Role of pressure dependent viscosity in measurements with falling cylinder viscometer

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Abstract The falling cylinder viscometer is frequently used in measuring the dependence of the viscosity on the pressure. The viscosity is calculated using an indirect procedure, namely by appealing to the linear relation between the time taken for the fall and the viscosity that is a consequence of assuming that inertial effects can be neglected. Under certain assumptions, the coefficient of proportionality can be derived analytically, and one gets the classical formula for the viscosity as a function of geometric parameters of the device, density of the fluid and the sinker, gravitational acceleration, and the distance and the time of the fall. Although the classical formula is valid only for fluids with constant viscosity, it is indiscriminately used even for fluids with pressure dependent viscosity. We investigate the role of variable viscosity, and we derive a heuristic correction to the classical formula for the case of fluids with pressure dependent viscosities. The systematic error introduced by the unwarranted application of the classical formula for fluids with pressure dependent viscosity is analysed, and it is shown it is measurable and it can in some cases significantly influence the experimental results.

1 Introduction

Starting from the pioneering experiments by Bridgman (1925, 1926, 1927), the falling cylinder viscometer is frequently used in measuring the dependence of the viscosity on the pressure. The reason is that in comparison to the other types of viscometers, the design of the falling cylinder viscometer allows one to easily maintain high pressures at which the experiments need to be carried out, see for example the discussion in Viswanath et al. (2007). Further, the measurements using the falling cylinder viscometer require a relatively small volume of the fluid that is being experimented on.

In this note we discuss a fundamental inconsistency in the classical approach of measuring the dependence of the viscosity on the pressure using the falling cylinder viscometer. The inconsistency in the classical approach has not been noted and discussed even in the recent studies, see for example Huang et al. (1966); Irving and Barlow (1971); Sen and Kiran (1990); Kiran and Sen (1992); Dindar and Kiran (2002a,b); Harris et al. (2005, 2006, 2007); Harris and Bair (2007); Schaschke et al. (2006, 2008); Zeng and Schaschke (2009); Paton and Schaschke (2009). The inconsistency may have a significant impact on the accuracy of the measurements, and rests in the fact that the pressure and conse-
quently the viscosity is not—contrary to what is assumed—uniform in the device\(^1\), see the discussion below.

Let us now briefly recall the classical theory for the falling cylinder viscometer. The principle of the measurement of the viscosity is the following, see Bridgman (1931):

The general idea of the method is very simple; in a steel cylinder [...] filled with the liquid under investigation, there is a steel cylindrical weight separated from the wall of the cylinder by a narrow annular space. The time of vertical fall of the weight from one end of the cylinder to the other is determined; the time is a measure of the viscosity.

The phrase “time is a measure of viscosity” means that the time the sinker needs to descend a fixed distance \( L \) is proportional to the viscosity. Indeed, under certain simplifying assumptions, it can be show that the falling cylinder (sinker) attains, after a certain transient period, a terminal velocity. In this regime, the viscosity can be calculated form the known (measured) parameters by the following well-known formula

\[
\mu_{\text{ref}} = \frac{(\rho_s - \rho_f) R_1^2}{2} \left( \ln \frac{R_2}{R_1} - \frac{R_2^2 - R_1^2}{R_2^2 + R_1^2} \right) \frac{T}{L} g, \quad (1.1)
\]

where \( \rho_s \) and \( \rho_f \) denote the densities of the sinker and fluid respectively, \( R_1 \) and \( R_2 \) denote radii of the sinker and the tube, see Figure 1 for the notation, \( T \) is the time taken to fall through the distance \( L \), and \( g \) is the gravitational acceleration. (For the derivation, see for example Lohrenz et al. (1960) and Cristescu et al. (2002).)

The derivation of (1.1) is based on the assumption that the constitutive relation\(^2\) for the fluid reads

\[
\mathbb{T} = -p \mathbb{I} + 2\mu \mathbb{D}, \quad (1.2)
\]

where \( \mathbb{T} \) is the Cauchy stress, \( p \) is the pressure and \( \mathbb{D} = \frac{1}{2}(\nabla v + \nabla v^\top) \) is the symmetric part of the velocity gradient. (Further it is assumed that the boundary conditions on the sinker and tube walls are the classical no-slip boundary conditions.)

