

MAXIMAL MARCINKIEWICZ MULTIPLIERS

PETR HONZÍK

ABSTRACT. Let us have a family of Marcinkiewicz multipliers $\mathcal{M} = \{m_j\}$ of sufficient uniform smoothness in \mathbb{R}^n . We show that the L^p norm ($1 < p < \infty$) of the related maximal operator

$$M_N f(x) = \sup_{1 \leq j \leq N} |\mathcal{F}^{-1}(m_j \mathcal{F}f)|(x)$$

is at most $C(\log(N+2))^{n/2}$. We show that this bound is sharp.

1. INTRODUCTION

Marcinkiewicz multiplier on \mathbb{R}^n is a Fourier multiplier with a symbol which satisfies a set of conditions

$$(1) \quad |\partial_{i_1} \cdots \partial_{i_k} m_j|(\xi) \leq A |\xi_{i_1}|^{-1} \cdots |\xi_{i_k}|^{-1},$$

for all $\{i_1, \dots, i_k\} \subset \{1, \dots, n\}$. We consider a family of N symbols m_1, \dots, m_N which satisfy these conditions uniformly. We form a maximal operator

$$M_N f(x) = \sup_{i=1, \dots, N} |\mathcal{F}^{-1}(m_i \hat{f})|(x).$$

We show that the norm of this operator grows as $C(\log(N+2))^{n/2}$. Previously, similar theorem was proved for Hörmander-Mikhlin symbols in [3] and [5]. The main difference is that in the case of Hörmander-Mikhlin symbols the bound is $C(\log(N+2))^{1/2}$ independently of dimension.

The smoothness of the symbol is not optimal here, as the Marcinkiewicz multiplier theorem may be formulated with a BV type condition. Also a different maximal theorem for Marcinkiewicz multipliers appeared recently in [7], with condition even weaker than BV . It is not clear to us if the smoothness of the symbol may be relaxed in our theorem.

Theorem 1. *Suppose $\mathcal{M} = \{m_j\}$ is a family of functions in \mathbb{R}^n such that*

$$(2) \quad |\partial_{i_1} \cdots \partial_{i_k} m_j|(\xi) \leq A |\xi_{i_1}|^{-1} \cdots |\xi_{i_k}|^{-1}$$

for all $\{i_1, \dots, i_k\} \subset \{1, \dots, n\}$. Then for any $N \in \mathbb{N}$ and $1 < p < \infty$ we have

$$\left\| \sup_{j \in \{1, \dots, N\}} |\mathcal{F}^{-1}(m_j \hat{f})|(x) \right\|_{L^p(x)} \leq AC_{n,p}(\log^{n/2}(N+2)) \|f\|_p.$$

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Also, for any given $N \geq 1$ and $1 < p < \infty$ there is a sequence $\mathcal{M} = \{m_j\}$ which satisfies (2) and a function g such that

$$\left\| \sup_{j \in \{1, \dots, N\}} |\mathcal{F}^{-1}(m_j \hat{g})|(x) \right\|_{L^p(x)} \geq A \tilde{C}_{n,p} (\log^{n/2}(N+2)) \|g\|_p.$$

One motivation for the study of the maximal Marcinkiewicz multipliers was the open problem of the maximal hyperbolic Bochner-Riesz means. We give a brief discussion of this problem at the end of the paper.

2. MULTIPLE MARTINGALES

In this section we introduce the dyadic multiple martingales, which we use as a tool to study the Marcinkiewicz multipliers. First, let us introduce the classical dyadic martingale. Consider an integrable function f on $[0, 1]$. Let us denote \mathcal{D}_k the set of dyadic intervals of length 2^{-k} and define the expectation operator

$$E_k f(x) = 2^k \int_{I: I \in \mathcal{D}_k, x \in I} f(y) dy.$$

We define the martingale differences as $D_k = E_k - E_{k-1}$ and the square function

