Adhesive contact of visco-elastic bodies and defect measures arising by vanishing viscosity

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Abstract: An adhesive unilateral contact of elastic bodies with a small viscosity in the linear Kelvin-Voigt rheology at small strains is scrutinized. The flow-rule for debonding the adhesive is considered rate-independent and unidirectional. The asymptotics for the viscosity or for external loading speed approaching zero is proved in some special cases, in particular when inertia is neglected or when delamination is in Mode II (pure shear). The solutions thus obtained involve certain defect-like measures recording in some sense natural additional energy dissipated in the bulk due to vanishing viscosity. Reflecting also the conventional engineering concept, the delamination is thus driven rather by stress than energy. An explicit example leading to a nontrivial defect measure is given.

Keywords: Adhesive contact, Kelvin-Voigt rheology, rate-independent delamination, vanishing viscosity limit.

AMS Subj. Class. 35K85, 49S05, 74A30, 74M15, 74R20.

1. Introduction, delamination problem. Quasistatic inelastic processes in (or between) solid bodies like damage, fracture, or delamination have important applications, and intensive engineering and mathematical scrutiny has been carried out during past decades. Often, the time scale of such processes is much faster than the external loading time scale, and such processes are then often modelled as rate independent, which may bring theoretical and computational advantages. Yet, the above mentioned inelastic phenomena typically lead to sudden jumps during evolution, which is related with the attribute of the governing stored energy not to be convex (with respect to a natural geometry, if exists, cf. also the non-convex term \( \int_{\Gamma_C} \frac{1}{2} z^2 \kappa u \cdot dS \) in (2.5d) below), and then it is not entirely clear which concept of solutions suits well for the desired specific application.

The “physically” safe way to cope with this problem is to reduce rate-independency on only such variables with respect to which the stored energy is convex, the remaining ones being subjected to viscosity or inertia. Anyhow, such viscosity or inertia can be very small, and it then makes a sense to investigate the asymptotics when they vanish. In the limit, we thus get some solutions of the underlying rate-independent system which, however, might (and, in specific applications, intentionally should) be different from solutions arising when viscosity or inertia are directly zero and global-energy-minimization principle is in play.

In this article, we want to demonstrate the above phenomena and concepts on a particular, but non-trivial and well motivated case of inelasticity on the surface, namely delamination of viscoelastic bodies at small strains. In contrast with bulk inelastic processes like damage or fatigue, this surface inelasticity benefits analytically from compactness of the trace operator, which simplifies (anyhow still complicated) some technicalities and is more lucid for demonstrating the desired effects. It should be also emphasized that the inelastic process itself (i.e. here the delamination) is considered fully rate-independent, in contrast to a usual vanishing-viscosity approach as e.g. in [5, 8, 10, 14, 17, 22, 23, 40].

Linear visco-elastic materials for vanishing viscosity naturally approximate in some sense the elastic materials. We consider the Kelvin-Voigt rheology, which is the simplest rheology which makes the desired effect of natural prevention of too early delamination, cf. Sect. 4. In principle, a combination of Maxwell and Kelvin-Voigt’s rheologies (called Jeffrey’s rheology) would serve equally but, e.g., mere Maxwell rheology is not “enough dissipative” to yield the asymptotical effects investigated below. We further consider a general so-called 2nd-grade nonsimple material, cf. e.g. [27, 41], because some finer results can be derived under this concept (namely a uniformly control of the velocity at least as a measure, cf. (3.3d) below, and disappearance of defect measures for regular solutions, cf. Proposition 3.5 below) although, as a particular case, we will address conventional simple materials, too.

For notational simplicity, we consider a single visco-elastic body occupying a bounded Lipschitz domain \( \Omega \) and the adhesive contact on the a part \( \Gamma_c \) of the boundary \( \partial \Omega \), so that we consider \( \partial \Omega = \Gamma_C \cup \Gamma_D \cup \Gamma_N \cup N \) with disjoint relatively open \( \Gamma_C, \Gamma_D, \) and \( \Gamma_N \) subsets of \( \partial \Omega \) and with measure (1) \( N = 0 \). All results are, however, valid equally for delamination on boundaries inside \( \Omega \), i.e. an adhesive contact between several elastic bodies. For readers’ convenience, let us summarize the notation used below:

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Table 1. Summary of the basic notation used throughout the paper.

The initial-boundary-value problem for the displacement $u$ on $\Omega$ and the so-called delamination parameter $z$ on $\Gamma_c$ valued in $[0, 1]$, representing Frémond’s concept [11] of delamination, considered in this paper is:

\[
\begin{align*}
(1.1a) & \quad \ddot{u} + \text{div}(C\varepsilon - \text{div } C_H \nabla \epsilon) = f, \quad \epsilon = \epsilon(u, \dot{u}) = \nu \epsilon(\dot{u}) + \epsilon(u) \quad \text{on } \Omega, \\
(1.1b) & \quad u = w_0 \quad \text{on } \Gamma_0, \\
(1.1c) & \quad \begin{cases} u \geq 0, \quad t(\epsilon) + u \nabla u \geq 0, \quad (t(\epsilon) + u \nabla u) \cdot u = 0, \\
\hat{\epsilon} \leq 0, \quad \hat{\nu} \leq \alpha, \quad \hat{\nu}(\alpha - \alpha) = 0, \\
\hat{\nu} \in \frac{1}{2} \nabla u + N_{[0, 1]}(z) \end{cases} \quad \text{on } \Gamma_c,
\end{align*}
\]

where we use the usual “dot-notation” for the time derivative, i.e. $(\cdot)' = \frac{\partial}{\partial t}$, where the set-valued mapping $N_{[0, 1]} : \mathbb{R} \rightrightarrows \mathbb{R}$ assigns $z \in \mathbb{R}$ the normal cone $N_{[0, 1]}(z)$ to the convex set $[0, 1] \subset \mathbb{R}$ and where the traction force $t(\epsilon)$ is defined on $\Gamma_c \cup \Gamma_n$ by the formula

\[
(1.1e) \quad t(\epsilon) = (C\varepsilon)[\nu] \nabla \epsilon - \text{div } (C_H \nabla \epsilon)[\nu] \nabla \epsilon - \text{div } (C_H \nabla \epsilon)[\nu] \nabla \epsilon,
\]

involving so-called hyper- or couple-stresses $C_H \nabla \epsilon$, cf. e.g. [27, 41] for this concept and for derivation of the natural boundary condition (1.1,e) and, thus, also (1.1d,e); cf. [29, Rem. 3.3.] for details in the latter case. We further consider the initial conditions

\[
(1.1f) \quad u(0) = u_0, \quad z(0) = z_0, \quad \dot{u}(0) = v_0.
\]

The parameter $\alpha > 0$ in (1.1d) is a given phenomenological number (or possibly a function of $x \in \Gamma_c$) with a physical dimension $J/m^2$ with the meaning of a specific energy needed (and thus deposited in the newly created surface) to delaminate $1m^2$ of the surface under adhesion or, equally, the energy dissipated by this delamination process; in fact, (2.7) below reflects the latter interpretation. In engineering, $\alpha$ is also called fracture toughness. Of course, the loading $f$, $g$, and $w_0$ in (1.1a-c) depend on time $t$. In (1.1e), $\nabla = \nabla(x)$ is the unit outward normal to $\Gamma := \partial \Omega$ and $\text{div}_s$ is the surface-divergence operator $\text{div}_s := \text{Tr}(\nabla s)$ with $\text{Tr}(\cdot)$ being the trace of a $(d-1)\times(d-1)$-matrix. The ordering “$\geq$” used in (1.1d) is induced by a closed convex cone $D(x) \subset \mathbb{R}^d$ in the sense $w \geq u$ if and only if $w - u \in D(x)$ at $x \in \Gamma_c$, and $\hat{\nu}$ is the dual ordering induced by the negative polar cone to $D$ in the sense $\hat{\nu} \geq 0$ if and only if $s(x) \cdot w \geq 0$ for all $w \in D(x)$ for a.a. $x \in \Gamma_c$. This cone is to handle a general contact conditions: it is natural to consider the Signorini unilateral contact which, however, is rather difficult in the dynamical setting. Therefore, we will consider also other situations where $D(x)$ is a linear subspace. The particular choices that can be described by a cone $D \subset \mathbb{R}^d$ depending possibly on $x \in \Gamma_c$ are typically

\[
(1.2a) \quad D(x) = \{ v \in \mathbb{R}^d; \quad v \cdot \nabla(x) \geq 0 \}, \quad \text{or}
\]

\[
(1.2b) \quad D(x) = \{ v \in \mathbb{R}^d; \quad v \cdot \nabla(x) = 0 \}.
\]

The case (1.2a) is the standard model a unilateral frictionless Signorini contact. The case (1.2b) fixes the normal jump of displacements at $x \in \Gamma_c$ to zero and allows thus only for a tangential slip along $\Gamma_c$, which may be a relevant model under high pressure when no cavity of $\Gamma_c$ can be expected anyhow. Such mode of delamination by mere shearing is called Mode II (or sometime also a bilateral contact, cf. [37, Sect.11.4]) and such situation occurs e.g. on lithospheric faults deep under the earth surface where cavities cannot occur due to the huge lithostatic pressure.

In conventional materials, the relaxation time $\nu > 0$ is mostly very small in comparison with external force loading time-scale, and it is worth studying the asymptotics for $\nu \to 0$. Also one can study the asymptotic for both $g \to 0$ and $\nu \to 0$, which, in the case $g \sim \nu^2$, has the meaning that the loading process becomes very slow and, after rescaling time on a fixed time interval, these parameters go to 0,
cf. Remark 5.3 below. Formally, the limit problem arising for \( \nu \to 0 \) is either a hyperbolic (if \( \rho > 0 \)) or a quasistatic (if \( \rho = 0 \) or if \( \rho \to 0 \)) problem which consists in replacing (1.1a) by

\[
\rho \ddot{u} - \text{div}(C(e(u)) - \text{div} C_H \nabla e(u)) = f \quad \text{on } \Omega, \tag{1.3}
\]

and in replacing (1.1c,d) with (1.1e) again from (1.1e). It should be emphasized that (1.3) in its classical formulation does not bear full information about energetics arising naturally from the above mentioned limit, as shown in Sect. 4.

The plan of the paper is as follows: First, in Section 2, we formulate the above problem (1.1) weakly.

### 2. Weak formulation of the viscous problem.

The weak formulation of the initial-boundary value problem (1.1) is a bit delicate due to the doubly-nonlinear structure of the flow rule for \( z \) on \( \Gamma_c \), without any compactness (i.e. without any gradient theory for \( z \)) and with both involved nonlinearities unbounded due to the constraints \( \dot{z} \leq 0 \) and \( z \geq 0 \) (while the third constraint \( z \leq 1 \) is essentially redundant if the initial condition satisfies it). This would make serious difficulties in proving existence of conventional weak solutions. Benefiting from rate-independency of the evolution rule for \( z \), we can cast a suitable definition by combining the conventional weak solution concept for \( u \) and the so-called energetic-solution concept [19, 25] of \( z \) as in [31].

Considering a fixed time horizon \( T > 0 \), we use the shorthand notation \( I = (0, T), \bar{I} = [0, T] \), \( Q = I \times \Omega, \bar{Q} = \bar{I} \times \bar{\Omega} \) with \( \bar{\Omega} \) the closure of \( \Omega \), and \( \Sigma_c = I \times \Gamma_c \). We will assume, without substantial restriction of generality of geometry of the problem, that

\[
dist(\Gamma_c, \Gamma_D) > 0, \quad \text{meas}_{d-1}(\Gamma_D) > 0, \quad \text{meas}_{d-1}(\Gamma_c) > 0. \tag{2.1}
\]

We first make a transformation of the problem to get homogeneous (and, in particular, time constant) Dirichlet condition. To this goal, we first consider a suitable prolongation \( u_D \) of \( u_0 \) defined on \( Q \), i.e. \( u_D|_{\Sigma_D} = w_D \). Then we shift \( u \) to \( u + u_D \), and rewrite (1.1) for such a shifted \( u \). Thanks to the first condition in (2.1), we can assume that \( u_D|_{\Sigma_c} = 0 \) and (1.1d) remains unchanged under this shift. The equations (1.1a-c) transform in such a way that the original right-hand sides \( f, g, \) and \( w_D \), as well as the original initial data \( u_0 \) and \( v_0 \) are respectively

\[
\begin{align*}
(2.2a) \quad f & \text{ replaced by } f + \text{div}(C(e_u) - \text{div} C_H \nabla e_u) \quad \text{with } e_u = e(\nu \dot{u}_D + u_D), \\
(2.2b) \quad g & \text{ replaced by } g + (C(e_u)|_{\Gamma} - \text{div} (C_H \nabla e_u))|_{\Gamma} \nu - \text{div}_d(C_H \nabla e_u \nu), \\
(2.2c) \quad w_D & \text{ replaced by } 0, \\
(2.2d) \quad u_0 & \text{ replaced by } u_D(0), \\
(2.2e) \quad v_0 & \text{ replaced by } v_0 - \dot{u}_D(0).
\end{align*}
\]

