On flows of fluids described by an implicit constitutive equation characterized by a maximal monotone graph

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Abstract

We study flows of incompressible fluids in which the deviatoric part of the Cauchy stress and the symmetric part of the velocity gradient are related through an implicit equation. Although we restrict ourselves to responses characterized by a maximal monotone graph, the structure is rich enough to include power-law type fluids, stress power-law fluids, Bingham and Herschel-Bulkley fluids, etc. We are interested in the development of (large-data) existence theory for internal flows subject to no-slip boundary conditions. We study first Stokes-like problems wherein the inertial effects are neglected, and later we consider the full balance of linear momentum that includes the inertial term.
1.1 Introduction

Incompressible Newtonian (or the Navier-Stokes) fluids are characterized by the constitutive equations that relates the symmetric part of the velocity gradient $D(v)$ and the deviatoric part of the Cauchy stress $T$ linearly, i.e.,

$$ T - \frac{1}{3} (tr T) I = 2\mu D(v), \quad (\mu > 0) \quad (1.1) $$

where the constant of proportionality $\mu$ is called the viscosity that in accordance with the second law of thermodynamics is positive. Recall that in the simple shear flow given by the velocity $v = u(y)e_1$ the only non-zero component of $D(v)$ is called the shear rate and the only non-zero component of the tensor $S := T - \frac{1}{3}(tr T)I$ is the shear stress.

It is known that the Navier-Stokes fluids are incapable of describing several phenomena exhibited by many real liquids. The simplest deviation from Newtonian behavior demonstrates itself through a non-linear relation between $S$ and $D(v)$. Shwedo, Trouton, Bingham, Ostwald, de Waele and Kincaid, Eyring, Stearn are some of the early developers of non-linear relations between the shear stress and the shear rate and various models of the power-law type were designed (and bear their names) to capture the observed experimental data concerning colloids, suspensions, polymeric liquids, etc. We refer the reader for example to Bird et al. (1987) for more details.

We are interested in investigating fluids for which $S$ and $D(v)$ are related implicitly, i.e., $D(v)$ and $S$ are given by a constitutive equation of the form

$$ G(D(v), S) = 0, \quad (1.2) $$

where we restrict ourselves to those responses that are described by maximal monotone graphs. In particular, we study the fluids that, in a simple shear flow, exhibit responses such as those drawn in Fig 1.1. As this picture indicates, the (admissible) responses are capable of capturing the presence of the activation criteria (such as yield stress). Notably, the activation can take place once the shear stress or the shear rate exceed certain critical values.

In Newtonian mechanics, force is clearly the cause and the kinematics the effect. As it would be natural to describe the effect in terms of the cause, it would be natural to provide constitutive definitions for the kinematical quantities in terms of the forces (stresses). In fact, in classical theories such as the linearized theory of elasticity, ideal gas law, linear viscoelasticity and even the classical Navier-Stokes theory,
one can express the constitutive function as a function for the kinematical quantity in terms of the stress. Even if one is unable to give an explicit constitutive definition for the kinematical quantity in terms of the stress, one should be able to provide “constitutive relations” between the kinematical quantities and the stress, or “relations” between kinematical quantities and their time derivatives and the stress and its time derivatives or for the history of appropriate kinematical quantities and the history of the stress. To provide constitutive definitions for the stress in terms of kinematical quantities seems to turn the idea of cause and effect, upside down. In fact, implicit constitutive equations have been used for quite some time to model the response of non-Newtonian fluids. These implicit models involve an implicit equation between the stress and various time derivatives of the stress and the symmetric part of the velocity gradient and various time derivatives of it. Invariably, the material moduli are assumed to be constant (for example the Oldroyd-B fluid, Burgers’ fluid, etc). However, implicit models that involve just the stress and the symmetric part of the velocity gradient, the simplest implicit model that one can think of, has not been studied in any detail. This simple class of models includes models wherein the material moduli can depend on the mean value of the normal stresses and the shear rate. In the case of incompressible fluids, such implicit models lead to material moduli that depend on the pressure and the shear rate, a model that has tremendous relevance in describing the response of geological and biological fluids. Recently, Rajagopal (see Rajagopal (2003,
Bulíček et al. (2006) has provided a systematic framework within which to study the response of such implicit fluid models. Such implicit models can provide a new way of looking at the turbulent flow of fluids wherein one takes into account perturbations in both the stresses and the velocities, unlike the present approach wherein one only considers perturbation in the velocity and the pressure. More importantly, until recently, in solid mechanics, implicit models have not been developed to describe the elastic response of materials. Implicit models to describe the non-dissipative response of solids has greatly extended the class of models to describe elastic materials (see Rajagopal (2007), Rajagopal & Srinivasa (2009)). Such implicit models for elastic bodies has provided a very novel approach to describe the fracturing of elastic solids (Rajagopal & Walton (2011)). Most importantly, such implicit theories can be studied systematically within a fully thermodynamic framework, see Rajagopal & Srinivasa (2008) or Málek & Rajagopal (2010). Our main interest in this paper is to consider a large class of implicit models wherein the stress and the symmetric part of the velocity gradient are given by “implicit constitutive relations”.

Introducing the graph $\mathcal{A} \subset \mathbb{R}_{\text{sym}}^{d \times d} \times \mathbb{R}_{\text{sym}}^{d \times d}$ through the natural characterization

$$(D,S) \in \mathcal{A} \iff G(D,S) = 0,$$  \hspace{1cm} (1.3)

we can specify precisely the class of admissible responses $G$ by requiring the following structural assumptions:

(A1) $\mathcal{A}$ includes the origin. $(0,0) \in \mathcal{A}$.

(A2) $\mathcal{A}$ is a monotone graph.

$$(S_1 - S_2) \cdot (D_1 - D_2) \geq 0 \text{ for all } (D_1,S_1),(D_2,S_2) \in \mathcal{A}.$$  

(A3) $\mathcal{A}$ is a maximal monotone graph. For $(D,S) \in \mathbb{R}_{\text{sym}}^{d \times d} \times \mathbb{R}_{\text{sym}}^{d \times d}$, if $(S - S) \cdot (D - D) \geq 0 \text{ for all } (D,S) \in \mathcal{A}$ then $(D,S) \in \mathcal{A}$.

(A4) $\mathcal{A}$ is a $\psi$ graph. There are positive $c_*,d_*$ and an $N$–function\footnote{Definition and properties of $N$–functions are recalled in Section 1.2 below. In particular, $\psi^*$ denotes the dual function to $\psi.$} $\psi$ such that

$$S \cdot D \geq c_*(\psi(D) + \psi^*(S)) - d_* \text{ for all } (D,S) \in \mathcal{A}.$$  

The graphs fulfilling (A1)–(A3) are called maximal monotone graphs (maximality refers to the fact that graph is continuous, the condition
(A1) states that "zero shear stress" and "zero shear rate" is an admissible "response" of the fluid. The condition (A4) can be motivated in several ways. For the Navier-Stokes fluid (1.1) we can write $S \cdot D = 2\mu |D|^2$ or $S \cdot D = \frac{1}{2\mu} |S|^2$ but also

$$S \cdot D = 2\mu |D|^2 + \frac{1}{4\mu} |S|^2,$$

which is (A4) with $\psi(D) = \mu |D|^2$ and $\psi^*(S) = \frac{1}{4\mu} |S|^2$. More generally, the following equivalence holds for proper convex functions:

$$S \cdot D = \psi(D) + \psi^*(S) \iff D \in \partial \psi^*(S) \iff S \in \partial \psi(D)$$

where $\partial \psi^*(S)$ and $\partial \psi(D)$ stand for sub-differentials of $\psi^*$ at $S$ and $\psi$ at $D$, respectively. Consequently, (A4) includes non-polynomial relations between $D$ and $S$. The assumption (A4) also gives a way of avoiding theories with non-standard growth conditions, etc.

Furthermore, the presence of positive constants $c_*$ (small) and $d_*$ (large) in (A4) permits one to consider much more general responses. For example, the Herschel-Bulkley\footnote{Herschel-Bulkley fluid reduces to a Bingham fluid for $r = 2$.} fluid which is usually described by the dichotomy

$$|S| \leq \tau^* \iff D = 0,$$
$$|S| < \tau^* \iff S = \tau^* \frac{D}{|D|} + 2\mu |D|^{r-2} D,$$  

(1.4)

where $\tau^* > 0$ is the yield stress and $r - 2$ is the power-law index ($r > 1$), can be rewritten as the implicit constitutive equation of the form (1.2), namely

$$2\mu |D|^{r-2}(\tau^* + (|S| - \tau^*)^+)D = (|S| - \tau^*)^+ S.$$  

(1.5)

Thus, it is not difficult to observe that (1.5) fulfills (A4) with $\psi(|D|) = \frac{1}{r} |D|^r$ and $\psi^*(|S|) = \frac{1}{r'} |S|^{r'}$, $r' = r/(r - 1)$, $r > 1$ with suitable $c_*$ and $d_*$. See Lemma 1.1 in Bulíček et al. (2011) for details.

Finally, the class of admissible fluids thus includes the classical power-law fluids and their generalizations of the form

$$S = 2\nu(|D|^2)D,$$

as well as the stress power-law fluids and their generalizations of the form

$$D = 2\alpha(|S|^2)S.$$
All these rheological relations are in particular very useful in geomechanical applications, see for example the models and references considered in Carey et al. (2004).

We aim to investigate large-data existence of flows of fluids responding to the external loading according to the implicit constitutive equation (1.2) specified by the assumptions (A1)–(A4). It means that for a given set of data

\[ A \] maximal monotone \( \psi \)-graph,
\[ T > 0 \] the length of time interval,
\[ \Omega \subset \mathbb{R}^d \] open bounded set with the boundary \( \partial \Omega \),
\[ v_0 \] the initial divergence-less velocity field,
\[ f \] the external body forces,
\[ \varrho^* \] the density of the fluid,

we look for \((v, p, S) : (0, T) \times \Omega \to \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d_{sym} \) satisfying

\[
\begin{align*}
0 &= \text{div } v \\
\varrho^* (\partial_t v + \text{div } (v \otimes v)) - \text{div } S &= -\nabla p + \varrho^* f \\
(D(v), S) &\in A
\end{align*}
\] in \((0, T) \times \Omega, \quad (1.6)\)

together with the initial and boundary conditions

\[
\begin{align*}
v(0, \cdot) &= v_0 \quad &\text{in } \Omega, \quad (1.7) \\
v &= 0 \quad &\text{on } (0, T) \times \partial \Omega. \quad (1.8)
\end{align*}
\]

This paper should be viewed as being complementary to the study by Bulíček et al. (2011), where the above problem has been analyzed. More precisely, in Bulíček et al. (2011), the long-time and large-data existence of weak solutions to (1.6) is established provided that the function \( \psi \) satisfying (A4) is of the form \( \tilde{\psi}(s) = \psi(s) \) and fulfills

\[
c_1 s^d - c_2 \leq \tilde{\psi}(s) \leq c_3 s^d + c_4 \quad (c_i > 0 \text{ for } i = 1, 2, 3, 4) \quad (1.9)
\]

and

\[
q > \frac{2d}{d+2}.
\]

Bulíček et al. (2011) assume that the internal flows that are considered satisfy for \( \Omega \in C^{1,1} \) the Navier’s slip boundary condition

\[
0 = v \cdot n \quad \text{and} \quad \lambda v_r + (1 - \lambda) \gamma^* (Sn)_r = 0 \quad \text{on } (0, T) \times \partial \Omega, \quad (1.10)
\]

where \( 0 \leq \lambda < 1 \) (eliminating the possibility of the fluid to adhere to
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the boundary), $z_\tau$ denotes the projection of $z$ into the tangent plane to the boundary and $\gamma^*$ is the slip parameter.

