NONABSOLUTELY CONVERGENT INTEGRALS AND SINGULAR INTEGRALS

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Abstract. We define the packing integral (a kind of non-absolutely convergent integral) with respect to distributions of arbitrary order. Then we show that singular integrals can be interpreted as packing integrals with respect to generating distributions. This allows us to consider singular integrals beyond L^1.

1. Introduction

This paper is a part of the program to study new nonabsolutely integrals with respect to distributions and distribution-like objects. These integrals are basically of Henstock–Kurzweil type [7], [9]. We study the so called packing integral [8], which coincides with the UC integral introduced in [10]. The packing integral with respect to the so-called metric distributions fits well in the setting of metric spaces.

As shown in [10], the one-dimensional packing integral contains the Denjoy-Perron integral [5], [11]. For an exposition of the Denjoy-Perron integral see [12]. Let us recall that the Denjoy-Perron integral, also called the *restricted Denjoy integral*, coincides with the Henstock-Kurzweil integral and with the MC-integral introduced in [2].

We postpone the definition of the packing integral to Section 3. Roughly speaking, to test that a distribution $\mathcal{F}$ is an indefinite packing integral of a function f with respect to $\mathcal{G}$, we estimate how far is $\langle \mathcal{F}, \varphi\rangle$ from $f(x)\langle \mathcal{G}, \varphi\rangle$ if φ is supported in a small ball around x and $\|\varphi\| \le 1$. Diverse variations of the concept arise depending on the choice of the norm $\|\cdot\|$. In [10], [8], we have considered a specific first order norm. In this paper we introduce the packing integrals adapted to *higher order distributions*, this means that we consider seminorms of higher order to measure test functions. For this goal we need a more subtle partition of unity than in [10]. The main result is then that this integral makes a sense, this means that it is unique, see Theorem 5.1. Concerning the existence, the higher order seminorms we consider, the wider is the class of integrable functions, and this contains all Lebesgue integrable functions already for the order 0, Theorem 6.1.

In Section 7 we use packing integrals to give an alternative sense to certain singular integrals. Indeed, if $\mathcal{T}^z$ is the distribution which associates the singular integral with a test function, the integral with respect to $\mathcal{T}^z$ serves as the general singular integral.

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2. Preliminaries

The ball with center at x and radius r is denoted by $B(x,r)$, especially $B = B(0,1)$. We write $|E|$ for the Lebesgue measure of a set E.

We use the barred integral symbol for integral averages, this means

$$\fint_E f\,d\mu = \frac{1}{\mu(E)}\int_E f\,d\mu.$$

The *principal value integral* of a function g over a set $E \subset \mathbb{R}^n$ is defined as

$$(\text{p.v.})\int_E g(y)\,dy = \lim_{\varepsilon\to 0}\int_{E\setminus B(0,\varepsilon)} g(y)\,dy.$$

We say that the principal value integral of g *converges* if the limit exists and is finite.

The total differential of the k-th order consists of all k-th order partial derivatives, we denote it by D^k.

If $\Omega \subset \mathbb{R}^n$ is an open set, $\mathcal{D}(\Omega)$ is the class of all infinitely differentiable functions supported in Ω (so-called "test functions"). Then $\mathcal{D}'(\Omega)$ is the family of all distributions on Ω. The action of a distribution $\mathcal{T}$ on a test function φ is denoted by $\langle \mathcal{T},\varphi\rangle$.

We use the symbol C for a generic constant which can change at each occurence.

3. The definitions of integral

3.1. **Packing integral.** We follow the ideas from [10] and [8], but we consider general seminorms of arbitrary order. Denote $B = B(0,1)$. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. Let $\mathscr{P}$ be a system of continuous seminorms on $\mathcal{D}(B)$. We assume that for each finite subsystem $\mathscr{P}'$ of $\mathscr{P}$ there exists $\boldsymbol{p} \in \mathscr{P}$ which majorizes every $\boldsymbol{p}' \in \mathscr{P}'$. For $\boldsymbol{p} \in \mathscr{P}$, $x \in \Omega$ and $r > 0$ with $B(x,r) \subset \Omega$ we write

$$\begin{aligned}
&\kappa_{x,r}(y) = x + ry, \qquad y \in B,\\
&\boldsymbol{p}_{x,r}(\varphi) = \boldsymbol{p}(\varphi\circ\kappa_{x,r}), \qquad \varphi \in \mathcal{D}(B(x,r)),\\
&\boldsymbol{p}^*_{x,r}(\mathcal{T}) = \sup\Big\{\langle \mathcal{T},\varphi\rangle:\ \varphi \in \mathcal{D}(B(x,r)),\ \boldsymbol{p}_{x,r}(\varphi) \le 1\Big\},\ \mathcal{T} \in \mathcal{D}'(\Omega).
\end{aligned}$$

