tilde{\zeta} = \frac{|\mathbf{S}|^2}{2\mu_* |\mathbf{D}^d|^{r-2}},
\]
(4.32)
where \( \mu_* > 0 \), we obtain
\[
\mathbf{D}^d = \frac{\mathbf{S}}{2\mu_* |\mathbf{D}^d|^{r-2}} \iff \mathbf{S} = 2\mu_* |\mathbf{D}^d|^{r-2}\mathbf{D}^d,
\]
(4.33)
and consequently
\[
\mathbf{T} = -p(\rho)\mathbf{I} + 2\mu_* |\mathbf{D}^d|^{r-2}\mathbf{D}^d.
\]
(4.34)
Note that (4.32) is meaningful for \( r \leq 2 \) (to avoid dividing by zero). The above model is interesting in that we have a compressible fluid which has a shear viscosity that depends on the deviatoric part of the symmetric part of the velocity gradient, but it has no bulk viscosity. Also, the mean normal stress \( m \) equals \(-p(\rho)\).

**Example 4.6.** In this example we start with a form for \( \tilde{\zeta} \) that is different from (4.32) and we end up with assumes for certain range of \( r \) the same constitutive relation for \( \mathbf{T} \) as in (4.34). Consider
\[
\tilde{\zeta} = \left( \frac{1}{2\mu_*} \right)^{\frac{1}{r-1}} |\mathbf{S}|^{\frac{r}{r-1}}.
\]
(4.35)
Again, \( \mu_* > 0 \). Note that (4.35) is defined for \( r \in (1, \infty) \). Using maximum rate of dissipation criterion we obtain
\[
\mathbf{D}^d = \left( \frac{1}{2\mu_*} \right)^{\frac{1}{r-1}} |\mathbf{S}|^{\frac{r}{r-1}} \mathbf{S},
\]
(4.36)
which after some elementary calculations leads again to (4.34). The difference between these two examples lies in value of \( r \) for which the given approach is applicable.
4.3. Compressible power-law fluids with activation: examples

In this part, we proceed as in the previous subsection focusing however on various responses connected with an activation or threshold criteria, thus, generalizing the Bingham and Herschel-Bulkley fluids within the context of compressible materials.

Example 4.7. Compressible Bingham fluid. Starting with the assumption

\[ \tilde{\zeta} = |D^d| \frac{|S|^2}{\tau_*}, \tag{4.37} \]

where \( \tau_* > 0 \) represents the threshold, we arrive at

\[ D^d = |D^d| \frac{S}{\tau_*}. \tag{4.38} \]

It follows from (4.38) that

If \(|S| < \tau_*\) then \(D^d \equiv 0\),

If \(|S| = \tau_*\) then \(D^d = \frac{S}{\tau_*} \) for \(D^d \neq 0\).

In the latter case, \(D = 0\) also solves (4.38). However, \(\zeta\) is maximal for \(D \neq 0\).

Also note that if \(|S| > \tau_*\) then \(T = -p(\rho)I + S\) fulfills the relationship (4.38). It is not possible to write the constitutive relation between \(T\) and \(D\) as an explicit equation for \(T\), or for \(D^d\). The relationships are truly implicit.

Example 4.8. Compressible viscous Bingham and Herschel-Bulkley fluids (Case I). Choosing the rate of dissipation to be

\[ \tilde{\zeta} = \frac{|D^d| |S|^2}{(\tau_* + 2\mu(|D^d|^2)|D^d|)}, \tag{4.39} \]

where again \(\tau_* > 0\) represents the threshold (the yield stress), we obtain

\[ D^d = |D^d| \frac{S}{(\tau_* + 2\mu(|D^d|^2)|D^d|)}. \tag{4.40} \]

It follows from (4.40) that

If \(|S| \leq \tau_*\) then \(D^d \equiv 0\),

If \(|S| > \tau_*\) then \(S = \frac{D^d}{|D^d|}(\tau_* + 2\mu(|D^d|^2)|D^d|) = \frac{D^d}{|D^d|}\tau_* + 2\mu(|D^d|^2)D^d. \tag{4.41} \)

Note also that if \(|S| > \tau^*\) then \(T = -p(\rho)I + \frac{D^d}{|D^d|}\tau_* + 2\mu(|D^d|^2)D^d. \)

Example 4.9. Compressible viscous Bingham and Herschel-Bulkley fluids (Case II). We set the rate of dissipation to be

\[ \tilde{\zeta} = \frac{1}{2\mu(|D^d|^2)(\tau_* + (|S| - \tau^*_+)^+ |S|^2 =: \frac{1}{2\mu(|D^d|^2)}f((|S| - \tau^*_+) |S|^2, \tag{4.42} \]

\[ (|S| - \tau^*_+) \]

\[ f((|S| - \tau^*_+) |S|^2, \tag{4.42} \]

\[ (|S| - \tau^*_+) \]

\[ f((|S| - \tau^*_+) |S|^2, \tag{4.42} \]
where again \( \tau_* > 0 \) represents the threshold (the yield stress), we obtain

\[
2\mu((D^d)^2)D^d = f((|\mathbf{S}| - \tau_*)^+)\mathbf{S}.
\]

(4.43)

Notice that (4.43) provides more detailed information in comparison to (4.40), namely,

\[
|\mathbf{S}| \leq \tau_* \iff D^d \equiv 0,
\]

\[
|\mathbf{S}| > \tau_* \iff \mathbf{S} = \frac{2\mu((|\mathbf{D}^d|^2)}{f((|\mathbf{S}| - \tau_*)^+)}D^d.
\]

(4.44)

References


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