Qualitative analysis of solutions to discrete static contact problems with Coulomb friction

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Abstract

We analyze properties of solutions to discrete contact problems with Coulomb friction which are parametrized by the coefficient of friction \( F \). Using a generalized variant of the implicit-function theorem we establish conditions under which there exists a local Lipschitz continuous branch of solutions around a reference point. Finally, a piecewise smooth continuation algorithm which allows to follow such branches of solutions is proposed.

Keywords: Coulomb friction, piecewise smooth continuation methods

1. Introduction

Contact mechanics is a branch of solid mechanics which studies the behavior of loaded deformable structures in mutual contact. Besides non-penetration conditions one should take into account the influence of friction on contacting surfaces to get a more realistic model. Although Coulomb friction is the classical one, its mathematical treatment remained open for a long time. The first mathematically justified existence proof for the static case was done in [1]. Later, results were extended to quasistatic and dynamic problems in [2]. Typically the existence of at least one solution is shown for sufficiently small coefficients of friction \( F \). On the other hand there is no information on the structure of solutions in a general case. Some partial results are known provided that the solution enjoys a priori given properties ([3], [4]). The situation is rather different in the discrete case. In examples with a small number of degrees of freedom one can find solutions "by hand". The structure of solutions is relatively complicated even for models with one degree of freedom ([5], [6], [7]). For models arising from finite element discretizations the situation is more involved but still we have certain knowledge on the qualitative behavior of their solutions which is not available in the continuous case. Indeed, the discrete model has at least one solution for any friction coefficient \( F \), and this solution is unique if \( F \) is small enough. On the other hand, the bounds on \( F \) guaranteeing uniqueness of the solution are mesh-dependent. In other words, the uniqueness property of the discrete problem depends on the size of the problem. Thus it may happen that a small change of \( F \) leads to a large change of the solution but this phenomenon appears only when sufficiently fine discretizations are used. This "catastrophic" behavior can be observed in numerical experiments ([8], [9]). The first step towards better understanding of the discrete model is to establish the existence of local Lipschitz continuous branches of solutions parametrized by \( F \). The present paper extends results from [10], where \( F \) was assumed to be constant, to the case when \( F \) depends on the spatial variable. Such results are important for the development of continuation algorithms, which is another goal of this paper.

The paper is organized as follows: in Sections 2 and 3 we formulate a 3D static contact problem with orthotropic Coulomb friction and solution-dependent friction coefficients. We use the fixed-point formulation of both, the continuous as well as the discrete problem. We show that the discrete problems have a solution for any friction matrix \( F \) belonging to an appropriate class and have a unique solution under additional assumptions on \( F \). Then we analyze how these assumptions depend on...
the size of the discrete problem, i.e. on the norms of finite element partitions. The new results extend and cover the classical ones for isotropic Coulomb friction with a coefficient which does not depend on the solution ([11]). In Section 4 we formulate sufficient conditions for isotropic Coulomb friction in 2D under which there exists a Lipschitz continuous solution branch in a vicinity of a reference value of $\mathcal{F}$. In Section 5 we propose a piecewise smooth variant of the Moore-Penrose continuation algorithm, which enables to follow the solution branch. Finally in Section 6 we apply this algorithm for solving very simple model examples with a small number of contact nodes.

Throughout the paper we shall use the following notation: $H^k(D)$, $k \geq 0$ integer, stands for the standard Sobolev space of functions in the set $D$ ($H^0(D) = L^2(D)$) equipped with the norm $\| \cdot \|_{k,D}$ and the scalar product $(\cdot, \cdot)_D$. For vectors, vector functions and matrices we use bold characters. The scalar product of two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^p$ is denoted by $\mathbf{x} \cdot \mathbf{y}$. The Euclidean norm in $\mathbb{R}^p$, which is frequently written as $\| \cdot \|$. The set of all $x \in \mathbb{R}^p$ such that $x_i \geq 0$, $x_i > 0$ for all $i = 1, \ldots, p$ will be denoted by $\mathbb{R}^p_+$ and $\mathbb{R}^p_{+*}$, respectively. Finally, the fact that a constant $C$ depends on parameters $t_1, \ldots, t_s$ will be emphasized by writing $C := C(t_1, \ldots, t_s)$.

2. Fixed-point formulation of contact problems with orthotropic Coulomb friction law

Let a deformable body be represented by a bounded domain $\Omega \subset \mathbb{R}^3$ whose sufficiently smooth boundary $\partial \Omega$ is decomposed into non-empty, non-overlapping parts $\Gamma_u$, $\Gamma_f$ and $\Gamma_c$. On $\Gamma_u$ the body is fixed, surface tractions of density $\mathbf{P} = (P_1, P_2, P_3)$ act on $\Gamma_f$. Along $\Gamma_c$ the body is unilaterally supported by a rigid half-space $S$. For simplicity of our presentation we shall suppose that $\Gamma_c$ is flat and there is no gap between $\Omega$ and $S$. Finally, body forces of density $\mathbf{F} = (F_1, F_2, F_3)$ are applied to $\Omega$.

Our aim is to find an equilibrium state of $\Omega$ taking into account effects of friction between $\Omega$ and $S$. By a classical solution of this problem we mean any displacement vector $\mathbf{u} = (u_1, u_2, u_3) : \Omega \rightarrow \mathbb{R}^3$ satisfying the following system of differential equations and boundary conditions (here and in what follows the summation convention will be used):

\begin{equation}
\partial_x \sigma_{ij}(\mathbf{u}) + F_i = 0 \quad \text{in } \Omega, \quad i = 1, 2, 3, \quad (2.1)
\end{equation}

where
\[
\sigma_{ij}(\mathbf{u}) = c_{ijkl}(\mathbf{u}), \quad \varepsilon_{kl}(\mathbf{u}) = \frac{1}{2} \left( \frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k} \right),
\]

$i, j, k, l = 1, 2, 3$, is given by linear Hooke’s law and $c_{ijkl}$ are elements of the bounded $4^\text{th}$ order elasticity tensor $C$ satisfying the usual symmetry and ellipticity conditions,

\text{(kinematical boundary conditions)}

\[
\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma_u, \quad (2.2)
\]

\text{(static boundary conditions)}

\[
\sigma_{ij} v_j = P_i \quad \text{on } \Gamma_f, \quad i = 1, 2, 3, \quad (2.3)
\]

where $\mathbf{v} = (v_1, v_2, v_3)$ is the outward unit normal vector to $\partial \Omega$.

\text{(non-penetration conditions)}

\[
u_e := \mathbf{u} \cdot \mathbf{v} \leq 0, \quad T_v(\mathbf{u}) := \sigma_{ij}(\mathbf{u}) v_j v_i \leq 0, \quad T_v(\mathbf{u}) u_i = 0 \quad \text{on } \Gamma_c, \quad (2.4)
\]

To formulate orthotropic Coulomb friction law, we introduce principal orthotropic axes $t_1 = [t_{1,1}, t_{1,2}, t_{1,3}]$, $t_2 = [t_{2,1}, t_{2,2}, t_{2,3}]$ on the tangent plane at a point on $\Gamma_c$ so that $[t_1, t_2, \mathbf{v}]$ forms a local orthonormal basis. Let $\mathcal{F}_1$, $\mathcal{F}_2$ be the respective coefficients of friction in the directions $t_1$ and $t_2$, respectively. By $\mathbf{u}_e \in \mathbb{R}^2$ we denote the vector whose components are the coordinates of $\mathbf{u}$ with respect to $t_1$ and $t_2$ on $\Gamma_c$, i.e. $\mathbf{u}_e = (u_{t_1}, u_{t_2})$. In what follows we shall suppose that both $\mathcal{F}_1$ and $\mathcal{F}_2$ may depend also on the Euclidean norm of $\mathbf{u}_e$ on $\Gamma_c$, i.e. $\mathcal{F}_i := \mathcal{F}_i(x, \|\mathbf{u}_e(x)\|)$, $x \in \Gamma_c$, $i = 1, 2$. Finally let $\mathcal{F} := \text{diag} \{\mathcal{F}_1, \mathcal{F}_2\}$ be the $(2 \times 2)$ diagonal matrix. To guarantee the existence of the inverse $\mathcal{F}^{-1}$ we suppose that $\mathcal{F}_1$ and $\mathcal{F}_2$ do not vanish. Orthotropic Coulomb friction law with the solution-dependent matrix of friction coefficients reads as follows:

\text{(orthotropic Coulomb friction law)}

\[
\mathbf{u}_e(x) = 0 \quad \Rightarrow \quad \|\mathcal{F}^{-1}(x, 0) T_v(\mathbf{u})(x)\| \leq T_v(\mathbf{u})(x), \quad x \in \Gamma_c, \quad (2.5)
\]

\[
\mathbf{u}_e(x) \neq 0 \quad \Rightarrow \quad \mathcal{F}^{-1}(x, \|\mathbf{u}_e(x)\|) \mathcal{F} \mathcal{F}^{-1}(x, \|\mathbf{u}_e(x)\|) \mathbf{u}_e(x) = T_v(\mathbf{u})(x), \quad x \in \Gamma_c
\]

where $T_v(\mathbf{u}) = (T_{v_1}(\mathbf{u}), T_{v_2}(\mathbf{u}))$, $T_{v_i}(\mathbf{u}) := \sigma_{ij}(\mathbf{u}) v_i t_{i,j}$, is the tangential contact stress.
To give the weak and fixed-point formulation of our problem we shall need the following function spaces and sets:

\[ V = \{ v \in (H^1(\Omega))^3 \mid v = 0 \text{ on } \Gamma_u \}, \]
\[ K = \{ v \in V \mid v_r \leq 0 \text{ on } \Gamma_c \}, \]
\[ X_r = \{ \varphi \in L^2(\Gamma_c) \mid \exists v \in V : \varphi = v_r \text{ on } \Gamma_c \}, \]
\[ X_r^\prime = \{ \varphi \in L^2(\Gamma_c) \mid \exists v \in V : \varphi = ||v|| \text{ on } \Gamma_c \}. \]

Further, let \( X_r^\prime \) be the dual space to \( X_r \) with the duality pairing denoted by \( \langle \cdot, \cdot \rangle \) and \( \Lambda_r \) be the cone of non-negative elements of \( X_r^\prime \):

\[ \Lambda_r = \{ \mu_r \in X_r^\prime \mid \langle \mu_r, v_r \rangle \leq 0 \ \forall v_r \in K \}. \]

The weak formulation of the contact problem with orthotropic Coulomb friction and the solution-dependent matrix of friction coefficients reads as follows:

Find \( u \in K \) such that

\[
\begin{align*}
\alpha(u, v - u) + j(\|u\|, -T_v(u), v_r) - j(\|u\|, -T_v(u), u_r) & \geq \ell(v - u) \ \forall v \in K, \\
\end{align*}
\]

where

\[
\begin{align*}
\alpha(v, w) &= \int_{\Omega} c_{ijkl} e_i(v) e_k(w) \, dx, \quad v, w \in V, \\
\ell(v) &= \int_{\Omega} F \cdot v \, dx + \int_{\Gamma_c} P \cdot v \, ds, \quad v \in V, \\
j(\varphi, g, v_r) &= \langle g, \|F(\varphi)v_r\| \rangle, \quad g \in \Lambda_r, \varphi \in X_{\varphi}, v_r \in V,
\end{align*}
\]

with \( F \in (L^2(\Omega))^3, P \in (L^2(\Gamma_c))^3 \).

Problem (P) is an implicit variational inequality of elliptic type. One possible way of proving the existence of a solution to (P) is to use the fixed-point approach. Let \( (\varphi, g) \in X_{\varphi} \times \Lambda_r \) be given and define the auxiliary problem

Find \( u := u(\varphi, g) \in K \) such that

\[
\begin{align*}
\alpha(u, v - u) + j(\varphi, g, v_r) - j(\varphi, g, u_r) & \geq \ell(v - u) \ \forall v \in K, \\
\end{align*}
\]

This is a contact problem with orthotropic friction of Tresca type and the fixed matrix of friction coefficients \( F(\varphi) \). Let \( \Psi : X_{\varphi} \times \Lambda_r \rightarrow X_{\varphi} \times \Lambda_r \) be defined by

\[ \Psi(\varphi, g) = (\|u\|, -T_v(u)), \quad (\varphi, g) \in X_{\varphi} \times \Lambda_r, \quad (2.6) \]

where \( u \) solves (P(\varphi, g)). Then \( u \in K \) solves (P) if and only if (\( \|u\|, -T_v(u) \)) is a fixed point of \( \Psi \).

For discretization purposes we introduce the equivalent definition of the mapping \( \Psi \) which uses the Lagrange multipliers associated with the unilateral constraint \( u \in K \). Instead of (P(\varphi, g)) we consider the following problem:

Find \( (u, \lambda_r) \in V \times \Lambda_r \) such that

\[
\begin{align*}
\alpha(u, v - u) + j(\varphi, g, v_r) - j(\varphi, g, u_r) & \geq \ell(v - u) - (\lambda_r, v_r - u_r) \ \forall v \in V, \\
(\mu_r - \lambda_r, u_r) & \leq 0 \ \forall \mu_r \in \Lambda_r.
\end{align*}
\]

It is well-known that \((\mathcal{M}(\varphi, g))\) has a unique solution \((u, \lambda_r) := (u(\varphi, g), \lambda_r(\varphi, g))\). In addition, \( u \) solves \((\mathcal{P}(\varphi, g))\) and \( \lambda_r = -T_v(u) \) on \( \Gamma_c \). Then \( \Psi \) defined by (2.6) takes the form

\[ \Psi(\varphi, g) = (\|u\|, \lambda_r). \quad (2.7) \]

3. Discretization of contact problems with orthotropic Coulomb friction. Existence and uniqueness results.

This section deals with a discretization of \((\mathcal{P})\) and the analysis of the resulting discrete problem. Unlike the continuous setting we prove that the discrete counterpart has a solution for any \( F = \text{diag}(F_1, F_2) \) satisfying (3.7). Moreover, it has a unique solution under the additional assumptions on \( F_1 \) and \( F_2 \). Furthermore, we shall investigate how the uniqueness result depends on the size of the discrete problem. The discretization of \((\mathcal{P})\) will be based on the fixed-point formulation for an appropriate discretization of the mapping \( \Psi \) given by (2.7). To get it we use a mixed finite element discretization of \((\mathcal{M}(\varphi, g))\).

Let \( T_h^\Omega, T_h^{\Gamma_c} \) be a partition of \( \overline{\Omega} \) and \( \Gamma_c \) into finite elements \( T \) and \( R \), respectively. The norm of \( T_h^\Omega, T_h^{\Gamma_c} \) is denoted by \( h \) and \( H \), respectively. We use the symbol \( H \) to point out that the partition of \( \Gamma_c \) is generally independent of \( T_h^\Omega \), but the case when \( T_h^{\Gamma_c} = T_h^{\Omega} \) is not excluded. With any \( T_h^\Omega \) and \( T_h^{\Gamma_c} \) we associate finite dimensional spaces \( V_h^\Omega \) and \( L_h^{\Omega} \) consisting of piecewise polynomial functions of degree less or equal \( k \) and \( s \), respectively:

\[ V_h^\Omega = \{ v^h \in C(\overline{\Omega}) \mid v^h|_T \in P_k(T) \forall T \in T_h^\Omega, v^h = 0 \text{ on } \Gamma_u \}, \]
\[ L_h^{\Omega} = \{ \mu^h \in L^2(\Gamma_c) \mid \mu^h|_e \in P_s(e) \forall e \in T_h^{\Gamma_c} \} \]

and set

\[ V_h^{\Gamma_c} = (V_h^\Omega)^3, \quad L_h^{\Gamma_c} = (L_h^{\Omega})^3, \quad \Lambda_h = \{ \mu^h \in L_h^{\Omega} \mid \mu^h \geq 0 \text{ on } \Gamma_c \}, \]
\[ W_h^{\Gamma_c} = W_h^{\Gamma_c}, \quad W_h^{\Gamma_c} = \{ \varphi^h \in W_h^{\Gamma_c} \mid \varphi^h \geq 0 \text{ on } \Gamma_c \}. \]
Clearly, $V^h$ and $\Lambda^H$ serve as the natural discretizations of $V$ and $\Lambda$, respectively (observe that $L^H$ may consist also of discontinuous functions). Next we shall suppose that the couple $(V^h, L^H)$ which will be used in the discretization is chosen in such a way that the following condition is satisfied:

$$\mu^H \in L^H & (\mu^H, v^h_0)_{0, \Gamma} = 0 \quad \forall v^h \in V^h$$

$$\implies \mu^H = 0. \quad (3.1)$$

**Remark 3.1.** Let us comment in short on the choice of $V^h$ and $L^H$ satisfying (3.1). If $L^H := L^h = W^h$, i.e. the space $L^H$ is formed by the restrictions of functions from $V^h$ on $\Gamma_c$, then (3.1) is automatically satisfied. If $V^h$, $L^H$ is the space of piecewise linear, piecewise constant functions over $T^h$ and $\mathcal{F}^H$, respectively, then (3.1) is satisfied provided that the ratio $H/h$ is sufficiently large, i.e. the partition $\mathcal{F}^H$ is coarser than the partition of $\Gamma_c$ given by $\mathcal{T}^\Omega_{\Gamma_c}$ (see [12]).