Here we identify $\mathcal{D}(B(x,r))$ with $\{\varphi \in \mathcal{D}(\Omega)\colon \operatorname{spt}\varphi \subset B(x,r)\}$. Let $\mathcal{F}$ be a distribution on Ω and $\mathscr{H} : \Omega \to \mathcal{D}'(\Omega)$ be a distribution-valued function. A function $\delta : \Omega \to (0,\infty)$ is termed a *gauge*. A finite system $(B(x_i,r_i)_{i=1}^m$ of balls is called a *packing* in Ω if the balls $B(x_i,r_i)$ are pairwise disjoint and contained in Ω, $i = 1,\dots,m$. (The number m of balls is finite but not limited.) We say that the packing is *δ-fine* if $r_i < \delta(x_i)$, $i = 1,\dots,m$. We say that $\mathcal{F}$ is an indefinite *$\mathscr{P}$-packing integral*, of $\mathscr{H}$ if there exist $\tau \in (0,1)$ and $\boldsymbol{p} \in \mathscr{P}$ such that for each $\varepsilon > 0$ there exists a gauge $\delta : \Omega \to (0,\infty)$ such that for each δ-fine packing $(B(x_i,r_i)_{i=1}^m$ in Ω we have

$$\sum_{i=1}^m \boldsymbol{p}^*_{x_i,\tau r_i}(\mathcal{F} - \mathscr{H}(x_i)) < \varepsilon.$$

We denote the indefinite integral of $\mathscr{H}$ by $\int \mathrm{d}\mathscr{H}$. If $\mathscr{H}$ has the form $\mathscr{H}(x) = f(x)\mathcal{G}$, where f is a function on Ω and $\mathcal{G}$ is a distribution independent of x, we call the process as integration of f with respect to $\mathcal{G}$; the resulting integral is denoted by $\int f\,\mathrm{d}\mathcal{G}$.

Remark 3.1. 1. It is easy to observe that the class of all distribution valued functions on Ω is a linear space and the operator of indefinite integration is linear.

2. If μ is a Radon measure on $\mathbb{R}^n$ and $f \in L^1(\mu)$, the task to integrate f with respect to μ corresponds to integration of the distribution valued function

$$\langle \mathscr{H}(x), \varphi \rangle = \int_{\mathbb{R}^n} f(x)\varphi(y)\, d\mu(y),$$

whereas the resulting indefinite integral should be

$$\langle \mathcal{F}, \varphi \rangle = \int_{\mathbb{R}^n} f(y)\varphi(y)\, d\mu(y).$$

3.2. **Definite integral.** There is a variety of "definite integrals" associated with our indefinite integral. Once found an indefinite integral $\mathcal{F}$, we need to "evaluate" $\langle \mathcal{F}, 1 \rangle$. Since $\varphi = 1$ is not a legitimate test function, it leads us to choose one of the many possibilities how to define the limit process $\varphi \to 1$.

Thus, a definition of a definite integral is a two step process: we combine a version of indefinite integral with a kind of the limit process. Apparently, the first step is the one which is more difficult. Therefore, in this paper, we focus our attention to this first step and we do not study the definite integrals.

We include only one example for illustration. We can say that a number $I \in \mathbb{R}$ is a definite integral of $\mathscr{H} : \mathbb{R}^n \to \mathcal{D}'(\mathbb{R}^n)$ if there exists an indefinite integral $\mathcal{F}$ of $\mathscr{H}$ and for each sequence $(\varphi_j)_j$ of test functions such that $\operatorname{spt} \varphi_j \subset \overline{B}(0, j)$ and $\varphi_j(0) = 1$ we have

$$(\boldsymbol{p}_{0,j}(\varphi_j))_j \text{ is bounded for each } \boldsymbol{p} \in \mathscr{P} \implies \langle \mathcal{F}, \varphi_j \rangle \to I.$$

4. Partition of unity

In this section, we consider the concrete family of seminorms

$$\mathscr{P} = \{\boldsymbol{p}^{(0)},\ \boldsymbol{p}^{(1)},\ \boldsymbol{p}^{(2)}, \dots\}, \tag{1}$$

where

$$\boldsymbol{p}^{(k)}(\varphi) = \|D^k \varphi\|_\infty, \qquad \varphi \in \mathcal{D}(B). \tag{2}$$

It is evident that $\boldsymbol{p}^{(k)}$ majorizes $\boldsymbol{p}^{(j)}$ if $j \le k$.

We need a careful partition of unity corresponding to a covering.