$$Sf(x) = \left(\sum_k (D_k f(x))^2 \right)^{1/2}.$$

The maximal martingale function f^* is defined as $f^*(x) = \sup_k |E_k f|(x)$. There is an equivalent representation of this object using Haar functions. For each dyadic interval I , we define a function h_I which is equal to $|I|^{-1/2}$ on the left half and to $-|I|^{-1/2}$ on the right half of the interval. These functions, together with the function $h_0(x) = 1$ form an orthonormal basis in $L^2([0, 1])$, and also a Schauder basis in $L^p([0, 1])$, $1 \leq p < \infty$. One sees that for $k \geq 1$

$$D_k f = \sum_{I \in \mathcal{D}_{k-1}} \langle h_I, f \rangle h_I.$$

The dyadic multiple martingales represent a tensored version of the above object. We stress that the multiple martingales are not martingales and in fact they lack many of the key properties of the classical martingales. Let us have a function g on the cube $[0, 1]^n$. For dyadic intervals I_1, \dots, I_n we define the function

$$h_{I_1, \dots, I_n}(x_1, \dots, x_n) = h_{I_1}(x_1) \cdots h_{I_n}(x_n).$$

Using these tensored Haar functions, one can define the multiple versions of operators E , D and S . In particular, we have

$$D_{k_1, \dots, k_n} g = \sum_{I_i \in \mathcal{D}_{k_i}} \langle h_{I_1, \dots, I_n}, g \rangle h_{I_1, \dots, I_n},$$

$$E_{k_1, \dots, k_n} = \sum_{-1 \leq m_1 \leq k_1, \dots, -1 \leq m_n \leq k_n} D_{m_1, \dots, m_n}$$

and

$$Sf(x) = S_{n+1}f(x) = \left(\sum_{k_1, \dots, k_n} (D_{k_1, \dots, k_n} f(x))^2 \right)^{1/2}.$$

If we fix $n - 1$ indices and $n - 1$ variables, then the expectations $E_{k_1, \dots, k_n} g(x_1, \dots, x_n)$ form a dyadic martingale, for example

$$\tilde{E}_j(y) = E_{k_1, \dots, k_{n-1}, j} g(x_1, \dots, x_{n-1}, y)$$

is a sequence of expectations of one dimensional dyadic martingale.

We use some fine results detailing the relationship of the square functions and maximal operators. In the case of the dyadic martingale, the sharp good lambda inequality

$$(3) \quad |\{f^* > 2\lambda, Sf < \epsilon\lambda\}| \leq Ce^{-c/\epsilon^2} |\{f^* > \lambda\}|$$

was proved by Chang, Wilson and Wolff [2]. In the case of double dyadic martingale, similar inequality was proved by Pipher [8]. The argument of Pipher extends to the higher dimensions as well by induction. Before we demonstrate this fact, we need to define intermediate square functions. We put for $m = 2, \dots, n + 1$

$$S_m f = \left(\sum_{k_1, \dots, k_{m-1}} \left(\sum_{k_m, \dots, k_n} D_{k_1, \dots, k_n} f(x) \right)^2 \right)^{1/2}.$$

and

$$S_m^* f = \sup_r \left(\sum_{k_1, \dots, k_{m-1}} \left(\sum_{k_m < r, k_{m+1}, \dots, k_n} D_{k_1, \dots, k_n} f(x) \right)^2 \right)^{1/2}.$$

We also define the following maximal function (replacing S_1^*)

$$M_1 f(x) = \sup_r \left| \sum_{m_1 \leq r, \dots, m_n} D_{m_1, \dots, m_n} f(x) \right|.$$

Standard arguments show that all the above operators are L^p bounded for $1 < p < \infty$.

In the case $m = 1, n = 2$ the good lambda inequality is proved by Pipher in [8] (fourth formula on page 76), but as the inequality is not stated exactly in the form we need it and we need a higher dimensional version, we feel that we need to reproduce the proof here. The starting point of the proof is the following lemma, proved in the paper [8] as Lemma 2.2 (here d_q^j is a difference of a dyadic martingale):

Lemma 1. *Suppose $X_N^j = \sum_{q=0}^N d_q^j$, $j = 1, \dots, M$ is a sequence of dyadic martingales and set*

$$SX_N^j = \left(\sum_{q=0}^N (d_q^j)^2 \right)^{1/2},$$

the square function of X_N^j . Then

$$\int \exp \left(\sqrt{1 + \sum_{j=1}^M (X_N^j)^2} - \sum_{j=1}^M (SX_N^j)^2 \right) dx \leq e.$$

The following lemma is the higher dimensional analogue of the Pipher good lambda inequality.

