Programs and Algorithms of Numerical Matematics 16
J. Chleboun, K. Segeth, J. Šístek, T. Vejchodský (Eds.)
Institute of Mathematics AS CR, Prague 2013

ERROR ESTIMATES FOR NONLINEAR CONVECTIVE PROBLEMS IN THE FINITE ELEMENT METHOD

Václav Kučera

Faculty of Mathematics and Physics, Charles University in Prague,
Sokolovská 83, 186 75 Praha 8, Czech Republic
vaclav.kucera@email.cz

Abstract

We describe the basic ideas needed to obtain apriori error estimates for a nonlinear convection diffusion equation discretized by higher order conforming finite elements. For simplicity of presentation, we derive the key estimates under simplified assumptions, e.g. Dirichlet-only boundary conditions. The resulting error estimate is obtained using continuous mathematical induction for the space semi-discrete scheme.

1. Continuous problem

Let $\Omega \subset \mathbb{R}^d, d \in \mathbb{N}$, be a bounded open polyhedral domain. We treat the following nonlinear convective problem. Find $u : \Omega \times (0,T) \to \mathbb{R}$ such that

$$\text{a)} \quad \frac{\partial u}{\partial t} + \operatorname{div} \mathbf{f}(u) = g \quad \text{in } \Omega \times (0,T), \tag{1}$$

$$\text{b)} \quad u\big|_{\partial\Omega \times (0,T)} = 0, \tag{2}$$

$$\text{d)} \quad u(x,0) = u^0(x), \quad x \in \Omega. \tag{3}$$

Here $g : \Omega \times (0,T) \to \mathbb{R}$ and $u^0 : \Omega \to \mathbb{R}$ are given functions. We assume that the *convective fluxes* $\mathbf{f} = (f_1, \cdots, f_d) \in (C_b^2(\mathbb{R}))^d = (C^2(\mathbb{R}) \cap W^{2,\infty}(\mathbb{R}))^d$, hence $\mathbf{f}$ and $\mathbf{f}' = (f_1', \cdots, f_d')$ are *globally Lipschitz continuous.*

By $(\cdot\,,\cdot)$ we denote the standard $L^2(\Omega)-$scalar product and by $\|\cdot\|$ the $L^2(\Omega)$-norm. By $\|\cdot\|_\infty$, we denote the $L^\infty(\Omega)$-norm. For simplicity of notation, we shall drop the argument Ω in Sobolev norms, e.g. $\|\cdot\|_{H^{p+1}}$ denotes the $H^{p+1}(\Omega)-$norm. We shall also denote the Bochner norms over the whole interval $[0,T]$ in concise form, e.g. $\|u\|_{L^\infty(H^{p+1})}$ denotes the $L^\infty(0,T;H^{p+1}(\Omega))$-norm.

2. Discretization

Let $\mathcal{T}_h$ be a triangulation of $\overline{\Omega}$, i.e. a partition into a finite number of closed simplexes with mutually disjoint interiors. We assume standard conforming properties: two neighboring elements from $\mathcal{T}_h$ share an entire face, edge or vertex. We set $h = \max_{K\in\mathcal{T}_h} \operatorname{diam}(K)$.

We consider a system $\{\mathcal{T}_h\}_{h\in(0,h_0)}$, $h_0 > 0$, of triangulations of the domain Ω which are shape regular and satisfy the inverse assumption, cf. [2]. Let $p \geq 1$ be an integer. The approximate solution will be sought in the space of globally continuous piecewise polynomial functions $S_h = \{v \in C(\overline{\Omega});\, v|_{\Gamma_D} = 0,\, v|_K \in P^p(K) \forall K \in \mathcal{T}_h\}$, where $P^p(K)$ denotes the space of polynomials on K of degree $\leq p$.

We discretize the continuous problem in a standard way. Multiply (1) by a test function $\varphi_h \in S_h$, integrate over Ω and apply Green's theorem.