We will use the standard notation \( W^{k,p}(\Omega) \) for the Sobolev space of functions with \( k \) derivatives in the corresponding \( L^p \)-spaces. If valued in \( \mathbb{R}^n \) with \( n \geq 2 \), we will write \( W^{k,p}(\Omega; \mathbb{R}^n) \), and furthermore we use the shorthand notation \( H^k(\bar{\Omega}; \mathbb{R}^n) = W^{k,2}(\Omega; \mathbb{R}^n) \). Occasionally, we also use a non-integer \( k \) for Sobolev-Sobodetski˘i spaces with fractional derivatives. We also use the notation of \( \cdot, \cdot, \cdot \), and \( \cdot \cdot \cdot \) for a scalar product of vectors, 2nd-order tensors, and 3rd-order tensors, respectively. Later, \( \text{Meas}(Q) \equiv C(Q)^* \) will denote the space of measures on the compact set \( Q \). For a Banach space \( X, L^p(I; X) \) denote the Bochner space of \( X \)-valued Bochner measurable functions with its norm in \( L^p(I, C_w(I; X)) \) the space of weakly continuous mappings \( I \to X \), and \( BV(I; X) \) the space of mappings \( u : I \to X \) with a bounded variations \( \text{Var}(u) := \sup_{0 \leq t_0 < t_1 < \cdots < t_n \leq T} \sum_{i=1}^n \| u(t_i) - u(t_{i-1}) \| \) where the supremum is taken over all finite partitions of the interval \( [0, T] \). To cover both simple and 2nd-grade nonsimple materials, let us define the Hilbert space

\[
\mathcal{H}(\Omega; \mathbb{R}^d) := \left\{ u \in H^1(\Omega; \mathbb{R}^d); \int_\Omega C_H \nabla e(u) : \nabla e(u) \, dx < \infty \right\}. \tag{2.3}
\]

Note that, if \( C_H = 0 \), we have simply \( \mathcal{H}(\Omega; \mathbb{R}^d) = H^1(\Omega; \mathbb{R}^d) \) while, if \( C_H \) is positive definite (we will write \( C_H \geq 0 \)), we have \( \mathcal{H}(\Omega; \mathbb{R}^d) \cong H^2(\Omega; \mathbb{R}^d) \). To accommodate the transformation (2.2) into the weak
formulation, we introduce the functional $\mathcal{J}(t) \in \mathcal{H}(\Omega; \mathbb{R}^d)^*$ by

$$\langle \mathcal{J}(t), v \rangle := \int_\Omega \left( f(t) - \varrho \ddot{u}_\nu(t) \right) \cdot v - \mathcal{C}_e(\varrho \dot{u}_\nu(t) + u_\nu(t)) : e(v) - \int_\Omega \mathcal{H} e(\varrho \dot{u}_\nu(t) + u_\nu(t)) : \nabla e(v) \, dx + \int_\Gamma g(t) \cdot v \, dS.$$  

**Definition 2.1.** The couple $(u_\nu, z_\nu)$ with $u_\nu \in H^1(I; \mathcal{H}(\Omega; \mathbb{R}^d))$ and $z_\nu \in BV(I; L^1(\Gamma_c)) \cap L^\infty(\Sigma_c)$ and, if $\varrho > 0$, also $\dot{u}_\nu \in C_w(I; L^2(\Omega; \mathbb{R}^d))$ is called an energetic solution to the initial-boundary-value problem (1.1) under the transformation (2.2) if

(i) the momentum equilibrium

$$\begin{align*}
\int_\Omega \mathcal{C}_e(\varrho \dot{u}_\nu + u_\nu) : e(v - u_\nu) + \mathcal{H} e(\varrho \dot{u}_\nu + u_\nu) : \nabla e(v - u_\nu) - \varrho \ddot{u}_\nu \cdot (v - \dot{u}_\nu) \, dxdt + \\
\int_{\Sigma_c} \varrho \dot{u}_\nu (v - u_\nu) \, dS(t) + \int_\Omega \varrho \ddot{u}_\nu (T) \cdot (v(T) - u_\nu(T)) \, dx \\
\geq \int_0^T \langle \mathcal{J}(t), v - u_\nu \rangle \, dt + \int_\Omega g_\nu (t) (v(0) - u_\nu(0)) \, dx
\end{align*}$$

with $\mathcal{J}$ defined in (2.4) holds for any $v \in H^1(I; \mathcal{H}(\Omega; \mathbb{R}^d))$ with $v|_{\Sigma_c} \geq 0$ and $v|_{\Sigma_\nu} = 0$,

(ii) the so-called semi-stability

$$\mathcal{K}u_\nu(x, t) \cdot u_\nu(t, x) \leq 2\alpha(x) \quad \text{or} \quad z_\nu(t, x) = 0 \quad \text{for a.a.} \ x \in \Gamma_c$$

holds for any $t \in [0, T]$,

(iii) and the energy equality

$$\begin{align*}
\mathcal{E}(t, u_\nu(t), z_\nu(t)) := & \int_\Omega \frac{\varrho}{2} |\dot{u}_\nu(t)|^2 \, dx + \int_0^t \int_\Omega \mathcal{C}_e(\varrho \dot{u}_\nu(t)) : e(u_\nu(t)) + \mathcal{H} e(\varrho \dot{u}_\nu(t)) : \nabla e(\varrho \dot{u}_\nu(t)) \, dxdt + \\
& + \int_{\Gamma_c} \alpha(z_\nu - z_\nu(t)) \, dS(t) = \mathcal{E}(0, u_\nu(0), z_\nu) + \int_0^t \int_\Omega \frac{\varrho}{2} |v_0|^2 \, dx + \int_0^t \langle \mathcal{J}, u_\nu \rangle \, dt \quad \text{with}
\end{align*}$$

$$\begin{align*}
\mathcal{E}(t, u, z) := & \int_\Omega \frac{\varrho}{2} |\dot{u}|^2 \, dx - \langle \mathcal{J}(t), u \rangle \quad \text{if} \quad u_\nu|_{\Gamma_c} \geq 0, \quad 0 \leq z_\nu \leq 1 \quad \text{on} \ \Gamma_c, \quad \\
& + \int_{\Gamma_c} \frac{1}{2} \mathcal{K}u_\nu \cdot dS(t) - \langle \mathcal{J}(t), u \rangle \quad \text{if} \quad u_\nu|_{\Gamma_c} = 0, \quad 0 \leq z_\nu \leq 1 \quad \text{on} \ \Gamma_c,
\end{align*}$$

(iv) the remaining initial conditions understood transformed as (2.2d,e) hold, i.e. $u_\nu(0) = u_0$ and $z_\nu(0) = z_0$.

This definition is indeed well selective in the sense that any smooth energetic solution solves also (1.1) in the classical sense. Note that the initial condition $\dot{u}(0) = 0$ is not involved in (iv) as it is contained already in (2.5a). Due to (2.5c), $\mathcal{E}(t, u_\nu(t), z_\nu(t)) < \infty$ so that it holds $u_\nu|_{\Sigma_c} \geq 0$, $u_\nu|_{\Sigma_\nu} = 0$, and $0 \leq z_\nu \leq 1$ if the initial conditions satisfies these constraints so that $\mathcal{E}(0, u_\nu(0), z_\nu) < \infty$. Note also that (1.1a-d) has an abstract structure of the triply nonlinear system of two evolution inclusions:

$$\begin{align*}
& \mathcal{F}' \dot{u} + [\mathcal{A}_\nu]_2 \dot{u} + \partial_u \mathcal{E}(t, u, z) \ni 0, \\
& \partial_z \mathcal{A}_\nu(z) + \partial_z \mathcal{E}(t, u, z) \ni 0
\end{align*}$$

with $(\cdot)'$ denoting the (partial) Gâteaux differentials and $\partial$ denoting the partial subdifferentials in the sense of convex analysis, with $\mathcal{E}$ from (2.5d), and with the kinetic energy $\mathcal{F}$ and the $\nu$-dependent (pseudo)potential of dissipative forces $\mathcal{A}_\nu$ given by

$$\begin{align*}
\mathcal{F}(\dot{u}) &= \int_\Omega \frac{\varrho}{2} |\dot{u}|^2 \, dx, \\
\mathcal{A}_\nu(\dot{u}, \dot{z}) &= \begin{cases} \int_\Omega \frac{\varrho}{2} \mathcal{C}_e(\dot{u}) : e(\dot{u}) + \frac{\varrho}{2} \mathcal{H} e(\dot{u}) : \nabla e(\dot{u}) \, dx \\
+ \int_{\Gamma_c} \alpha |\dot{z}| \, dS \quad \text{if} \quad \dot{z} \leq 0 \quad \text{a.e. on} \ \Gamma_c, \\
+ \infty \quad \text{otherwise.}
\end{cases}
\end{align*}$$
Occasionally, for \( \nu = 0 \) or \( \tilde{u} = 0 \), we will write briefly \( \mathcal{R}_0(\tilde{u}, \tilde{\nu}) = \mathcal{R}_0(0, \tilde{\nu}) =: \mathcal{R}_0(\tilde{\nu}) \). Existence of the solutions due to Definition 2.1 is known at least in particular cases: the quasi-static case \( \rho = 0 \) or the pure Mode-II case (1.2b), cf. the Appendix below. Also note that (2.5b) is equivalent to the integrated form of the abstract semistability \( \mathcal{E}(t, u_\nu(t), z_\nu(t)) \leq \mathcal{E}(t, u_\nu(t), \tilde{z}) + \mathcal{R}_0(\tilde{\nu} - z_\nu(t)) \) holding for any \( \tilde{z} \geq 0 \), i.e. here:

\[
(2.8) \quad \forall \tilde{z} \in L^\infty(\Gamma_C), \quad 0 \leq \tilde{z} \leq z_\nu(t) : \int_{\Gamma_C} (z_\nu(t) - \tilde{z})(K_{u_\nu}(t) - z_\nu(t) - 2\alpha) \, dS \leq 0.
\]

### 3. Quasistatic problems.

In this section, we neglect inertial effects from the very beginning by putting \( \rho = 0 \). Such a problem is called quasistatic. We will generally assume the following data qualification:

\[
\begin{align*}
(3.1a) & \quad C > 0 \ (= \text{positive definiteness}), \quad C_H \geq 0 \ (= \text{positive semi-definiteness}), \\
(3.1b) & \quad f \in W^{1,1}(I; H(\Omega; \mathbb{R}^d)^*), \quad u_0 \in H(\Omega; \mathbb{R}^d), \quad z_0 \in L^\infty(\Gamma_C), \\
(3.1c) & \quad u_0|_{\Gamma_C} \geq 0, \quad 0 \leq z_0 \leq 1 \ \text{a.e. on} \ \Gamma_C, \quad \text{and} \\
(3.1d) & \quad K_{u_\nu}(x) - u_0(x) \leq \alpha \quad \text{or} \quad z_0(x) = 0 \quad \text{for a.a.} \ x \in \Gamma_C;
\end{align*}
\]

note that the initial condition \( \tilde{u}(0) = v_0 \) is now irrelevant in this quasistatic case. Note that the qualification of \( f \) in (3.1b) represents, in fact, assumptions on \( f, g, \) and \( u_0 \) in the original problem (1.1), and that (3.1c,d) means semi-stability of the initial condition \((u_0, z_0)\). For finer results valid for nonsimple materials, we will additionally assume a “gentle” start and load in the sense:

\[
\begin{align*}
(3.2a) & \quad \int_{\Omega} C_{\nu} \nabla e(u_0) + \nabla v + C(H(u_0)) e(v) \, dx + \int_{\Gamma_C} z_0 K_{u_\nu} v \, dS = (f(0), v) \quad \forall v \in H^2(\Omega; \mathbb{R}^d), \\
(3.2b) & \quad f \in W^{1,1}(I; H^2(\Omega; \mathbb{R}^d)^*). 
\end{align*}
\]

**Lemma 3.1 (A-priori estimates).** Let the assumptions (2.1) and (3.1) be satisfied together with \( \rho = 0 \) and \( \nu > 0 \). Then any solution \((u_\nu, z_\nu)\) according to Definition 2.1 satisfies

\[
\begin{align*}
(3.3a) & \quad \|\tilde{u}_\nu\|_{L^2(I; H(\Omega; \mathbb{R}^d)^*)} \leq C/\sqrt{\gamma}, \\
(3.3b) & \quad \|u_\nu\|_{L^\infty(I; H(\Omega; \mathbb{R}^d))} \leq C, \\
(3.3c) & \quad \|z_\nu\|_{L^\infty(\Sigma_C) \cap BV(I; L^1(\Gamma_C))} \leq C
\end{align*}
\]

with \( C \) independent of \( \nu \). Moreover, if \( C_H > 0 \) and also (3.2) holds, then there is some solution according to Definition 2.1 which satisfies even

\[
(3.3d) \quad \|u_\nu\|_{BV(I; H^2(\Omega; \mathbb{R}^d))} \leq C.
\]

**Proof.** From (2.5c) when estimating the last term as \( \langle f, u_\nu \rangle \leq \|f\|_{H(\Omega; \mathbb{R}^d)^*}(\frac{1}{2} + \|u_\nu\|_{H(\Omega; \mathbb{R}^d)}) \) and then treating it by Gronwall’s inequality relying on (3.1b), we get \( \mathcal{E}(t, u_\nu(t), z_\nu(t)) \) uniformly bounded for \( t \in I \), from which \( \|u_\nu\|_{L^\infty(I; H(\Omega; \mathbb{R}^d))} \leq C \) and \( \|z_\nu\|_{L^\infty(\Sigma_C)} \leq C \), i.e. (3.3b) and the \( L^\infty \)-part of (3.3c). Using (2.5c) for \( t = T \), we get the dissipation bounded, which yields (3.3a) and also the rest of (3.3c).