Lemma 2. *Let $f \in L^1([0, 1]^n)$. Let us fix N and denote $g = E_{N, \dots, N} f$. Then we have for $0 < \epsilon < 1/2$ and $1 < m \leq n$ and $x_1, \dots, x_{m-1}, x_{m+1}, \dots, x_n$*

$$\begin{aligned} & |\{x_m : S_m^* g(x_1, \dots, x_n) > 2\lambda; S_{m+1} g(x_1, \dots, x_n) < \epsilon\lambda\}| \\ & \leq C e^{-c/\epsilon^2} |\{x_m : S_m^* g(x_1, \dots, x_n) > \lambda\}|. \end{aligned}$$

Moreover we have

$$\begin{aligned} & |\{x_1 : M_1 g(x_1, \dots, x_n) > 2\lambda; S_2 g(x_1, \dots, x_n) < \epsilon\lambda\}| \\ & \leq C e^{-c/\epsilon^2} |\{x_1 : M_1 g(x_1, \dots, x_n) > \lambda\}|. \end{aligned}$$

The constants C and c are independent of f, N, λ and ϵ .

Proof. The second inequality follows directly from the inequality of Chang, Wilson and Wolff (3), since for x_2, \dots, x_n fixed

$$M_1 g(x_1, \dots, x_n)$$

represents the dyadic martingale maximal function of the function

$$g_1(x_1) = g(x_1, \dots, x_n)$$

and

$$S_2 g(x_1, \dots, x_n)$$

represents its martingale square function $Sg_1(x_1)$.

To prove the first inequality, we use the same argument as J. Pipher [8] in the proof of Corollary 2.2a. We fix the m and $x_1, \dots, x_{m-1}, x_{m+1}, \dots, x_n$ and denote

$$d_q^{(j_1, \dots, j_{m-1})}(x_m) = \sum_{k_{m+1}, \dots, k_n} D_{j_1, \dots, j_{m-1}, q, k_{m+1}, \dots, k_n} g(x_1, \dots, x_n).$$

Denote $J = (j_1, \dots, j_{m-1})$, then

$$X_N^J = \sum_{q=0}^N d_q^J$$

is a sequence of martingales on the interval $0 \leq x_m \leq 1$, as in Lemma 1.

The set $\{S_m^* g > \lambda\}$ is composed of maximal dyadic intervals I such that for $x_m \in I$

$$\left(\sum_{k_1, \dots, k_{m-1}} \left(\sum_{k_m < r, \dots, k_n} D_{k_1, \dots, k_n} g(x) \right)^2 \right)^{1/2} > \lambda.$$

The r above is then minimal, and we fix a pair I, r . Assume $I \cup \{S_m^*g \geq 2\lambda\} \neq \emptyset$. Then we may find maximal dyadic intervals $I' \subset I$ such that for $x_m \in I'$

$$\left(\sum_{k_1, \dots, k_{m-1}} \left(\sum_{k_m < r', \dots, k_n} D_{k_1, \dots, k_n} g(x) \right)^2 \right)^{1/2} \geq 2\lambda.$$

We localize the $d_q^{(j_1, \dots, j_{m-1})}$ and X_N^J to the interval I' and apply a localized version of the Lemma 1, we get for any α and t

$$\int_{I'} \exp \left[\alpha \left(\sum_{k_1, \dots, k_{m-1}} \left(\sum_{r < k_m < t, \dots, k_n} D_{k_1, \dots, k_n} g(x) \right)^2 \right)^{1/2} - \alpha^2 \sum_{k_1, \dots, k_{m-1}, r < k_m < t} \left(\sum_{k_{m+1}, \dots, k_n} D_{k_1, \dots, k_n} g(x) \right)^2 \right] \leq e|I'|.$$