Definition 1. *We say that $u_h \in C^1([0,T]; S_h)$ is the space-semidiscretized finite element solution of problem (1) - (3), if $u_h(0) = u_h^0 \approx u^0$ and*

$$\frac{d}{dt}\big(u_h(t), \varphi_h\big) + b\big(u_h(t), \varphi_h\big) = l\big(\varphi_h\big)(t), \quad \forall \varphi_h \in S_h,\ t \in (0,T). \tag{4}$$

Here, we have introduced an approximation $u_h^0 \in S_h$ of the initial condition u^0 and the *convective* and *right-hand side forms* defined for $v, \varphi \in H^1(\Omega)$:

$$b(v,\varphi) = -\int_\Omega \mathbf{f}(v)\cdot \nabla\varphi \,\mathrm{d}x, \qquad l(\varphi)(t) = \int_\Omega g(t)\varphi \,\mathrm{d}x.$$

We note that a sufficiently regular exact solution u of problem (1) satisfies

$$\frac{d}{dt}\big(u(t), \varphi_h\big) + b\big(u(t), \varphi_h\big) = l\big(\varphi_h\big)(t), \quad \forall \varphi_h \in S_h,\ \forall t \in (0,T), \tag{5}$$

which implies the *Galerkin orthogonality property* of the error.

3. Key estimates of the convective terms

As usual in apriori error analysis, we assume that the weak solution u is sufficiently regular, namely

$$u, u_t \in L^2\big(0,T; H^{p+1}(\Omega)\big), \quad u \in L^\infty(0,T; W^{1,\infty}(\Omega)), \tag{6}$$

where $u_t := \frac{\partial u}{\partial t}$. For $v \in L^2(\Omega)$ we denote by $\Pi_h v$ the $L^2(\Omega)$-projection of v on S_h:

$$\Pi_h v \in S_h, \quad (\Pi_h v - v,\, \varphi_h) = 0, \qquad \forall\, \varphi_h \in S_h.$$

Let $\eta_h(t) = u(t) - \Pi_h u(t) \in H^{p+1}(\Omega)$ and $\xi_h(t) = \Pi_h u(t) - u_h(t) \in S_h$ for $t \in (0,T)$. Then we can write the error e_h as $e_h(t) := u(t) - u_h(t) = \eta_h(t) + \xi_h(t)$. By C we denote a generic constant independent of h, which may have different values in different parts of the text. Also, for simplicity of notation, we shall usually omit the argument (t) and subscript h in $\xi_h(t)$ and $\eta_h(t)$. In our analysis, we shall need the following standard inverse inequalities and approximation properties of η, (cf. [2]):

Lemma 1. *There exists a constant $C_I > 0$ independent of h s.t. for all $v_h \in S_h$*

$$|v_h|_{H^1} \leq C_I h^{-1} ||v_h||,$$
$$\|v_h\|_\infty \leq C_I h^{-d/2} \|v_h\|.$$

Lemma 2. *There exists a constant $C > 0$ independent of h s.t. for all $h \in (0, h_0)$*

$$\|\eta_h(t)\| \leq Ch^{p+1}|u(t)|_{H^{p+1}},$$
$$\|\tfrac{\partial \eta_h(t)}{\partial t}\| \leq Ch^{p+1}\big|\tfrac{\partial u(t)}{\partial t}\big|_{H^{p+1}},$$
$$\|\eta_h(t)\|_\infty \leq Ch|u(t)|_{W^{1,\infty}}.$$

Lemma 3. *There exists a constant $C \geq 0$ independent of h, t, such that*

$$b\big(u_h(t), \xi(t)\big) - b\big(u(t), \xi(t)\big) \leq C\Big(1 + \frac{\|e_h(t)\|_\infty}{h}\Big)\big(h^{2p+2}|u(t)|^2_{H^{p+1}} + \|\xi(t)\|^2\big). \quad (7)$$

Proof. The proof follows the arguments of [5], where similar estimates are derived for periodic boundary conditions or compactly supported solutions in 1D. The proof for mixed Dirichlet-Neumann boundary conditions is contained in [4]. We write

$$b(u_h, \xi) - b(u, \xi) = \int_\Omega \big(\mathbf{f}(u) - \mathbf{f}(u_h)\big)\cdot \nabla\xi \,\mathrm{d}x. \quad (8)$$