In case \( C_H > 0 \), to show (3.3d) at least heuristically (cf. the discussion at the end of this proof, referring to the Appendix for making this estimation rigorous), we differentiate the equation (1.1a) for \( u_\nu \) counting also the boundary conditions via (2.4) and then test it by \( \tilde{u}_\nu \). Then, using non-emptiness of \( \Gamma_D \) assumed in (2.1) and considering an equivalent norm \( \|u\|_{H^2(\Omega; \mathbb{R}^d)} := (\int_{\Omega} C e(u); e(u)) + C_H e(u); e(u) \) it gives

\[
\begin{align*}
(3.4) & \quad \|\tilde{u}_\nu\|_{H^2(\Omega; \mathbb{R}^d)} + \nu \|\tilde{u}_\nu\|_{H^2(\Omega; \mathbb{R}^d)} \frac{d}{dt} \|\tilde{u}_\nu\|_{H^2(\Omega; \mathbb{R}^d)} \leq c \|\tilde{u}_\nu\|_{H^2(\Omega; \mathbb{R}^d)} + \nu \frac{d}{dt} \|\tilde{u}_\nu\|_{H^2(\Omega; \mathbb{R}^d)} \\
& \quad \leq \int_{\Omega} \nu C e(\tilde{u}_\nu); e(\tilde{u}_\nu) + C e(\tilde{u}_\nu); e(\tilde{u}_\nu) + \nu C_H e(\tilde{u}_\nu); e(\tilde{u}_\nu) + C_H e(\tilde{u}_\nu); e(\tilde{u}_\nu) + C_H e(\tilde{u}_\nu); e(\tilde{u}_\nu) + C_H e(\tilde{u}_\nu); e(\tilde{u}_\nu) \, dx \\
& \quad + \int_{\Gamma_C} z_\nu K_{u_\nu} \tilde{u}_\nu \, dS \\
& \quad \leq \langle f, \tilde{u}_\nu \rangle - \int_{\Gamma_C} \tilde{z}_\nu K_{u_\nu} \tilde{u}_\nu \, dS \leq C \left( \|f\|_{H^2(\Omega; \mathbb{R}^d)} + \|\tilde{z}_\nu\|_{L^2(\Gamma_C)} \right) \|\tilde{u}_\nu\|_{H^2(\Omega; \mathbb{R}^d)}
\end{align*}
\]
for some $C$ depending on $|K|$ and on the norm of the trace operator $H^2(\Omega) \to L^\infty(\Gamma_c)$. Thus

$$
\|\ddot{u}_v\|_{H^2(\Omega; \mathbb{R}^d)} + \nu \frac{d}{dt} \|\dot{u}_v\|_{H^2(\Omega; \mathbb{R}^d)} \leq C \|\dot{f}\|_{H^2(\Omega; \mathbb{R}^d)} + C \|\dot{z}_{\nu}\|_{L^1(\Gamma_c)}.
$$

from which (3.3d) follows; here we used the qualification (3.2) and, in particular, we needed to have \(\nu \|\ddot{u}_v(0)\|_{H^2(\Omega; \mathbb{R}^d)}\) bounded uniformly in \(\nu > 0\) for which it is important that the condition (3.2a) together with the momentum equilibrium (3.6c) be satisfied. Details are quite technical and therefore omitted.

(ii) Any triple \((u, z, \mu)\) obtained in this way fulfills \(v|_{\Sigma_C} \geq 0\) and \(v|_{\Sigma_D} = 0\), the momentum equilibrium in the weak form, namely

\[
(3.6a) \quad u_v(t) \to u(t) \quad \text{in } \mathcal{H}(\Omega; \mathbb{R}^d) \quad \text{for a.a. } t \in [0, T],
\]

\[
(3.6b) \quad z_v(t) \overset{\ast}{\rightharpoonup} z(t) \quad \text{in } L^\infty(\Gamma_C) \quad \text{for all } t \in [0, T],
\]

\[
(3.6c) \quad \nu C_v \dot{u}_v + \nu C_H \nabla e(\dot{u}_v) : \nabla e(\dot{u}_v) \overset{\ast}{\rightharpoonup} \mu \quad \text{in } \mathcal{M}(\Omega).
\]

(iii) If \(C_H > 0\) holds together with (3.2) and \((u_v, z_v)\) complies with (3.3d), then \(u \in BV(I; H^2(\Omega; \mathbb{R}^d))\) and the above subsequence can be selected also to enjoy the convergence

\[
(3.8) \quad u_v(t) \to u(t) \quad \text{in } H^2(\Omega; \mathbb{R}^d) \quad \text{for all } t \in [0, T],
\]

and (3.7b) to hold even for all \(t \in [0, T]\).

Proof. For lucidity, let us divide the proof into five steps.

Step 1: Selection of converging subsequences: By the BV-estimate (3.3c) and Helly’s selection principle, we obtain (3.6b). By separability of the predual $C(\Omega)$ to $Meas(\Omega)$ and by Banach’s selection principle together with the a-priori estimate (3.3a), we obtain also (3.6c).

As for (3.6a), we use the strong monotonicity of $\mathcal{E}_0'(t, \cdot, z)$ and (2.5a):

\[
\int_0^T \langle \mathcal{E}_0'(t, u_v, z_v) - \mathcal{E}_0'(t, u, z_v), u_v - u \rangle dt
\]

\[
= \int_0^T \left( \int_{\Omega} \nu C_v e(\dot{u}_v) : e(u - u_v) + \nu C_H \nabla e(\dot{u}_v) : \nabla e(u - u_v) \right) dx
\]

\[
+ \int_{\Gamma_C} z_v \kappa u_v (u - u_v) dS - \langle \mathcal{E}_0'(t, u_v, z_v), u_v - u \rangle dt
\]

\[
\leq \int_0^T \left( \int_{\Omega} \nu C_v e(\dot{u}_v) : e(u) + \nu C_H \nabla e(\dot{u}_v) : \nabla e(u) \right) dx + \int_{\Gamma_C} z_v \kappa u_v (u - u_v) dS
\]

\[
- \langle \mathcal{E}_0'(t, u, z_v), u_v - u \rangle dt + \int_{\Omega} \nu C_v (u_0) e(u_0) + \nu C_H \nabla e(u_0) : \nabla e(u_0) dx 
\]

\[
\to 0.
\]
Here we used

\[
(3.10) \quad \|\nu C(\tilde{u}_\nu) : e(u)\|_{L^1(Q)} \leq \nu|C| \|e(\tilde{u}_\nu)\|_{L^1(Q;R^d)} \|e(u)\|_{L^1(Q;R^d)} = \Theta(\sqrt{\nu}) \to 0
\]

due to (3.3a), and a similar argument applies also for \(\|\nu C_H \nabla e(\tilde{u}_\nu) : \nabla e(u)\|_{L^1}\) if \(C_H > 0\).

Taking the limit of the \(\Gamma_c\)-term in (3.9) is a bit technical. For \(C_H = 0\), we need to show that

\[
(3.11a) \quad \lim_{\nu \to 0} \int_{\Sigma_{c^\nu}} z_\nu K u_{\nu} u \, dS dt = \int_{\Sigma_c} z K u u \, dS dt \quad \text{and}
\]

\[
(3.11b) \quad \limsup_{\nu \to 0} \int_{\Sigma_{c^\nu}} -z_\nu K u_{\nu} u \, dS dt = \lim_{\nu \to 0} \int_{\Sigma_c} (z - z_\nu) K u_{\nu} u \, dS dt
\]

\[-\liminf_{\nu \to 0} \int_{\Sigma_{c^\nu}} z K u_{\nu} u \, dS dt \leq \int_{\Sigma_c} z K u u \, dS dt.
\]

As for (3.11a), we may simply use \(u_{\nu} \to u\) in \(L^2(I; H^{1/2}(\Gamma_c; \mathbb{R}^d))\) and \(z_\nu \to z\) in \(L^2(I; H^{-1/2}(\Gamma_c))\) by the (generalized) Aubin-Lions’ lemma (cf. [33, Corollary 7.9]) relying here on the control of \(z_\nu\) in \(\text{Meas}(I; L^1(\Gamma_c))\) due to (3.3c), so that \(z_\nu K u_{\nu} \to z K u\) in \(L^1(\Sigma_c; \mathbb{R}^d)\) and, by a-priori estimates (3.1b,c), in \(L^2(\Sigma_c; \mathbb{R}^d)\), too. Thus \(z_\nu K u_{\nu} u \to z K u u\) in \(L^1(\Sigma_c; \mathbb{R}^d)\); in fact, it converges even in a smaller space. As for (3.11b), we used the weak lower-semicontinuity of the convex functional \(u \mapsto \int_{\Sigma_c} z K u u \, dS dt : L^2(I; H^1(\Omega; \mathbb{R}^d)) \to \mathbb{R}\) and that \((z_\nu - z) K u_{\nu} u\) in (3.11b) converges to zero weakly in \(L^1(\Sigma_c)\). This last fact follows from the boundedness of \(K u_{\nu} u\) in \(L^\infty(I; W^{1,q}(\Omega))\) with \(1 \leq q < d/(d-1)\) (in fact even for \(q = d/(d-1)\) if \(d \geq 3\), so that the traces are bounded in \(L^\infty(I; W^{1,1/q}(\Gamma_c))\) and from the strong convergence \(z_\nu - z \to 0\) in \(L^p(I; W^{1,1/q}(\Gamma_c))\) by the mentioned (generalized) Aubin-Lions’ lemma; here we use that \(W^{1,1/q}(\Gamma_c) \subset L^r(I; \Omega)\) compactly and densely for \(1 \leq r < (dq-q)/(d-q)\) so that \(W^{1,1/q}(\Gamma_c)\) contains compactly the reflexive space \(L^r/(r-1)(\Gamma_c)\) if \(r > 1\) is considered. Thus (3.11) is proved.

If \(C_H > 0\) and (3.3d) is at our disposal, we can use simply that \(\{u_{\nu}\}_{\nu > 0} \subset L^2(I; H^{2-\epsilon}(\Omega; \mathbb{R}^d))\) compactly and thus \(K u_{\nu} (u-u_{\nu}) \to K u (u-u) = 0\) in \(L^1(\Sigma_c)\) so that \(z_\nu K u_{\nu} (u-u_{\nu}) \to 0\) in \(L^1(\Sigma_c)\) to be used (3.9) for a selected subsequence.

Thus, in both cases, we proved the convergence to 0 in (3.9), and we then obtain \(u_{\nu} \to u\) strongly in \(L^1(I; H^1(\Omega; \mathbb{R}^d))\), and thus also (3.6a).

Step 2: the limit passage in the momentum equilibrium (2.5a): In most terms, we can use just weak upper-semicontinuity. As to the viscosity term, we use the limsup-estimate:

\[
(3.12) \quad \int_Q \nu C(\tilde{u}_\nu) : e(v-u_{\nu}) + \nu C_H \nabla e(\tilde{u}_\nu) : \nabla e(v-u_{\nu}) \, dx dt
\]

\[\leq \int_Q \nu C(\tilde{u}_\nu) : e(v) + \nu C_H \nabla e(\tilde{u}_\nu) : \nabla e(v) \, dx dt + \int_Q \nu C(0) : e(u_{\nu}) + \nu C_H \nabla e(0) : \nabla e(u_{\nu}) \, dx dt
\]

\[\leq \nu \|\tilde{u}_\nu\|_{L^1(Q;H^1(\Omega;\mathbb{R}^d))} \|v\|_{L^1(Q;H^{1/2}(\Omega;\mathbb{R}^d))} + \nu C \|u_{\nu}\|^2_{H^1(\Omega;\mathbb{R}^d)} = \Theta(\sqrt{\nu}) \to 0,
\]

where we used (3.3a).  

