GENERALIZATIONS OF THE NAVIER–STOKES FLUID FROM A NEW PERSPECTIVE

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Abstract. In this paper we study incompressible fluids described by constitutive equations from a different perspective, than that usually adopted, namely that of expressing kinematical quantities in terms of the stress, more particularly expressing the symmetric part of the velocity gradient in terms of a “power-law” of the stress. This stress power-law model automatically satisfies the constraint of incompressibility without our having to introduce a Lagrange multiplier to enforce the constraint. The model also includes the classical incompressible Navier–Stokes model as a special subclass. We compare the stress power-law model with the classical power-law models and we show that the stress power-law model exhibits qualitatively different response characteristics. Using a stress power law model we study several steady flow problems and obtain several new exact solutions, and we demonstrate the potential of the new approach with regard to constitutive modelling. The model leads to solutions that are markedly different from classical power-law fluids and the solutions possess interesting qualitative properties. Finally, we discuss the issue of the choice of boundary conditions, and we show that the choice of boundary conditions has, at least for one of the problems that we study, a profound impact on the solvability of the boundary value problem.

1. Introduction

The physical make-up of the classical incompressible Navier–Stokes and generalized Stokesian fluids\(^1\) are described by constitutive expressions for the Cauchy stress \(\mathbb{T}\) in terms of the symmetric part of the velocity gradient \(\mathbb{D}\). Such a specification of the stress in terms of a kinematical quantity seems to turn causality on its head as causality would lead one to express the deformation (rather kinematical quantities that reflect

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\(^1\)By an incompressible Stokesian fluid we mean a fluid whose constitutive expression takes the form \(\mathbb{T} = -\rho \mathbb{I} + f(\mathbb{D})\).

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the deformation) in terms of the force,

as the deformation is the effect of the force, which is the cause (see Rajagopal (2009) for a detailed discussion of the relevant issues).

Expressing the kinematical quantity in terms of the stress for an incompressible fluid leads to several interesting consequences in that the set of possible solutions in certain circumstances are simpler to obtain, while in other instances the converse is true. More interestingly, expressing the velocity gradient in terms of the stress does away with the need to introduce a spherical stress (the pressure) that is indeterminate, and it is not necessary that this constraint stress (the part of the stress that enforces the constraint) does no work as it is usually assumed in the classical approach that traces its roots to the ideas espoused by D’Alembert and Bernoulli, and popularized by Lagrange. The formal procedure that is currently in vogue can be found in the book by Truesdell and Noll (2004). For a criticism of such an approach and an alternative that stems from the work of Gauß (1829) in particle mechanics, we refer the reader to a paper by Rajagopal and Srinivasa (2005) and the references cited therein.

The model we are going to study is such that the velocity gradient automatically satisfies the constraint of incompressibility and there is no need to introduce the pressure in the classical sense. Such models have been discussed in a paper on implicit theories for fluids (see Rajagopal (2003) and also Rajagopal (2006)), and our paper is devoted to a demonstration of the potential of implicit constitutive theories and those wherein the kinematics is expressed in terms of the stress by considering several explicit solutions to some particular boundary value problems involving fluids characterized by the constitutive relation

\[
\mathbb{D} = \alpha (1 + \beta |T_\delta|^2)^n T_\delta^n,
\]

where \( T_\delta = \text{def} \ T - \left( \frac{1}{3} \text{Tr} \ T \right) \mathbb{I} \), \( |T_\delta| = \text{def} \ \sqrt{\text{Tr} (T_\delta T_\delta^\top)} \), \( \alpha \) and \( \beta \) are positive constants, \( n \) is a real number, and

\[
\mathbb{D} = \frac{1}{2} \left( \nabla \mathbf{v} + \nabla \mathbf{v}^\top \right),
\]

where \( \mathbf{v} \) is the velocity. As we shall see, the above model includes the incompressible Navier–Stokes model as a special case. The model \((1.1)\) is an explicit constitutive expression for the symmetric part of the velocity gradient in terms of the stress. Since the dependence of \( \mathbb{D} \) on \( T_\delta \) is polynomial, we call fluids characterized by the constitutive relation \((1.1)\) stress power law fluids. However, even more general implicit constitutive relations between the symmetric part of the velocity gradient and the stress are necessary if one has to model the response of fluids whose material moduli depend on both the symmetric part of the velocity gradient and the mean normal stress or for that matter any of the principal invariants of the stress or even just one component of the stress, see Rajagopal (2006).

The constitutive equation \((1.1)\) has to be solved in conjunction with the balance of linear momentum and the balance of mass. Since the balance of mass reduces to the constraint \( \text{Tr} \ \mathbb{D} = 0 \) because of the assumption of incompressibility and since the constitutive assumption \((1.1)\) automatically satisfies the constraint, we do not have to enforce it. The full system of governing equations is therefore

\[
\text{(1.3a)} \quad \mathbb{D} = \alpha (1 + \beta |T_\delta|^2)^n T_\delta^n,
\]

\[
\rho \frac{d \mathbf{v}}{dt} = \text{div} \ \mathbb{T} + \rho \mathbf{b},
\]

where \( \rho \) is the density and \( \mathbf{b} \) is the specific body force. Since \((1.3a)\) can not be, in general, inverted to obtain a constitutive relation of type \( \mathbb{T} = -p I + f(\mathbb{D}) \), one can not substitute for \( \mathbb{T} \) in \((1.3b)\) and get the classical system of partial differential equations

\[
\text{(1.4a)} \quad \rho \frac{d \mathbf{v}}{dt} = -\nabla p + \text{div} \ f(\mathbb{D}) + \rho \mathbf{b},
\]

\[
\text{(1.4b)} \quad \text{div} \ \mathbf{v} = 0,
\]

\(2\)In the case of the incompressible Navier–Stokes fluid it is a trivial matter to express \( \mathbb{D} \) in terms of \( \mathbb{T} \). This is not so in the case of a generalized Stokesian fluid as the relationship between \( \mathbb{T} \) and \( \mathbb{D} \) might not be irreversible.

\(3\)We need nonnegative constants in order to make the model consistent with the second law of thermodynamics.

\(4\)For a discussion on relation of model \((1.3a)\) to the classical models of type \( \mathbb{T} = -p I + f(\mathbb{D}) \) see Section 2.
as one does while dealing with the general Stokesian fluids. We note that the latter approach leads to three second order partial differential equations \(1.4a\) and one first order partial differential equation \(1.4b\) for the velocity \(v\) and the pressure \(p\), whereas system \(1.3\) is a system of first order partial differential equations for the stress \(T\) and the velocity \(v\)—the difference in the order of the partial differential equations to be solved can be of importance with respect to rigorous mathematical theory concerning the models, as one is now dealing with lower order equations.

The governing equations (1.3) have to be completed by supplying boundary conditions. One could use the classical boundary conditions specified in terms of the velocity, such as the no-slip boundary condition. This approach is however not consistent with the primary objective of the assumed constitutive theory that requires one to express the effects (the generated velocity field) in terms of causes (the forces/stresses acting on the system). A consistent approach to the problem would therefore require one to specify the force (traction, torque), not the velocity, on the boundary. We first solve the boundary value problems using the classical no-slip boundary condition, see Section 3, and then we discuss the boundary conditions specified in terms of forces, see Section 4.

We find that the right choice of the boundary condition is not a minor issue, and that it has, at least for one of the boundary value problems that we study, a profound impact on solvability of the boundary value problem.

In order to demonstrate features of the model (1.1), we present explicit solutions to some classical steady flow problems—counterparts of plane Couette and Poiseuille flow, Hagen–Poiseuille flow and cylindrical Couette flow.

2. COMPARISON OF STRESS POWER-LAW FLUIDS AND CLASSICAL POWER-LAW FLUIDS

Let us, before we proceed to the solution of the boundary value problems, present several qualitative observations concerning the model (1.1).

Let us first note that model (1.1) automatically satisfies the constraint of incompressibility

\[
\text{Tr } D = 0. \tag{2.1}
\]

Furthermore, it is obvious that \(v = 0\) implies \(D = 0\) and the stress in this case is given by \(T = -\frac{1}{3} (\text{Tr } T) \mathbb{I}\). Obviously when \(n = 0\), the model (1.1) reduces to \(D = -\frac{4}{3} (\text{Tr } T) \mathbb{I} + \alpha T\), which can be expressed as

\[
T = \frac{1}{3} (\text{Tr } T) \mathbb{I} + 2\mu D, \tag{2.2}
\]

where \(\mu = \frac{1}{2\alpha}\). We recall that the constitutive relation for the incompressible Navier–Stokes fluid is expressed as

\[
T = -p \mathbb{I} + 2\mu D, \tag{2.3}
\]

where \(p\) is the Lagrange multiplier that enforces the constraint (2.1). In virtue of (2.1), (2.3) can be expressed as (2.2).