The other key assumption in the derivation of (1.1) is that the entrance and exit effects can be ignored. This corresponds, among others, to the requirement that the gap between the sinker and the tube is small and that the height of the sinker is large compared to the diameter of the sinker. This requirement indeed guarantees the negligibility of the entrance and exit effects, see for example Lohrenz et al. (1960), Chen and Swift (1972), Wehbeh et al. (1993) and Gui and Irvine (1994). (See also the analytical investigation of the end effects by Borisov (1998).) If the end effects are not negligible, the right rand side of (1.1) is usually multiplied by a correction factor empirically found by calibrating the device using a fluid of known viscosity.

Simply when Bridgman (1926) refers to the pressure, he is referring to the pressure in the compressing medium and not in the fluid that is being tested. He is making two tacit assumptions, the first that the pressure in the fluid is uniform and second that this uniform pressure is the pressure in the compressing medium.

\(1\) Moreover when Bridgman (1926) refers to the pressure, he is referring to the pressure in the compressing medium and not in the fluid that is being tested. He is making two tacit assumptions, the first that the pressure in the fluid is uniform and second that this uniform pressure is the pressure in the compressing medium.

\(2\) We consider incompressible fluids.

Note that (1.3) is fundamentally different from the classical Navier-Stokes fluid model (1.2) since the former is an implicit relation between \( \mathbb{T} \) and \( \mathbb{D} \)—note that \( p = -\frac{1}{\rho} \mathbb{T} \mathbb{I} \), while the latter is an explicit expression for the \( \mathbb{T} \) in terms of \( \mathbb{D} \), see Rajagopal (2003, 2006); Rajagopal and Srinivasa (2008) for a detailed discussion of implicit constitutive theories.
for a fluid with pressure dependent viscosity. But the tacit assumption of the applicability of (1.1) for fluids with pressure dependent viscosity must be based on an \textit{a priori} estimate of the error that guarantees that the error is ignorable whereby the measurements are reasonable—this is the objective of the current study.

Further, the estimate of the error will give one the possibility to discuss its relevance compared to the other sources of error such as the error due to eccentricity, see for example Chen et al. (1968), end effects, see the citations above, precision in measurements of the dimensions of the sinker and the tube, see Wehbeh et al. (1993), compressibility of the sinker, see Bridgman (1926) and Schaschke et al. (2006), and the other effects.

### 2 Equation of motion for the sinker

The equation of motion for the sinker reads

\[
m \frac{d^2 x}{dt^2} = F_k e_z + F_f,
\]

where \(x\) denotes the position of the centre of mass, \(m = \pi R_k^2 h \rho_s\) the mass of the sinker, \(F_k = -mg\) is the gravitational force and \(F_f = \int_{\partial C} T n dS\) is the force exerted by the fluid on the sinker due to the flow. (Symbol \(n\) denotes the unit outward normal and \(\partial C\) denotes the surface of the sinker.)

Let us now make the standard assumptions and derive a formula for \(F_f\). We assume that the sinker moves only in the vertical direction and remains at the centre of the tube. Further, we ignore the end effects, which is a plausible assumption in the classical case, see the discussion above. This effectively means that the fluid is assumed to be at rest below and above the sinker. Finally, the motion of the fluid in the annular gap is assumed to be the same as if we considered the steady motion of the fluid between two infinite concentric cylinders. This means that even if the inner cylinder accelerates the flow field in the gap always corresponds to the steady flow between the infinite cylinders actuated by the inner cylinder moving with the actual velocity\(^4\).

The assumptions imply that the velocity of the fluid in the annular gap can be considered in the form \(v = v^\parallel (r) e_z\). The pressure in the gap is then assumed\(^5\) to be a function of \(z\) and \(r\) only, \(p = p(z, r)\). Under these assumptions the formula for \(F_f\) simplifies to

\[
\begin{align*}
\int_{\partial C} T n dS &= - \int_{\varphi=0}^{2\pi} \int_{r=0}^{R_1} T |_{z=z_h} e_z r dr d\varphi \\
+ \int_{\varphi=0}^{2\pi} \int_{r=0}^{R_1} T |_{z=z_h+h} e_z r dr d\varphi \\
+ \int_{\varphi=0}^{2\pi} \int_{z=z_h}^{z_h+h} T |_{r=R_1} e_z R_1 dz d\varphi &= (F_b + F_d) e_z,
\end{align*}
\]

(2.2)

where \(F_b\) denotes the drag and \(F_d\) the “buoyancy”.

\[
\begin{align*}
F_b e_z &= \int_{\varphi=0}^{2\pi} \int_{z=z_h}^{z_h+h} T |_{r=R_1} e_z R_1 dz d\varphi \\
F_d e_z &= \pi R_1^2 \left( p |_{z=z_h} - p |_{z=z_h+h} \right) e_z.
\end{align*}
\]

