Convergence of finite element methods for nonlinear convective problems

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Abstract

In this short note, we give an overview of the tools needed to estimate the error of finite element methods applied to nonlinear convective problems with smooth solutions. These results along with their generalizations to fully discrete explicit and implicit schemes represent a new, promising technique first outlined by [5] and extended in [4].

1 Continuous problem and discretization

Let $\Omega \subset \mathbb{R}^d, d \in \mathbb{N}$, be a bounded open polyhedral domain. We treat the following nonlinear convective problem. Find $u : \Omega \times (0,T) \to \mathbb{R}$ such that

$$\text{a)} \quad \frac{\partial u}{\partial t} + \operatorname{div} \mathbf{f}(u) = g \quad \text{in } Q_T, \tag{1}$$

$$\text{b)} \quad u\big|_{\Gamma_D \times (0,T)} = 0, \tag{2}$$

$$\text{d)} \quad u(x,0) = u^0(x), \quad x \in \Omega. \tag{3}$$

Here $g : Q_T \to \mathbb{R}$ and $u^0 : \Omega \to \mathbb{R}$ are given functions and $\Gamma_D \subset \partial\Omega$ has positive measure. We assume that the *convective fluxes* $\mathbf{f} = (f_1, \cdots, f_d) \in (C_b^2(\mathbb{R}))^d = (C^2(\mathbb{R}) \cap W^{2,\infty}(\mathbb{R}))^d$, hence $\mathbf{f}$ and $\mathbf{f}' = (f_1', \cdots, f_d')$ are *globally Lipschitz continuous*. The technique presented in [4] allows to generalize the results also to $\mathbf{f} = (f_1, \cdots, f_d) \in (C^2(\mathbb{R}))^d$, i.e. the locally Lipschitz case.

As for the boundary condition (2), we assume in our analysis that $\Gamma_N := \partial\Omega \setminus \Gamma_D$ is an outflow boundary for the exact or approximate solution, i.e. e.g. $\Gamma_N \subseteq \{x \in \partial\Omega;\, \mathbf{f}'(u(x,t)).\mathbf{n} \geq 0\}$.

We discretize problem (1)-(3) using the standard conforming p-order finite element method. Over a quasi-uniform, shape regular, conforming system of triangulations $\{\mathcal{T}_h\}_{h\in(0,h_0)}$, $h_0 > 0$ of $\overline{\Omega}$ we define the space of globally continuous piecewise p-order polynomial functions $S_h = \{v \in C(\overline{\Omega});\, v|_{\Gamma_D} = 0,\, v|_K \in P^p(K) \forall K \in \mathcal{T}_h\}$. We set $h = \max_{K\in\mathcal{T}_h} \operatorname{diam}(K)$. In this function space we introduce the space semidiscrete version of problem (1). We seek $u_h \in C^1([0,T]; S_h)$ such that $u_h(0) = u_h^0 \approx u^0$ and

$$\frac{d}{dt}\big(u_h(t), \varphi_h\big) + b\big(u_h(t), \varphi_h\big) = l\big(\varphi_h\big)(t), \quad \forall \varphi_h \in S_h,\, t \in (0,T). \tag{4}$$

Here, we have introduced the *convective* and *right-hand side forms* defined for $v, \varphi \in H^1(\Omega)$:

$$b(v,\varphi) = -\int_\Omega \mathbf{f}(v)\cdot \nabla\varphi \,\mathrm{d}x + \int_{\Gamma_N} \mathbf{f}(v)\cdot \mathbf{n}\,\varphi \,\mathrm{d}S, \qquad l(\varphi)(t) = \int_\Omega g(t)\varphi \,\mathrm{d}x.$$

We note that a sufficiently regular exact solution u of problem (1) also satisfies (4) for all $\varphi_h \in S_h$, i.e. we have *Galerkin orthogonality property* of the error.

2 Key estimates of the convective terms

As usual in apriori error analysis, we assume that the weak solution u is sufficiently regular:

$$u, u_t \in L^2\big(0, T; H^{p+1}(\Omega)\big), \quad u \in L^\infty(0, T; W^{1,\infty}(\Omega)).$$

Let $\eta_h(t) = u(t) - \Pi_h u(t) \in H^{p+1}(\Omega)$ and $\xi_h(t) = \Pi_h u(t) - u_h(t) \in S_h$, where $\Pi_h v$ is the $L^2(\Omega)$-projection of v on S_h. Then we can write the error e_h as $e_h(t) := u(t) - u_h(t) = \eta_h(t) + \xi_h(t)$. By C we will denote a generic constant independent of h. In our analysis, we shall need the following standard inverse inequalities

$$|v_h|_{H^1} \le C_I h^{-1} ||v_h||,$$
$$\|v_h\|_\infty \le C_I h^{-d/2} \|v_h\|$$

and approximation properties of η, (cf. [2]):

$$\|\eta_h(t)\| \le Ch^{p+1} |u(t)|_{H^{p+1}},$$
$$\big\|\tfrac{\partial \eta_h(t)}{\partial t}\big\| \le Ch^{p+1} \big|\tfrac{\partial u(t)}{\partial t}\big|_{H^{p+1}},$$

The key estimate of the convective terms is inspired by the work [5], originally derived for the DG method. A complete proof of our case can be found in [4].

