Reconstruction Operators: From Finite Volumes to Discontinuous Galerkin

V. Kučera

Department of Numerical Mathematics, Faculty of Mathematics and Physics
Charles University in Prague.

1 Introduction

In this work we use the methodology of higher order finite volume (FV) and spectral volume (SV) schemes to introduce a reconstruction operator into the discontinuous Galerkin (DG) method. This operator constructs higher order piecewise polynomial reconstructions from the lower order DG scheme. This allows us to increase the accuracy of existing DG schemes with a problem-independent reconstruction procedure. Unlike the FVM, the reconstruction stencil has minimal size independent of the approximation order. Such a procedure was proposed already in [1] based on heuristic arguments, however we provide a more rigorous derivation, which justifies the increased order of accuracy. Furthermore, we provide an alternative construction of the reconstruction procedure based on the SV method. Numerical experiments are carried out.

2 Problem formulation and notation

Let $\Omega \subset I\!R^d$ be a bounded domain with a Lipschitz-continuous boundary $\partial\Omega$. Let $Q_T := \Omega \times (0,T)$. We treat the following nonlinear nonstationary scalar hyperbolic equation:

$$\frac{\partial u}{\partial t} + \mathrm{div}\,\mathbf{f}(\mathrm{u}) = 0 \quad \text{in } \mathrm{Q_T} \tag{1}$$

along with an appropriate initial and boundary condition. Here $\mathbf{f} = (f_1, \cdots, f_d)$ and $f_s, s = 1, \ldots, d$ are Lipschitz continuous fluxes in the direction $x_s, s = 1, \ldots, d$.

Let $\mathcal{T}_h$ be a partition (triangulation) of $\overline{\Omega}$ into a finite number of closed simplices $K \in \mathcal{T}_h$. By ∂K we denote the boundary of an element $K \in \mathcal{T}_h$ and set $h = \max_{K\in\mathcal{T}_h}\mathrm{diam}(K)$. Let $K, K' \in \mathcal{T}_h$. We say that K and K' are *neighbours*, if they share a common *face* $\Gamma \subset \partial K$. By $\mathcal{F}_h$ we denote the system of all faces of all $K \in \mathcal{T}_h$. Further, we define the set of all interior and boundary faces, by $\mathcal{F}_h^I = \{\Gamma \in \mathcal{F}_h;\ \Gamma \subset \Omega\}$ and $\mathcal{F}_h^B = \{\Gamma \in \mathcal{F}_h;\ \Gamma \subset \partial\Omega\}$.

For each $\Gamma \in \mathcal{F}_h$ we define a unit normal vector $\mathbf{n}_\Gamma$, such that for $\Gamma \in \mathcal{F}_h^B$, $\mathbf{n}_\Gamma$ is the outer normal to $\partial\Omega$. For each $\Gamma \in \mathcal{F}_h^I$ there exist two neighbours $K_\Gamma^{(L)}$, $K_\Gamma^{(R)} \in \mathcal{T}_h$. We use the convention that $\mathbf{n}_\Gamma$ is the outer normal to $K_\Gamma^{(L)}$. For v piecewise H^1 on $\mathcal{T}_h$ and $\Gamma \in \mathcal{F}_h^I$ we introduce

$$v|_\Gamma^{(L)} = \text{ the trace of } v|_{K_\Gamma^{(L)}} \text{ on } \Gamma, \quad v|_\Gamma^{(R)} = \text{ the trace of } v|_{K_\Gamma^{(R)}} \text{ on } \Gamma, \quad [v]_\Gamma = v|_\Gamma^{(L)} - v|_\Gamma^{(R)}.$$

On boundary edges $\Gamma \in \mathcal{F}_h^B$, we define $v|_\Gamma^{(R)} = 0$, $[v]_\Gamma = v|_\Gamma^{(L)}$. Let $n \in \mathbb{N}$. The approximate solution will be sought in the space of discontinuous piecewise polynomial functions

$$S_h^n = \{v;\, v|_K \in P^n(K), \forall K \in \mathcal{T}_h\},$$

where $P^n(K)$ denotes the space of all polynomials on K of degree $\leq n$.

3 Discontinuous Galerkin (DG) formulation

We multiply (1) by some $\varphi_h^n \in S_h^n$, integrate over $K \in \mathcal{T}_h$ and apply Green's theorem. By summing over all $K \in \mathcal{T}_h$ and rearranging, we get

$$\frac{d}{dt}\int_\Omega u(t)\,\varphi_h^n\,dx + \sum_{\Gamma\in\mathcal{F}_h}\int_\Gamma \mathbf{f}(u)\cdot\mathbf{n}\,[\varphi_h^n]\,dS - \sum_{K\in\mathcal{T}_h}\int_K \mathbf{f}(u)\cdot\nabla\varphi_h^n\,dx = 0. \tag{2}$$