(2.3a) (2.3b)

Formula (2.3a) for the drag reduces, for the considered constitutive equations (1.3a) and (1.3b), to

\[
F_d = \int_{\varphi=0}^{2\pi} \int_{z=z_h}^{z_h+h} \mu ref \frac{\partial v^\parallel}{\partial r} \frac{dz}{dr} |_{r=R_1} R_1 dz d\varphi.
\]

(2.4)

---

\(^4\) A full justification of the assumptions (the neglect of the end effects, quasi-steady flow field) and verification of the negligence of the error introduced can be made only after the direct numerical simulation of the problem. We shall however focus on the analytical treatment of the problem under the standard simplifying assumptions, our aim is to derive a heuristic formula that provides a clue concerning the role of pressure dependent viscosity in measurements using the falling cylinder viscometer. (Concerning the drag in fluids with pressure dependent viscosity, see also the study by Chung and Vaidya (2010).)

The phrase “infinitely long cylinder” means that the cylinders are sufficiently long. (In infinite cylinders, the pressure and the viscosity will become infinite.) Consequently the solution in the large part of the cylindrical gap can be considered to be the solution in the form of unidirectional flow. Let us mention that the vague phrase “sufficiently long” can be treated in a rigorous manner—for a discussion on spatial decay estimates from “non-unidirectional flow” to “unidirectional flow” in steady pipe flow of Navier–Stokes fluids see Ames and Payne (1989).

\(^5\) We are essentially seeking a semi-inverse solution with such a special structure. It is important to recognize two facts. First, such a solution might not exist. Second, and more importantly, since the governing equations for the fluid flow are nonlinear, it is possible that there are other solutions that do not have the form chosen. In fact, as the problem is really a flow in a finite domain, it is necessary to solve the full partial differential equations in the flow domain and to verify that the approximate solution that we seek is close to the solution to the full problem. Unfortunately, this is not an easy task, especially for a fluid with pressure dependent viscosity. (See also Footnote 4 and Footnote 9.)
If the pressure and velocity field are known, one can substitute (2.3b) and (2.4) to (2.1) and get the following equation for the vertical position of the bottom of the sinker $z_b$,

$$\frac{d^2 z_b}{dt^2} = F_g + F_b + F_d. \tag{2.5}$$

Once the velocity and pressure field for the flow between two infinite concentric cylinders are known, (2.5) can be solved to find $z_b$ as a function of $t$.

### 3 Flow between infinite concentric cylinders

The governing equations for the motion of the fluid

$$\rho \frac{d}{dt} \mathbf{v} = \text{div} \mathbf{T} + \rho \mathbf{g}, \quad \text{div} \mathbf{v} = 0, \tag{3.1}$$

reduce, for the particular constitutive equation (1.3) and under the assumptions discussed above, to two nontrivial equations for the balance of momentum. Balance of momentum in the $e_r$ direction is given by

$$0 = -\frac{\partial p}{\partial r} + \mu_{ref} \varepsilon^0 (\rho - \rho_{ref}) \frac{\partial}{\partial z} \frac{dv^r}{dr}, \tag{3.2a}$$

and in the $e_z$ direction by

$$0 = -\frac{\partial p}{\partial z} + \mu_{ref} \varepsilon^0 (\rho - \rho_{ref}) \frac{\partial}{\partial r} \frac{dv^z}{dr} + \mu_{ref} \varepsilon^0 (\rho - \rho_{ref}) \frac{1}{r} \frac{d}{dr} \left( \frac{r \frac{dv^z}{dr}}{r} \right) - \rho \mathbf{g}. \tag{3.2b}$$

The solution must satisfy the following conditions\(^6\),

$$v^r|_{r=R_1} = -V, \quad v^z|_{r=R_2} = 0, \tag{3.3a}$$

$$\int_{\varphi=0}^{2\pi} \int_{r=R_1}^{R_2} v^r r \, dr \, d\varphi = Q, \quad p|_{r=R_1, z=z_0+h} = P, \tag{3.3b}$$

where $V$ is the velocity of the sinker, $Q$ is the volumetric flow rate through the slit and $P$ is the pressure at the given point. The first two conditions are the no-slip boundary conditions on the lateral walls of the sinker and the tube, the third condition specifies the required volumetric flow, and the last condition prescribes the pressure at “outlet”\(^7\).