Consider the set $A = I' \cap \{S_{m+1}g \leq \epsilon\lambda\}$. We have for $x \in A$

$$\left(\sum_{k_1, \dots, k_{m-1}} \left(\sum_{k_m=r, \dots, k_n} D_{k_1, \dots, k_n} g(x) \right)^2 \right)^{1/2} \leq \epsilon\lambda$$

and therefore

$$\sum_{k_1, \dots, k_{m-1}} \left(\sum_{k_m=r, \dots, k_n} D_{k_1, \dots, k_n} g(x) \right)^2 \geq 2\lambda - \lambda - \epsilon\lambda.$$

This gives

$$|A| \exp[-\alpha(1-\epsilon)\lambda + \alpha^2\epsilon^2\lambda^2] \leq e|I'|.$$

We take $\alpha = \frac{1-\epsilon}{2\epsilon^2\lambda}$ and sum the intervals I' to obtain the result. \square

3. PROOF OF THE POSITIVE RESULT

The main idea of the proof comes from the article [5], we apply the Lemma 2 in each variable separately. In some of the estimates we replace the usual maximal function by the strong maximal function, related to averages over rectangular parallelepipeds with sides parallel to axes.

In order to prove our theorem, we also need to refer to the the proof of the Marcinkiewicz theorem. We use the notation and method of the proof from [4]. We may assume that all the multipliers are supported in the positive cone $\{\xi : \xi_1 > 0, \dots, \xi_n > 0\}$. For a set $A \subset \mathbb{R}$ we introduce the coordinate cutoff operators

$$\Delta_A^{(j)} f = \mathcal{F}^{-1}(\chi_A(\xi_j)\hat{f}).$$

Let us fix $j \in \mathbb{Z}^n$ and take

$$\xi \in R_j = [2^{j_1}, 2^{j_1+1}] \times \dots \times [2^{j_n}, 2^{j_n+1}].$$

The following representation formula follows from the fundamental theorem of calculus.

$$\begin{aligned} m(\xi) &= m(2^{j_1}, \dots, 2^{j_n}) \\ &+ \sum_{k=1}^n \int_{2^{j_k}}^{\xi_{j_k}} \partial^k m(\xi_1, \dots, t, \dots, \xi_n) dt \\ &\quad \vdots \\ &+ \int_{2^{j_1}}^{\xi_{j_1}} \dots \int_{2^{j_n}}^{\xi_{j_n}} \partial^1 \dots \partial^n m(t_1, \dots, t_n) dt_1 \dots dt_n. \end{aligned}$$

Using this formula, we can write $T_m f$ as a sum of integral terms, with the leading term being

$$\sum_j \int_{2^{j_1}}^{2^{j_1+1}} \dots \int_{2^{j_n}}^{2^{j_n+1}} \partial^1 \dots \partial^n m(t_1, \dots, t_n) \Delta_{[t_1, 2^{j_1+1}]}^{(1)} \dots \Delta_{[t_n, 2^{j_n+1}]}^{(n)} f(x) dt_1 \dots dt_n$$

Therefore in order to establish L^p control of $\sup_{i \leq N} |T_{m_i} f(x)|$ we need to estimate

$$\begin{aligned} &\int_{2^{j_1}}^{2^{j_1+1}} \dots \int_{2^{j_n}}^{2^{j_n+1}} \sup_{i \leq N} |\partial^1 \dots \partial^n m(t_1, \dots, t_n) \Delta_{[t_1, 2^{j_1+1}]}^{(1)} \dots \Delta_{[t_n, 2^{j_n+1}]}^{(n)} f(x)| dt_1 \dots dt_n \\ &\leq \int_1^2 \dots \int_1^2 A \sup_{i \leq N} \left| \sum_{j \in \mathbb{Z}^n} a_{i,j}(t) \Delta_{[2^{j_1} t_1, 2^{j_1+1}]}^{(1)} \dots \Delta_{[2^{j_n} t_n, 2^{j_n+1}]}^{(n)} f(x) \right| dt_1 \dots dt_n, \end{aligned}$$

where for each $t \in [1, 2]$ we have $\|a_{i,j}(t)\|_\infty \leq 1$.