By the Taylor expansion of $\mathbf{f}$ with respect to u, we have

$$\mathbf{f}(u) - \mathbf{f}(u_h) = \mathbf{f}'(u)\xi + \mathbf{f}'(u)\eta - \frac{1}{2}\mathbf{f}''_{u,u_h} e_h^2, \quad (9)$$

where $\mathbf{f}''_{u,u_h}$ is the Lagrange form of the remainder of the Taylor expansion, i.e. $\mathbf{f}''_{u,u_h}(x,t)$ has components $f''_s\big(\vartheta_s(x,t)u(x,t) + (1-\vartheta_s(x,t))u_h(x,t)\big)$ for some $\vartheta_s(x,t) \in [0,1]$ and $s = 1, \cdots, d$. Substituting (9) into (8), we obtain

$$b(u_h, \xi) - b(u, \xi) = \underbrace{\int_\Omega \mathbf{f}'(u)\xi\cdot \nabla\xi \,\mathrm{d}x}_{Y_1} + \underbrace{\int_\Omega \mathbf{f}'(u)\eta\cdot \nabla\xi \,\mathrm{d}x}_{Y_2} - \underbrace{\frac{1}{2}\int_\Omega \mathbf{f}''_{u,u_h} e_h^2\cdot \nabla\xi \,\mathrm{d}x}_{Y_3}. \quad (10)$$

We shall estimate these terms individually.

(A) Term $\mathbf{Y_1}$: Due to Green's theorem and the boundedness of $\mathbf{f}''$ and the regularity of u, we have

$$\int_\Omega \mathbf{f}'(u)\xi\cdot \nabla\xi \,\mathrm{d}x = -\frac{1}{2}\int_\Omega \mathrm{div}\big(\mathbf{f}'(u)\big)\xi^2 \,\mathrm{d}x \leq C\|\xi\|^2.$$

(B) Term $\mathbf{Y_2}$: We define $\Pi_h^1 : (L^2(\Omega))^d \to (S_h^1)^d = \{\mathbf{v} \in (C(\overline{\Omega}))^d;\ \mathbf{v}|_{\Gamma_D} = 0,\ \mathbf{v}|_K \in (P^1(K))^d,\ \forall K \in \mathcal{T}_h\}$, the $(L^2(\Omega))^d$-projection onto the space of continuous piecewise linear vector functions. From standard approximation results (similar to those of Lemma 2, cf. [2]), we obtain

$$\|\mathbf{f}'(u) - \Pi_h^1(\mathbf{f}'(u))\|_\infty \leq Ch|\mathbf{f}'(u)|_{W^{1,\infty}} \leq Ch\|\mathbf{f}''\|_{L^\infty(\mathbb{R})}|u|_{L^\infty(W^{1,\infty})} = \tilde{C}h.$$

Furthermore, due to the definition of η, we have $\int_\Omega \Pi_h^1(\mathbf{f}'(u))\cdot\nabla\xi\,\eta\,\mathrm{d}x = 0$, since $\Pi_h^1(\mathbf{f}'(u))\cdot\nabla\xi \in S_h$. Therefore, by Lemmas 1, 2 and Young's inequality

$$|Y_2| = \Big|\int_\Omega \big(\mathbf{f}'(u) - \Pi_h^1(\mathbf{f}'(u))\big)\cdot\nabla\xi\,\eta\,\mathrm{d}x\Big| \le \|\mathbf{f}'(u) - \Pi_h^1(\mathbf{f}'(u))\|_\infty C_I h^{-1}\|\xi\|\|\eta\|$$
$$\le \tilde{C}hC_Ih^{-1}\|\xi\|\|\eta\| \le \|\xi\|^2 + Ch^{2p+2}|u(t)|^2_{H^{p+1}}.$$

(C) Term $\mathbf{Y_3}$: We apply Lemmas 1, 2 and Young's inequality:

$$|Y_3| \le C\|e_h\|_\infty\|e_h\|C_Ih^{-1}\|\xi\| \le Ch^{-1}\|e_h\|_\infty\big(Ch^{2p+2}|u(t)|^2_{H^{p+1}} + \|\xi\|^2\big).$$

□

4. Error analysis of the semidiscrete scheme

We proceed similarly as for a parabolic equation. By Galerkin orthogonality, we subtract (5) and (4) and set $\varphi_h := \xi_h(t) \in S_h$. Since $\big(\frac{\partial\xi_h}{\partial t}, \xi_h\big) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\xi_h\|^2$, we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\xi_h(t)\|^2 = b\big(u_h(t),\xi_h(t)\big) - b\big(u(t),\xi_h(t)\big) - \Big(\frac{\partial\eta_h(t)}{\partial t}, \xi_h(t)\Big).$$