Although there exist analytical solutions for flow of fluids with pressure dependent viscosity in the cylindrical geometry\(^8\), see Denn (1981), Renardy (2003) and Vasudeviah and Rajagopal (2005), it seems that a solution for the flow in the annular gap under the action of gravity (in the direction of the cylinder’s axes) can not be easily obtained. We will therefore solve the governing equations by a perturbation method—this is convenient and sufficient for our purpose since we want to discuss the departures from the classical case\(^9\).

The perturbation parameter is $\beta$. We shall in fact do the perturbation expansion of the dimensionless version of the governing equations, and the small parameter shall be $\beta^* = \beta p_{char}$ where $p_{char}$ is a characteristic pressure difference, say $h \beta g$, used for the non-dimensionalisation of $p - p_{ref}$. This will however introduce an unnecessary complexity to the notation. Further the perturbation will be formal without any analysis of the convergence of the expansion series. (Here we follow the common practice in the engineering applications of the perturbation method.)

Expanding the velocity and the pressure in powers of $\beta$, $v^r = v^r_0 + \beta v^r_1 + \beta^2 v^r_2 + \cdots, p = p_0 + \beta p_1 + \beta^2 p_2 + \cdots$, we get the following system of partial differential equations for the zeroth order

$$0 = -\frac{\partial p_0}{\partial r}, \tag{3.4a}$$

$$0 = -\frac{\partial p_0}{\partial z} + \frac{1}{\mu_{ref}} \frac{1}{r} \frac{d}{dr} \left( r \frac{dv^z_0}{dr} \right) - \rho \mathbf{g}, \tag{3.4b}$$

and the following equations for the first order correction

$$0 = -\frac{\partial p_1}{\partial r} + \mu_{ref} \frac{1}{\mu_{ref}} \frac{1}{r} \frac{d}{dr} \left( r \frac{dv^z_0}{dr} \right), \tag{3.5a}$$

and

$$0 = -\frac{\partial p_0}{\partial z} + \mu_{ref} \frac{1}{\mu_{ref}} \frac{1}{r} \frac{d}{dr} \left( r \frac{dv^z_0}{dr} \right) + \mu_{ref} \left( p_0 - p_{ref} \right) \frac{1}{r} \frac{d}{dr} \left( r \frac{dv^z_0}{dr} \right). \tag{3.5b}$$

The zeroth order solution $(v^r_0, p_0)$ must satisfy the conditions

$$v^r_0|_{r=R_1} = -V, \quad v^r_0|_{r=R_2} = 0, \tag{3.6a}$$

$$2\pi \int_{r=R_1}^{R_2} v^r_0 r \, dr \, d\varphi = Q, \quad p_0|_{r=R_1, z=z_0+h} = P. \tag{3.6b}$$

\(^6\) We assume that that the sinker is descending, hence the velocity of the sinker is, due to the choice of the coordinate system, negative. On the other hand, the net volumetric flux through the annular gap must be positive, since the fluid is, on the average, flowing upwards.

\(^7\) We can not expect the pressure to be uniform at outlet, thus in general $p(r_1, z) \neq p_0(R_2, z)$ for $r_1 \neq R_2$. Consequently the point $r$ where the pressure is specified is important. Here we assume that the radial variation of the pressure can be, in specifying the boundary conditions, neglected.

\(^8\) For analytical solution in other geometries see Hron et al. (2001) and the following discussion in Sudov and Tran (2008) and Hron et al. (2011). Another analytical solution can be found in Prüša (2010).

\(^9\) Note that Renardy (2003) has shown that the viscosity of the from (1.3b) is the only form of pressure dependent viscosity that admits, in Hagen–Poiseuille type flow, a solution in the form of the unidirectional flow. One should therefore consider the possibility that the solution for the current problem does not exist in the chosen form. Nevertheless, if we use the perturbation method, we can in principle assume that non-unidirectionality of the flow is a higher order effect, and apply the same procedure as above. Unfortunately a rigorous mathematical theory that can help in this case is missing, see for example the discussion in Bulíček et al. (2009).
and the first order correction \((v_1^z, p_1)\) shall satisfy
\[
\begin{align*}
  v_1^z |_{r=R_1} &= 0, \quad v_1^z |_{r=R_2} = 0, \quad (3.7a) \\
  2\pi \int_{r=R_1}^{R_2} v_1^z \, r \, dr &= 0, \quad p_1 |_{r=R_1, z=zh+h} = 0. \quad (3.7b)
\end{align*}
\]

Obviously, solution to (3.4) and (3.5) respectively is a function of the velocity \(V\) of the sinker, the pressure \(P\) at the outlet and the volumetric flow rate \(Q\).