One way of generalizing the classical incompressible Navier–Stokes fluid model (2.3) is to replace (2.3) by

\[
T = -p \mathbb{I} + f(D), \tag{2.4}
\]

where \(f\) is a tensorial function. Models

\[
T = -p \mathbb{I} + 2\mu_0 |D|^{2m} D, \tag{2.5a}
\]

\[
T = -p \mathbb{I} + 2\mu_0 (1 + \lambda |D|^2)^m D, \tag{2.5b}
\]

are often used, as several oils, food products, and geological materials can be approximated reasonably well by such models. In model (2.5b) one can, in contrast to (2.5a), allow \(m \leq -\frac{1}{2}\) without having problem with singular behaviour of the stress \(T\) for \(D\) tending to zero. Moreover in (2.5a), for \(m > 0\), the generalized

\[5\text{With regard to boundary value problems involving stress power law fluids, when we consider the same geometry and, where appropriate, the same boundary conditions as for plane Couette, Poiseuille, Hagen–Poiseuille and cylindrical Couette flow of the incompressible Navier–Stokes fluid, we use the same terminology (plane Couette and Poiseuille flow, Hagen–Poiseuille flow and cylindrical Couette flow) even though we are referring to flows of stress power law fluids. If we want to explicitly distinguish the flow of the Navier–Stokes fluid and the flow of stress power law fluid, we use the terms “classical plane Couette flow” and “counterpart of plane Couette flow”, etc.}
viscosity $\mu_0 |D|^{2m}$ degenerates, and for $m \in (-\frac{1}{2}, 0)$ it becomes singular for $D$ tending to zero. Although this kind of behaviour can be desirable in the description of certain materials, we will, for the reasons stated above, not consider model (2.5a), and we will focus solely on model (2.5b) that does not exhibit the singular behaviour. One can also see that (2.5a) admits the traceless part of the stress to be written as a function of $|T\delta|$, for $m \in (-\frac{1}{2}, 0)$ it allows shear thickening fluids, and for $m \in (-\frac{1}{2}, 0)$ it corresponds to shear thinning fluid. If $m = -\frac{1}{2}$ we get a limiting case, where we are, for the first time, dealing with a function that does not map $(0, +\infty)$ to $(0, +\infty)$, the norm of the traceless part of the Cauchy stress tends, for $|D| \rightarrow +\infty$, to a nonzero value $2\mu_0 |\alpha|$. If we take $m \in (-\infty, -\frac{1}{2})$ we get nonmonotone behaviour for $|T\delta|$ with $|D|$, and the norm of the symmetric part of the Cauchy stress tends to zero as $|D| \rightarrow +\infty$. All types of behaviour described above are relevant if one wants to characterize real materials—see Málek and Rajagopal (2005) for a discussion and further references.

Let us now consider (1.1) and let us once again plot the dependence of $|T\delta|$ on $|D|$ we obtain the results shown in Figure 1A. From the modelling point of view we can distinguish four different types of behaviour. If $m \in (0, +\infty)$ we get shear thickening fluids, $m = 0$ corresponds to the Navier–Stokes fluid, and $m \in (-\frac{1}{2}, 0)$ corresponds to shear thinning fluid. If $m = -\frac{1}{2}$ we get a limiting case, where we are, for the first time, dealing with a function that does not map $(0, +\infty)$ to $(0, +\infty)$, the norm of the traceless part of the Cauchy stress tends, for $|D| \rightarrow +\infty$, to a nonzero value $2\mu_0 |\alpha|$. If we take $m \in (-\infty, -\frac{1}{2})$ we get nonmonotone behaviour for $|T\delta|$ with $|D|$, and the norm of the symmetric part of the Cauchy stress tends to zero as $|D| \rightarrow +\infty$. All types of behaviour described above are relevant if one wants to characterize real materials—see Málek and Rajagopal (2005) for a discussion and further references.

Models of the class (2.5) have been, from a rigorous mathematical point of view, studied in great detail, the mathematical analysis of such models was initiated by Ladyzhenskaya (1967, 1969) and Lions (1969), and these early works have been followed by numerous studies—a detailed overview of the development of the mathematical theory of power-law fluids and an overview of the present state of art can be found in the recent surveys by Málek and Rajagopal (2005) and Málek (2008). Concerning more recent results not covered in the surveys, one should mention the results obtained by Diiening et al. (2008) and Diiening et al. (2008). The fluids with the generalized viscosity increasing even exponentially with increasing shear rate are analyzed in Gwiazda and Swierczewska-Gwiazda (2008).

On the other hand, models of the type where $D$ is a function of $T$ were introduced only recently, and we are aware of only one paper, that by Bulíček et al. (2009) that analyzes the internal steady flow of stress power-law like fluids, and establishes the existence of a weak solution to relevant homogeneous Dirichlet boundary value problem for such parameters that include (1.1) as a special case if $-\frac{1}{2} < n < 2$, see also Málek (2008).

3. Steady flows of stress power-law fluids

3.1. Plane Couette and Poiseuille flow. Let us consider the steady, fully developed flow between two infinite parallel plates at a distance $2h$ apart due to the application of a constant pressure gradient $\frac{\partial p}{\partial z} e_z = -\Delta e_z$ (counterpart of Poiseuille flow) and the flow that is maintained by the upper plate moving with a constant velocity $V = V_{top} e_z$ and the bottom plate moving with velocity $V = -V_{top} e_z$ (counterpart of Couette flow), see Figure 2c. In both cases we require the no-slip boundary condition on upper and bottom plate. Boundary conditions are constitutive specifications that depend on the nature of the material on either side of the boundary. Thus, for instance, the boundary condition between two fluids that are moving relative to each other would be different from that between a fluid flowing past a rigid boundary. The no-slip boundary condition seems to work reasonably well in the latter case, especially when flows are sufficiently slow. In this connection see the discussion concerning boundary conditions in Section 4.
Let $L$ be a characteristic length and $V$ a characteristic velocity, then we can rewrite equations (1.1) and (1.3b) in the form:

\[
\mathbb{D}^* = \left(1 + \mathcal{R}_2 \, \text{tr} \, \mathbb{T}^*_\delta \right)^n \mathbb{T}^*_\delta, \\
\frac{d\mathbb{v}^*}{dt} = \frac{1}{\mathcal{R}_1} \, \text{div} \, \mathbb{T}^*,
\]

We assume the fluid to be homogeneous, thus the density is constant and the constant is the same at all material points.
where \( \mathcal{R}_1 = \alpha \rho V L \), \( \mathcal{R}_2 = \frac{\partial V^2}{\alpha^2 L} \) are dimensionless parameters (\( \mathcal{R}_1 \) being an analogue of the Reynolds number), and star denotes dimensionless variables \( \mathbf{x}^* = \frac{1}{L} \mathbf{x} \), \( t^* = \frac{V}{L} t \), \( \mathbf{v}^* = \frac{1}{V} \mathbf{v} \), \( \mathcal{D}^* = \frac{L}{V} \mathcal{D} \), \( \mathbf{T}^* = \frac{\alpha L}{V} \mathbf{T} \), and \( \mathbf{T}_\delta^* = \text{def} \mathbf{T}^* - \left( \frac{1}{3} \text{Tr} \mathbf{T}^* \right) \mathbb{I} \) is the traceless part of \( \mathbf{T}^* \). In our case we can set \( L = h \), and \( V = V_{\text{top}} \) for Couette flow or \( V = \Delta \alpha L^2 \) for Poiseuille flow. From now on we will only use dimensionless variables, and we will omit the star to keep the notation simple.

We shall use a semi-inverse method and seek a stress field of the form

\[
\mathbf{T} = \left( \frac{1}{3} \text{Tr} \mathbf{T} \right) \mathbb{I} = T (\hat{e}_z \otimes \hat{e}_y + \hat{e}_y \otimes \hat{e}_z) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & T \\ 0 & T & 0 \end{bmatrix}, \tag{3.2}
\]

where \( \hat{e}_z \) and \( \hat{e}_y \) are unit vectors in the \( z \) and \( y \) co-ordinate directions, respectively, and \( T = T(y) \) is a scalar function. Consequently (3.1a) gives

\[
\mathcal{D} = \left( 1 + 2 \mathcal{R}_2 T^2 \right)^n T (\hat{e}_z \otimes \hat{e}_y + \hat{e}_y \otimes \hat{e}_z) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & (1 + 2 \mathcal{R}_2 T^2)^n T \\ 0 & (1 + 2 \mathcal{R}_2 T^2)^n T & 0 \end{bmatrix}, \tag{3.3}
\]

and since \( T \) is a function of \( y \) only, the velocity field that is consistent with the assumption concerning the form of the stress field must have the following form

\[
\mathbf{v} = v^z(y) \hat{e}_z. \tag{3.4}
\]

Using (3.2) and (3.4) it follows from the balance of linear momentum (3.1b) that, in the absence of body forces,

\[
0 = \frac{1}{\mathcal{R}_1} \frac{\partial}{\partial x} \left( \frac{1}{3} \text{Tr} \mathbf{T} \right), \tag{3.5a}
\]
\[
0 = \frac{1}{\mathcal{R}_1} \frac{\partial}{\partial y} \left( \frac{1}{3} \text{Tr} \mathbf{T} \right), \tag{3.5b}
\]
\[
0 = \frac{1}{\mathcal{R}_1} \left( \frac{\partial T}{\partial y} + \frac{\partial}{\partial z} \left( \frac{1}{3} \text{Tr} \mathbf{T} \right) \right). \tag{3.5c}
\]

From (3.5) it immediately follows that

\[
T(y) = Cy + E \quad \text{and} \quad - \frac{\partial}{\partial z} \left( \frac{1}{3} \text{Tr} \mathbf{T} \right) = C, \tag{3.6}
\]

where \( C \) and \( E \) are constants. Obviously, \( C \) is the gradient of the mean normal stress (the pressure) in the direction of the channel’s axis. By (3.4) we get

\[
\mathcal{D} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} \frac{\partial v^z}{\partial y} \\ 0 & \frac{1}{2} \frac{\partial v^z}{\partial y} & 0 \end{bmatrix},
\]

and using (3.3) it then follows that \( \frac{dv^z}{dy} = 2 \left( 1 + 2 \mathcal{R}_2 T^2 \right)^n T \). Substituting (3.6) into the previous expression leads to the following ordinary differential equation for the velocity

\[
\frac{dv^z}{dy} = 2 \left( 1 + 2 \mathcal{R}_2 (Cy + E)^2 \right)^n (Cy + E). \tag{3.7}
\]