Step 3: the limit passage in semistability (2.5b): As we do not have a pointwise convergence of \(z_\nu\), we use rather the integrated form of (2.5b), i.e. (2.8) holding for any \(0 \leq \tilde{z} \leq z_\nu(t)\). This requires to find explicitly a so-called mutual recovery sequence, i.e. here a sequence \(\{\tilde{z}_\nu\}_{\nu > 0}\) such that, for any \(0 \leq \tilde{z} \leq z(t)\), it holds \(z_\nu \to \tilde{z}\) weakly* in \(L^\infty(\Gamma_c)\) and simultaneously \(0 \leq \tilde{z}_\nu \leq z_\nu(t)\). This can be achieved by choosing

\[
(3.13) \quad \tilde{z}_\nu(x) := \begin{cases} z_\nu(t, x) \tilde{z}(x) / z(t, x) & \text{where } z(t, x) > 0, \\ 0 & \text{where } z(t, x) = 0. 
\end{cases}
\]

Note that the convergence \(z_\nu \to z\) weakly* in \(L^\infty(\Gamma_c)\) implies \(\tilde{z}_\nu \to \tilde{z}\) weakly* in \(L^\infty(\Gamma_c)\). Also note that \(0 \leq \tilde{z} \leq z(t)\) and \(z_\nu(t) \geq 0\) imply that \(\tilde{z}_\nu\) from (3.13) fulfills \(0 \leq \tilde{z}_\nu \leq z_\nu(t)\) a.e. on \(\Gamma_c\), so that we can use (2.8) with such a test delamination profile \(\tilde{z}_\nu\). By (3.6a), we have certainly \(u_{\nu}(t)_{|\Gamma_c} \to u(t)_{|\Gamma_c}\) in \(L^2(\Gamma_c)\) so that \((z_\nu(t) - \tilde{z})(K u_{\nu}(t) u_{\nu}(t) - 2\alpha) \to (z(t) - \tilde{z})(K u(t) u(t) - 2\alpha)\) weakly in \(L^1(\Gamma_c)\), and thus we can pass to the limit in (2.8) to obtain

\[
(3.14) \quad \forall \tilde{z} \in L^\infty(\Gamma_c), \quad 0 \leq \tilde{z} \leq z(t) : \quad \int_{\Gamma_c} (z(t) - \tilde{z})(K u(t) u(t) - 2\alpha) \, dS \leq 0.
\]

By disintegration, we get \((z(t) - \tilde{z})(K u(t) u(t) - 2\alpha) \leq 0\) for any \(\tilde{z} \in \mathbb{R}\) with \(0 \leq \tilde{z} \leq z(t)\) a.e. on \(\Gamma_c\), which is equivalent to (3.7b).
Step 4: the limit passage in energy equality (2.5c): For those \( t \)'s where (3.6a) holds, we have also \( \mathcal{E}(t, u_\nu(t), z_\nu(t)) \to \mathcal{E}(t, u(t), z(t)) \) by using also (3.6b) and (strong\( \times \)weak)-continuity of \( \mathcal{E}(t, \cdot, \cdot) \). Using also (3.6c), we directly obtain (3.7c).

Step 5: further results: For \( \mathbb{C}_H > 0 \), we can make some features stronger. By the BV-estimate (3.3d) and by Helly’s principle applied to the couple \((u(\cdot), z(\cdot))\), we obtain (3.6c,d) and (3.8). Then we can perform Step 3 for all \( t \in I \); note that, by compactness of the trace operator \( u \mapsto u|_{\Gamma_C} \), the weak convergence of \( u_\nu(t) \) is sufficient for this. ☐

The above assertion suggests the following:

**Definition 3.3.** A triple \((u, z, \mu)\) with \( u \in L^\infty(I; \mathcal{H}(\Omega; \mathbb{R}^d)) \), \( z \in BV(I; L^1(\Gamma_C)) \), and \( \mu \in \text{Meas}(\bar{Q}) \), \( \mu \geq 0 \), is called:

(i) a semi-energetic solution to the quasistatic rate-independent delamination problem (1.1) with \( q = 0 \) and \( \nu = 0 \) transformed by (2.2) if (3.7) holds for a.a. \( t \in I \) and \( z(0) = z_0 \).

(ii) If, in addition to (i), \((u, z, \mu)\) is attainable by a sequence of viscous solutions \( \{(u_\nu, z_\nu)\}_{\nu > 0} \) in the sense (3.6), the triple \((u, z, \mu)\) is called a Kelvin-Voigt-approximable solution.

**Corollary 3.4.** The Kelvin-Voigt-approximable solutions due to Definition 3.3(ii) do exist, and a fortiori also the semi-energetic solutions to the quasi-static rate-independent delamination problem (1.1) with \( q = 0 \) and \( \nu = 0 \).

The proof of the existence of Kelvin-Voigt-approximable solutions is due to Proposition 3.2 combined with Appendix.

The measure \( \mu \in \text{Meas}(\bar{Q}) \) occurring in Proposition 3.2 represents a certain, in general unspecified additional energy distributed over \( \bar{Q} \). Using such \( \mu \)'s, the concept of the semi-energetic solutions has been invented in [32]. In case of the Kelvin-Voigt-approximable solution, this additional energy arising by (3.6c) has got a physically justified meaning. Similar concept has been invented in various problems in continuum mechanics under the name of defect measures to reflect a possible additional energy dissipation of solutions lacking regularity and exhibiting various concentration effects in contrast to regular weak solutions where the defect measure vanishes, cf. [7, 12, 13, 9, 26]. Here, \( \mu \) reflects the possible additional dissipated energy of Kelvin-Voigt-approximable solutions comparing to so-called energetic solutions, cf. Remark 3.8. The latter ones are known to jump unphysically early in problems governed by non-convex stored energies \( \mathcal{E}(t, \cdot, \cdot) \) like considered also here in (2.5d) comparing to jumps governed by stresses, cf. Sect. 4 below, and also the discussion about energy versus stress or global versus local minimization in mathematical literature [4, 38] and in engineering [18], and also the examples [3, Sect. 9] or [21, Example 7.1]. Therefore a nonvanishing \( \mu \) is vitally important and rather desirable in the context of fracture mechanics, cf. also [35] for numerical experiments, in contrast to the other mentioned applications where the phenomenon of nonvanishing \( \mu \) is related “only” to a possible lack of regularity of weak solutions and is not entirely clear whether it has some physical justification and supported experimental evidence. One can also say that nonconvexity of the stored energy in our delamination problem leads generically to loss of regularity related to nonvanishing measures \( \mu \geq 0 \), in contrast e.g. to the mere bulk nonconvexity which, for nonsimple materials under smooth loading, may lead to regularity and in the limit no additional energy is dissipated via vanishing Kelvin-Voigt viscosity, as shown in [28]. In contrast to a scalar-valued total additionally dissipated energy occurring in temporary jumps as in [23, 38], our \( \mu \) records spatial distribution of such energy. Anyhow, it seems similarly related to a lack of temporal regularity of the solution, as the following assertion shows at least in a particular case:

**Proposition 3.5** (No defect measures in regular solutions). If nonsimple material delaminating in Mode II, i.e. \( \mathbb{C}_H > 0 \) and (1.2b), is considered and if \((u, z, \mu)\) in Proposition 3.2 would be so regular that \( \dot{u} \) would not concentrate, i.e. \( u \in W^{1,1}(I; H^2(\Omega; \mathbb{R}^d)) \), then \( \mu = 0 \) and the convergence (3.6c) would even be strong in \( L^1(\bar{Q}) \).

**Proof.** Argumenting like in [28, Prop. 2], we first use (2.5c) written now with \( q = 0 \) and \( t = T \) and re-organized, so that we have:

\[
(3.15) \quad \limsup_{\nu \to 0} \int_Q \nu \mathcal{C}(\dot{u}_\nu) : \varepsilon(\dot{u}_\nu) + \nu \mathbb{C}_H \nabla \varepsilon(\dot{u}_\nu) : \nabla \varepsilon(\dot{u}_\nu) \, dx \, dt = \mathcal{E}(0, u_0, z_0) - \liminf_{\nu \to 0} \mathcal{E}(T, u_\nu(T), z_\nu(T)) + \lim_{\nu \to 0} \int_{t=0}^T \langle T, u_\nu \rangle \, dt \leq \mathcal{E}(0, u_0, z_0) - \mathcal{E}(T, u(T), z(T)) + \int_{t=0}^T \langle T, u \rangle \, dt.
\]

Here we used (3.6b) and (3.8) for \( t = T \) and the (weak\( \times \)weak\*)-lower semicontinuity of \( \mathcal{E}(T, \cdot, \cdot, \cdot) \). Having \( \dot{u} \) at disposal, namely in \( L^1(I; \mathcal{H}(\Omega; \mathbb{R}^d)) \) by assumption, we can test the momentum equilibrium (3.7a)
written as equality with $v$ instead of $v-u(t)$ as (1.2b) is considered. It allows to make the by-part integration in time as far as $u$ concerns and, combined with the semi-stability (3.7b) holding a.e. and at the initial time by assumption (3.1c,d), we can obtain the lower energy inequality also for $z$-flow rule. Altogether,

$$
\tag{3.16} \mathcal{E}(T, u(T), z(T)) + \int_{I_C} \alpha(z_0-z(t)) \, dS \geq \mathcal{E}(0, u_0, z_0) + \int_0^T \langle \dot{f}, u \rangle \, dt.
$$

In fact, together with the upper energy balance (3.7c) written for $t = T$ (as we can if written as an inequality "\leq") and with $\mu$-term omitted (which again leads to the inequality "\leq" since $\mu \geq 0$), we have even equality in (3.16), cf. also [30, Step 4 in Sect.8] or [31, Prop.5.4]. Plugging (3.16) into (3.15) gives

$\lim_{t_\nu \to 0} \int_{\Omega} \nu \mathcal{C}c(u_\nu); e(\dot{u}_\nu) + \nu \mathcal{C}H e(\dot{u}_\nu); \nabla e(\dot{u}_\nu) \, dx \, dt = 0,$

from (2.5d) with $\kappa > 0$ and $r > d-1$; cf. [34, Sect.1.6.1]. This would simplify the arguments for (3.9) as

The assumption about Mode II in Proposition 3.5 can possibly be avoided by a regularization of the unilateral constraint and then another limit passage like in [29, Sect.6], yet, if successful, it would be very technical.

In any case, it should be emphasized that the assumption $u \in W^{1,1}(I; H^2(\Omega; \mathbb{R}^d))$ in Proposition 3.5 is merely speculative and $\mu \neq 0$ is to be preferred; cf. also the explicit example in Sect. 4 which works exactly equally for nonsimple material.

**Remark 3.6.** (Slow-loading limit.) One can interpret alternatively the above asymptotic analysis $\nu \to 0$ as rescaling time under very slow external loading of a fixed visco-elastic material when neglecting inertial effects, cf. also Remark 5.3 below.

**Remark 3.7.** Any BV-mapping, and in particular $z \in BV(\bar{I}; \mathcal{L}^1(I_C))$, is weakly continuous with the exception of at most countable number of time instances, say $t \in D$, i.e. $z \in C_{w^*}(\bar{I}; D; \mathcal{L}^\infty(I_C))$ when taken also the $\alpha$-priori $\mathcal{L}^\infty$-estimate into account with $D$ at most countable. Because of the (weak*-strong)-continuity of the mapping $I \times \mathcal{L}^\infty(I_C) \to \mathcal{H}(\Omega; \mathbb{R}^d): (t, z) \mapsto u := \text{solution to the Euler-Lagrange inequality} (3.7a)$ written with $u$ (resp. $z$) in place of $u(t)$ (resp. $z(t)$), we have also $u \in C(\bar{I}); \mathcal{H}(\Omega; \mathbb{R}^d))$. Besides, $t \mapsto \int_0^T \int_{\Omega} \mu \, dx \, dt$ is in $BV(\bar{I})$ and thus also in $C(\bar{I}; D)$ if $D$ is taken appropriately. As $\mathcal{E} : I \times \mathcal{H}(\Omega; \mathbb{R}^d) \times \mathcal{L}^\infty(I_C) \to \mathcal{R}$ is (strong*weak*)-continuous if restricted on $\{z \in \mathcal{L}^\infty(I_C); z(\cdot) \in [0, 1] \text{ a.e.}\}$, we have $\mathcal{E}(\cdot, u(\cdot), z(\cdot)) \in C(\bar{I}; D)$ for such $D$. Therefore, (3.7c) is valid even for all $t \in \bar{I}; D$. Also (3.6a) holds for all $t \in \bar{I}; D$.