The solution to the zeroth order equations is given by the following well-known formulae
\[
\begin{align*}
  v_0 &= \frac{C + \rho g \, r^2}{\mu_{ref}} + C_1 \ln r + C_2, \quad (3.8a) \\
  p_0 &= C z + \bar{C}, \quad (3.8b)
\end{align*}
\]
where
\[
\begin{align*}
  C' &= \gamma Q + \gamma V \, V, \\
  \bar{C} &= P - C (zh + h), \\
  C_1 &= -\frac{\ln (R_2^2 - R_1^2)}{2 \ln R^2_1 R^2_2}. \quad (3.9b)
\end{align*}
\]
(The formula for \(C_2\) can be found as well, but we do not need it in the subsequent calculations.) Symbols \(\gamma Q\) and \(\gamma V\) denote
\[
\begin{align*}
  \gamma Q &= \frac{2 \ln \frac{R_2}{R_1}}{\pi (R_2^2 - R_1^2) (R_2^2 - R_1^2) - (R_2^2 + R_1^2) \ln \frac{R_2}{R_1}}, \quad (3.10a) \\
  \gamma V &= \frac{(R_2^2 - R_1^2) - 2 R_1^2 \ln \frac{R_2}{R_1}}{(R_2^2 - R_1^2) - (R_2^2 + R_1^2) \ln \frac{R_2}{R_1}}. \quad (3.10b)
\end{align*}
\]

Having a solution to the zeroth order equations, it is easy to show that the first order correction reads
\[
\begin{align*}
  v_1^z &= 0, \quad (3.11a) \\
  p_1 (r_1, z) &= 4C' \mu_{ref} \int_{z=zh+h}^{z} ((C \xi + \bar{C}) - p_{ref}) \, d\xi. \quad (3.11b)
\end{align*}
\]

4 Calculation of drag and buoyancy

Now we can go back to (2.3b) and (2.4) and find the zeroth and first order formulae for the drag and “buoyancy”. Expanding the buoyancy and drag in powers of \(\beta\) yields
\[
\begin{align*}
  F_d(V, Q, P) &= F_{d0}(V, Q, P) + \beta F_{d1}(V, Q, P), \quad (4.1a) \\
  F_b(V, Q, P) &= F_{b0}(V, Q, P) + \beta F_{b1}(V, Q, P). \quad (4.1b)
\end{align*}
\]

where
\[
\begin{align*}
  F_{b0}(V, Q, P) &= \pi R_1^2 \left( p_0 \big|_{r=R_1, z=zh} - p_0 \big|_{r=R_1, z=zh+h} \right), \quad (4.2a) \\
  F_{b1}(V, Q, P) &= \pi R_1^2 \left( p_1 \big|_{r=R_1, z=zh} - p_1 \big|_{r=R_1, z=zh+h} \right), \quad (4.2b) \\
  F_{d0}(V, Q, P) &= \int_{r=0}^{2\pi} \int_{z=zh}^{zh+h} \mu_{ref} \frac{dv_1^z}{dr} \bigg|_{r=R_1} R_1 \, d\xi \, dz, \quad (4.2c) \\
  F_{d1}(V, Q, P) &= \int_{r=0}^{2\pi} \int_{z=zh}^{zh+h} \mu_{ref} (p_0 - p_{ref}) \frac{dv_1^z}{dr} \bigg|_{r=R_1} R_1 \, d\xi \, dz \quad + \int_{r=0}^{2\pi} \int_{z=zh}^{zh+h} \mu_{ref} \left( p_0 - p_{ref} \right) \frac{dv_1^z}{dr} \bigg|_{r=R_1} R_1 \, d\xi \, dz. \quad (4.2d)
\end{align*}
\]

Here we anticipate that the drag and the buoyancy to depend on the velocity \(V\), volumetric flow rate \(Q\) and the pressure \(P\).