In this study, we consider flows that exhibit no-slip on the boundary that we supposed to be Lipschitz continuous. Also, while Bulíček et al. (2011) considered the complete system (unsteady flows with convection), here we proceed from simpler to more complicated problems: we start with steady flows without convection in Section 1.3, continue with steady flows with convection in Section 1.4, and follow with unsteady flows without convection in Section 1.5 and end with a survey of results concerning unsteady flows with convection in Section 1.6. This scenario enables us to show how the mathematical tools involved in the analysis vary - this depends essentially on the fact whether $v$, a weak solution to (1.6), is the admissible test function in the weak form of the balance of linear momentum or not, and if not how is one to overcome this difficulty. In Bulíček et al. (2011), the authors use the properties of Lipschitz approximations to functions having the symmetric part of the velocity gradient in an Orlicz space (see also Frehse et al. (2003), Diening et al. (2008, 2010)) to overcome the mathematical complications, however the key restriction in Bulíček et al. (2011) is that the $N$-function $\psi$ depends only on the modulus of $D$, i.e., $\psi(D) = \psi(|D|)$. Here we present the cases where the use of Lipschitz approximations is not necessary; then the methods are simpler and allow one to include the general $N$-functions into the analysis. For the sake of completeness we however state theorems (without proof) that concern the complete system that either follow from or can be proved following the arguments advanced in Bulíček et al. (2011).

1.2 Mathematical tools

1.2.1 Orlicz spaces

In this subsection we recall several facts about $N$-functions and corresponding Orlicz spaces. We recall that $\psi: \mathbb{R}^{d \times d} \times \mathbb{R}_+ \to \mathbb{R}_+$ is an $N$-function if $\psi$ is an even continuous convex function satisfying

$$\lim_{|s| \to +0} \frac{\psi(s)}{|s|} = 0 \quad \text{and} \quad \lim_{|s| \to +\infty} \frac{\psi(s)}{|s|} = +\infty. \quad (1.11)$$

A function $\psi^*$ defined as

$$\psi^*(s) := \sup_{v \in \mathbb{R}^{d \times d} \times \mathbb{R}_+} (s \cdot v - \psi(v)) \quad (1.12)$$
is called a complementary (conjugate, dual) function to $\psi$. It follows from its definition that $\psi^*$ is also an $N$-function and $(\psi^*)^* = \psi$.

For any open bounded set $G \subset \mathbb{R}^k$, we define the Orlicz space $L^\psi(G)$ as a set of all measurable functions $u : G \to \mathbb{R}^{d \times d} \times \mathbb{R}^{d \times d}$ that satisfy

$$\lim_{\lambda \to 0} \int_G \psi(\lambda u) \, dz = 0.$$ 

This space equipped with the norm

$$\|u\|_{L^\psi} = \|u\|_\psi := \inf \left\{ \lambda > 0; \int_G \psi(\lambda^{-1} u) \, dz \leq 1 \right\}$$


An $N$-function $\psi$ satisfies $\Delta_2$ condition if there exist $C_1 > 0$ and $C_2 > 0$ such that for all $s \in \mathbb{R}^{d \times d} \times \mathbb{R}^{d \times d}$ we have

$$\psi(2s) \leq C_1 \psi(s) + C_2. \quad (1.13)$$

Next, if we assume that $\psi$ satisfies $\Delta_2$ condition then $L^\psi(G)$ is separable and moreover

$$(L^\psi(G))^* = L^{\psi^*}(G). \quad (1.14)$$

Otherwise, for any $N$-function $\psi$, we know that the predual space to $L^{\psi^*}(G)$ is the closure of bounded functions with respect to the norm in $L^\psi(G)$. We denote this space by $E^\psi(G)$, and recall that this is a separable space.

Finally, we formulate Young and Hölder inequalities for $N$-functions and Orlicz spaces (see e.g. Rao & Ren (1991)): the (Young) inequality reads

$$|a \cdot b| \leq \psi(a) + \psi^*(b) \quad \text{for all } a, b \in \mathbb{R}^{d \times d} \times \mathbb{R}^{d \times d}. \quad (1.15)$$

the (Hölder) inequality states that if $u \in L^\psi(G)$ and $v \in L^{\psi^*}(G)$, where $u$ and $v$ are real- or matrix-valued functions, then $u \cdot v \in L^1(G)$ and

$$\int_G u \cdot v \, dz \leq 2 \|u\|_\psi \|v\|_{\psi^*}. \quad (1.16)$$

In different parts of the paper we will assume that $\psi$ satisfies one or both from the following assumptions

(H1) the function $\psi$ satisfies $\Delta_2$-condition,

(H2) the function $\psi^*$ satisfies $\Delta_2$-condition.
Recall (see Rao & Ren (1991) for detailed proofs) that if \((H1)\) holds then there are positive constants \(c_3, c_4 > 0\) and \(r < \infty\) such that
\[
\psi(a) \leq c_3 |a|^r + c_4 \quad \text{for all} \quad a \in \mathbb{R}_\text{sym}^{d \times d} \times \mathbb{R}_\text{sym}^{d \times d} \quad (H1^*)
\]
and similarly \((H2)\) implies that there are positive constants \(c_1, c_2\) and \(q > 1\) such that if
\[
\psi(a) \geq c_1 |a|^q - c_2 \quad \text{for all} \quad a \in \mathbb{R}_\text{sym}^{d \times d} \times \mathbb{R}_\text{sym}^{d \times d} \quad (H2^*)
\]
Note that \((H1^*)\) and \((H2^*)\) are exactly the condition (1.9). In the rest of the paper the letters \(r, q\) are always related to \((H1^*)\) and \((H2^*)\) respectively, provided that either \((H1)\) or \((H2)\) hold.

1.2.2 Selections

Let \(A\) be a maximal monotone graph satisfying \((A1)-(A4)\). Consider mapping \(S_\ast : \mathbb{R}_\text{sym}^{d \times d} \to \mathbb{R}_\text{sym}^{d \times d}\) assigning to each \(B \in \mathbb{R}_\text{sym}^{d \times d}\) exactly one value \(S_\ast(B) \in \mathbb{R}_\text{sym}^{d \times d}\) so that \((B, S_\ast(B)) \in A\). Such a mapping \(S_\ast\) is called selection of \(A\). Obviously, each such a selection \(S_\ast\) defined on \(\mathbb{R}_\text{sym}^{d \times d}\) is monotone and due to \((A4)\) satisfies
\[
(A4^*) \quad S_\ast(D) \cdot D \geq c_4(\psi(D) + \psi^*(S_\ast(D))) - d_s \quad \text{for all} \quad D \in \mathbb{R}_\text{sym}^{d \times d}.
\]
It is also not difficult to observe (see also Alberti & Ambrosio (1999)) that the following condition replaces \((A3)\)
\[
(A3^*) \quad \text{For} \quad (D, S) \in \mathbb{R}_\text{sym}^{d \times d}:
\]
\[
\text{if} \quad (S - S_\ast(B), D - B) \geq 0 \quad \text{for all} \quad B \in \mathbb{R}_\text{sym}^{d \times d}, \quad \text{then} \quad (D, S) \in A.
\]
In general, \(S_\ast\) does not have to be a Borel function. On the other hand, there is a selection \(S_\ast\) that is a Borel function, see e.g. Aubin & Frankowska (2009), and only such a selection is considered in the remaining parts of the paper.

1.2.3 Convergence tools

In this subsection, we formulate simple criteria that allows us to verify that \(S\) and \(D(v)\) fulfill the nonlinear constitutive equation (1.2), which is equivalent to \((D(v), S) \in A\). The first criterion concerns in particular the case when \(\psi\) and \(\psi^*\) fulfill \((H1)\) and \((H2)\). The second criterion concerns the general (non-reflexive) case. We state the first lemma since it is sufficient for many applications and its proof is simpler than the proof for the general case.
Lemma 1.2.1 Let $\mathcal{A}$ be maximal monotone $\psi$-graph satisfying (A1)–(A4). Assume that there are sequences $\{S^n\}_{n=1}^{\infty}$ and $\{D^n\}_{n=1}^{\infty}$ defined on a measurable set $G \subset \mathbb{R}^k$ such the following holds:

\begin{align}
(D^n, S^n) &\in \mathcal{A} \quad \text{a.e. in } G, \quad (1.17) \\
D^n &\rightharpoonup D \quad \text{weakly in } L^\psi(G), \quad (1.18) \\
S^n &\righthpoonup S \quad \text{weakly in } L^{\psi^*}(G), \quad (1.19)
\end{align}

Then

\begin{align}
\limsup_{n \to \infty} \int_G S^n \cdot D^n \, dz &\leq \int_G S \cdot D \, dz. \quad (1.20)
\end{align}

Proof We first observe that (1.17)–(1.20) imply that

\begin{align}
\limsup_{n \to \infty} \int_G (S^n - S_*(D)) \cdot (D^n - D) \, dz &\leq 0. \quad (1.22)
\end{align}

Since the graph is monotone, (1.22) is equivalent to

\begin{align}
\limsup_{n \to \infty} \int_G \|(S^n - S_*(D)) \cdot (D^n - D)\| \, dz = 0. \quad (1.23)
\end{align}

Therefore, $(S^n - S_*(D)) \cdot (D^n - D)$ converges strongly in $L^1(G)$ and consequently weakly in $L^1(G)$, namely, we have for all $\varphi \in L^\infty(G)$

\begin{align}
\lim_{n \to \infty} \int_G (S^n - S_*(D)) \cdot (D^n - D) \varphi \, dz = 0. \quad (1.24)
\end{align}

From (1.24), (1.18) and (1.19) it follows that

\begin{align}
\lim_{n \to \infty} \int_G S^n \cdot D^n \varphi \, dz = \lim_{n \to \infty} \int_G S^n \cdot D \varphi \, dz = \int_G S \cdot D \varphi \, dz. \quad (1.25)
\end{align}

Consequently, since the graph is monotone, we observe that for an arbitrary fixed matrix $B \in \mathbb{R}^{d \times d}_{\text{sym}}$ and all nonnegative $\varphi \in L^\infty(G)$

\begin{align}
0 \leq \lim_{n \to \infty} \int_G (S^n - S_*(B)) \cdot (D^n - B) \varphi \, dz = \int_G (S - S_*(B)) \cdot (D - B) \varphi \, dz.
\end{align}

Since $\varphi \geq 0$ is arbitrary, this implies that

\begin{align}
(S - S_*(B)) \cdot (D - B) \geq 0 \quad \text{for all } B \in \mathbb{R}^{d \times d}_{\text{sym}} \text{ a.e. in } G, \quad (1.26)
\end{align}

and the assertion follows from $(A3^*)$. \qed
The next lemma covers the case when \( \{D^n\}_{n \in \mathbb{N}} \) is bounded in \( L^\psi(G) = (E^\psi(G))^* \) which implies only weak* convergence of \( D^n \) (analogously for \( \{S^n\}_{n \in \mathbb{N}} \)).