As in the FV method, we introduce a *Lipschitz continuous, consistent* and *conservative* numerical flux $H(u,v,\mathbf{n}) \approx \mathbf{f}(u)\cdot\mathbf{n}$. Thus, we obtain the following standard DG formulation

$$\frac{d}{dt}\big(u_h(t),\varphi_h^n\big) + b_h\big(u_h(t),\varphi_h^n\big) = 0, \quad \forall\varphi_h^n \in S_h^n,\ \forall t\in(0,T),$$

where $b_h(\cdot,\cdot)$ is the *convective form* defined for v piecewise H^1 on $\mathcal{T}_h$:

$$b_h(v,\varphi) = \int_{\mathcal{F}_h} H(v^{(L)},v^{(R)},\mathbf{n})[\varphi]\,dS - \sum_{K\in\mathcal{T}_h}\int_K \mathbf{f}(v)\cdot\nabla\varphi\,dx.$$

4 Reconstructed discontinuous Galerkin (RDG) formulation

For $v\in L^2(\Omega)$, we denote by $\Pi_h^n v$ the $L^2(\Omega)$-projection of v on S_h^n:

$$\Pi_h^n v \in S_h^n, \quad (\Pi_h^n v - v,\, \varphi_h^n) = 0, \qquad \forall\,\varphi_h^n \in S_h^n. \tag{3}$$

The basis of the RDG method lies in the observation that (2) can be viewed as an equation for the evolution of $\Pi_h^n u(t)$, where u is the exact solution of (1), since, due to (3),

$$\frac{d}{dt}\int_\Omega \Pi_h^n u(t)\,\varphi_h^n\,dx + \int_{\mathcal{F}_h}\mathbf{f}(u)\cdot\mathbf{n}\,[\varphi_h^n]\,dS - \sum_{K\in\mathcal{T}_h}\int_K \mathbf{f}(u)\cdot\nabla\varphi_h^n\,dx = 0. \tag{4}$$

Now, let $N > n$ be an integer. We assume, that there exists a piecewise polynomial function $U_h^N(t)\in S_h^N$, which is an approximation of $u(t)$ of order $N+1$, i.e.

$$U_h^N(x,t) = u(x,t) + O(h^{N+1}), \quad \forall x\in\Omega,\ \forall t\in[0,T]. \tag{5}$$

This is possible, if u is sufficiently regular in space, e.g. $u(t)\in W^{N+1,\infty}(\Omega)$. Now we incorporate the approximation $U_h^N(t)$ into (4): the exact solution u satisfies

$$\frac{d}{dt}\big(\Pi_h^n u(t),\varphi_h^n\big) + b_h\big(U_h^N(t),\varphi_h^n\big) = E(\varphi_h^n), \quad \forall\varphi_h^n\in S_h^n,\ \forall t\in(0,T), \tag{6}$$

where $E(\varphi_h^n)$ is an error term which may be estimated using standard DG arguments, cf [2]

$$E(\varphi_h^n) = b_h\big(U_h^N(t),\varphi_h^n\big) - b_h\big(u(t),\varphi_h^n\big) = O(h^N)\|\varphi_h^n\|_{L^2(\Omega)}. \tag{7}$$

Definition 4.1 (Reconstruction problem.) *Let $v:\Omega\to I\!R$ be sufficiently regular. Given $\Pi_h^n v\in S_h^n$, find $v_h^N\in S_h^N$ such that $v - v_h^N = O(h^{N+1})$ in Ω. We define the corresponding reconstruction operator $R: S_h^n\to S_h^N$ by $R\,\Pi_h^n v := v_h^N$.*

By setting $U_h^N(t) := R\,\Pi_h^n u(t)$ in (6)-(7), we obtain the following equation for $\Pi_h^n u(t)$:

$$\frac{d}{dt}\big(\Pi_h^n u(t),\varphi_h^n\big) + b_h\big(R\,\Pi_h^n u(t),\varphi_h^n\big) = O(h^N)\|\varphi_h^n\|_{L^2(\Omega)}, \quad \forall\varphi_h^n\in S_h^n. \tag{8}$$

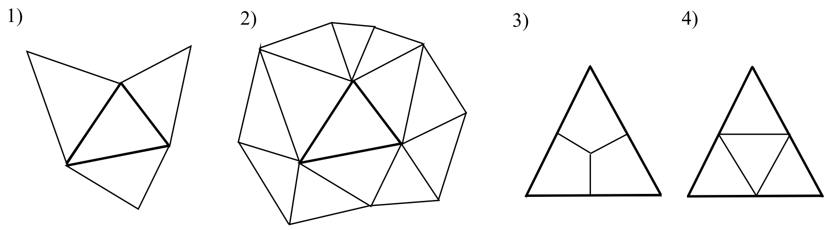

Figure 1: 1) FV stencil for linear reconstruction, 2) FV stencil for quadratic reconstruction, 3) Control volumes in a spectral volume for linear reconstruction, 4) Analogy to the SV approach for DG - partition of triangle into control volumes, e.g. cubic reconstruction from linear data.