In the light of these considerations we see that the proof of the theorem reduces to the obtaining the bound

$$(4) \quad \left\| \sup_{i \leq N} |\tilde{T}_i f|(x) \right\|_p \leq C(\log(N+2))^{n/2} \|f\|_p,$$

where

$$\tilde{T}_i f(x) = \sum_{j \in \mathbb{Z}} a_{i,j} \Delta_{[2^{j_1} t_1, 2^{j_1+1}]}^{(1)} \dots \Delta_{[2^{j_n} t_n, 2^{j_n+1}]}^{(n)} f(x),$$

each $t_l \in [1, 2]$ and $a_{i,j}$ is a sequence in l^∞ with norm 1. We note that the L^p boundedness of the operator \tilde{T}_i may be deduced from the boundedness of the square function

$$\tilde{S}f(x) = \left(\sum_{j \in \mathbb{Z}} |\Delta_{[2^{j_1} t_1, 2^{j_1+1}]}^{(1)} \dots \Delta_{[2^{j_n} t_n, 2^{j_n+1}]}^{(n)} f(x)|^2 \right)^{1/2}.$$

The estimates

$$\|\tilde{S}f\|_p \leq C\|f\|_p$$

and

$$\|\tilde{T}_i f\|_p \leq C\|\tilde{S}f\|_p$$

are proved in detail for example in [4].

We now fix $t = (t_1, \dots, t_n)$ and for each $j \in \mathbb{Z}^n$ introduce the notation

$$\Delta_j^b f = \Delta_{[2^{j_1} t_1, 2^{j_1+1}]}^{(1)} \dots \Delta_{[2^{j_n} t_n, 2^{j_n+1}]}^{(n)} f.$$

Let us recall the sharp maximal function. Let us denote \mathcal{R} the set of all rectangular parallelepiped with sides parallel to coordinate axes. We denote

$$M_s f(x) = \sup_{x \in R \in \mathcal{R}} \frac{1}{|R|} \int_R |f|(y) dy.$$

Let us introduce operator

$$G_r(f) = \left(\sum_{j \in \mathbb{Z}^n} (M_s(M_s(M_s|\Delta_j^b f|^r)))^{2/r} \right).$$

This operator is L^p bounded for $p > r$. This follows by repeated application of Fefferman-Stein theorem for vector valued maximal functions (see [10]) in each coordinate.

If we take dyadic cube Q with volume 1, we may adapt the dyadic multiple martingale operators to it. We need the following estimates:

Lemma 3. *For any dyadic cube Q with $|Q| = 1$ and $x \in Q$ we have for $1 < r < \infty$*

$$S(\tilde{T}_i f)(x) \leq C_r A G_r(f)(x).$$

Moreover, if $a_{i,j} = 0$ whenever some $|j_i| \leq N$, then we have

$$\left| \int_Q \tilde{T}_i f dy \right| \leq C_r A 2^{-N/r} G_r(f)(x).$$

Proof. Let us choose a Schwartz function b on \mathbb{R} with the following properties: $b(\xi) \neq 0$ for $\xi \in [1/2, 4]$ and \hat{b} is supported in $\{|x| \leq 1/4\}$. Then, we select a function a in C_c^∞ such that for $\xi \in [1, 2]$ we have $b^2(\xi)a(\xi) = 1$. For $\xi \in \mathbb{R}^n$ we put

$$\beta(\xi) = b(\xi_1) \cdots b(\xi_n)$$

and

$$\psi(\xi) = a(\xi_1) \cdots a(\xi_n).$$

For $k \in \mathbb{Z}^n$ we define operators

$$B_k f(\xi) = \mathcal{F}^{-1}(\beta(2^{-k_1} \xi_1, \dots, 2^{-k_n} \xi_n) \hat{f}(\xi))$$

and

$$L_k f(\xi) = \mathcal{F}^{-1}(\psi(2^{-k_1} \xi_1, \dots, 2^{-k_n} \xi_n) \hat{f}(\xi)).$$