**Lemma 1.2.2** Let \( A \) be maximal monotone \( \psi \)-graph satisfying (A1)–(A4). Assume that there are sequences \( \{S^n\}_{n = 1}^\infty \) and \( \{D^n\}_{n = 1}^\infty \) defined on a measurable set \( G \subset \mathbb{R}^k \) such that the following conditions hold:

\[
(D^n, S^n) \in A \quad \text{a.e. in } G, \\
D^n \rightharpoonup^* D \quad \text{weakly* in } L^\psi(G), \\
S^n \rightharpoonup^* S \quad \text{weakly* in } L^{\psi^*}(G), \\
\limsup_{n \to \infty} \int_G S^n \cdot D^n \, dz \leq \int_G S \cdot D \, dz.
\]

Then

\[
(D, S) \in A \quad \text{a.e. in } G.
\]

**Proof** Since the graph is monotone, we have

\[
\int_G (S_n(B) - S^n) \cdot (B - D^n) \, dz \geq 0 \quad \text{for all } B \in L^\infty(G).
\]

Note that if \( B \in L^\infty(G) \) then \( S_n(B) \in L^\infty(G) \). Indeed, assume the opposite, i.e., \( S_n(B) \) is unbounded. Then, it follows from (A4) and the fact that \( \psi \) is nonnegative that

\[
|B| \geq c_s \frac{\psi^*(S_n(B)) - d_*}{|S_n(B)|}.
\]

Since \( \psi^* \) is an \( N \)-function, then the right-hand side tends to infinity, which contradicts the fact that \( B \in L^\infty(G) \).

Next, rearranging (1.32) we have

\[
\int_G S^n \cdot D^n \, dz \geq \int_G S^n \cdot B \, dz + \int_G S_n(B) \cdot (D^n - B) \, dz.
\]

Letting \( n \to \infty \) in (1.33), we conclude from (1.28)–(1.30) that

\[
\int_G S \cdot D \, dz \geq \int_G S \cdot B \, dz + \int_G S_n(B) \cdot (D - B) \, dz
\]

and consequently

\[
\int_G (S_n(B) - S) \cdot (B - D) \, dz \geq 0 \quad \text{for all } B \in L^\infty(G).
\]
Next, we use a generalization of the Minty method to prove (1.31). For any \( j > 0 \) we define the set \( G_j := \{ z \in G; |D(z)| \leq j \} \) and use \( 1_{G_j} \) to denote the characteristic function of \( G_j \). Since \( |D| \in L^1(G) \) due to (1.28) we observe that
\[
|G \setminus G_j| \leq \frac{C}{j}, \quad (1.36)
\]
Next, let \( i, j \in \mathbb{N} \) be arbitrary such that \( 0 < j < i \). Then we take \( B \) in (1.35) of the form
\[
B := D1_{G_i} + hW1_{G_j},
\]
where \( h > 0 \) and \( W \in L^\infty(G) \) is arbitrary (note that \( B \) is bounded).
Doing so, we obtain
\[
\int_{G_j} (S_*(D + hW) - S) \cdot W \, dz \geq \frac{1}{h} \int_{G \setminus G_i} (S_*(0) \cdot D - S \cdot D) \, dz. \quad (1.37)
\]
Next, using (1.28)–(1.29), and the Hölder inequality we obtain
\[
\int_G |S_*(0) \cdot D - S \cdot D| \, dz < \infty.
\]
Consequently, we appeal to the Lebesgue dominated convergence theorem and (1.36) to let \( i \to \infty \) on the right-hand side of (1.37) and conclude that
\[
\lim_{i \to \infty} \int_{G \setminus G_i} (S_*(0) \cdot D + S \cdot D) \, dz = 0,
\]
which directly implies that
\[
\int_{G_j} (S_*(D + hW) - S) \cdot W \, dz \geq 0 \quad \text{for all } j \in \mathbb{N}. \quad (1.38)
\]
Our goal is to let \( h \to 0_+ \) and to show (1.31). Using the definition of \( G_j \) it is easy to see that for a subsequence
\[
S_*(D + hW) \to S \quad \text{weakly in } L^2(G_j),
\]
\[
D + hW \to D \quad \text{strongly in } L^2(G_j),
\]
\[
(D + hW, S_*(D + hW)) \in \mathcal{A} \quad \text{a.e. in } G_j.
\]
Therefore, we can use Lemma 1.2.1 (note that (1.20) is direct consequence of above convergence results) with \( L^\psi = L^2 \) and we observe that
\[
(D, S) \in \mathcal{A} \quad \text{a.e. in } G_j. \quad (1.39)
\]
Moreover, since $D$ is bounded in $G_j$ then also $\mathbf{S}$ is bounded in $G_j$ due to (1.39). Finally, taking the limit $h \to 0^+$ in (1.38), we have

$$\int_{G_j} (\mathbf{S} - \mathbf{S} - \mathbf{W}) \cdot \mathbf{z} \, dz \geq 0 \quad \text{for all } \mathbf{W} \in L^\infty(G_j).$$

Setting, $\mathbf{W} := \frac{\mathbf{S} - \mathbf{S} - \mathbf{S}}{|\mathbf{S} - \mathbf{S}|}$ yields

$$\int_{G_j} |\mathbf{S} - \mathbf{S}| \, dz \leq 0$$

and therefore (1.39) implies that $(D, S) \in A$ a.e. in $G_j$. But since $j$ was arbitrary, we use (1.36) and conclude that $(D, S) \in A$ a.e. in $G$.

### 1.3 Steady flows without convection

In this section we consider steady flows and ignore convection. It means that we neglect the first two terms on the left-hand side of (1.6). For the sake of simplicity we also set the density $\rho^* \equiv 1$, the results are however valid even when this assumption is not made. Since we want to have an optimal result it seems to be appropriate to assume that the body forces $f$ are given as the divergence of a symmetric tensor field $\mathbf{F}$, i.e., $f = -\text{div} \mathbf{F}$. Note that for smooth $f$ such tensor field can be easily constructed.

Hence, we consider the following problem: for given open bounded $\Omega \subset \mathbb{R}^d$, $A \subset \mathbb{R}^{d \times d \times d} \times \mathbb{R}^{d \times d \times d}$ and $\mathbf{F} : \Omega \to \mathbb{R}^{d \times d}$ to find $(\mathbf{v}, p, S) : \Omega \to \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^{d \times d}$ such that

$$-\text{div} \mathbf{S} + \nabla p = -\text{div} \mathbf{F}, \quad \text{div} \mathbf{v} = 0, \quad (D(v), S) \in A \quad \text{in } \Omega,$$

$$v = 0 \quad \text{on } \partial \Omega. \quad (1.40)$$

Even for such a relatively simple problem, difficulty may occur due to the fact we work with general $N$-functions $\psi$ that may not satisfy (H1) or (H2). To be more precise we will not be able construct the pressure in general and therefore we project (1.40)_1 onto the space of divergenceless functions and omit the construction of the pressure. Hence, we are led to the following definition.

**Definition 1.3.1.** Let $\Omega \subset \mathbb{R}^d$ be an open bounded set with Lipschitz boundary, $A$ satisfy the assumptions (A1)–(A4) and $\mathbf{F} \in L^{\psi^*}(\Omega)$. We
say that a couple \((v, S)\) is weak solution to Problem (1.40) if

\[
\begin{align*}
\mathbf{v} & \in W^{1,1}_{0,\text{div}}, \quad D(v) \in L^p(\Omega), \quad S \in L^{p^*}(\Omega), \\
&\int_{\Omega} S \cdot D(w) \, dx = \int_{\Omega} F \cdot D(w) \, dx \quad \text{for all } w \in C_0^{\infty,\text{div}}, \\
(D(v), S) & \in \mathcal{A} \quad \text{a.e. in } \Omega.
\end{align*}
\]  

We recall here the notation of function spaces used in Definition 1.3.1 and in the whole paper. For any \(p \in [1, \infty]\), we set

\[
W^{1,p}_{0,\text{div}}(\Omega) := \{ v : \Omega \to \mathbb{R}^d; \ v_i \in W^{1,p}_{0}(\Omega) \ \text{for all } i = 1, \ldots, d \},
\]

and define

\[
W^{1,p}_{0,\text{div}} := \{ v \in W^{1,p}_{0,\text{div}}(\Omega); \ \text{div} \, v = 0 \}, \quad L^2_{0,\text{div}} := L^2_{0,\text{div}} \| \cdot \|_2.
\]

The main theorem of this section is the following.

**Theorem 1.3.2** Let the assumption of Definition 1.3.1 be fulfilled. Let \(\psi\) in addition satisfy (H2). Then there exists a weak solution to Problem (1.40).