3.1.1. Classical Navier–Stokes fluid. When \( n = 0 \), the problem reduces to the classical problem for the Navier–Stokes fluid, and \( v^z(y) \) is given by

\[
v^z(y) = 2 \left( \frac{C}{2} y^2 + Ey \right) + F.
\]
Classical Couette flow. By Couette flow we mean a flow when there is no gradient of the mean normal stress, and in this case have $C = 0$. The no-slip boundary condition reads $v^z(-1) = -1$ and $v^z(1) = 1$, and we find

$$v^z(y) = y. \quad (3.8)$$

Classical Poiseuille flow. When there is a gradient of the mean normal stress along the axis of the pipe, we call the flow Poiseuille flow. In this case, we require $v^z(-1) = 0$ and $v^z(1) = 0$, and the gradient of the mean normal stress (the pressure) should be equal to $-\frac{\Delta \alpha L^2}{V}$ (the dimensionless counterpart of $-\Delta$), these requirements lead to solution

$$v^z(y) = \frac{\Delta \alpha L^2}{V} \left(1 - y^2\right) = (1 - y^2), \quad (3.9)$$

where we have used the particular choice of the characteristic velocity $V$.

3.1.2. Stress power law fluids. When $n \neq 0$, we need to solve (3.7), subject to appropriate boundary conditions. We immediately see that (3.7), for $n \neq -1$ and $C \neq 0$, can be rewritten as $\frac{dv^z}{dy} = \frac{1}{2C R_2(n+1)} \left(1 + 2 R_2 (Cy + E)^2\right)^{n+1}$, hence

$$v^z(y) = \frac{1}{2C R_2(n+1)} \left(1 + 2 R_2 (Cy + E)^2\right)^{n+1} + F, \quad (3.10)$$

where $F$ is a constant. In the case $n = -1$ we obtain $\frac{dv^z}{dy} = \frac{1}{2C R_2} \frac{d}{dy} \left(\ln \left(1 + 2 R_2 (Cy + E)^2\right)\right)$, and consequently

$$v^z(y) = \frac{1}{2C R_2} \ln \left(1 + 2 R_2 (Cy + E)^2\right) + F, \quad (3.11)$$

where $F$ is again a constant to be determined by the boundary conditions. Counterpart of Poiseuille flow. In the case of Poiseuille flow we require $v^z(-1) = 0$ and $v^z(1) = 0$, therefore we get (for $n \neq -1$) the following system of equations

$$\frac{1}{2C R_2(n+1)} \left(1 + 2 R_2 (-C + E)^2\right)^{n+1} + F = 0, \quad (3.12a)$$

$$\frac{1}{2C R_2(n+1)} \left(1 + 2 R_2 (C + E)^2\right)^{n+1} + F = 0. \quad (3.12b)$$

Subtracting equation (3.12b) from (3.12a) leads to

$$\frac{1}{2C R_2(n+1)} \left(\left(1 + 2 R_2 (C + E)^2\right)^{n+1} - \left(1 + 2 R_2 (-C + E)^2\right)^{n+1}\right) = 0,$$

If $n \in \mathbb{N}$ one can use the well known expansion formula $a^{n+1} - b^{n+1} = (a-b)(a^n + a^{n-1}b + a^{n-2}b^2 + \cdots + ab^{n-1} + b^n)$ with $a = 1 + 2 R_2 (C + E)^2$ and $b = 1 + 2 R_2 (C + E)^2$, and—utilizing the fact that $R_2 > 0$—reduce the equation to $(1 + 2 R_2 (C + E)^2) - (1 + 2 R_2 (-C + E)^2) = 0$, and hence

$$E = 0. \quad (3.13)$$

Moreover, since $R_2$ is positive, we can claim that (3.13) is (with the corresponding $F$) a unique solution to system (3.12). A similar approach can be used in the case $n \in \mathbb{Q}$. Indeed, if $n$ has form $n = \frac{p}{q}$, $p \in \mathbb{Z}$ and $q \in \mathbb{N}$, it is sufficient to multiply (3.12a) and (3.12b) by $(1 + 2 R_2 (\pm C + E)^2)^q$ respectively, and then one can apply the same approach as above, and we again get (3.13) for $E$. If $n \in \mathbb{R} \setminus \mathbb{Q}$ we can use the density of $\mathbb{Q}$ in $\mathbb{R}$, and we immediately see that (3.13) holds also in this case. Formula (3.13) is therefore valid for all $n \in \mathbb{R}$, $n \neq -1$. Substituting (3.13) back into (3.12a) gives $F = -\frac{1}{2C R_2(n+1)} \left(1 + 2 R_2 C^2\right)^{n+1}$, and finally we get

$$v^z(y) = \frac{1}{2C R_2(n+1)} \left(\left(1 + 2 R_2 C^2 y^2\right)^{n+1} - \left(1 + 2 R_2 C^2\right)^{n+1}\right). \quad (3.14)$$

By (3.6), the constant $C$ is equal to the (dimensionless) gradient of the mean normal stress (the pressure), $-\frac{\Delta \alpha L^2}{V}$. If we use $V = \Delta \alpha L^2$ as the characteristic velocity, then $C = -1$. In this case (3.14) reads

$$v^z(y) = -\frac{1}{2R_2(n+1)} \left((1 + 2 R_2 y^2)^{n+1} - (1 + 2 R_2)^{n+1}\right). \quad (3.15)$$
It is easy to see that (3.15) reduces to the classical formula (3.9) if \( n = 0 \) or if \( R_2 \to 0 \).

In the case \( n = -1 \) the counterpart of (3.12) is the system

\[
\frac{1}{2CR_2} \ln \left( 1 + 2R_2 (-C + E)^2 \right) + F = 0, \\
\frac{1}{2CR_2} \ln \left( 1 + 2R_2 (C + E)^2 \right) + F = 0.
\]

(3.16a)

(3.16b)

Subtracting (3.16b) for (3.16a), we again get an equation for the parameter \( E \) that admits an unique solution \( E = 0 \) and consequently we get \( F = \frac{1}{2CR_2} \ln \left( 1 + 2R_2 C^2 \right) \). Finally, fixing \( C = -1 \) (see the case \( n \neq 1 \) for discussion of this choice), we get

\[
v^\tau(y) = \frac{1}{2R_2} \ln \left( \frac{1 + 2R_2 y^2}{1 + 2R_2} \right).
\]

(3.17)

Plots of the velocity profiles (3.15) and (3.17) are given in Figure 3. It should be noticed that some of the velocity profiles have inflexion points. Indeed, the second derivative of (3.15) reads \( \frac{d^2v^\tau}{dy^2} = -2 \left( 1 + 2R_2 y^2 \right)^{n-1} \left( 1 + 2R_2 \right) \), and the second derivative vanishes if \( y^2 = -\frac{1}{2R_2(2n+1)} \), the equation has a real solution if \( n < -\frac{1}{2} \). For these values of \( n \) we therefore have velocity profiles with inflection points. The existence of inflection points is—in the case of a perfect fluid—intimately related to the question of the stability of the given velocity profile. An analysis of stability of the velocity profiles arising in model (1.1) for \( n < -\frac{1}{2} \) could be therefore highly interesting.

Counterpart of Couette flow. In the case of Couette flow, we fix \( C = 0 \) (no pressure gradient is present). Thus (3.7) reduces to \( \frac{dv^\tau}{dy} = 2 \left( 1 + 2R_2 E^2 \right)^n E \), and the formula for the velocity is

\[
v^\tau(y) = 2 \left( 1 + 2R_2 E^2 \right)^n E y + F,
\]

(3.18)

where \( F \) is a constant to be fixed by the boundary conditions. Finally, boundary conditions \( v^\tau(-1) = -1 \) and \( v^\tau(1) = 1 \) lead to

\[
v^\tau(y) = y.
\]

(3.19)

The velocity profiles for generalized fluids given by the constitutive relation (1.1) are therefore the same for arbitrary values of dimensionless parameters \( R_1, R_2 \) and for arbitrary values of the exponent \( n \), moreover the profiles are the same as that for the classical incompressible Navier–Stokes fluid.
3.1.3. Remarks. It is worth observing that in the case of the classical generalized Navier–Stokes fluid \((2.5b)\), if one assumes velocity field of the form \((3.4)\), the balance of linear momentum reduces to

\[
2\mu_0 \left(1 + 2\lambda \left(\frac{d v^z}{d y}\right)^2\right)^m \frac{d v^z}{d y} = C y + E.
\]  

If \(m > -\frac{1}{2}\), one can express \(\frac{d v^z}{d y}\) from \((3.20)\) as a function of \(C y + E\) in a unique way. For \(m < -\frac{1}{2}\) the above equation presents the possibility of more than one solution for \(\frac{d v^z}{d y}\). On the other hand, in the setting \((3.1a)\) we are considering here we directly get an explicit ordinary differential equation for the velocity, see \((3.7)\).

3.2. Hagen–Poiseuille flow. Let us consider the steady, fully developed flow due to the application of a constant pressure gradient \(\frac{\partial p}{\partial z} e^z\) in an infinite circular pipe of radius \(R\) (see Figure 2a), and let us require the no-slip boundary condition be satisfied on the pipe’s wall. (See discussion of the boundary conditions in Section 4.)