We have the following representation of the operator \tilde{T}_i :

$$\tilde{T}_i f(x) = \sum_j a_{i,j} B_j(B_j(L_j(\Delta_j^b(f))))(x).$$

Then, we get

$$D_k \tilde{T}_i f(x) = \sum_j a_{i,j} (D_k B_j)(B_j(L_j(\Delta_j^b(f))))(x).$$

Clearly, we have

$$|B_j L_j f|(x) \leq M_s f(x),$$

therefore in order to prove the Lemma, we need to obtain the following estimates

$$(5) \quad |D_k B_j|f(x) \leq C 2^{-\frac{\alpha}{r}} (M_s M_s |f|^r(x))^{1/r}$$

and

$$(6) \quad |E_k B_j|f(x) \leq C 2^{-\frac{\alpha}{r}} (M_s M_s |f|^r(x))^{1/r}$$

where $\alpha = \max\{|k_1 - j_1|, \dots, |k_1 - j_1|\}$ and $\tilde{\alpha} = \max\{j_1 - k_1, \dots, j_1 - k_1\}$.

The proof of (5) is simple in the case $\alpha = \max\{k_1 - j_1, \dots, k_1 - j_1\} = k_s - j_s$ for some s . Smoothness estimate in the variable x_s gives

$$|D_k B_j|f(x) \leq C 2^{-\alpha} M M f(x)$$

and the claim follows. If on the other hand $\alpha = \tilde{\alpha} = \max\{j_1 - k_1, \dots, j_1 - k_1\} = j_s - k_s$ for some s , the estimate (5) follows from (6), which we prove next. We write $f = g + h$ where

$$g = f \chi_{\{x: |x_s - l 2^{-k_s}| \leq 2^{-j_s} \text{ for some } l \in \mathbb{Z}\}}.$$

We observe that since $\int \hat{b}(x) dx = 0$ we get

$$|E_k B_j h(x)| = 0.$$

On the other hand

$$\begin{aligned} |E_k B_j|g(x) &\leq \frac{1}{|R_k(x)|} \int_{R_k(x) \cup \{x: |x_s - l 2^{-k_s}| \leq 2^{-j_s+1} \text{ for some } l \in \mathbb{Z}\}} |M_s g(y)| dy \\ &\leq C 2^{-\frac{\alpha}{r}} (M_s (M_s |f|)^r(x))^{1/r}, \end{aligned}$$

where $R_k(x) \in \mathcal{R}$ is the dyadic parallelepiped with dimensions given by k and containing x , the first inequality follows by the support properties of b and the second follows from Hölder inequality. This finishes the proof of (5) and (6) and the lemma is proved. \square

In order finish the proof, we need to estimate the measure of the set

$$\Omega_\lambda = \left\{ \sup_{1 \leq i \leq N} |\tilde{T}_i f|(x) > 2^{n+2} \lambda \right\}.$$

We may assume that f is compactly supported inside the positive cone and then, by a simple scaling, that $a_{i,j} = 0$ whenever some $|j_i| \leq N$.

We split $\Omega_\lambda = \Omega_\lambda^1 \cup \Omega_\lambda^2 \cup \Omega_\lambda^3$, where

$$\Omega_\lambda^1 = \left\{ \sup_{1 \leq i \leq N} |\tilde{T}_i f - E_0 \tilde{T}_i f|(x) > 2^{n+1} \lambda, G_r(f)(x) \leq \epsilon^n \lambda \right\},$$

$$\Omega_\lambda^2 = \{G_r(f)(x) > \epsilon^n \lambda\}$$

and

$$\Omega_\lambda^3 = \left\{ \sup_{1 \leq i \leq N} |E_0 \tilde{T}_i f|(x) > 2^{n+1} \lambda \right\}.$$

We have

$$\Omega_\lambda^1 \subset \bigcup_i \Omega_{\lambda,i}^1$$

where

$$\Omega_{\lambda,i}^1 = \{|\tilde{T}_i f - E_0 \tilde{T}_i f|(x) > 2^{n+1} \lambda, S \tilde{T}_i f(x) \leq \epsilon^n \lambda\}.$$