In addition, if \(\psi\) satisfies (H1) then there exists \(p \in L^1(\Omega)\) and the following holds

\[
\int_{\Omega} S \cdot D(w) \, dx = \int_{\Omega} F \cdot D(w) + p \\text{div} \, w \, dx \quad \text{for all } w \in W^{1,\infty}_{0,\text{div}}.
\]  

**Proof** Let \(S_\ast\) be a selection from the graph \(\mathcal{A}\) having all the properties discussed in Subsection 1.2.2 and let \(S_\ast^\varepsilon\) denote its mollification. It means

\[
S_\ast^\varepsilon(\xi) := (S_\ast \ast \rho^\varepsilon)(\xi) = \int_{\mathbb{R}^{d \times d}_{\text{sym}}} S_\ast(\zeta) \rho^\varepsilon(\xi - \zeta) \, d\zeta,
\]

where \(\rho^\varepsilon(\xi) = \frac{1}{\varepsilon^d} \rho \left( \frac{\xi}{\varepsilon} \right), \varepsilon > 0\) and \(\rho \in C_0^\infty(\mathbb{R}^{d \times d}_{\text{sym}})\) is a mollification kernel, i.e., a radially symmetric function with support in a unit ball \(B(0,1) \subset \mathbb{R}^{d \times d}_{\text{sym}}\) and \(\int_{\mathbb{R}^{d \times d}} \rho \, d\xi = 1\). It is not difficult to observe, using the convexity of \(\psi\) and \(\psi^\ast\) and by means of Jensen’s inequality, that the approximation \(S_\ast^\varepsilon\) satisfies a condition analogous to (A4).

Let \(\{w_i\}_{i=1}^n\) be a basis of \(W^{1,1}(\Omega)^d \cap W^{1,1}_{0,\text{div}} \hookrightarrow W^{1,\infty}_{0,\text{div}}\). We look for a vector-valued function \(v^{\varepsilon,n} := \sum_{i=1}^n c_i^{\varepsilon,n} w_i\) such that the coefficients \(c_i^{\varepsilon,n}\) solve the following system of \(n\) equations (Galerkin approximation)

\[
\int_{\Omega} S_\ast^\varepsilon(D(v^{\varepsilon,n})) : D(w_i) \, dx = \int_{\Omega} F \cdot D(w_i) \, dx, \quad i = 1, \ldots, n.
\]  

\[\text{(1.46)}\]
The existence of a solution to (1.46) can be shown by using a variant of the Brower fixed point theorem and the a priori estimates shown below. To get them, we multiply the $i$-th equation in (1.46) by $c^\epsilon_{i,n}$ and sum the result over $i = 1, \ldots, n$. We obtain

$$
\int_{\Omega} S^\epsilon_i(D(v^\epsilon,n)) \cdot D(v^\epsilon,n) \, dx = \int_{\Omega} F \cdot D(v^\epsilon,n) \, dx.
$$

(1.47)

Next, using (A4*) and the Young inequality we conclude from (1.47) that

$$
\int_{\Omega} \psi(D(v^\epsilon,n)) + \psi^*(S^\epsilon_i(D(v^\epsilon,n))) \, dx \leq C(F) \leq C.
$$

(1.48)

Obtaining the limit as $\epsilon \to 0^+$. Using (1.48) we see that $|c^\epsilon_{i,n}| \leq C(n)$ and therefore since $n$ is fixed and finite, we can find a (labeled again by $\epsilon$) subsequence such that

$$
c^\epsilon_{i,n} \to c^n_i \quad \text{strongly in } \mathbb{R} \quad \text{for all } i = 1, \ldots, n.
$$

(1.49)

Consequently, we use the definition of $v^\epsilon,n$, the fact that $w_i \in W^{1,\infty}_{0,\text{div}}$ for all $i \in \mathbb{N}$ and the properties of $\psi$ to get

$$
v^\epsilon,n \to v^n \quad \text{strongly in } w_i \in W^{1,\infty}_{0,\text{div}},
$$

(1.50)

$$
S^\epsilon_i(D(v^\epsilon,n)) \rightharpoonup S^i \quad \text{weakly* in } L^\infty(\Omega)^d \times d.
$$

(1.51)

Using (1.50)--(1.51) it is standard to take the limit $\epsilon \to 0$ in (1.46) and to show that $v^n = \sum_{i=1}^n c^n_i w_i$ and $S^n$ satisfy

$$
\int_{\Omega} S^n \cdot D(u_i) \, dx = \int_{\Omega} F \cdot D(u_i) \, dx \quad \text{for all } i = 1, \ldots, n.
$$

(1.52)

It remains to show that $(D(v^n),S^n) \in \mathcal{A}$. Since $S_*$ is the selection of the graph, then it is monotone and for all $\zeta, B \in \mathbb{R}^{d \times d}_{\text{sym}}$

$$
(S_*(\zeta) - S_*(B)) \cdot (\zeta - B) \geq 0.
$$

(1.53)

Multiplying this inequality by $\rho^\epsilon(D(v^\epsilon,n) - \zeta)$ and integrating the result with respect to $\zeta$ over $\mathbb{R}^{d \times d}_{\text{sym}}$, we observe that

$$
\int_{\mathbb{R}^{d \times d}_{\text{sym}}} (S_*(\zeta) - S_*(B)) \cdot (D(v^\epsilon,n) - B) \rho^\epsilon(D(v^\epsilon,n) - \zeta) \, d\zeta
\geq \int_{\mathbb{R}^{d \times d}_{\text{sym}}} (S_*(\zeta) - S_*(B)) \cdot (D(v^\epsilon,n) - \zeta) \rho^\epsilon(D(v^\epsilon,n) - \zeta) \, d\zeta.
$$

(1.54)
Since the difference \( (S_+(\zeta) - S_+(B))\) can be for \(|\zeta| \leq \|D(v^{\varepsilon,n})\|_\infty + \varepsilon \leq C(n)\) estimated simply by a constant dependent on \(B\), then the above estimate can be rewritten as
\[
\left( \int_{\mathbb{R}^{d \times d}_{\text{sym}}} S_+(\zeta) \rho^\varepsilon (D(v^{\varepsilon,n}) - \zeta) d\zeta - S_+(B) \right) \cdot (D(v^{\varepsilon,n}) - B) \geq -C_n(B) \int_{\mathbb{R}^{d \times d}_{\text{sym}}} |\zeta| \rho^\varepsilon (D(v^{\varepsilon,n}) - \zeta) d\zeta.
\]

Hence, using the strong convergence (1.50) we see that the right hand side of (1.55) tends to zero as \(\varepsilon \to 0\) and we get
\[
\liminf_{\varepsilon \to 0} (S^{\varepsilon,n} - S_+(B)) \cdot (D(v^{\varepsilon,n}) - B) \geq 0 \quad \text{a.e. in } \Omega,
\]
which due to the strong convergence of \(D(v^{\varepsilon,n})\) and weak* convergence of \(S^{\varepsilon,n}D(v^{\varepsilon,n})\) yields
\[
(S^n - S_+(B)) \cdot (D(v^n) - B) \geq 0 \quad \text{for all } B \in \mathbb{R}^{d \times d}_{\text{sym}} \text{ and a.e. in } \Omega. \quad (1.56)
\]

By (A3*), (1.56) implies that
\[
(D(v^n), S^n) \in A \quad \text{a.e. in } \Omega.
\]

**Obtaining the limit as } n \to \infty \text{.** We multiply the } i\text{-th equation in (1.52) by } c^i_n, \text{ sum the result over } i = 1, \ldots, n \text{ and obtain the energy equality}
\[
\int_{\Omega} S^n \cdot D(v^n) \, dx = \int_{\Omega} F \cdot D(v^n) \, dx. \quad (1.57)
\]

By (A4), this relation implies
\[
\int_{\Omega} \psi(D(v^n)) + \psi^*(S^n) \, dx \leq C(F) \leq C. \quad (1.58)
\]

Thus, we can extract a (labeled again by \(n\)) a subsequence of \(\{v^n, S^n\}\) such that
\[
D(v^n) \rightharpoonup D(v) \quad \text{weakly* in } L^\infty(\Omega), \quad (1.59)
\]
\[
S^n \rightharpoonup S \quad \text{weakly* in } L^{\infty'}(\Omega). \quad (1.60)
\]

Hence, having (1.59)–(1.60) we can let \(n \to \infty\) in (1.52) to obtain
\[
\int_{\Omega} S \cdot D(w_i) \, dx = \int_{\Omega} F \cdot D(w_i) \, dx \text{ for all } i \in \mathbb{N}. \quad (1.61)
\]

Consequently, since \(\{w_i\}_{i \in \mathbb{N}}\) is dense in \(W^{3,\infty}(\Omega)^d \cap W^{2,1}_{0,\text{div}}\) we deduce (1.42). Thus it remains to show that \((D(v), S) \in A \text{ a.e. in } \Omega\). To do
so we develop a generalization of the energy method. Since we are not allowed to use $\mathbf{v}$ as a test function in (1.42) we multiply $i$-th equation in (1.61) by $c^n_i$ and take a sum over $i = 1, \ldots, n$ and conclude that

$$\int_\Omega \mathbf{S} \cdot \mathbf{D}(\mathbf{v}^n) \, dx = \int_\Omega \mathbf{F} \cdot \mathbf{D}(\mathbf{v}^n) \, dx.$$  

(1.62)

Such a setting is possible since $\mathbf{v}^n$ is given as a finite linear combination of “smooth” functions $\mathbf{w}_i$. Consequently, comparing (1.57) and (1.62) we see that

$$\lim_{n \to \infty} \int_\Omega \mathbf{S}^n \cdot \mathbf{D}(\mathbf{v}^n) \, dx = \lim_{n \to \infty} \int_\Omega \mathbf{S} \cdot \mathbf{D}(\mathbf{v}^n) \, dx = \int_\Omega \mathbf{S} \cdot \mathbf{D}(\mathbf{v}) \, dx,$$  

(1.63)

where for the second equality we used (1.59)–(1.60) and the assumption (H2) that implies $(L^\psi)(\Omega))^* = L^\psi(\Omega)$. At this point, we apply Lemma 1.2.2 to conclude that $(\mathbf{D}(\mathbf{v}),\mathbf{S}) \in \mathcal{A}$ a.e. in $\Omega$.

The pressure $p \in (\mathcal{D}(\Omega))^*$ can be constructed in a standard way by means of de Rham theorem so that (1.44) holds for all $\mathbf{w} \in \mathcal{D}(\Omega)^d$. To provide also an uniform bound on the pressure, we however need to assume (H1). Indeed, having this (and consequently (H1*)–(H2*)) we see that $\nabla p$ in $(W^{1,r}_0(\Omega))^d_0$, which allows us to use the Nečas theorem and to conclude that $p \in L^r_0(\Omega)$, where subscript 0 means that $\int_\Omega p \, dx = 0$. We may refer the reader to Bulíček & Fišerová (2009) for details.

One can be also interested in analyzing the opposite case, i.e., the case when only (H1) holds. However, in this setting one must start to work with spaces of bounded deformations and we are not able to treat a general $\psi$. To be more precise, either $\psi = \psi(|\mathbf{D}|)$ or the upper and lower growth of $\psi$ has to be related, see Gwiazda et al. (2011) for details. Moreover, to the best of our knowledge, there is no theory in the case when neither (H1) nor (H2) hold.