Let $\psi^h \in W^h$ and $g^H \in \Lambda^H$ be given and define the problem:

$$\begin{align*}
\text{Find } (u^h, \lambda^H) & \in V^h \times \Lambda^H \text{ such that } \\
a(u^h, v^h) + j(\psi^h, g^H, v^h) - j(\lambda^H, v^h) & \geq (v^h - u^h) - (\lambda^H, v^h - u^h)_{0, \Gamma} \quad \forall v^h \in V^h, \\
(\mu^H - \lambda^H, v^h)_{0, \Gamma} & \leq 0 \quad \forall \mu^H \in \Lambda^H,
\end{align*}$$

(3.2)

where the meaning of all symbols has been introduced in Section 2. Because of (3.1), this problem has a unique solution $(u^h, \lambda^H) := (u^h(\psi^h, g^H), \lambda^H(\psi^h, g^H))$ for any $(\psi^h, g^H) \in W^h \times \Lambda^H$.

Before we give the definition of the discrete contact problem with orthotropic Coulomb friction we formulate assumptions needed in what follows. First we shall suppose that the vector field $x \mapsto (t_1(x), t_2(x))$ associating any $x \in \Gamma_c$ with the principal orthotropic axes is sufficiently smooth so that

$$v^h_i = (v^h_0, v^h_1) \in (H^1(\Gamma_c))^2 \quad \forall v^h \in V^h$$

and there exists a constant $c_1 > 0$ independent of $v^h \in V^h$ and $h$ such that

$$\|v^h_i\|_{1, \Gamma_c} \leq c_1\|v^h\|_{0, \Gamma_c} \quad \forall v^h \in V^h. \quad (3.3)$$

Since the function $\|v^h\|$ does not belong to $W^h$ we have to introduce a “return” operator $r_h : H^1(\Gamma_c) \rightarrow W^h$ possessing the following approximation and monotonicity property:

$$\|\varphi - r_h\varphi\|_{0, \Gamma_c} \leq c_h\|\varphi\|_{1, \Gamma_c} \quad \forall \varphi \in H^1(\Gamma_c), \quad (3.4)$$

$$\varphi \in H^1(\Gamma_c), \varphi \geq 0 \text{ on } \Gamma_c \implies r_h\varphi \in W^h, \quad (3.5)$$

where $c_h$ is a positive constant which does not depend on the norm $h\Gamma_c$ of $\mathcal{T}^\Omega_{h\Gamma_c}$. If $k = 1$ and $\mathcal{T}^\Omega_{h\Gamma_c}$ belongs to a regular family of partitions of $\Gamma_c$, then (3.4) and (3.5) are satisfied by the Clément interpolation operator ([13]).

We shall need also the satisfaction of the following inverse inequality for elements of $V^h$; there exists a positive constant $c_{inv}^{(1,0)}$ which does not depend on $h\Gamma_c$ such that

$$\|v^h\|_{1, \Gamma_c} \leq c_{inv}^{(1,0)} h^{-1}\|v^h\|_{0, \Gamma_c} \quad \forall v^h \in V^h. \quad (3.6)$$

Let us recall that (3.6) is satisfied provided that $\mathcal{T}^\Omega_{h\Gamma_c}$ belongs to a regular family of partitions of $\Gamma_c$ satisfying the so-called inverse assumption (see [14, Th. 3.2.6]). Finally, we shall suppose that the coefficients of friction $\mathcal{T}_1$ and $\mathcal{T}_2$ are continuous and bounded:

$$\begin{align*}
\mathcal{T}_1, \mathcal{T}_2 & \in C(\Gamma_c \times \mathbb{R}_+), \\
0 < \mathcal{T}_{\min} & \leq \mathcal{T}_1(x, \xi) \leq \mathcal{T}_{\max}, \quad (i = 1, 2, \chi(x, \xi) \in \Gamma_c \times \mathbb{R}_+),
\end{align*}$$

(3.7)

where $\mathcal{T}_{\min}$, $\mathcal{T}_{\max}$ are given positive numbers.

Let $\mathcal{M}_{h\Gamma} : W^h \times \Lambda^H \rightarrow W^h \times \Lambda^H$ be defined by

$$\mathcal{M}_{h\Gamma}(\psi^h, g^H) = (r_h\|\psi^h\|_{1, \Gamma_c}, \lambda^H),$$

$$\quad (\psi^h, g^H) \in W^h \times \Lambda^H, \quad (3.8)$$

where $(u^h, \lambda^H)$ is the unique solution of $(\mathcal{M}_{h\Gamma}(\psi^h, g^H))$. This definition is meaningful since (3.2) implies that $\|\psi^h\| \in H^1(\Gamma_c)$. The mapping $\mathcal{M}_{h\Gamma}$ serves as the approximation of $\mathcal{M}$.

Analogously to the continuous setting we say that $u^h \in V^h$ is a solution of the discrete contact problem with orthotropic Coulomb friction and the solution-dependent coefficients of friction iff the couple $(r_h\|\psi^h\|_{1, \Gamma_c}, \lambda^H)$ is a fixed point of $\mathcal{M}_{h\Gamma}$.

Next we shall examine the existence, eventually the uniqueness of the fixed points of $\mathcal{M}_{h\Gamma}$. To this end we introduce the following norm in $W^h \times L^H$:

$$\|\varphi^h, \mu^H\|_{W^h \times L^H} = \|\varphi^h\|_{0, \Gamma_c} + \|\mu^H\|_{-1/2, h},$$

$$\quad (\varphi^h, \mu^H) \in W^h \times L^H, \quad (3.9)$$

where

$$\|\mu^H\|_{-1/2, h} = \sup_{\varphi \in V^h} \frac{(\mu^H, v^h_0)_{0, \Gamma}}{\|v^h\|_{0, \Gamma_c}}. \quad (3.10)$$

In view of (3.1), the formula (3.10) defines a mesh-dependent norm in $L^H$.

To prove the existence of a fixed point we use Brouwer’s fixed-point theorem. Since the mapping $\mathcal{M}_{h\Gamma}$
is continuous in virtue of (3.7), it remains to show that it maps a closed convex set into itself. This property follows from the next lemma.