Substituting the formulae for the zeroth (3.8) and first order approximations (3.11) of the pressure and velocity field to (4.2) leads to
\[
\begin{align*}
  F_b(V, Q, P) &= F_a(V, Q, P) - \bar{F}_{b0}(V, Q, P) (1 + \beta \gamma \beta) \quad (4.3a) \\
  F_d(V, Q, P) &= F_{d0}(V, Q, P) (1 + \beta \gamma \beta). \quad (4.3b)
\end{align*}
\]

Here we denote
\[
\gamma \beta = P - p_{ref} - 2 \mu_{ref} h (\gamma Q + \gamma V) + \frac{h \rho g}{2}, \quad (4.4a)
\]
and
\[
\begin{align*}
  F_a(V, Q, P) &= \pi R_1^2 h \rho g, \\
  \bar{F}_{b0}(V, Q, P) &= 4 \mu_{ref} \pi R_1^2 h (\gamma Q + \gamma V), \quad (4.4c)
\end{align*}
\]
and finally
\[
\begin{align*}
  F_{d0}(V, Q, P) &= 2 \pi \mu_{ref} h \left( R_1^2 - \frac{R_2^2 - R_1^2}{\ln R_2^2} \right) (\gamma Q + \gamma V) - \frac{V}{\ln R_2^2}. \quad (4.4d)
\end{align*}
\]

Obviously (4.4b) is the part of buoyancy force due to the classical Archimedes law for buoyancy in the stationary fluid, and the other term \(\bar{F}_{b0}(1 + \beta \gamma \beta)\) is the buoyancy force due to the additional vertical pressure gradient caused by the motion of the fluid.
It remains to specify the velocity of the sinker $V$, the volumetric flow rate $Q$, and the pressure $P$ at the outlet. In virtue of the sign convention, the velocity $V$ is given by

$$V = -\frac{dz_b}{dt}. \tag{4.5a}$$

The volumetric flow rate through the slit between the sinker and the tube must be, in virtue of the incompressibility constraint, given by the formula

$$Q = -\pi R_1^2 \frac{dz_b}{dt}. \tag{4.5b}$$

Further, the pressure $P$ at $z = z_b + h$ (outlet) can be, ignoring the end effects, approximated by the hydrostatic pressure at the given depth. The pressure at the top of the sinker has two components—the confining pressure $P_a$ and the hydrostatic pressure,

$$P = P_a + (H - h - z_b) \rho_t g. \tag{4.5c}$$

Now we can substitute (4.5) to (4.3b) and (4.3a) and get an expression for the drag and buoyancy in terms of the position of the base of the sinker $z_b$ and the velocity of the sinker $\frac{dz_b}{dt}$. Note that (4.5a) and (4.5b) imply $\gamma Q + \gamma V = -\gamma C \frac{dz_b}{dt}$, where

$$\gamma C = \frac{1}{(R_2^2 - R_1^2) - (R_2^2 + R_1^2) \ln \frac{R_2}{R_1}}. \tag{4.6}$$

Further, if we go back to (4.4d) we see that

$$F_d = 2\pi \mu_{ref} h (R_2^2 - R_1^2) \gamma C \frac{dz_b}{dt}. \tag{4.7}$$

Finally, the factor $\gamma \beta$ that appears in (4.3a) and (4.3b) reads

$$\gamma \beta = \gamma P - \rho_t g z_b + 2\mu_{ref} h \gamma C \frac{dz_b}{dt}, \tag{4.8}$$

where

$$\gamma P = (P_a - \rho_{ref}) + \left( H - \frac{h}{2} \right) \rho_t g.$$

The first order approximation for the drag $F_d$, after substituting for $V$, $Q$ and $P$ into (4.3b), is

$$F_d = 2\pi \mu_{ref} h (R_2^2 - R_1^2) \gamma C \frac{dz_b}{dt} \times \left( 1 + \beta \left( \gamma P - \rho_t g z_b + 2\mu_{ref} h \gamma C \frac{dz_b}{dt} \right) \right). \tag{4.9a}$$

and the first order approximation for the buoyancy is

$$F_b = \pi R_2^2 h \rho_t g + 4\pi \mu_{ref} h R_2^2 \gamma C \frac{dz_b}{dt} \times \left( 1 + \beta \left( \gamma P - \rho_t g z_b + 2\mu_{ref} h \gamma C \frac{dz_b}{dt} \right) \right). \tag{4.9b}$$

If $\beta \neq 0$, formulae (4.9) for the drag and buoyancy respectively reduce to the classical formulae for the Navier–Stokes fluid and the drag is proportional to the velocity of the sinker. If $\beta = 0$, then the drag increases with depth and it is a nonlinear function of the velocity.