### 1.4 Steady flows with convection

In this subsection, we study the following problem: for a given $\Omega$, $\mathcal{A}$ and $\mathbf{F}$ to find $(\mathbf{v}, p, \mathbf{S}) : \Omega \to \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^{d \times d}_{\text{sym}}$ such that

$$\text{div}(\mathbf{v} \otimes \mathbf{v} - \mathbf{S}) + \nabla p = \text{div} \mathbf{F}, \quad \text{div} \mathbf{v} = 0, \quad (\mathbf{D}(\mathbf{v}),\mathbf{S}) \in \mathcal{A} \quad \text{in } \Omega,$$

$$\mathbf{v} = \mathbf{0} \quad \text{on } \partial \Omega.$$  

(1.64)

In the previous section we established the existence result for the graphs satisfying (A1)–(A4) provided that (H2) holds and we also discussed the cases in which (H2) is not necessary. Here, in addition, we have
to control the convective term and in order to, at the very least, give meaning to this term in the weak formulation we need to guarantee that \( v \in L^2(\Omega)^d \). For this reason we need to assume (H2) and consequently (H2\*\*) with \( q \) large enough so that \( W^{1,q}(\Omega) \) is continuously embedded into \( L^2(\Omega) \). This leads to the following definition of weak solution to to Problem (1.64).

**Definition 1.4.1** Let \( \Omega \subset \mathbb{R}^d \) be a bounded open set with Lipschitz boundary and \( F \in L^{\psi^*}(\Omega) \). Assume that \( \mathcal{A} \) is a maximal monotone \( \psi \)- graph fulfilling (A1)–(A4) and (H2\*\*) with \( q \geq \frac{3d}{d+2} \). We say that a couple \((v,S)\) is weak solution to Problem (1.64) if

\[
\int_{\Omega} (S - v \otimes v) \cdot \nabla w \, dx = \int_{\Omega} F \cdot \nabla w \, dx \quad \text{for all } w \in C_0^\infty(\Omega),
\]

(1.65)

\[
(D(v),S) \in \mathcal{A} \text{ a.e. in } \Omega.
\]

(1.66)

Next, we formulate three existence results. Particular statements differ in the dependence on the range of \( q \) and also on the additional structural assumptions on \( \psi \). The first existence result is the following.

**Theorem 1.4.2** Let all the assumptions of Definition 1.4.1 hold. In addition, let \( \psi \) satisfy (H2\*\*) with

\[
q \geq \frac{3d}{d+2}.
\]

(1.68)

Then there exists a weak solution to Problem (1.64). Moreover, if \( \psi \) satisfies (H1) then there exists a pressure \( p \in L^{r'}(\Omega) \) such that

\[
\int_{\Omega} (S - v \otimes v) \cdot \nabla w \, dx = \int_{\Omega} F \cdot \nabla w \, dx + p \div w \, dx
\]

for all \( w \in W_0^{1,\infty}(\Omega)^d \).

(1.69)

We do not give the proof of Theorem 1.4.2 since it follows along the same lines as in the previous section. The key point here is the assumption (1.68) that guarantees that the convective term is a lower order compact quantity and therefore the monotone operator theory can be simply used.

In the analysis of initial and boundary value problems in non-Newtonian fluid mechanics, there is large interest in establishing the existence result for \( q \)'s (see (1.70)) that are as small as possible. This is motivated by the attempt to capture the response of shear-thinning fluids with lower power-law index. The first extension presented here is the following.
Theorem 1.4.3 Let all the assumptions of Definition 1.4.1 hold. In addition, let \( \psi \) satisfy (H1\(^*\)) and (H2\(^*\)) with
\[
\frac{3d}{d+3} < q < \frac{3d}{d+2} \quad \text{and} \quad r < \frac{dq}{d-q}.
\] (1.70)
Then there exists a weak solution to Problem (1.64). Moreover, there exists a pressure \( p = p_1 + p_2 \) with \( p_1 \in L^{\frac{dq}{d-q}}(\Omega) \) and \( p_2 \in L^r(\Omega) \) such that (1.69) holds.

This is a new result and we give the proof at the end of this section. Since it is based on a new method, we describe it in detail. Here we mention only why the restriction (1.70) is assumed. During the proof we need at least formally to test (1.69) by \( w := v\eta \) with smooth \( \eta \) and after integration by parts to give meaning to all the terms. In particular we need to control \( |w|^3, S v \) and \( pv \) in \( L^1(\Omega) \) and by using the embedding theorem, we see that such a control takes place if the condition (1.70) holds.

The last theorem we present for the steady case is “the optimal” result. Here, optimality means that we cover the range of \( q \)'s for which the weak formulation is meaningful. However, we are not able to treat general \( N \)-functions \( \psi \); we have to restrict ourselves to \( \psi \) of the form \( \psi(s) = \tilde{\psi}(|s|) \).

Theorem 1.4.4 Let all the assumptions of Definition 1.4.1 hold. In addition, let \( \psi \) satisfy (H1\(^*\)) and (H2\(^*\)) with
\[
q > \frac{2d}{d+2}
\] (1.71)
and be of the form
\[
\psi(s) = \tilde{\psi}(|s|).
\] (1.72)
Then there exists a weak solution to Problem (1.64). Moreover, there exists a pressure \( p \in L^1(\Omega) \) such that (1.69) is satisfied.

The proof of Theorem 1.4.4 is based on the combination of the convergence Lemma 1.2.1 and properties of Lipschitz approximation to Sobolev functions. We refer the reader to Bulíček et al. (2011), where the proof for unsteady case with Navier’s slip boundary condition can be found; the proof can be modified in a straightforward manner to establish the result stated in Theorem 1.4.4.

The rest of this section is devoted to the proof of Theorem 1.4.3.
Proof [Proof of Theorem 1.4.3] First, for arbitrary \( n \in \mathbb{N} \), we approximate the problem (1.64) by adding the term \( \frac{1}{n} |v|^{2q'-2} v \) to the left-hand side of (1.64). Thus, we look for a couple \((v^n, S^n)\) solving

\[
v^n \in W^{1, q} \cap L^{2q'}(\Omega)^d, \quad D(v^n) \in L^q(\Omega), \quad S^n \in L^{r'}(\Omega),
\]

\[
\frac{1}{n} \int_\Omega |v^n|^{2q'-2} v^n \cdot w \, dx + \int_\Omega (S^n - v^n \otimes v^n) \cdot D(w) \, dx = \int_\Omega F \cdot D(w) \, dx \quad \text{for all } w \in C^\infty_0(\Omega).
\]

\[
(D(v^n), S^n) \in \mathcal{A} \text{ a.e. in } \Omega.
\]

(1.73)

(1.74)

For fixed \( n \in \mathbb{N} \) the existence of \((v^n, S^n)\) solving (1.73)–(1.75) can be established following the proof of Theorem 1.3.2. Moreover, it can be shown that the solution satisfies the energy identity

\[
\frac{1}{n} \int_\Omega |v^n|^{2q'} \, dx + \int_\Omega S^n \cdot D(v^n) \, dx = \int_\Omega F \cdot D(v^n) \, dx.
\]

(1.76)

Consequently, using the Young inequality, the assumption \((A4)\), the assumptions \((H1^*)\), \((H2^*)\) and the Korn inequality, we observe

\[
\int_\Omega \psi(D(v^n)) + \psi^*(S^n) + \frac{1}{n} |v^n|^{2q'} \, dx + \|\nabla v^n\|_q^2 + \|S^n\|_{r'} \leq C.
\]

(1.77)

For later use we also need to introduce the pressure \( p^n = p^n_1 + p^n_2 \) with \( p^n_1 \in L^{\frac{rd}{r'd-r}}(\Omega) \) and \( p^n_2 \in L^{r'}(\Omega) \) such that

\[
\frac{1}{n} \int_\Omega |v^n|^{2q'-2} v^n \cdot w \, dx + \int_\Omega (S^n - v^n \otimes v^n) \cdot D(w) \, dx = \int_\Omega p^n \, \text{div} w + F \cdot D(w) \, dx \quad \text{for all } w \in W_0^{1, \infty}(\Omega)^d.
\]

(1.78)

The existence of such a pressure can be again shown by using de Rham’s and Nečas’ theorems (see for example Bulíček & Fišerová (2009) for details). Here, we merely recall the uniform estimates that are available

\[
\int_\Omega |p^n_1|^{\frac{rd}{r'd-r}} \, dx \leq C \int_\Omega |v^n|^{\frac{rd}{r'd}} \, dx \leq C,
\]

\[
\int_\Omega |p^n_2|^{r'} \, dx \leq C \int_\Omega |S^n|^{r'} + |F|^{r'} \, dx \leq C.
\]

(1.79)
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As a consequence of (1.77) and (1.79) and the assumption (1.70), we can find a (again labeled by $n$) subsequence of \( \{ \mathbf{v}^n, \mathbf{S}^n, p^n_1, p^n_2 \} \) such that

\[
\begin{align*}
\mathbf{D}(\mathbf{v}^n) & \rightharpoonup \mathbf{D}(\mathbf{v}) \quad \text{weakly}^* \text{ in } L^\psi(\Omega), \\
\mathbf{v}^n & \rightharpoonup \mathbf{v} \quad \text{weakly in } W^{1,q}_0(\Omega), \\
\mathbf{v}^n & \rightrightarrows \mathbf{v} \quad \text{strongly in } L^p(\Omega)^d \cap L^q(\Omega)^d, \\
\mathbf{S}^n & \rightharpoonup \mathbf{S} \quad \text{weakly}^* \text{ in } L^\psi^*(\Omega), \\
\mathbf{S}^n & \rightharpoonup \mathbf{S} \quad \text{weakly in } L^{r'}(\Omega)^d, \\
\frac{1}{n}|\mathbf{v}^n|^{2\gamma-2}\mathbf{v}^n & \rightharpoonup \mathbf{0} \quad \text{strongly in } L^{\frac{2\gamma}{2\gamma-\gamma'}}(\Omega)^d, \\
p^n_2 & \rightharpoonup p_2 \quad \text{weakly in } L^{\frac{2\gamma}{2\gamma-\gamma'}}(\Omega), \\
p^n_1 & \rightharpoonup p_1 \quad \text{weakly in } L^{r'}(\Omega),
\end{align*}
\]

Having all these convergence results, it is then easy to let $n \to \infty$ in (1.78) and to obtain (1.69). Thus, it remains to show that \( (\mathbf{D}(\mathbf{v}), \mathbf{S}) \in \mathcal{A} \) a.e. in $\Omega$.