**Lemma 3.1.** Let (3.1)–(3.6) be satisfied. Then there exist positive constants $R_1$, $R_2$ which do not depend on $h$ and $H$ such that $\Psi_{\text{df}}$ maps $W^h_2 \times \Lambda^h_0 \cap B$ into itself, where $B$ is the ball

$$B = \{(\varphi^h, \mu^H) \in W^h_2 \times L^H_1 | \|\varphi^h\|_{0, \Omega}, \|\mu^H\|_{-1/2, h} \leq R_1, \|\mu^H\|_{-1, 2, h} \leq R_2\}.$$ 

**Proof.** Inserting $v^h = 0$, $2\varphi^h$ into the first inequality in $(M_{df}(\varphi^h, \mu^H))$ and using the $V$-ellipticity of $a$ with the constant of ellipticity $\alpha > 0$ we easily obtain that $\varphi^h$ is bounded:

$$3R := R(|F|_{0, \Omega}, |P|_{0, \Gamma_r}) > 0 : \alpha \|\varphi^h\|_{1, \Omega} \leq R. \quad (3.11)$$

Furthermore,

$$\|r_s|\varphi^h_{\Omega, s}\|_{0, \Omega} \leq \|r_s|a\varphi^h_{\Omega, s}\|_{0, \Omega} \leq \|r_s|\varphi^h_{\Omega, s}\|_{0, \Omega} + \|\varphi^h_{\Omega, s}\|_{0, \Omega},$$

**Remark 3.2.** It is worth mentioning that $R_1$ and $R_2$ are independent of $F$.

A natural question arises, namely under which conditions the fixed point of $\Psi_{\text{df}}$ is unique. To this end let us suppose that the friction coefficients $\mathcal{F}_1$, $\mathcal{F}_2$ are smooth enough and denote

$$L = \max_{\varepsilon \in [1, 2]} \max_{\xi, \xi' \in \mathbb{R}} \left\{ \frac{\partial \mathcal{F}_1(x, \xi)}{\partial \xi} \right\}$$

and

$$\kappa(\mathcal{F}) = \max_{\varepsilon \in [1, 2]} \frac{\max[\mathcal{F}_1(x, \xi), \mathcal{F}_2(x, \xi)]}{\min[\mathcal{F}_1(x, \xi), \mathcal{F}_2(x, \xi)]}.$$ 

The quantity $\kappa(\mathcal{F})$ can be considered as a "measure" of orthotropy (for the isotropic friction $\kappa(\mathcal{F}) = 1$). It can be shown (see [15]) that $\Psi_{\text{df}}$ is Lipschitz continuous in $W^h_2 \times \Lambda^h_0 \cap B$, i.e. there exists $C > 0$ such that

$$\|\Psi_{\text{df}}(\varphi^h, \mu^H) - \Psi_{\text{df}}(\varphi^\prime, \mu^H)\|_{W^h_2 \times L^H_1} \leq C \|\varphi^h - \varphi^\prime\|_{W^h_2 \times L^H_1} \quad (3.12)$$

holds for every $(\varphi^h, \mu^H) \in W^h_2 \times \Lambda^h_0 \cap B$, where $B$ is the same as in Lemma 3.1. Moreover, the Lipschitz constant $C$ in (3.12) is of the form $C = \max(C_1(\mathcal{F}_{\text{max}}, H), C_2(L, \kappa(\mathcal{F}), H, h_{\text{cr}}))$, where the constants $C_1, C_2$ enjoy the following properties:

$$\begin{align*}
(a) & \quad C_1(\mathcal{F}_{\text{max}}, H) \to 0 \text{ if } \mathcal{F}_{\text{max}} \to 0+ \text{ for any } H > 0 \\
& \quad \text{fixed}, \\
& \quad C_2(L, \kappa(\mathcal{F}), H, h_{\text{cr}}) \to 0 \text{ if } L \to 0+ \text{ for any } H, h_{\text{cr}} \text{ fixed and } \kappa(\mathcal{F}) \text{ bounded}; \\
(b) & \quad \text{if } \mathcal{F} \text{ does not depend on } \|\varphi^h\|_{0, \Omega}, \text{ i.e. } L \equiv 0, \text{ then } \\
& \quad C_2(0, \kappa(\mathcal{F}), H, h_{\text{cr}}) = 0 \text{ for any } \mathcal{F} \text{ and any } H, h_{\text{cr}} > 0; \\
(c) & \quad \text{if } \mathcal{F} \text{ is fixed then } C_1(\mathcal{F}_{\text{max}}, H), C_2(L, \kappa(\mathcal{F}), H, h_{\text{cr}}) \text{ behaves as } H^{1/2} \text{ and } (h_{\text{cr}})^{-1/2}, \text{ respectively, for } H, h_{\text{cr}} \to 0+ \text{ provided that the Babuška-Brezzi condition for } \{\varphi^h, \mu^H\} \text{ is satisfied:} \\
& \quad \sup_{0 \neq \varphi^h \in V^h} \frac{(\mu^H, \varphi^h)_{L^2(\Omega)}}{\|\varphi^h\|_{1, \Omega}} \geq \beta \|\mu^H\|_{1, \Gamma_r} \quad \forall \mu^H \in L^H \\
& \quad \text{where } \beta > 0 \text{ does not depend on } h \text{ and } H \\
& \quad \|\mu^H\|_{1, \Gamma_r} = \sup_{0 \neq \varphi^h \in V^h} \frac{(\mu^H, \varphi^h)_{L^2(\Omega)}}{\|\varphi^h\|_{1, \Omega}}.
\end{align*}$$

Suppose that $\kappa(\mathcal{F})$ is bounded. From (a) it follows that for any $H, h_{\text{cr}}$ fixed one can find $\mathcal{F}_{\text{crit}} := \mathcal{F}_{\text{crit}}(H) > 0$ and $L_{\text{crit}} := L_{\text{crit}}(H, h_{\text{cr}}) > 0$ such that if $\mathcal{F}_{\text{max}} \leq \mathcal{F}_{\text{crit}}$ and $L \leq L_{\text{crit}}$ then the mapping $\Psi_{\text{df}}$ is contractive. Therefore there exists a unique fixed point of $\Psi_{\text{df}}$ and the method of successive approximations converges. Recall that each iterative step of the method of successive approximations is represented by a contact problem with Tresca friction and in the case of solution-dependent friction coefficients we update not only the slip threshold but also the matrix $\mathcal{F}$. If $\mathcal{F}$ does not depend on the solution, i.e. $L \equiv 0$, then using (b) we recover the classical result (see [11]). If both $H$ and $h_{\text{cr}}$ tend to zero then to preserve the contractivity of $\Psi_{\text{df}}$, the parameters $\mathcal{F}_{\text{max}}, L$ have to decay at least as fast as $H^{1/2}$ and $(h_{\text{cr}})^{-1/2}$, respectively. This is a consequence of (c). The mesh dependency of discrete models can be read in two ways: either (i) the matrix $\mathcal{F}$ is fixed, then passing from coarser to finer meshes we may loose
unicity of the approximate solution or (ii) finite element meshes are fixed, then setting \( \mathcal{F}_c = \zeta_{\mathcal{F}} \), \( \zeta \geq 0 \), one can find \( \zeta_{\text{crit}} > 0 \) such that the discrete model has a unique solution for \( \zeta \leq \zeta_{\text{crit}} \) and eventually multiple solutions if \( \zeta > \zeta_{\text{crit}} \). This behavior has been observed in computations (see [9]).

4. Existence of local Lipschitz continuous branches of solutions

In this section we shall consider solutions to discrete contact problems to be a function of friction coefficients. We restrict ourselves to 2D-case and isotropic Coulomb friction with the coefficient \( \zeta \) depending on the spatial variable \( x \in \Gamma_c \). From the previous section we know that if \( \mathcal{F} \leq \mathcal{F}_c \) then there exists a unique solution of our problem. Moreover, it can be shown ([16]) that for such \( \mathcal{F} \) the mapping \( u^h : \mathcal{F} \mapsto u^h(\mathcal{F}) \) is Lipschitz continuous if (3.12) is satisfied. In what follows we shall investigate the case when \( \mathcal{F} > \mathcal{F}_c \). More precisely, if \( \mathcal{T}_0 > \mathcal{T}_{\text{crit}} \) is a reference point, under which conditions there exists a neighborhood \( U_\varepsilon(\mathcal{T}_0) \) such that the mapping \( u^h : \mathcal{T} \mapsto u^h(\mathcal{T}) \), \( \mathcal{T} \in U_\varepsilon(\mathcal{T}_0) \), has a Lipschitz continuous branch, implying (among others) local uniqueness of the respective solution. To this end we use the algebraic formulation of the discrete contact problem with Coulomb friction which involves two Lagrange multipliers: one releasing the unilateral constraint and the other regularizing the non-smooth frictional term. It reads as follows:

Find \( (u, \lambda, \lambda_t) \in \mathbb{R}^n \times \Lambda_+ \times \Lambda_+(\lambda_t) \) such that

\[
\begin{align*}
Au &= f - B^T \lambda - B^T F \lambda_t, \\
(\mu - \lambda_t) \cdot B u + F(\mu - \lambda_t) \cdot B u &\leq 0,
\end{align*}
\]