**Remark 3.8.** (Some other solution concepts.) The semi-energetic solutions with $\mu = 0$ even full stability, i.e.

$$
\tag{3.17} \forall (\bar{u}, \bar{z}) \in \mathcal{H}(\Omega; \mathbb{R}^d) \times \mathcal{L}^\infty(I_C), \quad 0 \leq \bar{z} \leq z(t) : \quad \mathcal{E}(t, u(t), z(t)) \leq \mathcal{E}(t, \bar{u}, \bar{z}) - \int_{I_C} \alpha(z_0-z(t)) \, dS,
$$

satisfied even for any $t \in \bar{I}$ and with (3.7c) holding even for all $t \in I$ are called the energetic solutions. The concept of energetic solutions has been invented in [19, 25]. Of course, (3.17) implies both (3.7a) and (3.7b) to hold for all $t \in I$. The existence of energetic solutions for our delamination problem is by [16] for $\mathcal{C}_H = 0$ and straightforwardly modified for $\mathcal{C}_H > 0$; actually, instead of the semi-implicit formula (6.1), one can pass to the limit in the fully-implicit formula used directly for $\rho = 0$ and $\nu = 0$. For $(u, z, \mu)$ with a general measure $\mu \geq 0$, the couple $(u, z)$ itself satisfied the modification of (3.7c), namely

$$
\tag{3.18} \mathcal{E}(t_2, u(t_2), z(t_2)) + \int_{I_C} \alpha(z(t_1)-z(t_2)) \, dS \leq \mathcal{E}(t_1, u(t_1), z(t_1)) + \int_{t_1}^{t_2} \langle \dot{f}, u \rangle \, dt
$$

for a.a. $0 \leq t_1 < t_2 \leq T$. The couples $(u, z)$ satisfying (3.7a,b) for a.a. $t \in I$ and (3.18) for all $t_1 < t_2$ are called local solutions, which is the concept invented in [40], cf. also [20]. It should be emphasized, however, that unfortunately we do not prove here (3.18) for all $t_1 < t_2$ because we are unable to prove $\mathcal{E}(t, u_\nu(t), z_\nu(t)) \to \mathcal{E}(t, u(t), z(t))$ for each $t$ but only for such $t$ where the convergence (3.6a) holds.

**Remark 3.9.** The concepts of solutions in Definition 3.3 and in Remark 3.8 possess at least generically (i.e. for a.a. time intervals, if not for all intervals) two important attributes pronounced in this context in [19, 20]: the so-called concatenation property (i.e. possibility of continuation of the evolution by mere gluing solutions on time intervals) and the restriction property (i.e. by shortening $T$ and possibly starting from a positive time $t$ with the corresponding initial condition $u(t)$ and $z(t)$, one again obtains the solution to that evolution system).

**Remark 3.10.** (Gradient delamination.) Analogously to the gradient theory for bulk damage [31, 39], one can consider a gradient theory for the delamination by adding a term $\int_{I_C} \kappa |\nabla z|^r |\nabla z|^\kappa \, dS$ to $\mathcal{E}$ from (2.5d) with $\kappa > 0$ and $r > d-1$; cf. [34, Sect.1.6.1]. This would simplify the arguments for (3.9) as
To be more precise, the formula (6.1) must be then modified; e.g. one can take \( \tilde{z}_\nu := (\tilde{z} - \|z_\nu - z(t)\|_{C^1(\Gamma_0)})^+ \). Also, it would allow for a nonlinear dependence \( \mathcal{K}(z) \) instead of \( z \mathcal{K} \), which might describe the so-called cohesive contacts (if \( \mathcal{K}(\cdot) \) would be convex) or weakening of fracture toughness if \( \mathcal{K}(\cdot) \) would be concave. If \( r \leq d-1 \), one would have to use a more sophisticated mutual recovery sequence [39].

Remark 3.11. (Cohesive contacts and their optimization.) If \( \mathcal{K}(\cdot) \) would be as in Remark 3.10 but in a special form \( \mathcal{K}(z) = \gamma_0(z)\mathcal{K}_0 \) with \( \gamma_0 \) strongly convex, one can think about making \( \mathcal{E}(t, u, \cdot) \) strongly convex by enhancing \( \mathcal{E} \) in (2.5d) by the term \( -\int_{\Gamma_c} \alpha z \, dS \) while forgetting the \( \alpha \)-term in \( \mathcal{R}_\nu \) and then by considering the transformation \( z \to \gamma(\tilde{z}) \) with a continuous monotone strongly concave \( \gamma = \gamma^{-1} : [0, 1] \to [0, 1] \) such that \( \gamma(0) = 0 \) and \( \gamma(1) = 1 \). Then, in terms of \( \tilde{z} \), the boundary term in \( \mathcal{E} \) takes the form \( \int_{\Gamma_c} \frac{1}{2} \tilde{z} \mathcal{K}_0 u + \kappa |\nabla\tilde{z}|^r - \alpha \gamma(\tilde{z}) \, dS - \langle f(t), u \rangle \). This makes still the transformed problem mechanically equivalent to the original one, although thermodynamically there is a difference in dissipated and stored energy if the heat production is considered, cf. [30]. Usually, the gradient term is not dominant (i.e. \( \kappa \) is small) and then it is not much big modification to neglect the \( \tilde{z} \)-dependence of \( \gamma' \) in this term, in particular if \( \gamma_0 \) is not far from to be affine so that \( \gamma' = 1/((\gamma_0 \circ \gamma_0) \sim 1) \). Then the new stored energy

\[
\mathcal{E}'(t, u, z) = \int_\Omega \frac{1}{2} C_{zz} e(u) e(u) + \frac{1}{2} C_{uu} \nabla e(u) \nabla e(u) \, dx + \int_{\Gamma_c} \frac{1}{2} \mathcal{K}_0 u + \kappa |\nabla z|^r - \alpha \gamma(z) \, dS - \langle f(t), u \rangle
\]

is strongly convex. The semi-implicit formula (6.1) then possesses uniquely determined discrete solutions which, in addition, depend continuously on the data of the problem. To be more precise, the formula (6.1) is to be modified correspondingly, namely by replacing \( \alpha \) with \( \alpha \gamma(z)^{z_+} \) and by adding a surface \( r \)-Laplacian \( -\div(z^r |\nabla z|^r z) \) into the flow rule on the last line of (6.1d); this can be easily seen by using the modified \( \mathcal{E} \) in (6.3). One can then think about some optimization or identification problem governed by such a delamination problem at least on the discretised level. After a further spatial discretisation, it yields a problem of the so-called Mathematical Programming with Evolution Equilibrium Constraints (MPEEC) for which one can evaluate a certain gradient information about the control-to-state mapping which can further be used for an efficient optimization strategies, cf. [15].

4. A “1-dimensional” example. It is interesting to compare the solution obtained by vanishing Kelvin-Voigt viscosity in the bulk due to Definition 3.3(ii) with the energetic solutions from Remark 3.8. In addition, we will compare it also with the solution obtained by vanishing viscosity in the adhesive (=no viscosity in the bulk, i.e. \( \nu = 0 \), but viscosity \( \nu_{ad} > 0 \) in the flow rule for \( z \) which, however, is again only small and eventually vanishes); this last concept means that \( \mathcal{E} \in \mathcal{K}_0 + \frac{1}{2} |\nabla u + N_{[0,1]}(z) | \) in (1.1d) is replaced by

\[
\mathcal{E}(t, u, z) = \int_\Omega \frac{1}{2} C_{zz} e(u) e(u) + \frac{1}{2} C_{uu} \nabla e(u) \nabla e(u) \, dx + \int_{\Gamma_c} \frac{1}{2} \mathcal{K}_0 u + \kappa |\nabla z|^r - \alpha \gamma(z) \, dS - \langle f(t), u \rangle
\]

which is, in fact, the rather standard way to introduce viscosity especially if \( u \) would be considered eliminated as (4.10) below, which was investigated in a general non-convex or fracture-mechanics context e.g. in [14, 17, 40] or in other mechanical context in [2, 3, 5, 10] or even in more abstract framework in [8, 22, 23].

Let us do it in a one-dimensional situation as on Fig. 1, essentially consisting from two springs in series, the left one undergoing a damage and representing thus the adhesive. Thus \( u \) and \( z \) are just scalar variables, the whole problem has only 2 degrees of freedom, and everything can be made quite explicit; in fact, as the inertia and thus possible waves are neglected, in fact this example is essentially 0-dimensional, which is why the quotes have been used in the section title. Let us make a simple experiment by considering the Dirichlet load starting from zero and growing in time with a constant speed \( v_0 > 0 \), i.e. \( u_0(t) = v_0 t \). We deal with the energies \( \mathcal{E} : I \times \mathbb{R} \rightarrow \mathbb{R} \cup \{+\infty\} \) and \( \mathcal{R}_0 : \mathbb{R} \rightarrow \mathbb{R} \cup \{+\infty\} \) given by

\[
\mathcal{E}(t, u, z) = \begin{cases} 
\frac{1}{2} K z u^2 + \frac{1}{2} C |u - v_0 t|^2 & \text{if } 0 \leq z \leq 1, \\
+\infty & \text{otherwise,} 
\end{cases} \quad \mathcal{R}_0(z) = \begin{cases} 
\alpha |z| & \text{if } z \leq 0, \\
+\infty & \text{otherwise,} 
\end{cases}
\]

with \( K > 0 \) and \( C > 0 \) just scalars. As we pull the springs, we do not need to consider a possible Signorini-type constraint \( u \geq 0 \) that would not be active in this regime anyhow. Our goal is to calculate the time when the damageable spring (=adhesive) breaks. Of course, we start with \( z(0) = z_0 = 1 \).
that we see in the limit the nontrivial defect measure, being spatially distributed uniformly in the bulk and
\[ \sigma \]

where \( \alpha \) is nontrivial, i.e. \( (4.3) \)

\[ \nu \]

Fig. 1. A one-dimensional delamination described by (4.2); a damageable spring (=adhesive) in series with visco-elastic bulk (=Kelvin-Voigt material) loaded by time-varying Dirichlet condition (=hard-device load); \( \epsilon \) means the strain provided the specimen has a unit length.

**Kelvin-Voigt-approximable solution:** Analyzing the semi-stability \( \delta(t, u(t), z(t)) \leq \delta(t, u(t), \tilde{z}) + \mathcal{R}_0(\tilde{z} - z(t)) \) for (4.2), i.e. \( \frac{1}{2}K(\tilde{z} - z)^2 + \alpha|\tilde{z} - z| \geq 0 \) for all \( \tilde{z} \leq z = 1 \), cf. also (2.8), we can see that the rupture will not happen for \( t < t_{KV} \) with \( t_{KV} \) as the time when the elastic energy of the fully bonded adhesive reaches the activation threshold \( \alpha \), i.e. \( \frac{1}{2}Ku(t)^2 = \alpha \). This means, by using (4.9) below for \( z = 1 \), that \( \frac{1}{2}K(v_0\mathcal{C}t_{KV}/(\mathcal{C}+\mathcal{K}))^2 = \alpha \), from which we can see that the delamination happens at the time

(4.3) \[ t_{KV} = \frac{\mathcal{K}+\mathcal{C}}{v_0\mathcal{C}}\sqrt{\frac{2\alpha}{\mathcal{K}}} \]

In this example, it is interesting that the defect measure can explicitly be identified and, in particular, it is nontrivial, i.e. \( \nu \neq 0 \). For \( \nu > 0 \), we can explicitly calculate that

(4.4) \[ u_{\nu}(t) = a_0t + b_\nu(1-e^{-t/t_\nu}), \quad a_0 = \frac{\mathcal{C}}{\mathcal{C}+\mathcal{K}}v_0, \quad b_\nu = -\nu\frac{\mathcal{C}K}{(\mathcal{C}+\mathcal{K})^2}v_0, \quad t_\nu = \nu\frac{\mathcal{C}}{\mathcal{C}+\mathcal{K}} \]

before the rupture happens, i.e. until \( \frac{1}{2}K\mathcal{C}(t)^2 \) with \( \epsilon_\nu(t) := v_0(t-u_{\nu}(t)) \) reaches the threshold \( \alpha \), or equivalently until the stress \( \sigma_\nu(t) := Ku_\nu(t) = \nu\mathcal{C}\epsilon_\nu(t) + \mathcal{C}\mathcal{C}_\nu(t) \) does not reach the threshold \( \sqrt{2\alpha} \), cf. Fig. 2 (left). Let us denote this time by \( t_{KV,\nu} \). In fact, \( t_{KV,\nu} \) is determined only rather implicitly as a solution of the transcendental equation \( (a_0t_{KV,\nu} + b_\nu(1-e^{-t_{KV,\nu}/t_\nu}))^2 = 2\nu/\mathcal{K} \). As \( b_\nu,t_\nu \to 0 \) for \( \nu \to 0 \), we have \( t_{KV,\nu} \to t_{KV} \) with \( t_{KV} \) from (4.3). Note also that \( \mathcal{C}\mathcal{C}_\nu(0) = \frac{\mathcal{K}}{\nu}v_0(1-e^{-t/\nu}) \) and that \( \mathcal{C}_\nu(0) = 0 \). As there is no cohesion in our model, the complete delamination occurs at one time, namely \( t_{KV,\nu} \), under the displacement \( u_{KV,\nu} := a_0t_{KV,\nu} + b_\nu(1-e^{-t_{KV,\nu}/t_\nu}) \). After the delamination, the displacement evolves as \( u(t) = v_0t - (u_{KV,\nu} - v_0t_{KV,\nu})e^{-(t-t_{KV,\nu})/\nu} \) or, in terms of the deformation of the bulk \( \epsilon_\nu \) from Fig. 1, as

(4.5) \[ \epsilon_\nu(t) = \epsilon_{KV,\nu}e^{(t-t_{KV,\nu})/\nu} \]

In view of (4.4) and (4.5), the rate of dissipated energy due to the viscosity before and after delamination is