First, we set $\mathbf{w} := \mathbf{v}^n \vartheta$ in (1.78), where $\vartheta \in D(\Omega)$ (such a setting is possible since all terms in (1.78) are well defined, see also the discussion in the next step, where such a procedure is proved rigorously). Then letting $n \to \infty$ we get

\[
\begin{align*}
\lim_{n \to \infty} \int_\Omega \mathbf{S}^n \cdot \mathbf{D}(\mathbf{v}^n) \vartheta \, dx & + \frac{1}{n} \int_\Omega |\mathbf{v}^n|^{2\gamma-2} \mathbf{v}^n \vartheta \, dx \\
& = \lim_{n \to \infty} \int_\Omega p^n \text{div}(\mathbf{v}^n \vartheta) + (\mathbf{v}^n \otimes \mathbf{v}^n) \cdot \mathbf{D}(\mathbf{v}^n) \vartheta \, dx \\
& + \lim_{n \to \infty} \int_\Omega -\mathbf{S}^n \cdot (\mathbf{v}^n \otimes \nabla \vartheta) + \mathbf{F} \cdot \mathbf{D}(\mathbf{v}^n) \vartheta \, dx.
\end{align*}
\]

Next, we study the limits of all terms on the right hand side. First, using (1.82) and integration by parts, we get (provided that $|\mathbf{v}|^3 \in L^1(\Omega)$)

\[
\begin{align*}
\lim_{n \to \infty} \int_\Omega (\mathbf{v}^n \otimes \mathbf{v}^n) \cdot \mathbf{D}(\mathbf{v}^n) \vartheta \, dx & = \frac{1}{2} \lim_{n \to \infty} \int_\Omega |\mathbf{v}^n|^2 \mathbf{v}^n \cdot \nabla \vartheta \, dx \\
& = \frac{1}{2} \int_\Omega |\mathbf{v}|^2 \mathbf{v} \cdot \nabla \vartheta \, dx.
\end{align*}
\]

Next, with help of (1.82) and (1.84) we deduce that (if $|\mathbf{S}| \in L^1(\Omega)$)

\[
\begin{align*}
\lim_{n \to \infty} \int_\Omega \mathbf{S}^n \cdot (\mathbf{v}^n \otimes \nabla \vartheta) \, dx & = \int_\Omega \mathbf{S} \cdot (\mathbf{v} \otimes \nabla \vartheta) \, dx.
\end{align*}
\]
Also, by using (1.82), (1.86) and (1.87) we get (we require $|p v| \in L^1(\Omega)$)

$$\lim_{n \to \infty} \int_{\Omega} p^n \text{div}(v^n \vartheta) \, dx = \lim_{n \to \infty} \int_{\Omega} p^n v^n \cdot \nabla \vartheta \, dx = \int_{\Omega} p \text{div}(v \vartheta) \, dx$$

and finally with help of (1.82) and (1.80) we have

$$\lim_{n \to \infty} \int_{\Omega} F \cdot D(v^n \vartheta) \, dx = \lim_{n \to \infty} \int_{\Omega} F \cdot (D(v^n) \vartheta + v^n \otimes \nabla \vartheta) \, dx = \int_{\Omega} F \cdot D(v \vartheta) \, dx$$

Hence, since the second term on the left-hand side of (1.88) is nonnegative, we conclude that for any smooth $\vartheta$ the following inequality holds

$$\limsup_{n \to \infty} \int_{\Omega} S \cdot D(v^n) \vartheta \, dx \leq -\int_{\Omega} S \cdot (v \otimes \nabla \vartheta) + \frac{1}{2} |v|^2 v \cdot \nabla \vartheta \, dx + \int_{\Omega} p \text{div}(v \vartheta) + F \cdot D(v \vartheta) \, dx \quad (1.90)$$

Recall that we have already established the validity of (1.69). Next we focus on setting $w := v \vartheta$ in (1.69). However, such a choice is not admissible in general. But making a special choice of $\vartheta$, we can make it possible.

To show this, we define $\Omega_K := \{x \in \Omega : M(|v(x)|) < K\}$, where $M(|v|)$ denotes the maximal function associated with $v$, i.e.,

$$M(|v(x)|) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |v(y)| \, dy.$$ 

Note that $|v(x)| \leq M(|v(x)|)$ a.e. in $\Omega$. Also note that $x \mapsto M(|v(x)|)$ is lower semicontinuous and then $\{\Omega_K\}_{K \in \mathbb{N}}$ is an increasing family of open sets, such that $\bigcup_{K \in \mathbb{N}} \Omega_K = \Omega$. Next, we consider that $\vartheta \in D(\Omega_K)$ is arbitrary and define a mollification $v^\epsilon := \varrho^\epsilon * v$, where $\varrho^\epsilon$ is a standard mollification kernel. Here, we assume that $v$ is extended by 0 outside $\Omega$. First, it is clear that $\text{div} v^\epsilon = 0$ and that (in virtue of (1.81)–(1.82))

$$v^\epsilon \to v \quad \text{strongly in } W^{1,q}(\Omega)^d, \quad (1.91)$$

$$v^\epsilon \to v \quad \text{strongly in } L^r(\Omega)^d \cap L^3(\Omega)^d. \quad (1.92)$$

Moreover, using the Jensen inequality, it is not difficult to observe that

$$\int_{\Omega} \psi(D(v^\epsilon)) \, dx \leq \int_{\Omega} \psi(D(v)) \, dx \leq C$$

and therefore, since $L^\psi(\Omega)$ is reflexive (it follows from (1.70)), we get
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for a (labeled by $\varepsilon$) subsequence $\{v^\varepsilon\}$

$$D(v^\varepsilon) \rightharpoonup D(v) \quad \text{weakly in } L^\psi(\Omega). \quad (1.93)$$

Then we set $w := v^\varepsilon \vartheta$ in (1.69) and let $\varepsilon \to 0_+$. Next, we discuss all terms arising from (1.69). Using (1.92) and (1.93) we observe that

$$\lim_{\varepsilon \to 0_+} \int_{\Omega} S \cdot D(v^\varepsilon \vartheta) \, dx = \int_{\Omega} S \cdot D(v \vartheta) \, dx \quad (1.94)$$

and similarly one can deduce that

$$\lim_{\varepsilon \to 0_+} \int_{\Omega} F \cdot D(v^\varepsilon \vartheta) \, dx = \int_{\Omega} F \cdot D(v \vartheta) \, dx \quad (1.95)$$

Next, using the same convergence results and the fact that $\text{div} \, v^\varepsilon = 0$ we deduce that

$$\lim_{\varepsilon \to 0_+} \int_{\Omega} p \text{div}(v^\varepsilon \vartheta) \, dx = \int_{\Omega} p \text{div}(v \vartheta) \, dx. \quad (1.96)$$

Finally, in the most critical (convective) term, we once again use (1.91)–(1.93) and the choice of $\varphi$. Indeed, since $\varphi \in D(\Omega_K)$ and $v$ is bounded in $\Omega_K$ we have

$$\lim_{\varepsilon \to 0_+} \int_{\Omega} (v \otimes v) \cdot D(v^\varepsilon \vartheta) \, dx = \lim_{\varepsilon \to 0_+} \int_{\Omega_K} (v \otimes v) \cdot D(v^\varepsilon \vartheta) \, dx$$

$$= \int_{\Omega_K} (v \otimes v) \cdot D(v \vartheta) \, dx = \frac{1}{2} \int_{\Omega} |v|^2 \cdot \nabla \vartheta \, dx \quad (1.97)$$

Consequently, using (1.94)–(1.97) in (1.69) and comparing the result with (1.90), we find that

$$\lim_{n \to \infty} \int_{\Omega} S^n \cdot D(v^n) \vartheta \, dx \leq \int_{\Omega} S \cdot D(v) \vartheta \, dx.$$

Since $\vartheta$ is arbitrary we can finally deduce that

$$\lim_{n \to \infty} \int_{\Omega_K} S^n \cdot D(v^n) \, dx \leq \int_{\Omega_K} S \cdot D(v) \, dx,$$

and therefore an application of Lemma 1.2.2 implies that $(D(v), S) \in A_{\Omega_K}$ a.e. in $\Omega_K$. But since $|\Omega \setminus \Omega_K| \to 0$ as $K \to \infty$, we see that $(D(v), S) \in A$ a.e. in $\Omega$.

1.5 Unsteady flows without convection

In this subsection we study another simplification of (1.6)–(1.8). We keep the time derivative in the balance of linear momentum, but omit
the presence of the convective term. Hence we would like to solve the
problem: for a given $Q := (0, T) \times \Omega$ with $\Omega \subset \mathbb{R}^d$ being an open bounded
set, for any $A \subset \mathbb{R}^{d \times d \times \mathbb{R}^{d \times d}}$, $v_0 : \Omega \rightarrow \mathbb{R}^d$ and $F : Q \rightarrow \mathbb{R}^{d \times d}$ such that

$$v_{,t} - \text{div} S + \nabla p = - \text{div} F, \text{ div } v = 0, \quad \text{in } Q,$$

$$(D(v), S) \in A \quad \text{in } Q,$$

$$v = 0 \quad \text{on } (0, T) \times \partial \Omega,$$

$$v(0, \cdot) = v_0 \quad \text{in } \Omega. \quad (1.98)$$

Before defining what we mean by a weak solution to (1.98), we discuss
briefly the possibility of introducing a globally integrable pressure. As in
the preceding subsection we solve first (1.98) projected onto the space
of divergenceless functions. Then a natural question is whether one can
find an integrable pressure $p \in L^1(Q)$ such that (1.98) holds without
the projections. For the steady case we know (as we have seen) that such
a formulation is available provided that $(H1)$–$(H2)$ hold. The same
result cannot be however proved for the unsteady case if $v$ is a weak
solution. Indeed, assuming $(H1)$ and $(H2)$, one can find a pressure $p$
of the form

$$p = p^1 + p^2,$$

with $p^1 \in L^1(Q)$, but with $p^2 \in L^\infty(Q)$ only. Thus, we see that $p$ is
only a distribution. This difficulty can be overcome in several ways.
First, one can modify the problem and replace the Dirichlet boundary
conditions by Navier’s boundary conditions (1.10). If this is the case,
one can conclude that $p^2 \equiv 0$ and consequently that $p \in L^1(Q)$ provided
that $\Omega \in C^{1,1}$, we refer the reader to Bulíček et al. (2007) for details.
Concerning the Dirichlet data, the situation is more delicate. Roughly
speaking, one can show that the function space for $p^2$ is of the same
character as the function space for $v$. Thus, to obtain that $p^2 \in L^p(Q)$
one needs to show that $v_{,t} \in L^p(Q)$, that is, one needs to improve the
regularity of the solution. For such an improvement, one has to assume
a better data and/or additional structure on $A$. To be more precise,
either one has to assume that there exists a potential $\eta : \mathbb{R}^{d \times d} \rightarrow \mathbb{R}_+$
such that if $(S, D(v)) \in A$ then $S \in \partial \eta(D)$ and that $D(v_0) \in L^p(\Omega)$, or
one has to replace the implicit form by the explicit one of the type $S(D)$
with $S$ smooth and require that the initial datum $v_0$ is smooth enough.\(^1\)

\(^1\)In the first case, the desired estimate on the time derivative is obtained by
multiplying the equation by $v_{,t}$. Consequently, the existence of a potential $\eta$ to $S$ is
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Having such difficulties with introducing the pressure, we omit it in what follows and formulate the definitions and all results without the pressure, considering the spaces of divergenceless functions and their duals and using them as test functions in the weak formulation of balance equations.