(4.1)

where \( A \) is an \((n \times n)\) stiffness matrix, \( u \in \mathbb{R}^n \) is the nodal displacement vector, \( B_m, B_t \) are \((p \times n)\) matrices representing the linear mappings \( u \mapsto u \cdot v \) and \( u \cdot t \), respectively, and \( p \) is the number of the contact nodes. We shall suppose that

\[
B^T \mu + B^T \mu_t = 0 \iff (\mu, \mu_t) = (0, 0) \in \mathbb{R}^p \times \mathbb{R}^p.
\]

Further, \( f \in \mathbb{R}^n \) is the load vector, \( F = \text{diag}(\mathcal{F}_1, \ldots, \mathcal{F}_p) \) is the diagonal matrix with \( \mathcal{F}_i \) being the value of \( \mathcal{F} \) at the \( i \)-th contact node, \( \Lambda_+ = \mathbb{R}_+^p \) and

\[
\Lambda_{\phi}(\mu) = \{ \mu \in \mathbb{R}^p | |\mu_i| \leq g_i, \forall i = 1, \ldots, p \}, \quad g \in \Lambda_+.
\]

It is well-known that \( \lambda_t \) and \( F \lambda_t \) is the opposite of the discrete normal and tangential contact stress, respectively.

Let \( f \in \mathbb{R}^n \) be fixed. Using tools of convex analysis, problem (4.1) can be equivalently written as the following system of generalized equations:

Find \( y \in \mathbb{R}^{n+2p} \) such that \( 0 \in C_f(\mathcal{F}, y) + Q(y) \),

(4.2)

where \( C_f : \mathbb{R}_+^p \times \mathbb{R}^{n+2p} \rightarrow \mathbb{R}^{n+2p} \) and \( Q : \mathbb{R}^{n+2p} \rightrightarrows \mathbb{R}^{n+2p} \) is the single-valued and the set-valued mapping, respectively, defined by

\[
C_f(\mathcal{F}, y) = \begin{pmatrix} A & B^T \lambda_t & B^T F \lambda_t \\ -B & 0 & 0 \\ -FB_t & 0 & 0 \end{pmatrix}
\]

\[
Q(y) = \begin{pmatrix} N_{\lambda}(\lambda_t) \\ N_{\Lambda_{\phi}(\mu)}(\lambda_t) \end{pmatrix},
\]

\[
\mathcal{F} := (\mathcal{F}_1, \ldots, \mathcal{F}_p) \in \mathbb{R}_+^p, \quad y := (u, \lambda, \lambda_t) \in \mathbb{R}^{n+2p},
\]

with \( N_{\lambda}(\mu), N_{\Lambda_{\phi}(\mu)}(\mu) \) denoting the normal cones of \( \Lambda_+ \) and \( \Lambda_{\phi}(\mu) \) at \( \mu \in \mathbb{R}_+^p \), respectively.

The system (4.2), in which \( \mathcal{F} \) plays the role of the perturbation parameter, can be analyzed by using the generalized implicit-function theorem ([16]).

Let \( S_f : \mathbb{R}_+^p \rightrightarrows \mathbb{R}^{n+2p} \) be the solution (multi-valued) mapping of (4.2):

\[
S_f(\mathcal{F}) = \{ y \in \mathbb{R}^{n+2p} | 0 \in C_f(\mathcal{F}, y) + Q(y) \}, \quad \mathcal{F} \in \mathbb{R}_+^p,
\]

and \( (\mathcal{F}_0, y^0) \in \mathbb{R}_+^p \times \mathbb{R}^{n+2p} \) be a reference point such that \( y^0 \in S_f(\mathcal{F}_0) \). Theorem 2.1 in [16] states that \( S_f \) has a local Lipschitz continuous branch passing through \( y^0 \) at \( \mathcal{F}_0 \) if the so-called strong regularity condition is satisfied at \( (\mathcal{F}_0, y^0) \). This condition pertains to the properties of the multi-valued mapping \( \Sigma_f : \mathbb{R}^{n+2p} \rightrightarrows \mathbb{R}^{n+2p} \), where

\[
\Sigma_f(\eta) = \{ y \in \mathbb{R}^{n+2p} | \eta \in C_f(\mathcal{F}_0, y^0) + \nabla_y C_f(\mathcal{F}_0, y^0)(y - y^0) + Q(y), \eta \in \mathbb{R}^{n+2p},
\]

i.e., \( \Sigma_f(\eta) \) is the solution set of the system of generalized equations arising from the partial linearization of the smooth part \( C_f \) in (4.2) at \( (\mathcal{F}_0, y^0) \) with respect to the second variable.

Taking \( \eta := (\eta_n, \eta_t, \eta) \in \mathbb{R}^{n+2p} \), \( \Sigma_f(\eta) \) consists of all \( y \) satisfying

\[
\begin{align*}
0 &= Au + B^T \lambda + B^T F \lambda_t - f - \eta_n, \\
0 &= -Bu - \eta_t + N_{\phi}(\lambda_t), \\
0 &= -F^T Bu - \eta + N_{\Lambda_{\phi}(\mu)}(\lambda_t),
\end{align*}
\]

(4.3)
respectively, and a single-valued Lipschitz continuous

Then there are neighborhoods $$\phi, y$$ of $$F^0$$ and $$y^0$$, respectively, and a single-valued Lipschitz continuous function $$\sigma_f : U \to Y$$ satisfying

$$\sigma_f(F^0) = y^0 \quad \& \quad \sigma_f(F) = S_F(F) \cap Y \quad \forall F \in U.$$  

This theorem says that locally the dependence of a solution on $$F$$ can be deduced from the dependence of the solution on the load vector keeping $$F$$ fixed. It is worth mentioning that the latter one is much simpler since the dependence on the load vector is piecewise linear. The local behavior of the set-valued mapping $$f \mapsto S_F(f)$$, $$f \in \mathbb{R}^n$$, for $$F \in \mathbb{R}^{n+p}_+$$ fixed can be studied in more details by writing (4.1) as a system of piecewise smooth equations (see (5.1)) and applying the implicit-function theorem for such a class of functions (Theorem 4.2.2 in [17]). From it one concludes that the existence of local Lipschitz continuous branches is not guaranteed only around points $$y \in S_F(f)$$ for $$f$$ belonging to the union of subspaces of dimension strictly lower than $$n$$. (For more details see [6].)

5. Numerical continuation of solution curves

In this section we shall propose the algorithm for tracing the (local) Lipschitz continuous branches of solutions, the existence of which was established in the previous section. More precisely, taking a smooth path $$\alpha \in I \mapsto F(\alpha) = (F_1(\alpha), \ldots, F_p(\alpha)) \in \mathbb{R}^p$$, $$I \subset \mathbb{R}^1$$ open, we shall approximate the curve consisting of all solutions to the discrete contact problem with Coulomb friction and the coefficient $$F(\alpha)$$ for $$\alpha$$ running over $$I$$.