Choosing a characteristic length \(L\) and a characteristic velocity \(V\), we can rewrite the governing equations \((1.1)\) and \((1.3b)\) in dimensionless form \((3.1)\). For Hagen–Poiseuille flow we can choose \(L = R\) and \(V = \frac{\Delta \alpha L^2}{2}\). From now on we will only use dimensionless variables, hence we can omit the star in order to keep the notation simple.

Let us suppose that the stress is given by

\[
T - \left(\frac{1}{3} \text{Tr } T\right) I = T (e_r \otimes e_z + e_z \otimes e_r) = \begin{bmatrix} 0 & 0 & T \\
0 & 0 & 0 \\
T & 0 & 0 \end{bmatrix},
\]

where \(e_r\) and \(e_z\) are unit vectors along the radial and axial directions, respectively, with respect to a cylindrical co-ordinate system \((r, \varphi, z)\), and \(T = T(r)\) is a scalar function. Substituting \((3.21)\) into the dimensionless counterpart of constitutive relation \((1.1)\) leads to

\[
\mathbb{D} = (1 + 2R_2 T^2)^n \begin{bmatrix} 0 & 0 & T \\
0 & 0 & 0 \\
T & 0 & 0 \end{bmatrix},
\]

and consequently, we are lead to assumption that the velocity field takes form \(v = v^z(r) e_z\). This particular choice for the form of the velocity field leads to

\[
\mathbb{D} = \begin{bmatrix} 0 & 0 & \frac{1}{2} \frac{d v^z}{d r} \\
0 & 0 & 0 \\
\frac{1}{2} \frac{d v^z}{d r} & 0 & 0 \end{bmatrix}
\]

and hence the form of the velocity field is “compatible” with the assumption concerning the form for the Cauchy stress. Using the assumptions it follows that the balance of linear momentum reduces to

\[
0 = \frac{1}{R_1} \frac{\partial}{\partial r} \left(\frac{1}{3} \text{Tr } T\right),
\]

\[
0 = \frac{1}{R_1} \frac{1}{r} \frac{\partial}{\partial \varphi} \left(\frac{1}{3} \text{Tr } T\right),
\]

\[
0 = \frac{1}{R_1} \left(\frac{dT}{dr} + \frac{T}{r} + \frac{\partial}{\partial z} \left(\frac{1}{3} \text{Tr } T\right)\right).
\]

From the first two equations, it immediately follows that \(\text{Tr } T\) does not depend on variables \(r\) and \(\varphi\). The last equation can be rewritten as

\[
\frac{dT}{dr} + \frac{T}{r} = -\frac{\partial}{\partial z} \left(\frac{1}{3} \text{Tr } T\right),
\]

where the left hand side depends only on the variable \(r\), and the right hand side is a function of \(z\) only. Consequently, the right hand side and left hand side of the previous equations must be equal a constant. If
we denote this constant as $C$, then $T$ obeys the equation \( \frac{dT}{dr} + \frac{r}{2} = C \), that can be rewritten as \( \frac{1}{r} \frac{dT}{dr} (rT) = C \), and the solution to this equation is
\[
T = \frac{Cr}{2} + \frac{D}{r},
\]
(3.25)
where $D$ is an arbitrary constant. Obviously, $C = -\frac{\partial}{\partial z} \left( \frac{1}{3} \text{Tr} \mathbb{T} \right)$ is the (dimensionless) gradient of the mean normal stress (the pressure) in the direction of the pipe’s axis—a quantity given as the datum of the problem. Now we can substitute (3.25) back into the constitutive relation (3.22), and we get the following ordinary differential equation for the velocity
\[
\frac{1}{2} \frac{dv^z}{dr} = \left( 1 + 2 \mathcal{R}_2 \left( \frac{Cr}{2} + \frac{D}{r} \right)^2 \right) \left( \frac{Cr}{2} + \frac{D}{r} \right). \tag{3.26}
\]

3.2.1. Classical Navier–Stokes fluid. For the classical Navier–Stokes fluid ($n = 0$) the equation for the velocity is reduced to \( \frac{dv^z}{dr} = (Cr + 2D/r) \), and consequently $v^z(r) = \left( \frac{Cr^2}{2} + 2D \ln r + E \right)$, where $E$ is a constant. Since we require the velocity to be finite, we get
\[
v^z(r) = \left( \frac{Cr^2}{2} + E \right). \tag{3.27}
\]

We already know that $C = -\frac{\partial}{\partial z} \left( \frac{1}{3} \text{Tr} \mathbb{T} \right)$, therefore—in the dimensionless variables—we have $C = -\frac{\Delta \alpha L^2}{V}$. Now we can fix the remaining constant $E$ by applying the boundary conditions. The no-slip boundary condition $v|_{r=1} = 0$ is in our case reduced to $v^z(1) = 0$, and application of this boundary condition results in the following formula for the velocity $v^z(r) = \frac{\Delta \alpha L^2}{2V} (1 - r^2)$. The characteristic velocity was chosen to be $V = \frac{\Delta \alpha L^2}{2V}$ (therefore we get $C = -2$), and hence we recover the well-known formula\footnote{The formula first appeared in papers by Hagen (1839) and Poiseuille (1847). Derivation of the formula directly form the Navier–Stokes equations is accredited to Hagenbach (1860), although Stokes gave the right formula at the time of publication of his well-known paper (1845). See Sutera and Skalak (1993) for a detailed history of Poiseuille law.} for Hagen–Poiseuille flow
\[
v^z(r) = (1 - r^2). \tag{3.28}
\]

3.2.2. Stress power law fluids. Let us now solve (3.26) for $n \neq 0$. We require the stress to be finite\footnote{If we are willing to consider $n < -\frac{1}{2}$, then, obviously, even if the stress is infinite for $r \to 0+$, the right hand side of (3.26) is well defined even for $r \to 0+$. In the present study we however do not consider the possibility of having infinite stress.} for $r \to 0+$, therefore we must take $D = 0$ in (3.26), and consequently the governing equation (3.26) reduces to
\[
\frac{1}{2} \frac{dv^z}{dr} = \left( 1 + 2 \mathcal{R}_2 \left( \frac{Cr}{2} \right)^2 \right) \left( \frac{Cr}{2} \right). \tag{3.29}
\]

We immediately see that the equation, for $n \neq -1$ and $C \neq 0$, can be rewritten in form \( \frac{dv^z}{dr} = \frac{1}{\mathcal{R}_2 C(n+1)} \frac{d}{dr} \left( 1 + 2 \mathcal{R}_2 \left( \frac{Cr}{2} \right) \right)^{n+1} \), and therefore
\[
v^z(r) = \frac{1}{\mathcal{R}_2 C(n+1)} \left( 1 + 2 \mathcal{R}_2 \left( \frac{Cr}{2} \right)^2 \right)^{n+1} + F, \tag{3.30}
\]
where $F$ is a constant to be fixed by the boundary conditions. Let us once again use the classical no-slip boundary condition on the pipe’s wall $v|_{r=1} = 0$ and fix $C = -2$ (see the discussion for the Navier–Stokes fluid), then it is obvious that the solution is
\[
v^z(r) = -\frac{1}{2 \mathcal{R}_2 (n+1)} \left( 1 + 2 \mathcal{R}_2 r^2 \right)^{n+1} - \left( 1 + 2 \mathcal{R}_2 \right)^{n+1}. \tag{3.31}
\]
It is easy to see that (3.31) reduces to (3.28) if $n = 0$ or if $\mathcal{R}_2 \to 0+$. 
If \( n = -1 \) and \( C \neq 0 \), then (3.25) can be rewritten in form
\[
\frac{dv^z}{dr} = \frac{1}{R_2C} \frac{d}{dr} \ln \left( 1 + 2R_2 \left( \frac{Cr}{2} \right)^2 \right),
\]
and therefore \( v^z = \frac{1}{R_2C} \ln \left( 1 + 2R_2 \left( \frac{Cr}{2} \right)^2 \right) + F \). If we use the classical no-slip boundary condition and the particular choice of the characteristic velocity already introduced above, we get
\[
v^z(r) = -\frac{1}{2R_2} \ln \left( 1 + 2R_2r^2 \right).
\]

Since the functions (3.31) and (3.33) are the same as functions (3.15) and (3.17), we refer the reader who wants visualize the corresponding velocity profiles to Figure 4.

### 3.3. Cylindrical Couette flow

Let us consider the steady, fully developed flow in the annulus between two concentric cylinders of radii \( R_1 \) and \( R_2 \), \( R_1 < R_2 \) rotating with angular frequencies \( \Omega_1 \) and \( \Omega_2 \) about the common axis (see Figure 2b for the problem setting). We will once again solve the governing equations in the dimensionless form (3.1). In cylindrical Couette flow we set the characteristic length to be the radius of the inner cylinder \( L = R_1 \) and the characteristic velocity \( V = R_1 \Omega_1 \), therefore we get \( R_1 = \alpha \rho \Omega_1 R_1^2 \) and \( R_2 = \frac{\beta R_1^2}{\alpha^2} \). Moreover, following the classical treatises by Taylor (1923) and Chandrasekhar (1958), we introduce two dimensionless parameters \( \mu = \frac{\Omega_2}{\Omega_1} \) and \( \eta = \frac{R_2}{R_1} \) in order to get a complete set of dimensionless parameters. Obviously \( \eta \in (0, 1) \). From now on we will work with the dimensionless variables, and—in order to keep the notation simple—we will omit the star that denotes dimensionless variables.