(4.6) \[ \text{rate of viscous dissipation} = \nu\mathcal{C}\mathcal{C}_\nu^2 \]

\[ = \begin{cases} \frac{\mathcal{K}^2}{(\mathcal{C}+\mathcal{K})^2}v_0^2(1-e^{-t/\nu})^2 & \text{for } t < t_{KV,\nu}, \\ \frac{1}{\nu}\mathcal{C}\mathcal{C}_\nu e^{-2(t-t_{KV,\nu})/\nu} & \text{for } t > t_{KV,\nu}. \end{cases} \]

cf. Fig. 2 (right). Before the delamination, the viscous dissipation rate can obviously be estimated as \( \nu\mathcal{C}\mathcal{C}_\nu^2 v_0^2(1-e^{-t/\nu})^2 \leq \nu\frac{\mathcal{K}^2}{(\mathcal{C}+\mathcal{K})^2}v_0^2 = \delta(\nu) \to 0 \) for \( \nu \to 0 \), so that the overall viscous dissipation on the a-priori bounded interval before delamination \( [0, t_{KV,\nu}] \subset [0, t_{KV}] \) tends to zero for \( \nu \to 0 \), and therefore there is no contribution to the defect measure before the delamination occurs. The dissipation rate converges to 0 pointwise also after delamination, i.e. at each \( t > t_{KV,\nu} \), but the overall viscous energy dissipated after delamination on the interval \( (t_{KV,\nu}, +\infty) \) remains positive, namely

(4.7) \[ \int_{t_{KV,\nu}}^{+\infty} \nu\mathcal{C}\mathcal{C}_\nu^2 dt = \int_{t_{KV,\nu}}^{+\infty} \frac{1}{\nu}\mathcal{C}\mathcal{C}_\nu e^{-2(t-t_{KV,\nu})/\nu}dt = \int_{0}^{+\infty} \frac{1}{\nu}\mathcal{C}\mathcal{C}_\nu e^{-2t/\nu}dt \]

\[ = \frac{1}{2}\mathcal{C}\mathcal{C}_\nu e^{-2t/\nu} \bigg|_{t=0}^{+\infty} = \frac{1}{2}\mathcal{C}\mathcal{C}_\nu = \frac{\mathcal{K}}{\mathcal{C}}\alpha \neq 0, \]

where \( \mathcal{C}\mathcal{C}_\nu \) is the deformation of the inviscid bulk at time \( t_{KV} \) when rupture occurs again under the stress \( \sigma = \sqrt{2\alpha} \), cf. Fig. 2 (left), i.e. \( \mathcal{C}\mathcal{C}_\nu = \sigma = \sqrt{2\alpha} \), from which the last equality in (4.7) follows. Therefore, although there is no contribution to the defect measure on the intervals \( (t, +\infty) \) for \( t > t_{KV} \), we see in the limit the nontrivial defect measure, being spatially distributed uniformly in the bulk and
but concentrated in time at a single time of rupture \( t = t_{KV} \), cf. Fig. 2 (right). In this inviscid limit, the energetical picture during rupture is now clear: above, we have shown that

\[
\frac{1}{2} C_\varepsilon t_{KV}^2 = \alpha \frac{K}{C} \quad \text{and} \quad \frac{1}{2} K \alpha^2 = \alpha,
\]

which says that all energy stored in the bulk is recorded in the defect measure during the rupture and all energy stored in the adhesive is dissipated by the delamination. Moreover, the latter equation in (4.8) clearly reveals that this Kelvin-Voigt-approximable solution counts with stress-driven delamination rather than energy-driven one. This is perfectly in accord with conventional engineering handling of fracture mechanics (which, however, is typically performed without any attempt of rigorous analysis and with energy balance completely ignored).

**Energetic solution:** Let us analyze the stability (3.17) at time \( t \), which takes the form of minimizing \( (u, z) \mapsto \frac{1}{2} K z u^2 + \frac{1}{2} C |u - v_0| t^2 - \alpha z \) subject to \( u \geq 0 \) and \( 1 \geq z \geq 0 \). We can easily calculate \( u = \Omega(t, z) \) from the optimality condition as

\[
(4.9) \quad u = \Omega(t, z) = \frac{v_0 C t}{K + C}.
\]

So we face the minimization of \( z \mapsto J(t, z) - \alpha z \) subject to \( z \in [0, z] \) for the initially undamaged adhesive \( z = 1 \) with the reduced functional

\[
(4.10) \quad J(t, z) = J(t, \Omega(t, z), z) = \frac{v_0^2 K C^2 z t^2 + v_0^2 K^2 C z^2 t^2}{2(K + C)^2}.
\]

One can calculate

\[
(4.11) \quad J_z(t, z) = v_0^2 t^2 \left( \frac{K C^2 + 2 K^2 C z}{2(K + C)^2} - \frac{K^2 C^2 z + K^3 C z^2}{(K + C)^3} \right)
\]

\[
= v_0^2 t^2 \left( \frac{K^2 C^2 z + 2 K^2 C z^2 + KC + 2 K^2 C z^2 - 2 K^2 C z^2 - 2 K^3 C z^2}{2(K + C)^3} \right)
\]

\[
= v_0^2 t^2 \frac{2 K C}{2(K + C)^3}
\]

and then

\[
(4.12) \quad J_{zz}(t, z) = \frac{v_0^2 t^2 K^2 C z}{(K + C)^3} < 0,
\]

so that \( J(t, \cdot) \) is concave on \([0, 1]\) for any \( t \geq 0 \). Obviously, \( J(t, 0) = 0 \) and \( J(\cdot, 1) \) is increasing, so the rupture occurs when \( J(t, 1) = \alpha \), let us denote such \( t \) by \( t_{res} \); here the concavity of \( J(t, \cdot) \) is also used. The condition \( J(t_{res}, 1) = \alpha \) gives

\[
(4.12) \quad t_{res} = \sqrt{\frac{2 \alpha K + 2 \alpha C}{v_0^2 K C}}.
\]

The energetics during the rupture works, in contrast to (4.8), in such a way that the total stored energy \( \frac{1}{2} K u_{ES}^2 + \frac{1}{2} C \varepsilon_{ES}^2 \) reaches the threshold \( \alpha \), where naturally \( u_{ES} \) and \( \varepsilon_{ES} \) are determined by the kinematic constraint \( u_{ES} + \varepsilon_{ES} = v_0 t_{res} \) and the stress equilibrium \( K u_{ES} = C \varepsilon_{ES} \). This means that all this stored energy is dissipated to delamination. It reveals that the energetic-solution concept makes, as intuitively
related to its name, the delamination energy-driven rather than stress-driven and explains why always $t_{es} < t_{kv}$, as obviously seen also from (4.3) compared to (4.12).

**Vanishing-viscosity solution** (i.e. viscosity in the adhesive): Analyzing the local stability, we can see that the delamination occurs at the time, let us denote it by $t_{vv}$, when $\mathcal{F}_2(t_{vv},1) = \alpha$. Evaluating $\mathcal{F}_2(t,z) = (t_{vv},1)$, cf. (4.11), gives again

$$
(4.13)
\quad t_{vv} = \frac{K+C}{v_0 C} \sqrt{\frac{2\alpha}{K}} = t_{kv}.
$$

It is known that, in general, $t_{vv} \leq \frac{K+C}{v_0 C} \sqrt{2\alpha/K}$ as the local stability itself is not decisive without energy identity but here one can see that the flow rule for $z$, i.e. the initial-value problem $\nu \dot{z} + N_{[-\infty,\alpha]}(\dot{z}) \in \mathcal{F}_2(t,z) = \frac{1}{2}v_0^2 t^2 K C^2/(K z + C)^2$ with $z(0) = 1$ will have a constant solution $z = 1$ on $[0, \frac{K+C}{v_0 C} \sqrt{2\alpha/K}]$ no matter how $\nu \alpha > 0$ is small, so also the vanishing-viscosity limit must be constant at least up to the time $t_{vv}$.

Note that, considering physical dimensions in this 1-dimensional case as $[\alpha] = J$, $[\nu] = m$, $[K] = [C] = J/m^2$, and $[v_0] = m/s$, all formulas (4.12), (4.3), and (4.13) give indeed a correct physical dimension of time, i.e. $[t_{es}] = [t_{kv}] = [t_{vv}] = s$. To summarize, we always have

$$
(4.14)
\quad t_{es} < t_{kv} = t_{vv}.
$$

In particular, energetic solutions always jump earlier than any of these viscous solutions.

**Remark 4.1.** *(Large-bulk effect.)* Fixing the material properties of the adhesive, i.e. $\alpha$ and $K$, we can have a look into the asymptotics for $C \to 0$. Inspecting the above formulas (4.12), (4.3), and (4.13), we can see that

$$
(4.15)
\quad t_{es} \sim \sqrt{\frac{2\alpha}{v_0^2 C}} \quad \text{and} \quad t_{kv} = t_{vv} \sim \sqrt{\frac{2\alpha K}{v_0 C}} \quad \text{for } C \to 0.
$$

Note that, asymptotically, the difference between the energetic and the viscous solutions is very big because $t_{kv}/t_{es} \sim \sqrt{K/C} \to \infty$ for $C \to 0$. This supports the (not surprising) engineering observation that very large bulk with capacity to deposit very much elastic energy even under small stress (which here means a long spring, i.e. $C$ small) causes indeed too early delamination when the concept of energetic solution applies. In other words, we can also say that this effect applies for a very brittle adhesive, i.e. $K \to \infty$, and in particular to the brittle delamination which can be obtained as a limit if $K \to \infty$, cf. [36]. It is in agreement with the formula (4.7) for the energy taken by the defect measure, i.e. $\alpha K/C$, which may be very large if $K$ is large or $C$ is small.

**Remark 4.2.** *(Weak-adhesive effect.)* Still keeping $\alpha$ and $K$ fixed, we can have a look into the opposite asymptotics, namely $C \to \infty$. It yields

$$
(4.16)
\quad t_{es} \sim \sqrt{\frac{2\alpha}{v_0^2 K}} \sim t_{kv} = t_{vv} \quad \text{for } C \to \infty.
$$

Thus for a very rigid bulk (or, in other words, for a very “weak” adhesive of a “bubble-gum” type), the rupture time (asymptotically) does not depend on $C$ (because the Dirichlet load is nearly immediately transferred to the adhesive) and all the above concepts give (asymptotically) the same result. Due to (4.7), the energy of the defect measure is then very small only.

**Remark 4.3.** If the semistability would represent a nonconvex problem, one could distinguish between $t_{kv}$ and $t_{vv}$. This can be realized by letting $z$ to be involved nonlinearly in $\mathcal{F}$ from (4.2). The nonconvexity of $\mathcal{F}(t,u,\cdot)$ is then related with a delamination weakening, cf. also Remark 3.10. Then we can expect $t_{kv} < t_{vv}$. In multidimensional problems, even if $\mathcal{F}(t,u,\cdot)$ would be convex, one can also expect different responses for viscosity in the bulk versus viscosity in the adhesive, like in the finite-dimensional example in [38, Sec. 2.2].

**5. An excursion to dynamical problems in Mode II.** Dynamical problems (i.e. a fixed $g > 0$) bring necessity of certain modifications of the previous results/arguments. In particular, we keep certain information about $u_\nu$ even in the limit $\nu \to 0$ even in simple materials but, on the other hand, e.g. the arguments (3.4) and (3.9) do not work. We confine ourselves to Mode II, i.e. (1.2b), for both existence of solutions due to Definition 2.1 and for the limit passage.

**Proposition 5.1 (Vanishing viscosity limit).** Let $g > 0$, (1.2b), (2.1), and (3.1) hold, $g > 0$ is fixed, $\nu > 0$, and let also $v_0 \in L^2(\Omega;\mathbb{R}^d)$. Then the weak solutions $(u_\nu, z_\nu)$ exist and satisfy, together with (3.3),
also

\[(5.1a) \quad \| \hat{u}_\nu \|_{L^\infty(I; L^2(\Omega; \mathbb{R}^d))} \leq C,\]

\[(5.1b) \quad \| \ddot{u}_\nu \|_{L^2(I; H^{-1}(\Omega; \mathbb{R}^d))} \leq C.\]

Moreover, there is \(u \in L^\infty(I; H(\Omega; \mathbb{R}^d))\) with \(\hat{u} \in C_0(I; L^2(\Omega; \mathbb{R}^d))\), \(z \in BV(I; L^1(\Gamma_c)) \cap L^\infty(\Sigma_c)\), and \(\mu \in \text{Meas}(Q)\) such that, for a subsequence as \(\nu \to 0\), the convergence (3.6b,c) holds, the following modification of (3.6a) holds:

\[(5.2a) \quad u_\nu(t) \to u(t) \quad \text{in} \ H(\Omega; \mathbb{R}^d),\]

\[(5.2b) \quad \dot{u}_\nu(t) \to \dot{u}(t) \quad \text{in} \ L^2(\Omega; \mathbb{R}^d)\]

for all \(t \in [0, T]\). Moreover, any \((u, z, \mu)\) obtained by this way satisfies the momentum equilibrium

\[(5.3a) \quad \int_Q C(e(u); e(v)) + C_H \nabla e(u) : \nabla e(v) - \nu \dot{u} \cdot \dot{v} \, dx \, dt + \int_\Omega \hat{u}(T) \cdot v(T) \, dx + \int_\Sigma_C z \mathbb{K} u \cdot v \, dS = \int_0^T \langle \dot{f}, v \rangle \, dt + \int_\Omega \omega_0(v(0)) \, dx\]

and, for all \(t \in [0, T]\), the semistability (3.7b) and the energy inequality holds:

\[(5.3b) \quad E(t, u(t), z(t)) + \frac{\alpha}{4} \int_\Omega \| \dot{u}(t) \|^2 \, dx + \int_0^t \int_\Omega \mu(\mu \, dx \, dt) + \int_{\Gamma_c} \alpha(\gamma_0 - z(t)) \, dS\]

\[\leq E(0, u_0, z_0) + \int_\Omega \frac{\alpha}{2} |v_0|^2 \, dx + \int_0^t \langle \dot{f}, u \rangle \, dt.\]

**Proof.** For any \(\nu > 0\), existence of a weak solution \((u_\nu, z_\nu)\) is a special case of [30]; cf. also Appendix. Here (1.2b) is used.