Next we use the equivalent formulation of (4.1) as the system of non-smooth equations involving the projection mappings $$P_{\lambda_0} : \mathbb{R}^p \to \Lambda_0$$ and $$P_{\lambda_0}(\mathbb{F}(\alpha)) : \mathbb{R}^p \to \Lambda(\mathbb{F}(\alpha))$$, where $$\Lambda_0 = \mathbb{R}^p$$ and

$$\lambda_i(F \mathbf{g}) = |\mu_i| \leq F_i \mathbf{g}, \quad i = 1, \ldots, p, \quad \mathbf{g} \in \Lambda_0.$$  

Taking $$F := F(\alpha), \alpha \in I,$$ this system becomes:

Find $$x \in \mathbb{R}^{n+2p} \times I$$ such that $$\mathcal{H}(x) = 0,$$  

where $$\mathcal{H} : \mathbb{R}^{n+2p} \times I \to \mathbb{R}^{n+2}$$ is defined by

$$\mathcal{H}(x) = \begin{pmatrix} Au + B^T \lambda_0 + B^T \lambda_i - f \\ \lambda_i - P_{\Lambda_0}(\mathbb{F}(\alpha)) \mathbf{g} - \mathbf{h} \end{pmatrix},$$  

where $$r > 0$$ is a parameter and the components of $$P_{\lambda_0}$$, $$P_{\lambda_0}(\mathbb{F}(\alpha))$$ are given by

$$P_{\lambda_0}(\mu) = P_{(0,\ldots,0)}(\mu_0),$$

$$P_{\lambda_0}(\mathbb{F}(\alpha)) = \begin{cases} P_{(-\mathcal{F}(\alpha),\mathcal{F}(\alpha))}(\mu) & \text{if } \lambda_{ij} \geq 0, \\
-\mathcal{F}(\alpha,\mathcal{F}(\alpha)) \mathcal{F}(\alpha,\mathcal{F}(\alpha)) \mu & \text{if } \lambda_{ij} < 0, \\
\end{cases} \quad i = 1, \ldots, p, \quad \mu \in \mathbb{R}^p,$$
with \( P_{[0, +\infty)} \), \( P_{(a, b]} \) being the projections of \( \mathbb{R}^1 \) onto \([0, +\infty)\) and \([a, b), -\infty < a \leq b < +\infty\), respectively.

The meaning of the remaining symbols in the definition of \( \mathcal{H} \) is the same as in (4.1).

Note that \( \mathcal{H} \) is a piecewise smooth function, i.e. for every \( \tilde{x} \in \mathbb{R}^{e+p} \times I \) there exists an open neighborhood \( O \subset \mathbb{R}^{e+p} \times I, \tilde{x} \in O \), and a finite number of smooth functions \( \mathcal{H}^{(i)} : O \rightarrow \mathbb{R}^{e+p}, i = 1, \ldots, l \), (the so-called selection functions) such that \( \mathcal{H}(x) \in \{ \mathcal{H}^{(1)}(x), \ldots, \mathcal{H}^{(l)}(x) \} \) for every \( x \in O \). The set

\[
I_{\mathcal{H}}(\tilde{x}) \equiv \{ i \in [1, \ldots, l] | \mathcal{H}(\tilde{x}) = \mathcal{H}^{(i)}(\tilde{x}) \}
\]

is known as the active index set at \( \tilde{x} \) and the functions \( \mathcal{H}^{(i)}, i \in I_{\mathcal{H}}(\tilde{x}) \), are termed the active selection functions for \( \mathcal{H} \) at \( \tilde{x} \).

Classical continuation techniques require \( \mathcal{H} \) in (5.1) to be smooth. Next, we shall show how these techniques can be adapted to our non-smooth case. In particular, we shall modify the Moore-Penrose continuation, which is presented e.g. in [18]. This continuation procedure consists of two steps. In the correction, we use the piecewise smooth Newton method (7.2.14 Algorithm in [19]) instead of the smooth one when the gradient \( \nabla \mathcal{H} \) is replaced by the gradient of one of its active selection functions if necessary.

On the other hand, a generalization of the prediction is more complicated. Indeed, if one tries to use the tangential direction of the form

\[
\mathcal{H}'(x^j; \tau^j) = 0, \quad ||\tau^j|| = 1, \quad j = 1, 2, \ldots
\]

where \( \mathcal{H}'(x^j; \tau^j) \) denotes the directional derivative of \( \mathcal{H} \) at \( x^j \) in the direction \( \tau^j \), the continuation may fail when approaching a point of non-differentiability on the solution curve. It is caused by the fact that the Newton corrections are only locally convergent and one has to take a suitable tangential vector to reach their zone of convergence (see Figure 1 for illustration). This is why we have to develop a special approach to pass through such points.

![Figure 1: Necessity of a good tangential prediction.](image)

Clearly, the non-differentiability of \( \mathcal{H} \) is caused by the functions:

\[
\begin{align*}
\mathcal{H}(x) &= \{ \lambda_i - P_{\Lambda}(\lambda_i + rB_iu_i), \quad \mathcal{H}(x) = \{ \lambda_i - P_{\Lambda}(\lambda_i + rB_iu_i) \}\}
\end{align*}
\]

(5.2)

With (5.2) the following selection functions will be associated:

\[
\begin{align*}
\lambda_i &\mapsto \lambda_i, \quad x \mapsto -rB_iu_i, \\
\lambda_i &\mapsto -\mathcal{F}_i(\lambda_i)\lambda_i, \quad x \mapsto -rB_iu_i, \\
\lambda_i &\mapsto 2\lambda_i + rB_iu_i, \quad x \mapsto -rB_iu_i,
\end{align*}
\]

\( i = 1, \ldots, p, \) respectively, representing all possible values of the mappings in (5.2). Furthermore, we define the so-called test functions \( \theta^i = (\theta^i_1, \ldots, \theta^i_3) : \mathbb{R}^{e+p} \times I \rightarrow \mathbb{R}^p \), \( k = 1, 2, 3, \) by

\[
\begin{align*}
\theta^i_1(x) &= (\lambda_i + rB_iu_i), \\
\theta^i_2(x) &= (\lambda_i + rB_iu_i) - \mathcal{F}_i(\lambda_i)\lambda_i, \\
\theta^i_3(x) &= (\lambda_i + rB_iu_i) + \mathcal{F}_i(\lambda_i)\lambda_i,
\end{align*}
\]

\( i = 1, \ldots, p, x \in \mathbb{R}^{e+p} \times I. \)

From their definition it immediately follows that there is the one-to-one correspondence between the signs of the components of \( \theta^i(x) \), \( \theta^i(x) \) and \( \theta^i(x) \) and the selection functions for \( \mathcal{H} \) which are active at \( x \). (Possible zero components indicate that more than one selection function is active.) Suppose that \( x \in \mathbb{R}^{e+p} \times I \) is a point where only one selection function is active, i.e. all components of the test functions are nonzero there. Assembling the signs of the test functions into a \( (p \times 3) \) array in such a way that the \( i \)-th column corresponds to \( \theta^i(x), i = 1, 2, 3 \), we see that every selection function for \( \mathcal{H} \) can be represented by a \( (p \times 3) \) array and this representation is unique.

Let \( x^j \) be a current point which is close to a point \( \tilde{x} \) of non-differentiability of \( \mathcal{H} \) as illustrated in Figure 1. Assume that exactly two selection functions \( \mathcal{H}^{(1)} \) and \( \mathcal{H}^{(2)} \) are active at \( \tilde{x} \) and there exists a piecewise smooth curve of solutions passing through \( \tilde{x} \) which consists of two smooth branches belonging to the solution sets to \( \mathcal{H}^{(1)}(x) = 0 \) and \( \mathcal{H}^{(2)}(x) = 0 \). Supposing that \( x^j \) is a root of \( \mathcal{H}^{(1)} \), we shall describe how to reach the unknown smooth branch of the curve corresponding to \( \mathcal{H}^{(2)}(x) = 0 \).

Clearly, one of the test functions, say \( \theta^i \), has a zero component at \( \tilde{x} \), say the \( m \)-th one, and this component changes its sign when passing through \( \tilde{x} \). Moreover, continuity of \( \theta^i \) ensures that \( \theta^i_m(x^j) \) is close to zero. If the \( (p \times 3) \) array represents \( \mathcal{H}^{(1)} \) then changing the sign
of $\theta^j$ we obtain the representative of the selection function $\mathcal{H}^{(0)}$. This leads us to the following choice of the tangential vector $\tau^j$ at $x^j$:

$$\nabla \mathcal{H}^{(0)}(x^j)\tau^j = 0, \quad \|\tau^j\| = 1.$$ 

To select the direction of this vector, the notion of orientation is adapted from the theory of smooth continuations.

**Definition 5.1.** Let $\mathcal{H}$ be smooth at a point $x \in \mathbb{R}^{n+2p+1}$. The tangential vector $\tau \in \mathbb{R}^{n+2p+1}$ satisfying

$$\nabla \mathcal{H}(x)\tau = 0, \quad \|\tau\| = 1,$$

is called positively oriented if

$$\det \left( \frac{\nabla \mathcal{H}(x)}{\tau^j} \right) > 0.$$

In the opposite case it is called negatively oriented.