Let us suppose that the stress is given by
\[
T = \left( \frac{1}{3} \text{Tr} \ T \right) I = T (e_\rho \otimes e_\phi + e_\rho \otimes e_\phi) = \begin{bmatrix} 0 & T & 0 \\ T & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},
\]
where \( e_\rho \) and \( e_\phi \) are the unit vectors along the radial and circumferential directions, respectively, with respect to the cylindrical co-ordinate system \((r, \phi, z)\), and \( T = T(r) \) is a scalar function. Further, let us suppose that
\[
v = v^\phi(r)e_\phi.
\]

Under the assumptions above, it follows that the balance of linear momentum reduces to
\[
-\frac{v^z v^z}{r} = \frac{1}{R_1} \left( \frac{\partial}{\partial r} \left( \frac{1}{3} \text{Tr} \ T \right) \right),
\]

\[
0 = \frac{1}{R_1} \left( \frac{1}{r} \frac{\partial}{\partial r} \left( \frac{1}{3} \text{Tr} \ T \right) + \left( \frac{dT}{dr} + \frac{2T}{r} \right) \right),
\]

\[
0 = \frac{1}{R_1} \left( \frac{\partial}{\partial z} \left( \frac{1}{3} \text{Tr} \ T \right) \right).
\]

From (3.36c) it follows that the trace of the Cauchy stress tensor does not depend on \( z \). Furthermore (3.36b) can be rewritten as
\[
-\frac{\partial}{\partial \phi} \left( \frac{1}{3} \text{Tr} \ T \right) = r \left( \frac{dT}{dr} + \frac{2T}{r} \right),
\]
where the right hand side is a function of \( r \) only, and the left hand side can depend on \( r \) and \( \phi \). Consequently, we get \( r \left( \frac{dT}{dr} + \frac{2T}{r} \right) = K \) and \( -\frac{\partial}{\partial \phi} \left( \frac{1}{3} \text{Tr} \ T \right) = K \), where \( K \) is a constant, and we can conclude that the left hand side of (3.37) is therefore a linear function in \( \phi \). However, it must also be periodic with respect to \( \phi \) (due to the axial symmetry of the problem), hence \( K = 0 \). From (3.36b) it therefore follows that
\[
T(r) = -\frac{B}{r^2},
\]
where \( B \) is a constant, and that \( \text{Tr} \ T \) is a function of \( r \) only.
Next, it follows from (3.38) and (3.39) that
\[
\mathbb{D} = (1 + 2\mathcal{R}_2 T^2)^n T \left( e_r \otimes e_\phi + e_\phi \otimes e_r \right) = \begin{bmatrix}
0 & (1 + 2\mathcal{R}_2 T^2)^n T & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}, \tag{3.39}
\]
where, by the definition of $\mathbb{D}$,
\[
\mathbb{D} = \begin{bmatrix}
0 & \frac{1}{2} \left( \frac{dv^\phi}{dr} - \frac{v^\phi}{r} \right) & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}. \tag{3.40}
\]
Combining (3.38), (3.39) and (3.40) we arrive at the following ordinary differential equation for the velocity
\[
dv^\phi = -2B \left( 1 + \frac{2\mathcal{R}_2 B^2}{r^4} \right) \frac{1}{r^2}. \tag{3.41}
\]

3.3.1. Classical Navier–Stokes fluid. When $n = 0$, (3.41) reduces to $\dv^\phi = -\frac{2\mu}{r^2}$. Obviously, the velocity is then given by the following formula
\[
v^\phi(r) = Ar + \frac{B}{r}, \tag{3.42}
\]
where $A$ is a constant. The no-slip boundary condition on both walls, $v^\phi|_{r=1} = 1$ and $v^\phi|_{r=\frac{1}{2}} = \frac{A}{\eta}$, leads to
\[
A = \frac{1 - \frac{\mu}{\eta^2}}{1 - \frac{1}{\eta^2}}, \tag{3.43a}
\]
\[
B = \frac{1 - \mu}{1 - \eta^2}. \tag{3.43b}
\]

3.3.2. Stress power law fluids. Let us now focus on the general case $n \neq 0$, and let us assume that the solution to (3.41) takes form $v^\phi = v^\phi(\phi(r))$, where $\phi(r) = \frac{1}{r}$. We can take $\phi$ as a new variable, and it follows that $\dv^\phi = -\phi^2 \dv^r$, and consequently $\dv^\phi = -\phi \dv^r \left( \phi v^r \right)$, (3.41) therefore reads
\[
- \dv^r \left( \phi v^r \right) = -2B \left( 1 + \frac{2\mathcal{R}_2 B^2}{r^4} \right) \frac{1}{r^2}. \tag{3.44}
\]
Obviously, once we get a primitive function for the right hand side of (3.44), the problem (3.41) can be solved. Finding a primitive function for the right hand side of (3.44) is easy, indeed
\[
\int 2B \left( 1 + \frac{2\mathcal{R}_2 B^2}{r^4} \right) \frac{1}{r^2} \dv^r = \frac{B}{\sqrt{2\mathcal{R}_2 B^2}} \int (1 + u^2)^n du
\]
\[
= \frac{B}{\sqrt{2\mathcal{R}_2 B^2}} u F \left( \left[ -n, \frac{1}{2} \right] ; \left[ \frac{3}{2} \right] ; u^2 \right) = B \phi^2 F \left( \left[ -n, \frac{1}{2} \right] ; \left[ \frac{3}{2} \right] ; -2\mathcal{R}_2 B^2 \phi^4 \right), \tag{3.45}
\]
where we have used substitution $u = \sqrt{2\mathcal{R}_2 B^2} \phi^2$, $du = 2\sqrt{2\mathcal{R}_2 B^2} \phi \ dv^r$, and $F \left( [a, b] ; [c] ; z \right)$ denotes the hypergeometric function\textsuperscript{9}. Since we are mainly interested in exponents of the type $n = \frac{m}{2}$, $m \in \mathbb{Z}$ we can in

\textsuperscript{9}The problem was, to our best knowledge, first solved, as were many other problems in fluid mechanics, by Stokes (1845). Later Conetot (1890) used the flow between (finite) concentric cylinders rotating along common axis for measurements of fluid viscosities. For a short introduction of the history of the problem, we refer the reader to Donnelly (1991).

\textsuperscript{10}See for example Abramowitz and Stegun (1964) or Erdélyi et al. (1953) for the definition and various properties of the hypergeometric function. We note that the hypergeometric function is basically a sum of an infinite series, and one should therefore always verify that the parameters $a$, $b$, $c$ and $z$ lead to a convergent series.
this case find simple recursive formulae for the primitive function, indeed if \( k \in \mathbb{N} \), then

\[
\int (1 + u^2)^{k/2} \, du = \frac{u (1 + u^2)^{k/2}}{k + 1} + \frac{k}{k + 1} \int (1 + u^2)^{(k-1)/2} \, du,
\]

\[ (3.46a) \]

\[
\int \frac{1}{(1 + u^2)^{k/2}} \, du = \frac{u}{(k-2) (1 + u^2)^{(k-3)/2}} - \frac{k - 3}{k - 2} \int \frac{1}{(1 + u^2)^{(k-3)/2}} \, du,
\]

\[ (3.46b) \]

with obvious initial terms \( \int \frac{1}{1 + u^2} \, du = \arcsinh u \) and \( \int \frac{1}{1 + u^2} \, du = \arctan u \). We will use (3.46) especially in Section 3.3.3. Having (3.45), we can go back to (3.44), and it follows that in the original variable \( r \) we obtain

\[
v^\phi (r) = \frac{B}{r} F \left( \left[ -n, \frac{1}{2} \right]; \left[ \frac{3}{2} \right]; -\frac{2R_2B^2}{r^4} \right) + Ar,
\]

\[ (3.47) \]

where \( A \) is a constant to be fixed by the boundary conditions. If we require the classical no-slip boundary condition on both walls, \( v^\phi |_{r=1} = 1 \) and \( v^\phi |_{r=\frac{1}{n}} = \frac{\mu}{\eta} \), we get the following system

\[
BF \left( \left[ -n, \frac{1}{2} \right]; \left[ \frac{3}{2} \right]; -2R_2B^2 \right) + A = 1,
\]

\[ (3.48a) \]

\[
\eta BF \left( \left[ -n, \frac{1}{2} \right]; \left[ \frac{3}{2} \right]; -2R_2B^2 \eta^4 \right) + \frac{A}{\eta} = \frac{\mu}{\eta},
\]

\[ (3.48b) \]

for constants \( A \) and \( B \). It is easy to see that a solution to (3.48) is

\[
A = \frac{F \left( \left[ -n, \frac{1}{2} \right]; \left[ \frac{3}{2} \right]; -2R_2B^2 \eta^4 \right) - \frac{\mu}{\eta^2} F \left( \left[ -n, \frac{1}{2} \right]; \left[ \frac{3}{2} \right]; -2R_2B^2 \right)}{F \left( \left[ -n, \frac{1}{2} \right]; \left[ \frac{3}{2} \right]; -2R_2B^2 \eta^4 \right) - \frac{1}{\eta^2} F \left( \left[ -n, \frac{1}{2} \right]; \left[ \frac{3}{2} \right]; -2R_2B^2 \right)}.
\]

\[ (3.49a) \]

\[
B = \frac{1 - \mu}{F \left( \left[ -n, \frac{1}{2} \right]; \left[ \frac{3}{2} \right]; -2R_2B^2 \eta^4 \right) - \eta^2 F \left( \left[ -n, \frac{1}{2} \right]; \left[ \frac{3}{2} \right]; -2R_2B^2 \right)}.
\]

\[ (3.49b) \]

where (3.49b) is an implicit equation\(^{11}\) for \( B \). Since \( F \left( \left[ -n, \frac{1}{2} \right]; \left[ \frac{3}{2} \right]; z \right) \) is, for \( n = 0 \), by definition equal to one, we can see that (3.47) and (3.49) reduce to the well known formulae (3.42) and (3.43) when \( n = 0 \).