The estimates (3.3) follow as before, and now the boundedness of the kinetic energy yields also (5.1a). The estimate (5.1b) follows from the identity

\[
\| \ddot{u}_\nu \|_{L^2(I; H^{-1}(\Omega; \mathbb{R}^d))} = \sup_{\|v\|_{L^2(I; H(\Omega; \mathbb{R}^d))} \leq 1} \left( \ddot{u}_\nu, v \right)
\]

\[
\leq \sup_{\|v\|_{L^2(I; H(\Omega; \mathbb{R}^d))} \leq 1} \int_Q C(e(u_\nu + \nu \dot{u}_\nu); e(v)) + C_H \nabla e(u_\nu + \nu \dot{u}_\nu) : \nabla e(v) \, dx \, dt + \int_\Sigma_C z \mathbb{K} u \cdot v \, dS - \int_0^T \langle \dot{f}, v \rangle \, dt
\]

and then from the already obtained estimates (3.3); in particular, \(\int_Q C(e(\nu \dot{u}_\nu); e(v)) \, dx \, dt\) can be estimated as (3.10) before, and analogously also \(\int_\Omega C_H \nabla e(\nu \dot{u}_\nu) : \nabla e(v) \, dx \, dt\). Here again (1.2b) has been used. Having the a-priori estimates, we use Banach’s and Helly’s selection principles to obtain converging subsequence fulfilling (3.6b,c) and (5.2); note that for (5.2b), we have used the estimate (5.1a) together with the information about \(\ddot{u}_\nu\) from (5.1b).

As before, for any \(v\) smooth, we have again (3.12) now simply with \(v - u_\nu\). Thus the limit passage in the momentum balance (2.5a) again with \(v\) instead of \(v - u_\nu\) is simple. The terminal term \(\nu \hat{u}(T) \cdot v(T)\) in (2.5a) converges as \(\dot{u}_\nu(T) \to \dot{u}(T)\) in \(L^2(\Omega; \mathbb{R}^d)\), cf. (5.2b). Thus we obtain (5.3a).

Passage to the limit in the energy balance (2.5c) is simple by weak lower semicontinuity of the convex form \((u, z, \dot{u}) \to \mathcal{F}(\dot{u}) + \mathcal{E}(t, u, z)\) and the convergence (3.6b) and (5.2). Using also (3.6c), we obtain the inequality (5.3b) for any \(t\).

Passage in the semi-stability remains as in Step 3 of the proof of Proposition 3.2. □

**Remark 5.2.** The definition of a weak solution based on (5.3) with (3.7b) would unfortunately lose the restriction property, in contrast to Definition 3.3 and Remark 3.9. Yet, sometimes, even such concepts are considered, cf. so-called dissipative trajectories in [38].

**Remark 5.3. (Slow-loading limit at Mode II.)** Like Remark 3.6, the quasi-static rate-independent model considered in Sections 3–4 can also be obtained by scaling for slow loading even if both \(g > 0\) and \(\nu > 0\) are considered fixed. Up to change of physical units, we may consider the original material parameters \(g = 1\) and \(\nu = 1\). Then, scaling time as \(\nu t\) with \(\nu\) in the position of a scaling coefficient (or to be understood also as “loading frequency”) results to

\[(5.4) \quad \nu^2 \ddot{u}_\nu - \text{div}(C_{\nu, u} - \text{div}C_H \nabla \epsilon_{\nu}) = f, \quad \epsilon_{\nu} = \epsilon(u_\nu, \dot{u}_\nu) = \nu \epsilon(\dot{u}_\nu) + \epsilon(u_\nu)\]
on $Q$ with analogous scaling in (1.1c.d). For passing the loading frequency $\nu$ to 0, we expectedly obtain the quasistatic rate-independent situation $\varrho \to 0$ and $\nu = 0$. In contrast to Remark 3.6, (5.4) does not neglect inertial forces and again we will confine ourselves on Mode II, i.e. (1.2b).

We have again the a-priori estimates (3.3) and, instead of (5.1), we now have only

$$\|\dot{u}_\nu\|_{L^\infty(I; L^2(\Omega; \mathbb{R}^d))} \leq C/\nu.$$  

The convergence (3.6) again holds. For this, it is important that the strong convergence of $u_\delta$ again holds as in Proposition 3.2, in contrast to Proposition 5.1. Yet, the arguments (3.9) must be modified. We have to consider a regularization $u_\delta \in H^1(I; H(\Omega; \mathbb{R}^d))$ of $u$ and then, instead of (3.9), estimate

$$\int_0^T \langle \mathcal{E}'_u(t, u_\delta, z_\delta) - \mathcal{E}'_u(t, u, z_\delta), u_\delta - u \rangle \, dt$$

$$\leq \int_0^T \left( \nu \mathcal{C}(\dot{u}_\nu) : e(u_\delta - u) + \nu \mathcal{C}_H \nabla e(\dot{u}_\nu) : \nabla e(u_\delta - u) \right) \, dx$$

$$+ \int_{\Gamma_\mathcal{C}} z_\delta K_{u_\nu}(u_\delta - u_\nu) \, dS - \int_0^T \nu^2 \dot{u}_\nu(T) \cdot (u_\delta(T) - u_\nu(T)) - \nu^2 v_0 \cdot (u_\delta(0) - u_\nu(0)) \, dx$$

$$\leq \int_0^T \left( \nu \mathcal{C}(\dot{u}_\nu) : e(u_\delta) + \nu \mathcal{C}_H \nabla e(u_\nu) : \nabla e(u_\delta) \right) \, dx + \int_{\Gamma_\mathcal{C}} z_\delta K_{u_\nu}(u_\delta - u_\nu) \, dS$$

$$- \int_0^T \nu^2 \dot{u}_\nu(T) \cdot (u_\delta(T) - u_\nu(T)) - \nu^2 v_0 \cdot (u_\delta(0) - u_\nu(0)) + \nu \mathcal{C}(u_\nu) : e(u_\nu) + \nu \mathcal{C}_H \nabla e(u_\nu) : \nabla e(u_\nu) \, dx$$

$$\rightarrow \int_0^T \left( \mathcal{E}'_u(t, u, z, u_\delta - u_\nu) + \int_{\Gamma_\mathcal{C}} z \mathbb{K} u_\nu(u_\delta - u_\nu) \, dS \right) \, dt.$$  

We used the estimate (3.3a), which allows for estimation

$$\left| \int_Q \nu^2 \dot{u}_\nu(T) \cdot (u_\delta(T) - u_\nu(T)) \, dx \right| \leq \nu^2 N \|\dot{u}_\nu\|_{L^2(I; H(\Omega; \mathbb{R}^d))} \|u_\delta - u_\nu\|_{L^2(I; H(\Omega; \mathbb{R}^d))} = O(\nu) \rightarrow 0$$

for $\nu \rightarrow 0$ with $N$ the norm of the embedding $H(\Omega) \subset L^2(\Omega)$, and also we used (5.5), which, together again with (3.3a), allows for estimation

$$\left| \int_\Omega \nu^2 \dot{u}_\nu(T) \cdot (u_\delta(T) - u_\nu(T)) \, dx \right| \leq \nu^2 N \|\dot{u}_\nu\|_{L^2(I; H(\Omega; \mathbb{R}^d))} \|u_\delta(T) - u_\nu(T)\|_{H(\Omega; \mathbb{R}^d)} = O(\sqrt{\nu}) \rightarrow 0.$$  

Eventually, we can make the last expression in (5.6) arbitrarily close to 0 when passing $u_\delta \rightarrow u$ in $L^2(I; H(\Omega; \mathbb{R}^d))$. As a result, we have recovered the strong convergence (3.6a).

The energy equality corresponding to (5.4) involves the kinetic energy $\frac{1}{2} \int_\Omega \nu^2 |u_\nu(t)|^2 \, dx \geq 0$ which does not seem to allow for any limit passage, and forgetting this term gives (5.3b) now without $\varrho$-terms but again only as an inequality inspite that we now have the strong convergence (3.6a). The limit passage in the equilibrium is simple, noting that the inertial term can be estimated as

$$\int_Q \nu^2 \dot{u}_\nu \cdot \dot{v} \, dxdt \leq \nu^2 \|\dot{u}_\nu\|_{L^\infty(I; L^2(\Omega; \mathbb{R}^d))} \|\dot{v}\|_{L^1(I; L^2(\Omega; \mathbb{R}^d))} = O(\nu) \rightarrow 0$$

due to (5.5). Note that again Mode II has been exploited here, otherwise we would have the term $\int_Q -\nu^2 \dot{u}_\nu \cdot (\dot{v} - \ddot{u}_\nu) \, dxdt$, cf. (2.5a), which could not be limited. Passage in the semi-stability remains as in Step 3 of the proof of Proposition 3.2.

Altogether, the limit couple $(u, z)$ with $u \in L^\infty(I; H(\Omega; \mathbb{R}^d))$ and $z \in BV(\bar{I}; L^1(\Gamma_\mathcal{C})) \cap L^\infty(\Sigma_c)$ satisfies
Thus consider: facilitate the a-priori estimates, we again consider the transformation (2.2). Using a time step according to Definition 2.1 can be obtained rather constructively by a semi-implicit time discretisation case of a cohesive contact as in Remark 3.11, we have even (6.1a)

\[
\text{minimize } E(ku) \text{ subject to } u \in H(\Omega; \mathbb{R}^4), \quad u|_{\Gamma} = 0.
\]

(6.1b) \( u_{\tau}^k = 0 \) on \( \Gamma_{0} \),

(6.1c) \( \tilde{t}(\epsilon_\tau) = g_\tau^h \)

\[
\begin{cases}
\alpha & \leq \tau \\
u_\tau^k & = 0,
\end{cases}
\]

(6.1d) \( \dot{z}_\tau^k \leq z_{\tau}^{k-1}, \quad \ddot{z}_\tau^k \leq \alpha, \quad \ddot{z}_\tau^k = 0, \quad \ddot{z}_\tau^k = 0 \)

with \( \epsilon_k \) from (1.1e) and with \( f_\tau^k = f(k\tau) \) and \( g_\tau^h = g(k\tau) \) with \( f \) and \( g \) from (2.2), and proceeding recursively for \( k = 1, \ldots, T/\tau \in \mathbb{N} \) with starting for \( k = 1 \) from

\[
\begin{align*}
0^0 & = u_0, \\
0^0 & = z_0, \\
0^0 & = u^{0}_{\tau} = \tau v_0.
\end{align*}
\]

The adjective “semi-implicit” is related with usage of \( z_{\tau}^{k-1} \) in the first complementarity problem in (6.1d), instead of \( z_{\tau}^{k-1} \) which would lead to a fully implicit formula. Such usage of \( z_{\tau}^{k-1} \) leads to the decoupling of the problem: first we can solve (6.1a-c) with the first complementarity problem in (6.1d) for \( u_{\tau}^k \) and only after the rest of (6.1d) for \( z_{\tau}^k \). In addition, we can employ the variational structure of both decoupled problems. We thus obtain two convex minimization problems: first, we are to solve

\[
\begin{align*}
\text{minimize } & \, \mathcal{E}(k\tau, u, z_{\tau}^{k-1}) + \mathcal{R}^h(u - u_{\tau}^{k-1}, 0) + \tau^2 \mathcal{J}(u - 2u_{\tau}^{k-1} + u_{\tau}^{k-2}, \tau^2) \\
\text{subject to } & \, u \in H(\Omega; \mathbb{R}^4), \quad u|_{\Gamma} = 0, \quad u|_{\Gamma_{\tau}} \geq 0
\end{align*}
\]

and, denoting its unique solution by \( u_{\tau}^k \), then we solve

\[
\begin{align*}
\text{minimize } & \, \mathcal{E}(k\tau, u_{\tau}^k, z) + \mathcal{R}_0(z - z_{\tau}^{k-1}) \\
\text{subject to } & \, z \in L^\infty(\Gamma_{\tau}), \quad 0 \leq z \leq z_{\tau}^{k-1}
\end{align*}
\]
with the kinetic energy $\mathcal{T}$ from (2.7), and with the stored energy $\mathcal{E}$ and the dissipation (pseudo)potential $\mathcal{R}_{\nu}$ defined here by

$$
\mathcal{E}(t, u, z) = \int_{\Omega} \frac{1}{2} C e(u) : e(u) + \frac{\nu}{2} C_H \nabla e(u) : \nabla e(u) \, dx + \int_{\Gamma_c} \frac{1}{2} z \kappa w : dS - \langle f(t), u \rangle,
$$

$$
\mathcal{R}_{\nu}(\tilde{u}, \tilde{z}) = \int_{\Omega} \frac{1}{2} C \tilde{e}(\tilde{u}) : e(\tilde{u}) + \frac{\nu}{2} C_H \nabla e(\tilde{u}) : \nabla e(\tilde{u}) \, d\tilde{x} - \int_{\Gamma_c} a \tilde{z} : dS.
$$

Note that the constraints $u|_{\Gamma_c} \geq 0$, $0 \leq z \leq 1$, and $\tilde{z} \leq 0$, originally contained in $\mathcal{E}$ and $\mathcal{R}_{\nu}$ in (2.5d) and (2.7), are now included in (6.3) so that we can equivalently use the smooth functionals $\mathcal{E}(t, \cdot, \cdot)$ and $\mathcal{R}_{\nu}$ in (6.4). Also note that $\mathcal{R}_{\nu}(\tilde{u}, \cdot)$ is degree-1 homogeneous so that the factor $\tau$ does not show up in the functional in (6.3), in contrast to the degree-2 homogeneous functionals $\mathcal{T}$ and $\mathcal{R}_{\nu}(\cdot, \cdot, \cdot)$ in (6.3a).