In our non-smooth case, we determine the direction of $\tau^j$ to preserve the orientation in the following sense:

$$\det \left( \frac{\nabla \mathcal{H}^{(0)}(x^j)}{(\tau^j)^T} \right) \det \left( \frac{\nabla \mathcal{H}^{(0)}(x^i)}{(\tau^i)^T} \right) > 0.$$

Let us note that the expounded procedure can be also applied when the point of non-differentiability $\bar{x}$ is met exactly, i.e. $x^j = \bar{x}$. Nevertheless, this situation is highly improbable.

On the basis of the above considerations we propose the following algorithm.

**Algorithm 5.1.** (Piecewise smooth variant of the Moore-Penrose continuation)

**Data:** $e, e' > 0$, $h_{\text{max}} \geq h_{\text{init}} \geq h_{\text{min}} > 0$, $h_{\text{inc}} > 1 > h_{\text{dec}} > 0$, $k_{\text{dec}} > 0$, $m_{\text{dec}} := m_{\text{dec}} + 1$ and $\mathcal{H}^{(0)} \in \mathbb{R}^{n+2p+1}$ satisfying:

$$\|\mathcal{H}(x^0)\| < e, \quad \mathcal{H}'(x^0; \tau^0) = 0, \quad \|\tau^0\| = 1.$$

**Step 1:** Set $h_0 := h_{\text{init}}$, $j := 0$.

**Step 2:** Set $m_{\text{dec}} := 0$.

**Step 3 (prediction):** Set $X^0 := x^j + h_j \tau^j$, $T^0 := \tau^j$, $k := 0$.

**Step 4 (correction):** Select an index $i_k$ in $I_{ef}(X^k)$ and set:

$$B := \left( \frac{\nabla \mathcal{H}^{(0)}(X^k)}{(T^k)^T} \right), \quad R := \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad Q := \left( \mathcal{H}(X^k) \right), \quad \bar{T} := B^{-1}R, \quad \bar{T}^{k+1} := \frac{\tilde{T}}{\|\bar{T}\|}, \quad X^{k+1} := X^k - B^{-1}Q.$$

**Step 5:** If $\|\mathcal{H}(X^{k+1})\| < e$ and $\|X^{k+1} - X^k\| < e'$, set $x^{j+1} := X^{k+1}$, $\tau^{j+1} := \bar{T}^{k+1}$ and go to Step 9.

**Step 6:** If $k < k_{\text{max}}$, set $k := k + 1$ and go to Step 4.

**Step 7:** If $h_j > h_{\text{min}}$, set $h_j := \max(h_{\text{dec}} h_j, h_{\text{min}})$, $m_{\text{dec}} := m_{\text{dec}} + 1$ and go to Step 3.

**Step 8:** According to a component of $\theta^1(x^j)$, $\theta^2(x^j)$ or $\theta^3(x^j)$ close to 0 select a function $\mathcal{H}^{(0)}$ which is likely to be active in a vicinity of $x^j$ and compute a vector $\tau^j$ satisfying

$$\nabla \mathcal{H}^{(0)}(x^j)\tau^j = 0, \quad \|\tau^j\| = 1$$

and preserving the orientation. Set $h_j := h_{\text{init}}$ and go to Step 2.

**Step 9:** Set

$$h_{j+1} := \begin{cases} \min(h_{\text{inc}} h_j, h_{\text{max}}) & \text{if } k < k_{\text{thr}} \\
 h_j & \text{otherwise} \end{cases}$$

and $j := j + 1$; go to Step 2.

Here $e$ and $e'$ are convergence tolerances, $h_{\text{min}}$, $h_{\text{max}}$ and $h_{\text{init}}$ is the minimal, maximal and initial step length, respectively, and $h_{\text{inc}}$, $h_{\text{dec}}$, $k_{\text{dec}}$ are the scale factors for adjustment of the step length. Further, $k_{\text{max}}$ stands for the maximal number of corrections allowed and $m_{\text{dec}}$ denotes the number of the step length reductions of the current value of $h_j$.

In Step 7 the current step length is shortened in case of non-convergence of the corrections. Step 9 defines the step length for the prediction in the next iteration. The new step length $h_{j+1}$ can be larger than $h_j$ only if the number of corrections (Step 4) does not exceed $k_{\text{thr}}$ given a priori and $m_{\text{dec}} = 0$. These parts of the routine together with the prediction and the corrections are taken directly from the classical Moore-Penrose continuation. Step 8 is added to handle the situation when the corrections do not converge even for $h = h_{\text{min}}$. Then we determine a new tangential prediction, making use of our test functions, and return to the classical part of the procedure.

**Remark 5.1.** (i) This algorithm can be also used to pass through a point where more than two selection functions for $\mathcal{H}$ are active. In this case, however, one has more possibilities to choose a new selection function when “switching” between smooth branches.

(ii) Control of changes of the signs of components of $\theta^1$, $\theta^2$ and $\theta^3$ during the continuation routine enables us to detect points of non-differentiability with an arbitrarily chosen accuracy.
6. Numerical examples

The last section is devoted to the application of Algorithm 5.1. We shall restrict ourselves to very simple model examples with one and two contact nodes (see Figures 2 and 3).

In the forthcoming computations, we will use the following parameter setting: \( \varepsilon = \varepsilon' = 10^{-6} \), \( h_{\min} = 10^{-5} \), \( 0.1 \leq h_{\max} \leq 5 \), \( h_{\min} = 0.05 \), \( h_{\text{inc}} = 1.3 \), \( h_{\text{dec}} = 0.5 \), \( k_{\max} = 10 \), \( k_{\text{thr}} = 4 \).

6.1. One contact node

This subsection deals with an elementary problem involving a single linear triangular finite element shown in Figure 2. This problem was suggested and analyzed in [10]: let \( f = (f_\nu, f_t) \) be the load vector, \( x = (u_\nu, u_t, \lambda_\nu, \lambda_t, \alpha) \) be the solution of the piecewise smooth system (5.1) with \( n = 2 \), \( p = 1 \), \( F(\alpha) = \alpha > 0 \) and \( \lambda, \mu > 0 \) be the Lamé coefficients. Then

(j) if \( ((\lambda + 3\mu)f_\nu + (\lambda + \mu)f_t \leq 0 \& f_t \leq 0) \vee ((\lambda + 3\mu)f_\nu + (\lambda + \mu)f_t > 0) \) then there exists just one solution branch;

(jj) if \( ((\lambda + 3\mu)f_\nu + (\lambda + \mu)f_t < 0 \& f_t < 0) \) then there exist two separated solution branches;

(jjj) if \( ((\lambda + 3\mu)f_\nu + (\lambda + \mu)f_t = 0 \& f_t > 0) \) then there exist two solution branches which meet one another in a bifurcation point.

The respective solution branches are given explicitly in [10]. For comparison, we will compute them numerically. The following three examples pertain to the cases (jj), (j) and (jjj) with \( \lambda = \mu = 1 \).

The respective solution branches are given explicitly in [10]. For comparison, we will compute them numerically. The following three examples pertain to the cases (jj), (j) and (jjj) with \( \lambda = \mu = 1 \).

Example 6.1. Data: \( f = (1.5, -4) \). There exist two disjoint solution branches (1 and 2), see Figure 4 at the top. We always plot only the relation \( \alpha \mapsto \lambda_\nu \), since the components \( u_\nu, u_t \) and \( \lambda_t \) are uniquely determined by \( \lambda_\nu \). The performance of Algorithm 5.1 is illustrated at the bottom of Figure 4: the continuation starts at two points on each branch marked by the asterisks. We follow the solution in both positive and negative directions (the arrows always mark the positive directions). The solutions obtained by Algorithm 5.1 are represented by the dots •.
The solution points are classified as no contact (meaning that $u < 0$), contact-stick ($\lambda > 0$ & $|\lambda| < \mathcal{F}(\lambda)$) and contact-slip points ($\lambda > 0$ & $u \neq 0$). The sets of such points will be denoted here and in what follows by solid, dashed and dash-dotted curves, respectively. Along these curves, just one selection function is active. The points marked by the circle refer to the transition points between the contact modes: at such points at least two selection functions are active and Step 8 of Algorithm 5.1 has to be executed.