For the last right-hand side term, we use the Cauchy and Young's inequalities and Lemma 2 and Lemma 3 for the convective terms. We integrate from 0 to $t \in [0,T]$,

$$\|\xi_h(t)\|^2 \le C\int_0^t\Big(1+\tfrac{\|e_h(\vartheta)\|_\infty}{h}\Big)\Big(h^{2p+1}|u(\vartheta)|^2_{H^{p+1}}+h^{2p+2}|u_t(\vartheta)|^2_{H^{p+1}}+\|\xi_h(\vartheta)\|^2\Big)\mathrm{d}\vartheta, \quad (11)$$

where $C \ge 0$ is independent of h, t. For simplicity, we have assumed that $\xi_h(0) = 0$, i.e. $u_h^0 = \Pi_h u^0$. Otherwise we must assume e.g. $\|\xi_h(0)\|^2 \le Ch^{2p+1}|u^0|^2_{H^{p+1}}$ and include this term in the estimate.

We notice that if we knew *apriori* that $\|e_h\|_\infty = O(h)$ then the unpleasant term $h^{-1}\|e_h\|_\infty$ in (11) would be $O(1)$. Thus we could simply apply the standard Gronwall lemma to obtain the desired error estimates. We state this formally:

Lemma 4. *Let $t \in [0,T]$ and $p \ge d/2$. If $\|e_h(\vartheta)\| \le h^{1+d/2}$ for all $\vartheta \in [0,t]$, then there exists a constant C_T independent of h, t such that*

$$\max_{\vartheta\in[0,t]}\|e_h(\vartheta)\|^2 \le C_T^2h^{2p+1}. \quad (12)$$

Proof. The assumptions imply, by the inverse inequality and estimates of η, that

$$\|e_h(\vartheta)\|_\infty \le \|\eta_h(\vartheta)\|_\infty + \|\xi_h(\vartheta)\|_\infty \le Ch|u(t)|_{W^{1,\infty}} + C_Ih^{-d/2}\|\xi_h(\vartheta)\| \quad (13)$$
$$\le Ch + C_Ih^{-d/2}\|e_h(\vartheta)\| + C_Ih^{-d/2}\|\eta_h(\vartheta)\| \le Ch + Ch^{p+1-d/2}|u(\vartheta)|_{H^{p+1}(\Omega)} \le Ch,$$

where the constant C is independent of h, ϑ, t. Using this estimate in (11) gives us

$$\|\xi_h(t)\|^2 \le \tilde{C}h^{2p+1} + C\int_0^t\|\xi_h(\vartheta)\|^2\,\mathrm{d}\vartheta, \quad (14)$$

where the constants $\widetilde{C}, C$ are independent of h, t. Gronwall's inequality applied to (14) states that there exists a constant $\widetilde{C}_T$, independent of h, t, such that

$$\max_{\vartheta\in[0,t]} \|\xi_h(\vartheta)\|^2 + \frac{1}{2}\int_0^t |\xi_h(\vartheta)|^2_{\Gamma_N}\,\mathrm{d}\vartheta \le \widetilde{C}_T h^{2p+1},$$

which allong with similar estimates for η gives us (12). □

Now it remains to get rid of the *apriori* assumption $\|e_h\|_\infty = O(h)$. In [5] this is done for an explicit scheme using mathematical induction. Starting from $\|e_h^0\| = O(h^{p+1/2})$, the following induction step is proved:

$$\|e_h^n\| = O(h^{p+1/2}) \quad\Longrightarrow\quad \|e_h^{n+1}\|_\infty = O(h) \quad\Longrightarrow\quad \|e_h^{n+1}\| = O(h^{p+1/2}). \tag{15}$$

For the method of lines we have continuous time and hence cannot use mathematical induction straightforwardly. However, we can divide $[0,T]$ into a finite number of sufficiently small intervals $[t_n, t_{n+1}]$ on which "e_h *does not change too much*" and use induction with respect to n. This is essentially a *continuous mathematical induction* argument, a concept introduced in [1], which has many generalizations, cf. [3].

Lemma 5 (Continuous mathematical induction). *Let $\varphi(t)$ be a propositional function depending on $t \in [0,T]$ such that*

(i) $\varphi(0)$ *is true,*

(ii) $\exists\delta_0 > 0:\ \varphi(t)$ *implies* $\varphi(t+\delta),\ \forall t \in [0,T]\ \forall\delta \in [0,\delta_0] : t+\delta \in [0,T]$.

Then $\varphi(t)$ holds for all $t \in [0,T]$.