The discrete analog of (2.5a) is by summing the optimality conditions for (6.3a) written at $u = u^k_{\tau}$, i.e.

$$
\int_0^1 \frac{u^k_{\tau} - 2u^{k-1}_{\tau} + u^{k-2}_{\tau}}{\tau^2} (v - u^k_{\tau}) + C e(u^k_{\tau}) : e(v - u^k_{\tau}) + C_H \nabla e(u^k_{\tau}) : \nabla e(v - u^k_{\tau}) \, dx
$$

$$
+ \int_{\Gamma_c} z^{k-1}_{\tau} \kappa u^k_{\tau} : (v - u^k_{\tau}) \, dS \geq \langle f^k_{\tau}, v - u^k_{\tau} \rangle
$$

with $e^k_{\tau}$ from (6.1a) and $f^k_{\tau} = \langle f(k \tau) \rangle$, tested by an arbitrary test-function $v = v^k_{\tau}$ and making by-part summation of the inertial term. The discrete analog of the semistability (2.5b) is by comparison of values of (6.3b) at $z^k_{\tau}$ and an arbitrary $\tilde{z}$:

$$
\mathcal{E}(k \tau, u^k_{\tau}, z^k_{\tau}) \leq \mathcal{E}(k \tau, u^k_{\tau}, \tilde{z}) + \mathcal{R}_0(\tilde{z} - z^k_{\tau} - 1) - \mathcal{R}_0(z^k_{\tau} - z^{k-1}_{\tau} - 1) \leq \mathcal{E}(k \tau, u^k_{\tau}, \tilde{z}) + \mathcal{R}_0(\tilde{z} - z^{k-1}_{\tau} - 1),
$$

where we also used the degree-1 homogeneity and the convexity of $\mathcal{R}_0(\cdot)$ which yields the triangle inequality $\mathcal{R}_0(\tilde{z} - z^{k-1}_{\tau} - 1) \leq \mathcal{R}_0(z^k_{\tau} - z^{k-1}_{\tau}) + \mathcal{R}_0(\tilde{z} - z^{k}_{\tau})$. A discrete analog of (2.5c) as an inequality $\leq \mathcal{E}$ can be obtained by testing the optimality conditions for (6.3a) and (6.3b) respectively by $u^k_{\tau} - u^{k-1}_{\tau}$ and $z^k_{\tau} - z^{k-1}_{\tau}$ (which, in fact, means plugging $v = u^k_{\tau} - 1$ into (6.5) for the former test), and by adding it, benefiting from the cancellation of the terms $\pm \mathcal{E}(k \tau, u^k_{\tau}, z^k_{\tau} - 1)$ and by the separate convexity of $\mathcal{E}(k \tau, \cdot, \cdot)$.

The discrete analog of the estimation (3.4) arises by summing (6.5) written for $v = u^k_{\tau} - 1$ with (6.5) written with $k - 1$ instead of $k$ for $v = u^k_{\tau}$. Considering $\rho = 0$ as in (3.4), this gives

$$
\int_{\Omega} \nu C e\left(\frac{u^k_{\tau} - 2u^{k-1}_{\tau} + u^{k-2}_{\tau}}{\tau^2}\right) : e\left(\frac{u^k_{\tau} - u^{k-1}_{\tau}}{\tau}\right) + C e\left(\frac{u^k_{\tau} - u^{k-1}_{\tau}}{\tau}\right) : e\left(\frac{u^k_{\tau} - u^{k-1}_{\tau}}{\tau}\right)
$$

$$
+ \nu C_H \nabla e\left(\frac{u^k_{\tau} - 2u^{k-1}_{\tau} + u^{k-2}_{\tau}}{\tau^2}\right) : \nabla e\left(\frac{u^k_{\tau} - u^{k-1}_{\tau}}{\tau}\right) + C_H \nabla e\left(\frac{u^k_{\tau} - u^{k-1}_{\tau}}{\tau}\right) : \nabla e\left(\frac{u^k_{\tau} - u^{k-1}_{\tau}}{\tau}\right) \, dx
$$

$$
+ \int_{\Gamma_c} z^{k-1}_{\tau} \kappa \frac{u^k_{\tau} - u^{k-1}_{\tau}}{\tau} : \frac{u^k_{\tau} - u^{k-1}_{\tau}}{\tau} \, dS \leq \langle f^k_{\tau} - f^{k-1}_{\tau}, u^k_{\tau} - u^{k-1}_{\tau} \rangle - \int_{\Gamma_c} z^{k-1}_{\tau} \kappa \frac{u^k_{\tau} - u^{k-1}_{\tau}}{\tau} : \frac{u^k_{\tau} - u^{k-1}_{\tau}}{\tau} \, dS,
$$

which obviously is just what was used in the heuristical variant (3.4). Using the discrete by-part integration (i.e. summation), we obtain the discrete analog of the a-priori estimate (3.3d); here we need to have $\|u^k_{\tau} - u^{k-1}_{\tau}\|_{H^2(\Omega ; \mathbb{R}^d)}$ bounded for $k = 0$. For this, we use (3.2a) which implies that $u_0$ solves (6.3a) with $\mathcal{T} \equiv 0$ and which further allows us to consider simply

$$
u^1 := u_0 = u_0.
$$

Limit passage for $\tau \to 0$ towards an energetic solution according Definition 2.1 can be proved if $\rho = 0$ or if $D(x)$ is a linear space for a.a. $x \in \Gamma_c$ like, e.g., in (1.2b); this restriction is needed to keep the energy balance in the rate-dependent part, i.e.

$$
\int_{\Omega} \frac{1}{2} C e(u_0(t)) : e(u_0(t)) + \frac{1}{2} C_H \nabla e(u_0(t)) : \nabla e(u_0(t)) + \frac{\rho}{2} \|\dot{u}_0(t)\|^2 \, dx - \langle f(0), u_0(t) \rangle
$$

$$
+ \int_0^t \frac{\nu}{2} C e(\dot{u}_0(t)) : e(\dot{u}_0(t)) + \frac{\nu}{2} C_H \nabla e(\dot{u}_0(t)) : \nabla e(\dot{u}_0(t)) \, dx \, dt
$$

$$
= \int_{\Omega} \frac{1}{2} C e(u_0) : e(u_0) + \frac{1}{2} C_H \nabla e(u_0) : \nabla e(u_0) + \frac{\rho}{2} \|u_0\|^2 \, dx - \langle f(0), u_0 \rangle + \int_0^t \langle \dot{f}, u_0 \rangle \, dt;
$$

in Mode II, this equality is simple while for a Signorini contact, it can be proved by limiting a regularization if $\rho = 0$, cf. [30, Step 4 in Sects. 8-9]. Therefore, we excluded dynamical Signorini contact for which this is
not known, cf. also [30, Remark 5.3]. Then the only two essential points are the proof of the semistability (2.5b) and the “≥” inequality in (2.5c). The former proof is again by an explicit construction of a mutual recovery sequence, which here can be exactly the one in (3.13) but now parametrized by \( \tau \) instead of \( \nu \); cf. also [24, Formula (4.35)] or [36, Formula (3.71)]. Then, having (2.5b) proved, the latter point leading, together with (6.8) the total energy balance (2.5c) is by the Riemann-sum trick as devised in [6] and used in the context of rate-independent sub-system coupled with rate-dependent part in [31] and in particular in the context in adhesive contacts in [30]; here the semi-stability of the initial condition (3.1c,d) is used.

Remark 6.1. (Joint convergence.) Having the time-discrete viscous scheme, one may think about a convergence for both \( \nu \rightarrow 0 \) and \( \tau \rightarrow 0 \) simultaneously to obtain the Kelvin-Voigt-approximable solution to the quasistatic rate-independent problem. Yet, only conditional convergence can be expected because obviously \( \lim_{\nu \rightarrow 0} \nu C e(u_{\nu,\tau}) + \nu C \nabla e(u_{\nu,\tau}) = 0 \) for any \( \tau > 0 \) fixed, where \( u_{\nu,\tau} \) denotes the piecewise affine continuous interpolant of the values \( (u^{k}_{\nu,\tau})_{k=0}^{T/\tau} \); in fact, this convergence is even strong in \( W^{1,\infty}(I; L^{1}(\Omega)) \). Therefore clearly \( \lim_{\tau \rightarrow 0} \lim_{\nu \rightarrow 0} \nu C e(u_{\nu,\tau}) + \nu C \nabla e(u_{\nu,\tau}) = 0 \) and the energy balance (3.7c) would be obtained with \( \mu = 0 \) as an inequality only. Therefore, only some conditional convergence will lead to the desired \( \mu \) and the energy equality (3.7c) as before. Obviously, we must consider rather \( \lim_{\nu \rightarrow 0} \lim_{\tau \rightarrow 0} \), which yields an implicit stability criterion \( \mathcal{T} : \mathbb{R}^{\tau} \rightarrow \mathbb{R}^{\tau} \) so that

\[
\nu C e(u_{\nu,\tau}) + \nu C \nabla e(u_{\nu,\tau}) = \mu,
\]

cf. the arguments in the proof of [1, Cor. 4.8(ii)]. In fact, this implicit stability criterion should also control the discrete energy imbalance in order to obtain in the limit the energy equality (3.7c), cf. the numerical experiments in [35]. For abstract quasistatic problems, an explicit stability criterion, namely \( \tau \leq o(\nu) \), i.e. \( \tau/\nu \rightarrow 0 \), was derived in [23, Thm. 4.11] but a different concept (of the so-called BV-solutions) was used and the rate-independent part of the dissipation potential was finite and coercive, which does not precisely fit with our \( \mathcal{R}_{\nu} \).

Remark 6.2. (Numerical implementation.) The decoupled problems (6.3) yield (after further spatial discretisation) constructive numerical strategy as it results to two convex linear-quadratic mathematical programs for which efficient numerical solvers are available. As no gradient of \( \mathcal{R}_{\nu} \) is available, after a suitable discretisation of \( \Gamma_{c} \), decoupling on particular boundary elements. An \( \mathcal{R}_{\nu}(\cdot, z) \) and \( \mathcal{T}(\cdot) \) are quadratic functionals, (6.3a) is a quadratic problem with the only constraint on \( \Gamma_{c} \) which are linear. This allows for using efficiently the boundary-element method (BEM) combined with linear-quadratic programming treating variables on \( \Gamma_{c} \). If \( \nu = 0 \) is considered as in Sect. 3, we can furthermore use BEM with the same Poincaré-Steklov operators as in the static case for a new variable \( v^{\nu}_{\tau} = u^{\nu}_{\tau} + \nu(u^{\nu}_{\tau} - u^{\nu}_{\tau})/\tau \); then, in terms of this new variable, one obviously has the Kelvin-Voigt strain \( e^{\nu}_{\tau} = e(v^{\nu}_{\tau}) \), the velocity \( (u^{\nu}_{\tau} - u^{\nu}_{\tau})/\tau = (v^{\nu}_{\tau} - v^{\nu}_{\tau})/\tau + \nu \), and the displacement \( u^{\nu}_{\tau} = (\tau v^{\nu}_{\tau} + \nu u^{\nu}_{\tau})/\tau + \nu \), which is used in (6.1) and then in the corresponding minimization problems (6.3), too.

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REFERENCES