By neglecting the right-hand side and approximating $u_h^n(t) \approx \Pi_h^n u(t)$, we arrive at the following definition of the *reconstructed discontinuous Galerkin* (RDG) scheme. We seek u_h^n such that

$$\frac{d}{dt}\big(u_h^n(t), \varphi_h^n\big) + b_h\big(Ru_h^n(t), \varphi_h^n\big) = 0, \quad \forall \varphi_h^n \in S_h^n,\ \forall t \in (0,T). \tag{9}$$

There are several points worth mentioning:

- We have followed the methodology of higher order finite volume schemes and spectral volume schemes, cf. [5]. The basis of these schemes is an equation for the evolution of averages of u on individual elements (i.e. an equation for $\Pi_h^0 u(t)$). Equation (8) is a generalization to the case of higher order $L^2(\Omega)$-projections $\Pi_h^n u(t)$, $n \geq 0$.
- Both $u_h^n(t)$ and φ_h^n lie in S_h^n. Only $Ru_h^n(t)$ lies in the larger space S_h^N. Despite this fact, equation (8) indicates, that we may expect $u - Ru_h^n = O(h^{N+1})$, although $u - u_h^n = O(h^{n+1})$.
- Numerical quadrature must be used to evaluate integrals in (9). Since test functions are in S_h^n, as compared to S_h^N in the corresponding Nth order standard DG scheme, we may use lower order (i.e. more efficient) quadrature formulae as compared to standard DG.
- In practice, an explicit time discretization must be applied to (9). The upper limit on stable time steps, given by a CFL-like condition, is more restrictive with growing N. However, in the RDG scheme, stability properties are inherited from the lower order scheme, thus a larger time step is possible as compared to standard DG.

4.1 Construction of the reconstruction operator

'Standard' approach. We use a stencil (group of neighboring elements and the element under consideration) to build an Nth-degree polynomial approximation to u on the element under consideration ([3]). In the FV method, the von Neumann neighborhood of an element is used as a stencil to obtain a piecewise linear reconstruction, Figure 4, 1). For higher orders, the size of the stencil increases dramatically, Figure 4, 2), rendering higher degrees than quadratic impractical. For the RDG scheme, we need not increase the stencil size to obtain higher order accuracy, it suffices to increase the order of the underlying DG scheme.

As in the FV method, the reconstruction operator R is constructed so that $R\Pi_h^n$ is in some sense *polynomial preserving*. Specifically, for each element K and its stencil S, we require that

$$\Big(\big(R\Pi_h^n\big)\big|_S\, p\Big)\Big|_K = p\big|_K, \qquad \forall p \in P^N(S). \tag{10}$$

This allows us to study approximation properties of R using the Bramble-Hilbert technique as in standard finite element methods. The disadvantage of this approach is that for unstructured

N	$\|\|e_h\|\|_{L^\infty(\Omega)}$	α	$\|\|e_h\|\|_{L^2(\Omega)}$	α	$\|e_h\|_{H^1(\Omega,\mathcal{T}_h)}$	α
4	5.82E-03	–	3.49E-03	–	3.65E-02	–
8	7.53E-05	6.27	4.43E-05	6,30	1.06E-03	5,11
16	9.07E-07	6.38	5.95E-07	6,22	3.58E-05	4,89
32	1.82E-08	5.64	8.70E-09	6,10	1.16E-06	4,95
64	3.41E-10	5.74	1.33E-10	6,03	3.67E-08	4,98

Table 1: 1D advection, P^1 RDG scheme with P^5 reconstruction.

N	$\|\|e_h\|\|_{L^\infty(\Omega)}$	α	$\|\|e_h\|\|_{L^2(\Omega)}$	α	$\|e_h\|_{H^1(\Omega,\mathcal{T}_h)}$	α
4	2.90E-03	–	1.85E-03	–	1.63E-02	–
8	7.75E-06	8.55	3.56E-06	9.02	1.03E-04	7.30
16	2.10E-08	8.53	6.64E-09	9.07	4.34E-07	7.89
32	7.21E-11	8.18	4.02E-11	7.37	1.76E-09	7.94

Table 2: 1D advection, P^2 RDG scheme with P^8 reconstruction.

meshes, the coefficients of the reconstruction operator must be stored for each individual stencil.

Spectral volume approach. We start with a partition of Ω into so-called *spectral volumes* S, e.g. triangles in 2D. The triangulation $\mathcal{T}_h$ is formed by subdividing each spectral volume S into *control volumes* K, [5]. In the FV method, the order of accuracy of the reconstruction determines the number of control volumes to be generated in each spectral volume. For example, for linear reconstruction on a triangle, S is divided into three control volumes, Figure 4, 3). Again, in the RDG scheme, we may use only the smallest partition into control volumes, and increase the accuracy by increasing the order of the underlying DG scheme, cf. Figure 4, 4). Again, as in (10), R should be polynomial preserving. The advantage of this approach is that we may construct R only on a reference spectral volume, since all simplices are affine equivalent.

Numerical experiments. We treat the periodic advection of a 1D sine wave on uniform meshes. Experimental orders of accuracy α in various norms on meshes with N elements are given in Tables 1 and 2. The increase in accuracy due to reconstruction is clearly visible.

Acknowledgement: The work was supported by the project P201/11/P414 of the Czech Science Foundation.

References

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