Having an expression for the velocity, we can go back to (3.39a) and, by a simple integration, get a formula for the mean normal stress, \( \frac{1}{3} \text{Tr} \mathbb{T} \). Knowledge of \( \frac{1}{3} \text{Tr} \mathbb{T} \) does not play a role in the problem of determining the velocity field, and hence we omit the calculation. We will, however, briefly discuss the role of the mean normal stress in Section 4.1.

Plots of the velocity profiles given by (3.47) for various values of \( n \), dimensionless parameters \( \mu, \eta \) and \( R_2 \) are shown in Figure 4, Figure 5 and Figure 6. (Recall that \( n = 0 \) corresponds to the Navier–Stokes fluid.) The implicit equation for \( B \), (3.49b), for \( n < -\frac{1}{2} \), possesses two distinct solutions \( B_1 \) and \( B_2 \), therefore, for these values of \( n \), we plot both solutions in Figures 5 and 6. The nonuniqueness and even nonexistence of solution to (3.49b) will be thoroughly discussed in the next section.

3.3.3. Solvability of the implicit equation (3.49b) for \( B \). Since \( B \) is given by a complicated implicit equation (3.49b), one should be cautious in assuming that the equation has a solution. We will analyze the solvability of (3.49b) for general exponent \( n \) in the form \( n = \frac{m}{2} \), \( m \in \mathbb{Z} \), and the general admissible parameter values \( \eta \in (0, 1) \), \( \mu \in \mathbb{R} \) and \( R_2 \in \mathbb{R}^+ \). For the sake of simplicity—we do not want to juggle with plus and minus signs—we can assume \( \mu > 1 \), analysis for \( \mu \leq 1 \) will be a straightforward adaptation of the approach used below. Moreover, a generalization of the analysis shown below can be used to get similar results for \( n \in \mathbb{Q} \) or, if necessary, even for \( n \in \mathbb{R} \).

Let us first note that the formula for the velocity (3.47) can be rewritten as

\[
v^\phi (r) = \frac{Br}{\sqrt{2R_2B^2}} P_m \left( \frac{\sqrt{2R_2B^2}}{r^2} \right) + Ar,
\]

\[ (3.50) \]

\(^{11}\)We yet do not know whether this equation has a solution. This question will be addressed in Section 3.3.3.
Figure 4: Velocity profiles for co-rotating and counter-rotating cylindrical Couette flow according to formula (3.48); nonnegative exponent \( n \). \((\eta = \frac{1}{2}, R_2 = 200)\)

(a) Co-rotating cylinders, \( \mu = 2 \).

(b) Counter-rotating cylinders, \( \mu = -\frac{1}{2} \).

Figure 5: Velocity profiles for counter-rotating cylindrical Couette flow according to formula (3.48), for nonpositive exponent \( n \). Implicit equation (3.49b) has two solutions, \( B_1 \) and \( B_2 \) for the given values of the parameters and \( n < -\frac{1}{2} \). Hence we get two distinct velocity profiles shown in Figures (a) and (b). Unique velocity profiles for \( n = 0 \) and \( n = -\frac{1}{2} \) are portrayed in both Figures for reference. \((\eta = \frac{1}{2}, R_2 = \frac{1}{100}, \mu = -\frac{1}{2})\)

where \( P_m(y) = \text{def} \int_0^y (1 + \xi^2)^{\frac{m}{2}} \, d\xi \). Function \( P_m(y) \) is for all \( m \) an increasing function of \( y \), and it can be shown that for \( s \in (0, 1) \) we have

\[
\lim_{y \to +\infty} P_m(y) - P_m(sy) = \begin{cases} +\infty, & m \geq 0, \\ -\ln s, & m = -1, \\ 0, & m < -1. \end{cases}
\]  (3.51)
Figure 6: Velocity profiles for co-rotating cylindrical Couette flow according to formula (3.48), for nonpositive exponent \( n \). Implicit equation (3.49b) has two solutions, \( B_1 \) and \( B_2 \) for the given values of the parameters and \( n < -\frac{1}{2} \). Hence we get two distinct velocity profiles shown in Figures (a) and (b). Unique velocity profiles for \( n = 0 \) and \( n = -\frac{1}{2} \) are portrayed in both Figures for reference. (\( \eta = \frac{1}{2}, R_2 = 100, \mu = 2 \))

If we use (3.50) as the formula for the velocity, the boundary conditions \( v^\phi \bigg|_{r=1} = 1 \) and \( v^\phi \bigg|_{r=\frac{1}{\eta}} = \frac{\mu}{\eta} \) lead to the following system:

\[
\frac{B}{\sqrt{2R_2B^2}} P_m \left( \sqrt{2R_2B^2} \right) + A = 1, \tag{3.52a}
\]

\[
\frac{B}{\eta\sqrt{2R_2B^2}} P_m \left( \sqrt{2R_2B^2\eta^2} \right) + \frac{A}{\eta} = \frac{\mu}{\eta}. \tag{3.52b}
\]

that is a counterpart of (3.48). The implicit equation for \( B \) is then

\[
B \left( P_m \left( \sqrt{2R_2B^2} \right) - P_m \left( \sqrt{2R_2B^2\eta^2} \right) \right) = \sqrt{2R_2B^2} (1 - \mu). \tag{3.53}
\]

Since \( \mu > 1 \) and \( \eta \in (0, 1) \) and \( P_m(y) \) is an increasing function, we see that a nontrivial solution to (3.53) must be negative, (3.53) therefore reads

\[
f_m(B; R_2, \eta) = -\sqrt{2R_2} (1 - \mu), \tag{3.54}
\]

where we have ignored the trivial solution \( B = 0 \), and we have denoted

\[
f_m(B; R_2, \eta) = \det P_m \left( -B\sqrt{2R_2} \right) - P_m \left( -B\sqrt{2R_2\eta^2} \right). \tag{3.55}
\]

In order to get information on the solvability of (3.54), we need to carefully look at the behaviour of \( f_m(B; R_2, \eta) \). The function \( f_m(B; R_2, \eta) \) is always positive, and one can easily show—see also (3.51)—that \( \forall R_2 \in \mathbb{R}^+ \) we get the same asymptotic behaviour for \( B \) tending to minus infinity and zero respectively,

\[
\lim_{B \to -\infty} f_m(B; R_2, \eta) = \begin{cases} +\infty, & m \geq 0, \\ -2 \ln \eta, & m = -1, \\ 0, & m < -1, \end{cases} \tag{3.56a}
\]

\[
\lim_{B \to 0^+} f_m(B; R_2, \eta) = 0. \tag{3.56b}
\]
Let us now calculate the first derivative of \( f_m(B; \mathcal{R}_2, \eta) \) with respect to \( B \). We will consider four qualitatively different cases, \( m < -1 \), \( m = -1 \), \( m = 0 \), and \( m > 0 \).

Let us start with \( m < -1 \). If \( m = -k \), \( k \in \mathbb{N} \setminus \{0, 1\} \), then

\[
- \frac{1}{\sqrt{2\mathcal{R}_2^2}} \frac{d}{dB} f_{-k}(B; \mathcal{R}_2, \eta) = \frac{1}{(1 - (\sqrt{2\mathcal{R}_2 B \eta})^2)^{\frac{k}{2}}} - \frac{\eta^2}{(1 - (\sqrt{2\mathcal{R}_2 B \eta})^2)^{\frac{k}{2}}}
\]

\[
= \frac{1}{(1 - (\sqrt{2\mathcal{R}_2 B \eta})^2)^{\frac{k}{2}}} \left( 1 + (\sqrt{2\mathcal{R}_2 B \eta})^2 \right)^{\frac{k-1}{2}} \eta^{\frac{k}{2}} \left( 1 + (\sqrt{2\mathcal{R}_2 B \eta})^2 \right)^{\frac{1}{2}} + \eta^2 \left( 1 + (\sqrt{2\mathcal{R}_2 B \eta})^2 \right)^{\frac{1}{2}}
\]

and the derivative therefore vanishes for

\[
B_{\text{max},k}^2 = \frac{1 - \eta^4}{2\mathcal{R}_2 (\eta^4 - \eta^4)}.
\] (3.57)

The term on the right hand side of (3.57) is always positive, since, for \( \eta \in (0, 1) \) and \( k \in \mathbb{N} \), we have \( 1 - \eta^4 \geq 0 \) and \( \eta^4 - \eta^4 \geq 0 \). Moreover, because we are interested only in negative \( B \), (3.57) gives an unique \( B_{\text{max},k} \) at which the derivative vanishes.