Using the formulae for the drag (4.9a) and buoyancy (4.9b), the equation of motion (2.5) reads

$$\frac{d^2 z_b}{dt^2} = \frac{\rho_t - \rho_s}{\rho_s} g + \frac{2\mu_{ref}}{R_1^4 \rho_s} \left( R_2^2 + R_1^2 \right) \gamma C \frac{dz_b}{dt} \times \left( 1 + \beta \left( \gamma P - \rho_t g z_b + 2\mu_{ref} h \gamma C \frac{dz_b}{dt} \right) \right). \tag{4.10}$$

If $\beta = 0$ then the sinker reaches, for $t \to +\infty$, a negative (sinker is descending) terminal velocity $v_{\text{term}}^\beta$, and the terminal velocity is given by the well-known formula

$$v_{\text{term}}^\beta = \left( \rho_s - \rho_t \right) \frac{R_2^2 \left( \left( R_2^2 - R_1^2 \right) - \left( R_2^2 + R_1^2 \right) \ln \frac{R_2}{R_1} \right) \pi}{2\mu_{ref}} \frac{R_2^2 + R_1^2}{R_1^2}. \tag{4.11}$$

Further, if $L$ denotes the distance covered in time interval $T$, then it immediately follows that the viscosity can be calculated by the classical formula (1.1). (Note that the distance $L$ must be taken, due to the sign convention we use, with the minus sign—the sinker is descending.)

If $\beta \neq 0$ then we deal with a nonlinear ordinary differential equation, and the sinker does not attain a nonzero terminal velocity.

## 5 Results

Let us now estimate how strong the effects are due to the pressure dependent viscosity in a real setting. Table 1 summarizes dimensions of some falling cylinder type viscometers for which we were able to find all the necessary geometrical parameters. Table 2 illustrates parameter values in formula (1.3b) for some real fluids. (Note that although the viscosity range is for the given fluids very broad, the magnitude of the pressure viscosity coefficient $\beta$ for all these fluids is of the same order. The densities take values in the range from 800 kg/m$^3$ to 2500 kg/m$^3$.)

Let us assume that the confining pressure$^{10}$ is fixed at the value $P_a$. This pressure will be used as the reference pressure $P_{ref}$ in the formula (1.3b). Choice $P_{ref} = P_a$ yields

$$\mu = \mu_0 e^{\beta P_a e^{\beta (p - P_a)}} = \mu_{ref} e^{\beta (p - P_{ref})}, \tag{5.1}$$

where $\mu_{ref} = \mu_0 e^{\beta P_a}$. The term $p - P_{ref}$ now corresponds to the pressure variation in the device relative to the ambient pressure. As we have discussed before, it is not considered (or even thought over) in the experiments.

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$^{10}$ The pressure measured by a sensor at the top of the outer cylinder, thus the pressure at height $H$. 

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Let us now compare two cases. In the first one we will consider a sinker falling in a fluid with constant viscosity \( \mu_{\text{ref}} \). This corresponds to the classical treatment of the problem. The second case will be the same sinker falling in a fluid with pressure dependent viscosity of the form (5.1). This corresponds to the treatment where the pressure variation in the device is not neglected.

The time of the fall is being measured from the instant when the sinker reaches the depth \( z_b|_{t=0} = H - 11h \), and we will assume that at this point the sinker has attained the velocity \( \frac{d}{dt} v_{\text{term}}|_{t=0} = \frac{\beta}{\rho} \beta_{\text{term}} \), thus \( \frac{d}{dt} v_t \) of the terminal velocity in a Navier–Stokes fluid, see formula (4.11). This means that the transient effects due to the release of the sinker from rest are assumed to be smeared out after the fall of length corresponding to ten heights of the sinker. These assumptions are consistent with the experimental evidence\(^{11}\), and after reaching this depth the sinker can be assumed to enter the regime where the motion is described by the equation (4.10).

Now we will, using (4.10), compute the time needed to fall through the distance \( L \). We will investigate the relative difference between the time of fall \( t_{\text{NSE}} \) obtained by the “constant” viscosity model \( \mu = \mu_{\text{ref}} \) and the time of the fall \( t_{\text{pressure}} \) obtained by the correct\(^{12}\) model \( \mu = \mu_{\text{ref}} e^{\beta (p - p_{\text{ref}})} \). The relative difference \( \frac{t_{\text{pressure}} - t_{\text{NSE}}}{t_{\text{NSE}}} \) is shown in Figure 2.