Lemma 2.1. *There exists a constant $C \ge 0$ independent of h, t, such that*

$$b\big(u_h(t), \xi(t)\big) - b\big(u(t), \xi(t)\big) \le C\Big(1 + \frac{\|e_h(t)\|_\infty}{h}\Big)\big(h^{2p+1}|u(t)|^2_{H^{p+1}} + \|\xi(t)\|^2\big). \tag{5}$$

Proof. The key trick of the estimate is performing a Taylor expansion of $\mathbf{f}$ with respect to u:

$$\mathbf{f}(u) - \mathbf{f}(u_h) = \mathbf{f}'(u)\xi + \mathbf{f}'(u)\eta - \frac{1}{2}\mathbf{f}''_{u,u_h} e_h^2,$$

where $\mathbf{f}''_{u,u_h}$ is the Lagrange form of the remainder of the Taylor expansion. Substituting into the definition of $b(\cdot,\cdot)$, we obtain the interior terms

$$\int_\Omega \mathbf{f}'(u)\xi \cdot \nabla\xi \,\mathrm{d}x + \int_\Omega \mathbf{f}'(u)\eta \cdot \nabla\xi \,\mathrm{d}x - \frac{1}{2}\int_\Omega \mathbf{f}''_{u,u_h} e_h^2 \cdot \nabla\xi \,\mathrm{d}x.$$

Estimating these terms by (5) is straightforward, using the inverse inequalities and estimates of η. A similar procedure is done for the boundary terms of $b(\cdot,\cdot)$. □

3 Error analysis of the semidiscrete scheme

We proceed similarly as for a parabolic equation. By Galerkin orthogonality, we subtract the equations for u and u_h and set $\varphi_h := \xi_h(t) \in S_h$. Since $\big(\frac{\partial \xi_h}{\partial t}, \xi_h\big) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\xi_h\|^2$, we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\xi_h(t)\|^2 = b\big(u_h(t), \xi_h(t)\big) - b\big(u(t), \xi_h(t)\big) - \Big(\frac{\partial \eta_h(t)}{\partial t}, \xi_h(t)\Big).$$

For the last right-hand side term, we use the Cauchy and Young's inequalities and estimates of η and Lemma 2.1 for the convective terms. We integrate from 0 to $t \in [0, T]$,

$$\|\xi_h(t)\|^2 \le C\int_0^t \Big(1 + \tfrac{\|e_h(\vartheta)\|_\infty}{h}\Big)\Big(h^{2p+1}|u(\vartheta)|^2_{H^{p+1}} + h^{2p+2}|u_t(\vartheta)|^2_{H^{p+1}} + \|\xi_h(\vartheta)\|^2\Big)\mathrm{d}\vartheta, \tag{6}$$

where $C \geq 0$ is independent of h, t. For simplicity, we have assumed that $\xi_h(0) = 0$, i.e. $u_h^0 = \Pi_h u^0$. Otherwise we must assume e.g. $\|\xi_h(0)\|^2 \leq Ch^{2p+1}|u^0|^2_{H^{p+1}}$ and include this term in the estimate.

We notice that if we knew *apriori* that $\|e_h\|_\infty = O(h)$ then the unpleasant term $h^{-1}\|e_h\|_\infty$ in (6) would be $O(1)$. Thus we could simply apply the standard Gronwall lemma to obtain the desired error estimates. We state this formally:

Lemma 3.1. *Let $t \in [0, T]$ and $p \geq d/2$. If $\|e_h(\vartheta)\| \leq h^{1+d/2}$ for all $\vartheta \in [0, t]$, then there exists a constant C_T independent of h, t such that*

$$\max_{\vartheta\in[0,t]} \|e_h(\vartheta)\|^2 \leq C_T^2 h^{2p+1}. \tag{7}$$

Proof. The assumptions imply, by the inverse inequality and estimates of η, that

$$\begin{aligned}\|e_h(\vartheta)\|_\infty &\leq \|\eta_h(\vartheta)\|_\infty + \|\xi_h(\vartheta)\|_\infty \leq Ch|u(t)|_{W^{1,\infty}} + C_I h^{-d/2}\|\xi_h(\vartheta)\| \\ &\leq Ch + C_I h^{-d/2}\|e_h(\vartheta)\| + C_I h^{-d/2}\|\eta_h(\vartheta)\| \leq Ch + Ch^{p+1-d/2}|u(\vartheta)|_{H^{p+1}(\Omega)} \leq Ch,\end{aligned}$$