**Example 6.2.** Data: $f = (1.5, 7)$. The solution set consists of just one branch, see Figure 5.

**Example 6.3.** Data: $f = (1, -2)$. The solution set is depicted at the top of Figure 6: for $\alpha = (\lambda + 3\mu)/(\lambda + \mu) = 2$ branch 1 bifurcates and the corresponding contact problem possesses continuum of solutions represented by the vertical segment.

The actual computed paths when starting on branches 1 and 2 are depicted in Figure 6 in the center and at the bottom, respectively. Although the whole branch 1 consists of the grazing contact points, i.e. $u = \lambda = 0$, and two selection functions are active on it, the algorithm copes even with this situation. Observe that the direction of continuation during the transition between

![Figure 5: Example 6.2.](image)

![Figure 6: Example 6.3.](image)
branches 1 and 2 is determined in such a way that the orientation is preserved. Nevertheless, if one changes the orientation in the corresponding transition point, the other part of branch 1 is revealed.

It is worth mentioning that an arbitrarily small perturbation of $f$ destroys the bifurcation. For example, if we set $f = (0, 0.98, -2)$ then we arrive at (j): the solution set consists of two distinct branches, see Figure 7.

6.2. Two contact nodes

Next we shall consider a discrete contact problem with two contact nodes (see Figure 3), i.e. $n = 4, p = 2$ in (5.1). We set $F(\alpha) = (\alpha, \alpha), \alpha > 0, f = (f_1, f_2), f_1 = (f_{1i}, f_{2i}), i = 1, 2$, and $\lambda = \mu = 1$. Unlike to the previous case, this time the exact solutions are not at our disposal. The following figures are constructed on the basis of the solutions computed by Algorithm 5.1.

**Example 6.4. Data:** $f_1 = (1, -3), f_2 = (1, -1)$. We found numerically three branches depicted in Figures 8 and 9 (one has to be aware of different scaling of the figures). Again we plot only the relation $\alpha \mapsto \lambda_{i,1}, \lambda_{i,2}$, i.e. the normal stresses in the first and the second contact node. Solutions of the system (5.1) can be deduced from the figures in the following way: for a fixed $\alpha$ they consist of pairs $(\lambda_{i,1}, \lambda_{i,2})$ whose components are the intersection of the vertical line at $\alpha$ with the same branch $i$ associated with the first and the second contact node. Nevertheless, when coupling the components $\lambda_{i,1}$ and $\lambda_{i,2}$ from the same branch, one has to take into account also the order in which they were obtained during the continuation. Let us take $\alpha = 5$, for example. Then (5.1) has (at least) five solutions — one on branch 1, two on branch 2 (the upper intersection in the first component with the upper intersection in the second component and the lower one with the lower one) and two on branch 3 (the lower intersection in the first component with the upper intersection in the second component and vice-versa).
The performance of Algorithm 5.1 is illustrated in Figure 10: the continuation is initialized at the points marked by the asterisk corresponding to the value $\alpha = 5$ and each point is computed with a guaranteed precision given by the tolerances $\varepsilon = \varepsilon' = 10^{-6}$.

The crucial problem is to find starting points $x^0$ for the continuation routine. In general, we can fix $\alpha > 0$ and compute a corresponding solution to (5.1) by the piecewise smooth Newton method, see e.g. [19].

**Remark 6.1.** In order to find all roots, we proceed as follows: let us fix $\alpha > 0$. We determine all particular selection functions for $H$. For each selection function, which is linear, we solve (5.1). Finally, we sort out the classes of the roots. Obviously, such an algorithm has the exponential complexity.

**Example 6.5.** Data: $f_1 = (0.4, -1.4), f_2 = (1.5, -1.3)$. We found three branches depicted in Figures 11 and 12. It is worth mentioning that the points on branch 3 in the graph $\alpha \mapsto \lambda_{\nu,1}$ has to be viewed twice, i.e. there are
two solutions with the same $\lambda_{\nu,1}$ but different $\lambda_{\nu,2}$. The numerical performance is illustrated in Figure 13.

Next, we give an example of a bifurcation, similar to that one shown in Figure 6.

**Example 6.6.** Data: $f_1 = (2, -3)$, $f_2 = (1, -1)$. In Figure 14, we observe the bifurcation phenomenon: branch 1 bifurcates at $\alpha = 1.3333$ (for better understanding, the transition points are numbered). Note that the abscissa between transition points 3 and 4 in the graph $\alpha \mapsto \lambda_{\nu,1}$ is vertical and it represents together with the corresponding singleton in the graph $\alpha \mapsto \lambda_{\nu,2}$ continuum of solutions of the respective contact problem. Furthermore, the whole branch 1 consists of points corresponding to grazing contact of the first contact node and no contact of the second contact node. The second contact node comes into contact in transition point 3 on branch 2. The output of the numerical continuation of branch 2 when starting from $\alpha = 5$ is shown in Figure 15.
Remark 6.2. It is worth mentioning that a simple adaptation of Algorithm 5.1 enables us to continue along a loading path \( \alpha \in I \mapsto f(\alpha) \) keeping \( F \) fixed, see \[20\].

Example 6.7. Data: \( \mathcal{F} = (4, 4), f(\alpha) = (f_1(\alpha), f_2(\alpha)) \) with \( f_1(\alpha) = (-0.1\alpha + 0.4, 1.1\alpha + 0.2), f_2(\alpha) = (0.2\alpha + 1.8, 0.8\alpha - 0.1), -3 < \alpha < 3 \). The linear loading path results in the piecewise linear solution branch (see Figure 16). Again, it suffices to plot the relation \( \alpha \mapsto \lambda_{\nu,1}, \lambda_{\nu,2} \). Note that we encounter at most five solutions of the corresponding contact problem for \( \alpha \) fixed. For example, there are three intersections of the branch with the vertical line at \( \alpha = -1.6 \) at the first contact node and five intersections at the second contact node. However, any point between transition points 1 and 2 of the first contact node has to be counted three times as seen from Figure 16.

Conclusions

Mathematical models of contact problems with Coulomb friction in elastostatics have at least one solution provided that the coefficient of friction \( \mathcal{F} \) is sufficiently small. On the other hand there is no information on the structure of solutions (uniqueness, multiplicity) in a general case. The situation is somewhat different for discrete versions of these problems. It is well-known that appropriate discretizations have a solution for any \( \mathcal{F} \) belonging to a suitable class of coefficients and this solution is unique for \( \mathcal{F} \) small enough. The aim of this paper was twofold: \( a) \) to show how the uniqueness property depends on the number of degrees of freedom, i.e. on the norm of finite element meshes used to get the discrete model; \( b) \) how (eventually) multiple solutions to the discrete problems can be captured. Sections 2 and 3 of this paper were devoted to \( a) \). We considered the orthotropic Coulomb friction law with friction coefficients which may depend on the solution.
itself. The discrete solutions are represented by fixed points of the mapping $Ψ_{hf}$ acting on the contact zone. This mapping was defined by means of the mixed finite element formulation of auxiliary contact problems with Tresca friction. We presented conditions under which $Ψ_{hf}$ is contractive, implying not only the uniqueness of the fixed point but also convergence of the method of successive approximations. Unfortunately, these conditions are mesh-dependent. To fulfill them, the coefficient $F$ has to decay in an appropriate rate depending on the mesh norms. The rest of the paper deals with the fixed point but also convergence of the method of successive approximations. To understand better the structure of discrete solutions, we analyzed conditions ensuring the existence of local Lipschitz continuous branches of solutions assumed to be a function of $F$. This was done in Section 4 by using a variant of the implicit-function theorem for generalized equations. In a standard way of solving contact problems with Coulomb friction one finds some solution (which depends for example on the choice of the initial approximation) but without any further information on its position within other potential solutions. This is why we proposed a piecewise smooth variant of the Moore-Penrose continuation in Section 5 which allows us to follow branches of solutions parametrized by the friction coefficient $F$, the load vector $f$, etc. Unlike the classical Moore-Penrose continuation for smooth (differentiable) problems, some modifications in the prediction step ensuring the transition through points of non-differentiability have to be done. Using this technique, various graphs of solutions were obtained in the model examples presented in Section 6. Therefore, the proposed method seems to be promising for solving larger models arising from finite element discretizations of contact problems.

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References