Finally, we observe that we may choose $j = (j_1, \dots, j_n)$ such that for each i

$$(7) \quad \|\tilde{T}_i f - E_j \tilde{T}_i f\|_p \leq \frac{\|f\|_p}{N}.$$

We may therefore write $\Omega_{\lambda,i}^1 \subset \Gamma_{\lambda,i} \cup \Gamma'_{\lambda,i}$ where

$$\Gamma'_{\lambda,i} = \{|\tilde{T}_i f - E_j \tilde{T}_i f|(x) > 2^n \lambda\}$$

and

$$\begin{aligned} \Gamma_{\lambda,i} \subset & \{|E_j \tilde{T}_i f - E_0 \tilde{T}_i f|(x) > 2^n \lambda, S_1 E_j \tilde{T}_i f \leq 2^{n-1} \epsilon \lambda\} \\ & \cup \bigcup_{\nu=1}^{n-1} \{|S_\nu E_j \tilde{T}_i f|(x) > 2^{n-\nu} \epsilon^\nu \lambda, S_{\nu+1} E_j \tilde{T}_i f \leq 2^{n-1} \epsilon^{\nu+1} \lambda\}. \end{aligned}$$

We apply the Lemma 2 and obtain

$$\begin{aligned} |\Gamma_{\lambda,i}| \leq & C e^{-c/\epsilon^2} (|\{M_1(E_j \tilde{T}_i f - E_0 \tilde{T}_i f)|(x) > 2^{n-1} \lambda\}| \\ & + \sum |\{S_\nu^* E_j \tilde{T}_i f|(x) > 2^{n-\nu-1} \epsilon^\nu \lambda\}|). \end{aligned}$$

We set $\epsilon = C/\log^{1/2}(N+2)$, collect all the previous estimates and integrate them with respect to λ^{p-1} .

This yields

$$\begin{aligned} \left\| \sup_{1 \leq i \leq N} |\tilde{T}_i f| \right\|_p^p & \leq C_{n,p} \int \lambda^{p-1} (\Omega_\lambda^2 + \Omega_\lambda^3 + \sum_i |\Gamma_{\lambda,i}| + |\Gamma'_{\lambda,i}|) d\lambda \\ & \leq C_{n,p} (\log^{n/2}(N+2) \|G_r(f)\|_p + \sum_i \|E_0 \tilde{T}_i f\|_p \\ & \quad + \frac{1}{N} \sum_i \|M_1(E_j \tilde{T}_i f - E_0 \tilde{T}_i f)\|_p \\ & \quad + \frac{\log^{n/2}(N+2)}{N} \sum_i \sum_\nu \|S_\nu^* E_j \tilde{T}_i f\|_p \\ & \quad + \sum_i \|\tilde{T}_i f - E_j \tilde{T}_i f\|). \end{aligned}$$

Here we observe, that the first, third and fifth term is bounded from the boundedness of the operators in question, the second is bounded by Lemma 3 and the fourth is bounded by (7). Together this finishes the proof of the first part of the theorem.

4. EXAMPLE

Here we construct the sequence from the second part of the theorem. The example is very similar to the one from [3], the key difference is the observation that the tensored products have less freedom of oscillations than one dimensional sequences.

Let us take a smooth function $\phi_0(x)$ such that $\phi(x) = 1$ for $x \in [5/4, 7/4]$ and $\phi(x) = 0$ for $x \in \mathbb{R} \setminus (1, 2)$ and a smooth nonzero function ψ_0 supported in $[5/4, 7/4]$. For $\xi \in \mathbb{R}^n$ we then define

$$\psi(\xi) = \psi_0(\xi_1) \dots \psi_0(\xi_n).$$

Suppose $(20n)^{n(K+1)} \geq N > (20n)^{nK}$. For $x \in \mathbb{R}^n$ we define

$$g(x) = \sum_{j_1, \dots, j_n \in \{1, \dots, K\}} e^{2\pi i(2^{j_1}, \dots, 2^{j_n}) \cdot x} (\mathcal{F}^{-1} \psi)(x).$$

A square function argument shows that $\|g\|_p \approx K^{n/2}$.