where the constant C is independent of h, ϑ, t. Using this estimate in (6) gives us

$$\|\xi_h(t)\|^2 \leq \tilde{C}h^{2p+1} + C\int_0^t \|\xi_h(\vartheta)\|^2 \,\mathrm{d}\vartheta, \tag{8}$$

where the constants $\widetilde{C}, C$ are independent of h, t. Gronwall's inequality applied to (8) states that there exists a constant $\widetilde{C}_T$, independent of h, t, such that

$$\max_{\vartheta\in[0,t]} \|\xi_h(\vartheta)\|^2 \leq \widetilde{C}_T h^{2p+1},$$

which along with similar estimates for η gives us (7). $\square$

Now it remains to get rid of the *apriori* assumption $\|e_h\|_\infty = O(h)$. For an explicit scheme, this can be done using mathematical induction. Starting from $\|e_h^0\| = O(h^{p+1/2})$, we prove:

$$\|e_h^n\| = O(h^{p+1/2}) \quad\Longrightarrow\quad \|e_h^{n+1}\|_\infty = O(h) \quad\Longrightarrow\quad \|e_h^{n+1}\| = O(h^{p+1/2}).$$

For the method of lines we have continuous time and hence cannot use mathematical induction straightforwardly. However, we can use some continuous version of mathematical induction, cf. [1], [3]. In our case, we can use the simplest version:

Lemma 3.2 (Continuous mathematical induction)**.** *Let $\varphi(t)$ be a propositional function depending on $t \in [0, T]$ such that*

(*i*) $\varphi(0)$ *is true,*
(*ii*) $\exists\delta_0 > 0 :\ \varphi(t)$ *implies* $\varphi(t + \delta)$, $\forall t \in [0, T]\ \forall\delta \in [0, \delta_0] : t + \delta \in [0, T]$.

Then $\varphi(t)$ holds for all $t \in [0, T]$.

Theorem 1 (Semidiscrete error estimate)**.** *Let $p > (1 + d)/2$. Let $h_1 > 0$ be such that $C_T h_1^{p+1/2} = \frac{1}{2}h_1^{1+d/2}$, where C_T is the constant from Lemma 3.1. Then for all $h \in (0, h_1]$ we have the estimate*

$$\max_{\vartheta\in[0,T]} \|e_h(\vartheta)\|^2 \leq C_T^2 h^{2p+1}.$$

Proof. Since $p > (1+d)/2$, h_1 is uniquely determined and $C_T h^{p+1/2} \le \frac{1}{2} h^{1+d/2}$ for all $h \in (0, h_1]$. We define the propositional function φ by

$$\varphi(t) \equiv \Big\{ \max_{\vartheta \in [0,t]} \|e_h(\vartheta)\|^2 \le C_T^2 h^{2p+1} \Big\}.$$

We shall use Lemma 3.2 to show that φ holds on $[0, T]$, hence $\varphi(T)$ holds.

(i) $\varphi(0)$ holds, since this is the error of the initial condition.

(ii) *Induction step*: We fix an arbitrary $h \in (0, h_1]$. Due to the regularity assumptions, the functions $u(\cdot), u_h(\cdot)$ are *uniformly continuous* function from $[0, T]$ to $L^2(\Omega)$. Therefore, there exists $\delta_0 > 0$, such that if $t \in [0, T), \delta \in [0, \delta_0]$, then $\|e_h(t+\delta) - e_h(t)\| \le \frac{1}{2} h^{1+d/2}$. Now let $t \in [0, T)$ and assume $\varphi(t)$ holds. Then $\varphi(t)$ implies $\|e_h(t)\| \le C_T h^{p+1/2} \le \frac{1}{2} h^{1+d/2}$. Let $\delta \in [0, \delta_0]$, then by uniform continuity

$$\|e_h(t+\delta)\| \le \|e_h(t)\| + \|e_h(t+\delta) - e_h(t)\| \le \tfrac{1}{2} h^{1+d/2} + \tfrac{1}{2} h^{1+d/2} = h^{1+d/2}.$$

This and $\varphi(t)$ implies that $\|e_h(s)\| \le h^{1+d/2}$ for $s \in [0, t] \cup [t, t+\delta] = [0, t+\delta]$. By Lemma 3.1, φ holds on $[0, t+\delta]$. This proves the "induction step" $\varphi(t) \Longrightarrow \varphi(t+\delta)$ for all $\delta \in [0, \delta_0]$. ☐

4 Conclusion

We gave a simple overview of the concepts used to obtain error estimates of smooth solutions of nonlinear convective problems. The results can be extended much further beyond this expository account. For example, for a fully discrete implicit scheme, similar estimates can be obtained after introducing a suitable continuation of the discrete solution. As mentioned, the technique can be extended to locally Lipschitz continuous nonlinearities as well. We refer to [4] for details.

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