In what follows we use the standard notation for Bochner spaces. In order to provide a proper meaning to \( v, t \) in a proper dual space we introduce the Gelfand triplet \( V, H, V^* \) as follows

\[
V := W^{3,d}(\Omega)^d \cap W^{1,1}_{0,\text{div}} \cap L^2_{0,\text{div}}, \quad H := L^2_{0,\text{div}}.
\]

Then \( V \) is a separable reflexive Banach space. Since \( V \hookrightarrow H \) densely, then \( H \hookrightarrow V^* \). Thus, for any \( u, w \in V \) we identify

\[
\langle u, w \rangle_{V^*, V} := (u, v)_{H}
\]

and we also find that the following “integration by parts” formula

\[
\langle w, t(w) \rangle_{V^*, V} = \frac{1}{2} \frac{d}{dt} \| w(t) \|_H^2 \quad \text{a.e. in } (0, T)
\]

holds provided that \( w(t) \in V \) and \( w, t(t) \in V^* \) on the time interval of interest.

Having such a Gelfand triplet we finally introduce a definition of a weak solution to (1.98) as follows.

**Definition 1.5.1** Let \( \Omega \subset \mathbb{R}^d \) be an open bounded set with Lipschitz boundary, \( T > 0 \) be arbitrary, \( A \) satisfy the assumptions (A1)–(A4), \( v_0 \in L^2_{0, \text{div}} \) and \( F \in L^{\psi'}(Q) \). We say that a couple \( (v, S) \) is a weak solution to the Problem (1.98) if

\[
\left\{ \begin{align*}
\int_0^T \langle v, t, w \rangle_{V^* \times V} \, dt + \int_Q S \cdot D(w) \, dx \, dt = \int_Q F \cdot D(w) \, dx \, dt, \\
\lim_{t \to 0^+} \| v(t) - v_0 \|_2^2 = 0, \\
(D(v), S) \in A \quad \text{a.e. in } Q.
\end{align*} \right.
\]

required and we need to control \( \int_0^T \eta(D(w_0)) \, dx \). In the second case, the estimate is achieved by applying time derivative to the equations and multiplying the result by \( v, t \). In this case no potential structure of \( S \) is needed but we have to assume that \( S \) is a sufficiently smooth function of \( D \).
Next, we formulate the main theorem of the paper for $\psi$ satisfying (H2). For $\psi$ satisfying only (H1), see remark stated in Subsection 1.3.

**Theorem 1.5.2** Let the assumption of Definition 1.5.1 be fulfilled. Let $\psi$ in addition satisfy (H2). Then there exists a weak solution to Problem (1.98).

**Proof** The proof is very similar to the proof of Theorem 1.3.2, where the steady case was treated. Here, however, other difficulties appear due to the presence of the time derivative. Hence, assume that 

$$\{w_i\}_{i=1}^\infty$$

is a basis of $V$ that is orthogonal in $L^2$. Due to the separability of $V$ such a basis surely exists. Next, we look for a Galerkin approximation of $v$ of the form $v^{\varepsilon,n} := \sum_{i=1}^n c_i^{\varepsilon,n}(t)w_i$ that solves the following system of $n$ ordinary differential equations

$$
\int_\Omega v_i^{\varepsilon,n} \cdot w_i \, dx + \int_\Omega S_i^{\varepsilon,n}(D(v^{\varepsilon,n})) \cdot D(w_i) \, dx = \int_\Omega F \cdot D(w_i) \, dx,
$$

for all $t \in (0,T)$ and all $i = 1,\ldots,n$, \hspace{1cm} (1.104)

$$v^{\varepsilon,n}(0,\cdot) = P^n v_0. \hspace{1cm} (1.105)$$

Here, $S_i^{\varepsilon,n}$ denotes a mollification of the selection $S_i$ (see the proof of Theorem 1.3.2) and $P^n$ denotes the orthogonal projection of $H$ on the span\{$w_1,\ldots,w_n$\}. Note that $\|P^n v_0 - v_0\|_H \to 0$ as $n \to \infty$.

Using the standard Carathéodory theory it is not difficult to obtain a solution to (1.104) at least for some short time interval $[0,T^*)$. This solution can be however extended onto the whole time interval $[0,T]$ provided that we show an uniform bound on $v^{\varepsilon,n}$ that does not depend on the length of the time interval. Moreover, following the proof of Theorem 1.3.2 it is also not difficult to let $\varepsilon \to 0_+$ and to conclude that there exist $(v^n,S^n)$ such that $v^n := \sum_{i=1}^n c_i^n(t)w_i, (D(v^n),S^n) \in A$ almost everywhere in $Q$ and that

$$
\int_\Omega v_i^n \cdot w_i \, dx + \int_\Omega S_i^n \cdot D(w_i) \, dx = \int_\Omega F \cdot D(w_i) \, dx,
$$

for all $t \in (0,T)$ and all $i = 1,\ldots,n$, \hspace{1cm} (1.106)

$$v^n(0,\cdot) = P^n v_0. \hspace{1cm} (1.107)$$

**Obtaining the limit** as $n \to \infty$. First, we derive uniform estimates. For any $t \in (0,T)$ we denote $Q_t := (0,t) \times \Omega$. Then we multiply the $i$-th equation in (1.106) by $c_i^n(t)$, sum the result over $i = 1,\ldots,n$ and...
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integrate over \((0, t)\) to get the energy identity

\[
\frac{1}{2} \|v^n(t)\|_2^2 + \int_{Q_t} S^n \cdot D(v^n) \, dx \, d\tau = \int_{Q_t} F \cdot D(v^n) \, dx \, d\tau + \frac{1}{2} \|P^n v_0\|_2^2.
\]

Consequently, using \((A4)\), the Young inequality and the assumptions on \(v_0\) and \(F\) we find that

\[
\sup_{t \in (0, T)} \|v^n(t)\|_2^2 + \int_Q \psi(D(v^n)) + \psi^*(S^n) \, dx \, dt \leq C.
\]

Therefore, using (1.109), (H2\(^*\)) and the properties\(^1\) of N-functions, we can find a (again labeled by \(n\)) subsequence of \(\{v^n, S^n\}\) such that

\[
\begin{align*}
v^n & \rightharpoonup v \quad \text{weakly* in } L^\infty(0, T; L^2_{0, \text{div}}), \quad (1.110) \\
D(v^n) & \rightharpoonup D(v) \quad \text{weakly* in } L^\psi(Q), \quad (1.111) \\
v^n & \to v \quad \text{weakly in } L^1(0, T; W^{1, q}_0, \text{div}), \quad (1.112) \\
S^n & \rightharpoonup S \quad \text{weakly in } L^\psi^*(Q), \quad (1.113) \\
S^n & \to S \quad \text{weakly in } L^1(0, T; L^1(\Omega)^{d \times d}). \quad (1.114)
\end{align*}
\]

The next step concerns the uniform estimate on the time derivative of \(v^n\) and its weak convergence. To get such an uniform bound we use (1.106) and the fact that \(\{w_i\}_{i \in \mathbb{N}}\) are orthogonal in \(L^2(\Omega)^d\), and observe that

\[
\begin{align*}
\|v^n_t(t)\|_{V^*} = \sup_{u, \|u\|_{V^*} = 1} \langle v^n, P^n(u) \rangle_H & \leq \sup_{u} \int_\Omega (|F| + |S^n|) |\nabla P^n(u)| \, dx \quad (1.115) \\
& \leq C \int_\Omega (|F| + |S^n|) \, dx
\end{align*}
\]

Consequently, integration w.r.t. \(t \in (0, T)\) and (1.109) implies that

\[
\int_0^T \|v^n_t\|_{V^*} \, dt \leq C. \quad (1.116)
\]

Hence, we see that \(v^n_t\) converges in spaces of measures with values in \(V^*\). To improve this convergence result, we use (1.114) that implies that \(S^n\) is uniformly equi-integrable (which is equivalent to weak convergence in

\(^1\)To be precise, we use the fact that \(\psi\) is superlinear at infinity to obtain the weak convergence of \(S^n\) in \(L^1\).
Therefore, (1.115) implies that \( v^n \) is uniformly equi-integrable sequence of functions with values in \( V^* \) and therefore using the reflexivity of \( V^* \) we can conclude

\[
v^n \rightharpoonup v \quad \text{weakly in } L^1(0,T; V^*).
\]  

(1.117)

Hence, using (1.110)–(1.114) and (1.117) and letting \( n \to \infty \) in (1.106) it is easy to to obtain (1.101). Moreover, the the standard method implies that (1.102) holds (see for example Málek & Rajagopal (2010)).