Remark 1 Due to the regularity assumptions, the functions $u(\cdot), u_h(\cdot)$ are continuous mappings from $[0,T]$ to $L^2(\Omega)$. Since $[0,T]$ is a compact set, $e_h(\cdot)$ is a *uniformly continuous* function from $[0,T]$ to $L^2(\Omega)$. By definition,

$$\forall\epsilon > 0\ \exists\delta > 0:\ s, \bar{s} \in [0,T], |s-\bar{s}| \le \delta \implies \|e_h(s) - e_h(\bar{s})\| \le \epsilon.$$

Theorem 6 (Semidiscrete error estimate). *Let $p > (1+d)/2$. Let $h_1 > 0$ be such that $C_T h_1^{p+1/2} = \frac{1}{2}h_1^{1+d/2}$, where C_T is the constant from Lemma 4. Then for all $h \in (0, h_1]$ we have the estimate*

$$\max_{\vartheta\in[0,T]} \|e_h(\vartheta)\|^2 \le C_T^2 h^{2p+1}. \tag{16}$$

Proof. Since $p > (1+d)/2$, h_1 is uniquely determined and $C_T h^{p+1/2} \le \frac{1}{2}h^{1+d/2}$ for all $h \in (0, h_1]$. We define the propositional function φ by

$$\varphi(t) \equiv \Big\{ \max_{\vartheta\in[0,t]} \|e_h(\vartheta)\|^2 \le C_T^2 h^{2p+1} \Big\}.$$

We shall use Lemma 5 to show that φ holds on $[0,T]$, hence $\varphi(T)$ holds, which is equivalent to (16).

(i) $\varphi(0)$ holds, since this is the error of the initial condition.
(ii) *Induction step*: We fix an arbitrary $h \in (0, h_1]$. By Remark 1, there exists $\delta_0 > 0$, such that if $t \in [0, T), \delta \in [0, \delta_0]$, then $\|e_h(t+\delta) - e_h(t)\| \le \frac{1}{2}h^{1+d/2}$. Now let $t \in [0, T)$ and assume $\varphi(t)$ holds. Then $\varphi(t)$ implies $\|e_h(t)\| \le C_T h^{p+1/2} \le \frac{1}{2}h^{1+d/2}$. Let $\delta \in [0, \delta_0]$, then by uniform continuity

$$\|e_h(t+\delta)\| \le \|e_h(t)\| + \|e_h(t+\delta) - e_h(t)\| \le \tfrac{1}{2}h^{1+d/2} + \tfrac{1}{2}h^{1+d/2} = h^{1+d/2}.$$

This and $\varphi(t)$ implies that $\|e_h(s)\| \le h^{1+d/2}$ for $s \in [0, t] \cup [t, t+\delta] = [0, t+\delta]$. By Lemma 4, φ holds on $[0, t+\delta]$. As a special case, we obtain the "induction step" $\varphi(t) \Longrightarrow \varphi(t+\delta)$ for all $\delta \in [0, \delta_0]$. □

5. Conclusion

We have presented the basic ideas behind the apriori analysis of nonlinear convective problems. To keep things as simple as possible, we have presented the analysis only for a space-semidiscrete scheme, with Dirichlet boundary conditions only. The extension to mixed boundary conditions, the extension to implicit schemes via continuation, derivation of improved estimates under the assumption $\mathbf{f} \in (C_b^3(\mathbb{R}))^d$ and the generalization to *locally Lipschitz* $\mathbf{f} \in (C^2(\mathbb{R}))^d$ can be found in [4].

Acknowledgements

The work was supported by the project P201/11/P414 of the Czech Science Foundation.

References

[1] Chao, Y.R.: *A note on "Continuous mathematical induction"*, Bull. Amer. Math. Soc., **26**(1), 17–18, 1919.

[2] Ciarlet, P.G: *The Finite Element Method for Elliptic Problems,* North-Holland, Amsterdam, 1979.

[3] Clark, P.L.: *Real induction,* available online `http://citeseer.ist.psu.edu/viewdoc/summary?doi=10.1.1.187.3514`.

[4] Kučera, V.:Finite element error estimates for nonlinear convective problems, *The Preprint Series of the School of Mathematics,* preprint No. MATH-knm-2012/1, `http://www.karlin.mff.cuni.cz/ms-preprints/prep.php`. Submitted to Numer. Math, 2012.

[5] Zhang, Q., and Shu, C.-W.: *Error estimates to smooth solutions of Runge–Kutta discontinuous Galerkin methods for scalar conservation laws*, SIAM J. Numer. Anal., **42**(2), 641–666, 2004.