The notation \( B_{\text{max},k} \) is justified, since \( f_{-k}(B; \mathcal{R}_2, \eta) \) attains its maximum at this point. Indeed, the fact that the stationary point is a maximum follows from the fact that there is only one stationary point, the function is smooth and we have the asymptotic behaviour (3.56). Function \( f_{-k}(B; \mathcal{R}_2, \eta) \) contains—with respect to variable \( B \) and parameter \( \mathcal{R}_2 \)—only the combination \( \sqrt{2\mathcal{R}_2 B} \), and, due to (3.57), it is clear that \( \sqrt{2\mathcal{R}_2 B_{\text{max},k}} \) does not depend on \( \mathcal{R}_2 \) and the maximal value \( C_{-k}(\eta) = \max_{B \in \mathbb{R}^+} f_{-k}(B; \mathcal{R}_2, \eta) \) is therefore uniform with respect to \( \mathcal{R}_2 \). Finally, summing up all we know about properties of \( f_{-k}(B; \mathcal{R}_2, \eta) \) and its first derivative, we can conclude that

\[
\forall \mathcal{R}_2 \in \mathbb{R}^+: \quad f_{-k}(B; \mathcal{R}_2, \eta) : \mathbb{R}^+ \mapsto (0, C_{-k}(\eta)].
\] (3.58)

Whether (3.59) has a solution therefore depends on whether the right hand side of (3.54) falls within the interval \((0, C_{-k}(\eta)]\). There are three qualitatively different possibilities:

\[
\mathcal{R}_2 \in \begin{cases} 
(0, \frac{C_{-k}(\eta)}{\mu-1}) : & \text{(3.52)} \text{ has two solutions, } B_{1,k} \text{ and } B_{2,k}, \ B_{1,k} < B_{\text{max},k} < B_{2,k}, \\
\left\{ \frac{C_{-k}(\eta)}{\mu-1} \right\} : & \text{(3.52)} \text{ has one solution, } B_{1,k} = B_{2,k} = B_{\text{max},k}, \\
\left( \frac{C_{-k}(\eta)}{\mu-1}, +\infty \right) : & \text{(3.52)} \text{ has no solution.}
\end{cases}
\] (3.59)

If \( m = -1 \), then

\[
- \frac{1}{\sqrt{2\mathcal{R}_2^2}} \frac{d}{dB} f_{-1}(B; \mathcal{R}_2, \eta) = \frac{1}{(1 - (\sqrt{2\mathcal{R}_2 B \eta})^2)^{\frac{1}{2}}} - \frac{\eta^2}{(1 - (\sqrt{2\mathcal{R}_2 B \eta})^2)^{\frac{1}{2}}}
\]

\[
= \frac{1}{(1 - (\sqrt{2\mathcal{R}_2 B \eta})^2)^{\frac{1}{2}}} \left( 1 + (\sqrt{2\mathcal{R}_2 B \eta})^2 \right)^{\frac{1}{2}} + \eta^2 \left( 1 + (\sqrt{2\mathcal{R}_2 B \eta})^2 \right)^{\frac{1}{2}}
\]

The derivative is therefore nonvanishing and negative, furthermore we see that \( \lim_{B \to -\infty} \frac{d}{dB} f_{-1}(B; \mathcal{R}_2, \eta) = 0 \). Using the information we have on the derivative, smoothness of \( f_{-1}(B; \mathcal{R}_2, \eta) \), and asymptotic behaviour (3.56), we can conclude that \( f_{-1}(B; \mathcal{R}_2, \eta) \) maps—for all \( \mathcal{R}_2 \in \mathbb{R} \)—the negative real axis to the

\[\text{In the manipulations we are using several times the well known identity } a^n - b^n = (a - b) (a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \cdots + ab^{n-2} + b^{n-1}).\]

\[\text{For visualization of (3.59) and properties of functions } f_{-k}(B; \mathcal{R}_2, \eta) \text{ see Figure 4a that contains a plot of } f_{-2}(B; \mathcal{R}_2, \eta).\]
therefore have two possibilities

\[ f_{-1}(B; R_2, \eta) = -2 \ln \eta \]

Equation (3.54) therefore always has a solution, and this solution is unique.

If \( m = 0 \), then (3.54) reduces to \( B \sqrt{2R_2} (1 - \eta^2) = -\sqrt{2R_2} (1 - \mu) \). We are dealing with the classical Navier–Stokes fluid, and the problem has, as we already know, an unique solution.

If \( m = k, k \in \mathbb{N} \setminus \{0\} \), then

\[ -\frac{1}{\sqrt{2R_2}} \frac{df_k(B; R_2, \eta)}{dB} = \left(1 + (-\sqrt{2R_2} B)^2\right) \frac{1}{2} - \eta^2 \left(1 + (-\sqrt{2R_2} B \eta)^2\right)^{\frac{1}{2}}. \]

Function \( \left(1 + (-\sqrt{2R_2} B)^2\right)^{\frac{1}{2}} \) is an increasing function of \( B \) and since \( \eta \in (0, 1) \) it is clear that the derivative for \( B \in \mathbb{R}^- \) is nonvanishing and always negative. Hence the function \( f_k(B; R_2, \eta) \) maps \( \mathbb{R}^- \) to \( \mathbb{R}^+ \) and the mapping is one-to-one. Equation (3.54) therefore always has a solution, and this solution is unique.

Plots of functions \( f_m(B; R_2, \eta) \) for \( m \in \{-2, -1, 0, 1\} \) (thus for all possible types of qualitative behaviour) are given in Figure 7.

The possible nonexistence of the solution of course does not imply that the governing equations (3.1) do not have a solution, we have only shown that the solution (for certain parameter values) does not have the form given by (3.34) and (3.35). Furthermore, we would like to emphasize that the (non)existence or multiplicity of solutions for certain values of the material parameter \( R_2 \) and corresponding controllable parameters \( \eta, \mu \) is a feature of the cylindrical Couette flow problem. If we consider for example Hagen–Poiseuille flow, see Section 3.2.2, then the solution exists and is unique for arbitrary material parameter \( R_2 \) and controllable parameter \( \Delta \) (the forcing pressure gradient). Interestingly, if we specify boundary conditions in terms of the torque acting on the cylinders, we reach a completely different conclusion on the solvability of the problem, see Section 4.1 for details.

4. Boundary conditions

Concerning the boundary value problems discussed in the previous section, we were not following our basic idea that the force (the stress) is the primary quantity and that the kinematics depends on the force. To make our approach consistent we shall formulate the boundary conditions in terms of stress or related
quantities such as torque or surface traction and not in terms of the velocity. We will start the discussion of the role of boundary conditions with the case of cylindrical Couette flow, where the choice of the boundary condition has a significant impact on the qualitative characteristics of the solution.

4.1. Cylindrical Couette flow. In real experiments, cylindrical Couette flow is used for the measurement of fluid viscosity, and in fact the quantity that is directly measured is the torque exhibited on the cylinders, not the velocity of the cylinders, see for example Couette (1890). We can therefore claim that giving the boundary condition in terms of torque is a natural approach to the problem.

Let us first calculate the torque $\mathbf{M}$ exhibited by the fluid on the inner and outer cylinder. The force acting on the outer surface (unit normal $\mathbf{e}_r$) of a cylindrical segment of the cylinder of radius $r$ is $\mathbf{F} = F_\parallel \mathbf{e}_\parallel + F_\perp \mathbf{e}_\perp = T_\parallel r d\varphi dz \mathbf{e}_\parallel + \frac{1}{2} (\mathbf{TrT})(r d\varphi dz \mathbf{e}_\perp)$, where $dz$ is the height of the segment and $r d\varphi$ is the width of the segment. Since we assume that the relevant quantities are independent of $z$, it will be more convenient to work with densities. Using the expression for the force, we see that the density of the force is $f = f_\parallel \mathbf{e}_\parallel + f_\perp \mathbf{e}_\perp = T_\parallel r d\varphi \mathbf{e}_\parallel + \frac{1}{2} (\mathbf{TrT})(r d\varphi \mathbf{e}_\perp)$. Consequently, the density of the torque $\mathbf{m}$ exhibited by the fluid to the whole linear segment of the cylinder of radius $r$ is

$$\mathbf{m} = \int r \times f = \int r \mathbf{e}_r \times f = \left( \int_{\varphi=0}^{2\pi} T_\parallel r^2 d\varphi \right) \mathbf{e}_z = -2\pi B \mathbf{e}_z,$$

(4.1)

where we have used (3.35) for the stress. Hence the density of the torque exhibited by the fluid on the inner cylinder is $\mathbf{m}_1 = -2\pi B \mathbf{e}_z$ and for the outer cylinder (due to the opposite orientation of the unit outward normal) we get $\mathbf{m}_2 = -\mathbf{m}_1$.

Therefore, if we want to specify the torque $\mathbf{M}$ (or more precisely its density $\mathbf{m}$) on both cylinders, we can not, if we do not want to give up our assumptions on the form of the stress, prescribe $\mathbf{m}_1$ and $\mathbf{m}_2$ arbitrarily. This observation has no relevance to the constitutive model for the fluid filling the gap between the cylinders, not, if we do not want to give up our assumptions on the form of the stress, prescribe $\mathbf{m}_1$ and $\mathbf{m}_2$ arbitrarily.

If the boundary conditions are specified in terms of velocity, (3.49) and (3.52) respectively are systems that are used to find $A$ and $B$ from known values of $\mu$ and $\eta$. On the other hand, if we specify torque $\mathbf{m}_1 = -\mathbf{m}_2$, then $B$—see (4.1)—and $\eta$ are given, and we have to find $A$ and $\mu$. For this purpose we need to specify the behaviour of the fluid velocity on the interface between the cylinders, it is only a consequence of our particular form for the stress field, indeed if we were solving the cylindrical Couette flow problem using the standard approach, see for example Berker (1963), we would get the same restriction on the torque.