First, we see that the fall time predicted by the model with pressure dependent viscosity is higher that the fall time for the model with constant viscosity. Using the classical formula (1.1) instead of a correct variant for fluids with pressure dependent viscosity therefore leads to an overestimation of the viscosity \( \mu_{\text{ref}} \) at the reference pressure \( p_{\text{ref}} \). Further, we see that for the particular geometry used in the computations, the difference between fall times predicted by the two models is far below 1% unless we deal with high density fluids with relatively high pressure viscosity coefficient \( \beta \). (High density fluids are for example liquid bromine—3120 kg/m\(^3\) at 25°C—or liquid iodine—4927 kg/m\(^3\) at 25°C. Unfortunately, we did not find any pressure viscosity coefficient measurements for these fluids.) This shows that the effects due to the pressure variation are measurable but relatively small. Therefore, for practical purposes in most of the fluids these effects can be ignored. The possible exceptions are high precision measurements of high density fluids with expected high pressure viscosity coefficients.

Note also that the derived formulae show that increasing the length of the fall \( L \) in order to maximise the fall time—and consequently minimise the relative error in measurement of the fall time—is potentially a dangerous practice if one deals with fluid with pressure dependent viscosity.

Here the longer fall length contributes to the higher variation of the pressure (and consequently the viscosity) which has an impact on the fall time.

Further, the pressures \( p_{\text{ref}} \) and \( \beta p_{\text{ref}} \) in the annular region \([R_1, R_2] \times [z_h, z_b + h]\) at the time when the sinker leaves the domain, thus at position \( z_h = z_{\text{ref}}|_{t=0} = L \), is shown in Figure 3. We see, that the zeroth order pressure term \( p_{\text{ref}} \) indeed dominates the first order correction \( \beta p_{\text{ref}} \). The variation of the pressure in the domain is mainly due to the hydrostatic pressure. The hydrostatic pressure is therefore the dominant factor that contributes to the variation of the viscosity. We also see that the variation of the pressure \( \beta p_{\text{ref}} \) in the radial direction is very small compared to the variation in the \( z \) direction, this justifies the assumption discussed in Footnote 7.

### Table 1: Parameter values for some falling cylinder viscometers.

<table>
<thead>
<tr>
<th>Fluid</th>
<th>( \mu_0 ) [Pa·s]</th>
<th>( \beta ) [1/GPa]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Octamethyldisiloxane(^a)</td>
<td>0.12 \times 10^{-3}</td>
<td>13</td>
</tr>
<tr>
<td>Vegetable biodiesel(^b)</td>
<td>7.5 \times 10^{-3}</td>
<td>12</td>
</tr>
<tr>
<td>Diisodocyl phthalate(^c)</td>
<td>123 \times 10^{-3}</td>
<td>26</td>
</tr>
<tr>
<td>Paraffinic oils(^d)</td>
<td>810 \times 10^{-3}</td>
<td>34</td>
</tr>
<tr>
<td>PO(_{\text{MSAN}})(^e)</td>
<td>1.08 \times 10^{-3}</td>
<td>35</td>
</tr>
</tbody>
</table>

\(^a\) Data taken from H. E. King et al. (1992). (Our fit of original tabulated data.)
\(^b\) Data taken from Paton and Schaschke (2009). Controlled temperature of 20°C.
\(^c\) Data taken from Harris and Bair (2007), sample B at 20°C. The authors have fitted the data fit for the formula \( \mu = e^{a_0 + a_1 p + a_2 p + a_3 p^3} \). Here we have calculated \( \mu_{\text{ref}} \) and \( \beta \) from \( a_0 \) and \( a_1 \).
\(^d\) Data taken from Neale (1973). (Generic cylinder paraffinic oil, temperature 30°C.)
\(^e\) Data taken from Cardinaels et al. (2007). (Our fit of original data in Figure 8, temperature 210°C, shear stress level 270 kPa.)

11 The damping term in (4.10) is indeed dominant in all relevant situations, hence the terminal velocity is attained very quickly.
12 In the sense that the variation of the pressure in the device is not neglected.
The perturbation method we have used in the derivation of the governing equation (4.10) means that we have been effectively dealing with a pressure dependent viscosity in the form $\mu = \mu_{\text{ref}} \left(1 + \beta \left(p - p_{\text{ref}}\right)\right)$. The results are therefore valid for any particular fluid with pressure dependent viscosity that admits the same linearisation.

The fact that the effects due to the pressure dependent viscosity can be in some cases significant in measurements using the falling cylinder viscometer calls for direct numerical simulation of the full problem. Such simulation shall help to determine and verify the limitations of the current approach based on the large set of restrictive assumptions.

### References


Dindar, C. and E. Kiran (2002a, DEC 11). High-pressure viscosity and density of polymer solutions at the critical polymer concen-