To finish this subsection, we need to show that \((\mathbf{D}(v), \mathbf{S}) \in \mathcal{A}\). To show it, we define for all \( t \in (2\delta, T-2\delta) \) and any \( i, j \in \mathbb{N} \) with \( i \leq j \)

\[
(c^j_i)^\delta := \varrho^\delta * c_i^j, \quad \mathbf{v}^{\delta,j} := \varrho^\delta * \varrho^\delta * \mathbf{v}, \quad \mathbf{v}^{\delta,j} := \varrho^\delta * \mathbf{v}.
\]

Here, \( \varrho^\delta \) is a mollifier of radius \( \delta \) depending only on time and \( * \) denotes the standard convolution operator with respect to the time variable. Next, assume that \( n, i,j \in \mathbb{N} \) such that \( i \leq j \leq n \), multiply the \( i \)-th equation in (1.106) by \((c^j_i)^\delta\) and sum w.r.t. \( i = 1, \ldots, j \) to observe that

\[
\int_{\Omega} v^n_i \cdot v^{\delta,j} \, dx + \int_{\Omega} \mathbf{S}^n \cdot \mathbf{D}(\mathbf{v}^{\delta,j}) \, dx = \int_{\Omega} \mathbf{F} \cdot \mathbf{D}(\mathbf{v}^{\delta,j}) \, dx.
\]  

(1.118)

Next, we integrate (1.118) with respect to time over \((s_0, s_1) \subset (2\delta, T-2\delta)\), and use the definition of the convolution to get

\[
\int_{s_0}^{s_1} \int_{\Omega} v^n_i \cdot v^{\delta,j} \, dx + \int_{\Omega} \mathbf{S}^n \cdot \mathbf{D}(\mathbf{v}^{\delta,j}) \, dx \, dt = \int_{s_0}^{s_1} \int_{\Omega} \mathbf{F} \cdot \mathbf{D}(\mathbf{v}^{\delta,j}) \, dx \, dt.
\]  

(1.119)

Since, we “mollify” the time derivative, we can use (1.110) and (1.113) and let \( n \to \infty \) in (1.119) to obtain

\[
\int_{s_0}^{s_1} \int_{\Omega} \mathbf{v}^\delta \cdot v^{\delta,j} \, dx + \int_{\Omega} \mathbf{S} \cdot \mathbf{D}(\mathbf{v}^{\delta,j}) \, dx \, dt = \int_{s_0}^{s_1} \int_{\Omega} \mathbf{F} \cdot \mathbf{D}(\mathbf{v}^{\delta,j}) \, dx \, dt,
\]  

(1.120)

where

\[
\mathbf{v}^\delta := \varrho^\delta \ast \mathbf{v}, \quad \mathbf{v}^{\delta,j} := \varrho^\delta \ast \varrho^\delta \ast \mathbf{v}.
\]

Our next goal is to let \( j \to \infty \) in (1.120). First, it is a consequence of (1.110) that

\[
\mathbf{v}^{\delta,j} \rightharpoonup \mathbf{v}^\delta \quad \text{weakly in } L^2(s_0, s_1, L^2_{0,\text{div}})
\]  

(1.121)
and consequently,
\[
\lim_{j \to \infty} \int_{s_0}^{s_1} \int_{\Omega} \mathbf{v}^\delta \cdot \mathbf{v}^{\delta,j} \, dxdt = \int_{s_0}^{s_1} \int_{\Omega} \mathbf{v}^\delta \cdot \mathbf{v}^\delta \, dxdt = \frac{1}{2} (\|\mathbf{v}^\delta(s_1)\|_2^2 - \|\mathbf{v}^\delta(s_0)\|_2^2). \tag{1.122}
\]

Moreover, using (1.122) one can deduce that
\[
\mathbf{v}^{\delta,\delta,j} \rightharpoonup \mathbf{v}^{\delta,\delta} \quad \text{weakly in } L^q(s_0, s_1; W^{1,q}_{0,\text{div}}). \tag{1.123}
\]

In addition, using the Jensen inequality we can deduce that (on setting \(Q_{0,1} := (s_0, s_1) \times \Omega\))
\[
\int_{Q_{0,1}} \psi(D(\mathbf{v}^{\delta,\delta,j})) \, dx \, dt = \int_{Q_{0,1}} \psi \left( \int_{\mathbb{R}} D(\mathbf{v}^{\delta,j}(\tau)) g^\delta(t - \tau) \, d\tau \right) \, dx \, dt \\
\leq \int_{Q_{0,1}} \int_{\mathbb{R}} \psi(D(\mathbf{v}^{\delta,j}(\tau))) g^\delta(t - \tau) \, d\tau \, dx \, dt \\
= \int_{Q_{0,1}} \int_{\mathbb{R}} \psi(D(\mathbf{v}^{\delta,j}(t + z))) g^\delta(z) \, dz \, dx \, dt \\
\leq \int_{Q} \psi(D(\mathbf{v}^{\delta,j})) \, dx \, dt \leq \int_{Q} \psi(D(\mathbf{v}^\delta)) \, dx \, dt.
\]

Thus, using the uniqueness of a weak limit, we see that we can extract a (labeled again by \(j\)) subsequence of \(\{\mathbf{v}^{\delta,\delta,j}\}\) such that
\[
D(\mathbf{v}^{\delta,\delta,j}) \rightharpoonup^* D(\mathbf{v}^{\delta,\delta}) \quad \text{weakly}^* \text{ in } L^q((s_0, s_1) \times \Omega). \tag{1.124}
\]

Consequently, letting \(j \to \infty\) in (1.120), using that \(S \in L^{q*}(Q)\), and using (H2) that implies \((L^{q*}(Q))^* = L^q(Q)\), we deduce with help of (1.122) that
\[
\frac{1}{2} \|\mathbf{v}^\delta(s_1)\|_2^2 + \int_{s_0}^{s_1} \int_{\Omega} S \cdot D(\mathbf{v}^{\delta}) \, dx \, dt = \int_{s_0}^{s_1} \int_{\Omega} F \cdot D(\mathbf{v}^{\delta}) \, dx \, dt + \frac{1}{2} \|\mathbf{v}^\delta(s_0)\|_2^2. \tag{1.125}
\]

Next, we let \(\delta \to 0_+\) in (1.125). First, we can deduce that
\[
\mathbf{v}^\delta \rightharpoonup^* \mathbf{v} \quad \text{weakly}^* \text{ in } L^q_{loc}(0, T; L^2_{0,\text{div}}), \tag{1.126}
\]
\[
\mathbf{v}^\delta \to \mathbf{v} \quad \text{strongly in } L^2_{loc}(0, T; L^2_{0,\text{div}}). \tag{1.127}
\]

Moreover, using a procedure similar to that used above we can deduce that
\[
\int_{Q_{0,1}} \psi(D(\mathbf{v}^{\delta,\delta,j})) \, dx \, dt \leq \int_{Q} \psi(D(\mathbf{v})) \, dx \, dt \leq C.
\]
and therefore, from the uniqueness of the weak limit we can find a (again not relabeled) subsequence of $\{v^\delta,\delta\}$ such that

$$D(v^\delta,\delta) \rightharpoonup^* D(v) \quad \text{weakly}^* \text{ in } L^V(Q_{0,1}). \quad (1.128)$$

Hence we see from (1.125) that for a.a. $s_0 < s_1 < T$ there holds

$$\frac{1}{2}\|v(s_1)\|^2_2 + \int_{s_0}^{s_1} \int_\Omega S : D(v) \, dx \, dt$$

$$= \int_{s_0}^{s_1} \int_\Omega F : D(v) \, dx \, dt + \frac{1}{2}\|v(s_0)\|^2_2 \quad (1.129)$$

and taking lim inf as $s_0 \to 0+$ and using (1.102) we deduce that for a.a. $t \in (0,T)$

$$\frac{1}{2}\|v(t)\|^2_2 + \int_{Q_t} S : D(v) \, dx \, dt = \int_{Q_t} F : D(v) \, dx \, dt + \frac{1}{2}\|v_0\|^2_2. \quad (1.130)$$

Finally, we let $n \to \infty$ in (1.108) and appeal to weak-lower semicontinuity to deduce that

$$\limsup_{n \to \infty} \int_{Q_t} S^n : D(v^n) \, dx \, dt$$

$$= \limsup_{n \to \infty} \left( -\frac{1}{2}\|v^n(t)\|^2_2 + \frac{1}{2}\|v^n(0)\|^2_2 + \int_{Q_t} F : D(v^n) \, dx \, d\tau \right)$$

$$\leq -\frac{1}{2}\|v(t)\|^2_2 + \frac{1}{2}\|v_0\|^2_2 + \int_{Q_t} F : D(v) \, dx \, d\tau$$

$$= \int_{Q_t} S : D(v) \, dx \, dt. \quad (1.131)$$

Hence, we can apply Lemma 1.2.2 to observe that $(D(v),S) \in A$ a.e. in $Q_t$ and consequently also in $Q$, so the proof is complete. \qed

1.6 Full problem

In the last subsection we discuss the existence result for the full problem. Here the word “full” means that we want to solve the problem: for a given $Q := (0,T) \times \Omega$ with $\Omega \subset \mathbb{R}^d$ being an open bounded set, for any $A \subset \mathbb{R}_{sym}^{d \times d} \times \mathbb{R}_{sym}^{d \times d}$, $v_0 : \Omega \to \mathbb{R}^d$ and $F : Q \to \mathbb{R}_{sym}^{d \times d}$ to find
(v, p, S) : Q → R^d × R × R^{d×d}_{sym} such that
\[ v_t - \text{div}(v \otimes v) - \text{div} S + \nabla p = -\text{div} F, \quad \text{in } Q, \]
\[ \text{div} v = 0, \quad \text{in } Q, \]
\[ (D(v), S) \in A \quad \text{in } Q, \quad (1.132) \]
\[ v = 0 \quad \text{on } (0, T) \times \partial \Omega, \]
\[ v(0, \cdot) = v_0 \quad \text{in } \Omega. \]

**Definition 1.6.1** Let \( \Omega \subset \mathbb{R}^d \) be an open bounded set with Lipschitz boundary, \( T > 0 \) be arbitrary, \( A \) satisfy the assumptions (A1)–(A4), \( v_0 \in L^2_0 \), \( \text{div} \) and \( F \in L^{\infty}(Q) \). We say that a couple \((v, S)\) is a weak solution to Problem (1.132) if

\[ v \in C_w(0, T; H) \cap L^1(0, T; W^{1,1}_{0, \text{div}}), D(v) \in L^{\psi}(Q), S \in L^{\psi^*}(Q), \quad (1.133) \]
\[ \int_0^T \langle v_t, w \rangle_{V^* \times V} dt + \int_Q (S - v \otimes v) \cdot D(w) \, dx \, dt \]
\[ = \int_Q F \cdot D(w) \, dx \, dt, \quad \text{for all } w \in L^\infty(0, T; V), \quad (1.134) \]
\[ \lim_{t \to t_0} \| v(t) - v_0 \|^2_2 = 0, \quad (1.135) \]
\[ (D(v), S) \in A \quad \text{a.e. in } Q. \quad (1.136) \]

Next, we formulate the existence theorem for the full problem. As discussed in preceding subsection, we need to consider several restrictions for the \( \psi \) function. First, similarly as in Subsection 1.4 we need to be able to handle the convective term, therefore we need to assume that \( (H2^\ast) \) holds.

**Theorem 1.6.2** Let the assumption of Definition 1.6.1 be fulfilled. Let \( \psi \) in addition be of the form \( \psi(s) = \tilde{\psi}(|s|) \) and satisfy \( (H1) \) and \( (H2) \) and

\[ q > \frac{2d}{d + 2}, \quad (1.137) \]

Then there exists a weak solution to Problem (1.132).

We do not prove Theorem 1.6.2 here. The same result was proved in Bulíček et al. (2011) for Navier’s boundary conditions. However, combining the method developed in Bulíček et al. (2011) with those described in Wolf (2007) one can prove an identical result also for homogeneous Dirichlet data.
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