From a technical point of view we can once again use the system (3.49) and (3.52) respectively, but now we have to solve it for $A$ and $\mu$. Interestingly, in this case the system will always have a solution even in the case $n < -\frac{1}{2}$. Indeed, given $B$ and $\eta$, (3.49b) and (3.53) respectively give an unique value of $\mu$, and (3.49a) and (3.52a) respectively give an unique value of $A$. Thus if we specify the boundary conditions in terms of the stress (torque) we get existence and uniqueness of solution to the cylindrical Couette problem. The uniqueness must be nevertheless understood as uniqueness in the class of special solutions of form (3.34) and (3.35) respectively, since it is possible that there exist solutions that do not have the form we used in our analysis.

In order to get a full solution to the problem, we should also specify the density of the normal force $f_\perp$ acting on the cylinders—this is necessary to determine the integration constant for the mean normal stress, $\frac{1}{3} \mathbf{TrT}$, that arises by integration of (3.36a). But since it is usually assumed that the cylinders are rigid and hence can bear any normal stress (no change in geometry can occur due to the flow), and because knowledge of the mean normal stress is not necessary to determine the velocity and shear stress field, there is—as far as we are not interested in the mean normal stress—no need to specify $f_\perp$.

---

14 By linear segment we mean the circle of radius $r$ lying on a plane perpendicular to the $z$-axis.

15 See also the end of Section 3.3.2 for a discussion on calculation of the mean normal stress.
4.2. Plane Couette and Poiseuille flow. We note that the classical approach to Poiseuille flow is somehow odd in that the boundary condition on the upper and bottom plates are specified in terms of velocity, whereas the mean normal stress (the pressure) is specified on the lateral sides. Components of the Cauchy stress $\mathbb{T}$ are therefore already used even in the classical approach.

Let us again start with calculation of the force $\mathbf{F}$ acting on the plates. As in the previous case it will be convenient to work with surface density of the force, $\mathbf{F} = \int \mathbf{f} \, dS$, where $dS$ is the area of the surface element. We recall that the traceless part of the Cauchy stress is given by (3.2) and, in the light of (3.6), we have

$$\mathbb{T}_\delta = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & Cy + E \\ 0 & Cy + E & 0 \end{bmatrix},$$

and the constant $C$ is given by $-\frac{\partial}{\partial z} \left( \frac{1}{3} \text{Tr} \mathbb{T} \right)$. The force $\mathbf{f}$ acting on the surface element with unit outward normal $\mathbf{e}_y$ is

$$\mathbf{f} = f_\parallel \mathbf{e}_z + f_\perp \mathbf{e}_y = T_\parallel \mathbf{e}_y + \frac{1}{3} \text{Tr} \mathbb{T} \mathbf{e}_y,$$

and (4.2) therefore gives, for the upper plate, $f_{\parallel,1} = -(C + E)$, and $f_{\parallel,2} = -(C + E)$ for the bottom plate. We are interested only in $f_\parallel$ since we assume that the plates are rigid and $f_\perp$ therefore can not induce any change in the problem geometry, and irrespective of the magnitude of the force, the plates will remain in the same position. This assumption will deprive us from the possibility of fixing the integration constant for $\frac{1}{3} \text{Tr} \mathbb{T}$, but since this integration constant does not have an influence on the velocity field, we can make such assumption. This situation in fact also occurs if we study the classical Navier–Stokes fluid, since the pressure is given up to a constant. If necessary, it would be however easy to include the normal force and fix the mean normal stress (the pressure).

In the case of Couette flow we have

$$f_{\parallel,1} = -E, \quad f_{\parallel,2} = E,$$

and for the Poiseuille flow we get\textsuperscript{16}

$$f_{\parallel,1} = -C, \quad f_{\parallel,2} = -C.$$ \hspace{1cm} (4.3a) \hspace{1cm} (4.3b)

We can now use $C = -\frac{\partial}{\partial z} \left( \frac{1}{3} \text{Tr} \mathbb{T} \right)$ and (4.3)—Poiseuille flow—or $C = 0$ and (4.4)—Couette flow—as a starting point for our boundary value problem. We note that (4.3) and (4.4) are straightforward consequences of our assumptions on the form of the stress and do not have any relevance to a particular constitutive model. The fact that if we want to get flow in the form (5.1) and consequently the traceless part of the Cauchy stress in form (3.2)—or vice versa—then we have to require (4.3) and (4.4) respectively, is as well valid for the classical Navier–Stokes fluid, and (4.3) and (4.4) are therefore in no sense artificial.

If we start with (4.3) and (4.4) instead of the classical boundary conditions, we can repeat step by step the derivation presented in Section 3.1 and we finally end up with the formula for the velocity, (3.10) and (3.11) respectively. In these formulae the constants $C$ and $E$ are already fixed by (4.3) and (4.4) respectively. What remains is to fix the constant $F$. This constant, however can not be fixed by specifying the forces (stresses), since the same forces (stresses) generate an unique\textsuperscript{17} velocity field up to an uniform rectilinear motion (choice of the reference frame). Obviously, the nature of constant $F$ is precisely of this kind.

In order to determine the velocity in an unique way, we have to say, for example, that in our particular reference frame there is no motion in the sense that the plates stay at rest. Then we have to specify a constitutive relation for the fluid-solid interface (see the discussion above). We can for example require that—on the boundary—the fluid velocity is always equal to the solid velocity, in other words the no-slip

\textsuperscript{16}We can, without loss of generality, fix $E = 0$, otherwise we would get a superposition of Couette and Poiseuille flow. Fixing $E = 0$ allows us to strictly distinguish between these two flows, each of them being driven by a different physical mechanism—sliding plates in the case of Couette flow and pressure gradient in the case of Poiseuille flow.

\textsuperscript{17}If we indeed have uniqueness of solution. But let us for the moment assume that the uniqueness can be proved.
boundary condition. Under these assumptions we can, from the technical point of view, use the same
equations to determine the value of \( F \). If we do so, we recover the results already presented.

4.3. Hagen–Poiseuille flow. Discussion of the boundary conditions for Hagen–Poiseuille flow is similar to
that for plane Poiseuille flow, therefore we will only briefly review the main points.

Forc e acting on the inner surface (unit normal \(-\mathbf{e}_r\)) of a cylindrical segment of a cylinder of radius \( r \) is
\[
\mathbf{F} = F_\perp \mathbf{e}_r + F_\parallel \mathbf{e}_z = -\left( \frac{1}{3} \text{Tr} \mathbf{T} \right) r \mathrm{d} \varphi \mathrm{d} z \mathbf{e}_r - T_{r z} r \mathrm{d} \varphi \mathrm{d} z \mathbf{e}_z,
\]
where \( \mathrm{d} z \) is the height of the segment and \( r \mathrm{d} \varphi \) is the
width of the segment. We see, that due to the formula for the stress, it is convenient to work with a density
of the force \( f = f_\perp \mathbf{e}_r + f_\parallel \mathbf{e}_z = -\left( \frac{1}{3} \text{Tr} \mathbf{T} \right) r \mathrm{d} \varphi \mathbf{e}_r - T_{r z} r \mathrm{d} \varphi \mathbf{e}_z \). Using (3.30) with \( D = 0 \) we get
\[
f_\parallel = -\int_0^{2\pi} \frac{C}{2} \mathrm{d} \varphi = -\pi C
\]
(4.5)
for the total force acting on the whole linear segment of the cylinder of radius \( R \). (Recall that in the
dimensionless variables we have \( R = 1 \).) The perpendicular component of the force, \( f_\perp \), will be again ignored
on grounds of the same arguments as in the previous section.

Now we can use \( C = -\frac{\partial}{\partial r} \left( \frac{1}{3} \text{Tr} \mathbf{T} \right) \) and (4.3) as boundary conditions for the pipe flow problem. If we do so,
we can repeat step by step the derivation presented in Section 3.2, and we finally end up with formula (3.30)
or its appropriate counterpart in the singular case \( n = -1 \). The constant \( F \) that appears in these formulæ
can be fixed by the same arguments as those in the previous section—in order to fix the constant we require
the pipe wall to stay at rest in our chosen reference frame, and we want the no-slip boundary condition to
hold on the fluid-solid interface. This requirement technically leads to the same equations as in the classical
approach, and we therefore recover the solution found in Section 3.2.

5. Conclusion

We have found analytical solutions to several boundary value problems for “stress power law” fluids—plane
Couette and Poiseuille flow, Hagen–Poiseuille flow and cylindrical Couette flow. The velocity profiles that
have been obtained are different from the velocity profiles obtained using the classical Navier–Stokes fluid,
but this is to be expected. Moreover the velocity profiles for certain parameter values exhibit interesting
features like existence of inflection points and multiplicity of solutions.

The boundary value problems were solved using the classical no-slip boundary condition for the velocity
and also using boundary conditions specified in terms of force (torque, stress). Specifying the boundary
conditions in terms of forces follows the basic idea that when solving a problem it is preferable to specify
the input data in terms of causes (force, stress, torque) and not in terms of effects (generated velocity field).
In some cases there is no difference between the classical approach and the current approach, but at least
the example of the cylindrical Couette flow shows that the different point of view concerning the boundary
conditions can lead to results that are different from those we obtain in the case of the usual specification.
Specifying the boundary conditions in terms of forces might at first seem unnatural, but we think that it
is only due to the fact that specifying the boundary conditions in terms of velocity is so deeply burnt into
our minds that one rarely doubts its status in spite of the fact that the question of the choice of the right
boundary conditions is a very delicate issue.

The paper provides a starting point for further research focused on the class of fluids considered here and
more generally on fluids, defined through implicit constitutive relations for the stress \( \mathbf{T} \) and the symmetric
part of the velocity gradient \( \mathbf{D} \).

References

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