We take $\{\omega_1, \dots, \omega_{20n}\} = \sqrt[20n]{1}$ to be the set of the complex roots of 1. Let us take a sequence $a \in \{1, \dots, 20n\}^K$. We define for $\eta \in \mathbb{R}$

$$\tilde{m}_a(\eta) = \sum_{i=1}^K \omega_{a(i)} \phi(\eta - 2^i).$$

For sequence $b = (a_1, \dots, a_n)$ and $\xi \in \mathbb{R}^n$ we define

$$m_b(\xi) = \tilde{m}_{a_1}(\xi_1) \cdots \tilde{m}_{a_n}(\xi_n).$$

Now we define

$$Mf(x) = \sup_b |\mathcal{F}^{-1}(m_b \hat{f})|(x).$$

Now let us fix $x \in \mathbb{R}^n$ and select sequence $b = (a_1, \dots, a_n)$ such that

$$|\omega_{a(j)} e^{2\pi i 2^j x_j} - 1| \leq 2\pi/10n.$$

It is easy to check that

$$|\mathcal{F}^{-1}(m_b \hat{g})|(x) \geq |\Re \mathcal{F}^{-1}(m_b \hat{g})(x)| \geq CK^n |\Re \mathcal{F}^{-1}(\psi)(x)|.$$

Therefore

$$\|Mg\|_p \geq CK^{n/2} \|g\|_p.$$

and we are finished.

5. MAXIMAL HYPERBOLIC BOCHNER–RIESZ

One motivation for the study of the maximal Marcinkiewicz multipliers was the open problem of the boundedness of the maximal hyperbolic Bochner–Riesz operator

$$Mf(x) = \sup_k |(\mathcal{F})^{-1}(m_\lambda(2^k \cdot) \hat{f})|(x),$$

where

$$m_\lambda(\xi_1, \xi_2) = (1 - |\xi_1 \xi_2|)_+^\lambda.$$

While the boundedness of the operator $(\mathcal{F})^{-1}(m_\lambda \hat{f})$ was settled by El-Cohen [6] and Carbery [1], the boundedness of the maximal operator remains open for any λ . Our result gives logarithmic growth with respect to the number of the dilations in case $\lambda > 1$, but the full solution remains elusive (see also [9].)

REFERENCES

- [1] Carbery, A. *A note on the "hyperbolic" Bochner-Riesz means*. Proc. Amer. Math. Soc. 92 (1984), no. 3, 397400.
- [2] Chang, S. Y. A.; Wilson, J. M. and Wolff, T. H. *Some weighted norm inequalities concerning the Schrödinger operators*. Comment. Math. Helv. 60 (1985), no. 2, 217–246
- [3] Christ, M.; Grafakos, L.; Honzík, P.; Seeger, A. *Maximal functions associated with Fourier multipliers of Mihlin-Hörmander type*. Math. Z. 249 (2005), no. 1, 223–240.
- [4] Grafakos, L. *Classical and modern Fourier analysis*. Pearson Education, Inc., Upper Saddle River, NJ, 2004.
- [5] Grafakos, L.; Honzík, P.; Seeger, A. *On maximal functions for Mihlin-Hörmander multipliers*. Adv. Math. 204 (2006), no. 2, 363–378.

- [6] El Kohen, A. *On the hyperbolic Riesz means*. Proc. Amer. Math. Soc. 89 (1983), no. 1, 113-116.
- [7] Oberlin, R. *A Marcinkiewicz maximal-multiplier theorem*. arXiv:1110.1065v1
- [8] Pipher, J. *Bounded double square functions*. Ann. Inst. Fourier (Grenoble) 36 (1986), no. 2, 69–82.
- [9] Seeger, A.; Wright, J. *Problems on averages and lacunary maximal functions*. Józef Marcinkiewicz Centenary Volume. Banach Center Publications, vol. 95 (2011), 235-250
- [10] Stein, E. M. *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*. *With the assistance of Timothy S. Murphy*. Princeton Mathematical Series, 43. Monographs in Harmonic Analysis, III. Princeton University Press, Princeton, NJ, 1993.

PETR HONZÍK, FACULTY OF MATHEMATICS AND PHYSICS, CHARLES UNIVERSITY, SOKOLOVSKÁ 83
180 00 PRAGUE 8 CZECH REPUBLIC
E-mail address: honzik@gmail.com