Theorem 4.1. *There exists a sequence C_k of constants with the following property. Suppose that $(B_i)_{i=1}^m$ be a finite pairwise disjoint system of balls in $\mathbb{R}^n$, $B_i = B(x_i, r_i)$, and $G = \bigcup_i B(x_i, 5r_i)$. Then there exists a system ω_i of functions on $\mathbb{R}^n$ such that*

(a) $\omega_i \in \mathcal{D}(\mathbb{R}^n)$, $i = 1, \dots, m$,
(b) $\omega_i \ge 0$, $i = 1, \dots, m$,
(c) $\operatorname{spt} \omega_i \subset B(x_i, 10r_i)$, $i = 1, \dots, m$,
(d) $\chi_G \le \sum_i \omega_i \le 1$,
(e) $\boldsymbol{p}^{(k)}_{x_i,10r_i}(\omega_i) \le C_k, \qquad i = 1, \dots, m, \quad k \in \mathbb{N}.$

Proof. We may assume that $r_1 \geq r_2 \geq \cdots \geq r_m$. Let $\eta \in \mathcal{D}(\mathbb{R}^n)$ be a fixed function such that $\operatorname{spt} \eta \subset B(0,10)$ and $\eta = 1$ on $B(0,5)$. Set

$$\begin{aligned}
\psi_i &= 1 - \eta \circ \kappa_{x_i,r_i}^{-1}, \\
\rho_i &= \psi_1 \psi_2 \cdots \psi_i, \\
\omega_1 &= 1 - \rho_1, \\
\omega_i &= \rho_{i-1} - \rho_i, \quad i = 2, 3, \dots .
\end{aligned}$$

Then $\psi_i = 0$ on $B(x_i, 5r_i)$, $\psi_i = 1$ on $\mathbb{R}^n \setminus B(x_i, 10r_i)$, the properties (a),…,(d) are evidently satisfied and it remains to prove the estimate (e). We fix $q \in \{1, \dots, m\}$. We need to show that there exists a constant C_k such that

$$\boldsymbol{p}^{(k)}_{x_q,10r_q}(1-\rho_q) \leq \frac{1}{2} C_k.$$

We decompose

$$\{1, \dots, q\} = \bigcup_{j=1}^{\infty} I_j,$$

where

$$I_j = \big\{ i \in \{1, \dots, q\} :\ 2^{j-1} r_q \leq r_i < 2^j r_q \big\}.$$

Denote

$$\xi_j = \prod_{i \in I_j} \psi_i, \qquad j = 1, 2, \dots .$$

Then there is $\beta \in \mathbb{N}$ such that

$$\rho_q = \xi_1 \xi_2 \dots \xi_\beta .$$

Given $x \in \mathbb{R}^n$, let $I_j(x)$ be the set of indices $i \in I_j$ for which $x \in \overline{B}(x_i, 10r_i)$. From the assumption that the balls $B(x_i, r_i)$ are pairwise disjoint we infer that the number of indices in $I_j(x)$ is limited by a constant N depending only on the dimension n. A simple scaling argument shows that there exist constants $\tilde{C}_0, \dots, \tilde{C}_k$ such that

$$\|D^s \xi_j\|_\infty \leq \tilde{C}_s (2^j r_q)^{-s}, \qquad s = 0, \dots, k, \quad j = 1, \dots, \beta.$$

(Here and in the sequel we do not mention dependence of constants on the dimension and on the choice of the function η). We find a constant $A = A_k > 0$ such that

$$\tilde{C}_s \leq A^{-s}, \quad s = 0, \dots, k,$$

so that

$$\|D^s \xi_j\|_\infty \leq 2^{-js} (A r_q)^{-s}, \qquad s = 0, \dots, k, \quad j = 1, \dots, \beta. \tag{3}$$

We claim that

$$\|D^s (\xi_1 \dots \xi_\alpha)\|_\infty \leq \big(1 - 2^{-\alpha}\big)^s (A r_q)^{-s}, \qquad s = 0, \dots, k, \quad \alpha = 1, 2, \dots . \tag{4}$$

We prove the claim by induction. For $\alpha = 1$ it has the form

$$\|D^s (\xi_1)\|_\infty \leq \big(1 - 2^{-1}\big)^s (A r_q)^{-s}, \qquad s = 0, \dots, k,$$

this is just (3). Assume that $j > 1$ and the claim is true for $\alpha = j-1$, then by the Leibniz rule we compute

$$\begin{aligned}\|D^k((\xi_1 \dots \xi_{j-1})\xi_j)\|_\infty &\le \sum_{s=0}^{k} \binom{k}{s} \|D^s((\xi_1 \dots \xi_{j-1})\|_\infty \|D^{k-s}\xi_j\|_\infty \\ &\le \sum_{s=0}^{k} \binom{k}{s} \big(1-2^{-j+1}\big)^s (Ar_q)^{-s}\, 2^{-j(k-s)} (Ar_q)^{s-k}.\end{aligned}$$

Using the binomial rule we simplify the estimate as

$$\|D^k(\xi_1 \dots \xi_j)\|_\infty \le \big((1-2^{-j+1}) + 2^{-j}\big)^k (Ar_q)^{-k},$$

which proves (4) for $\alpha = j$ and $s = k$. Similarly we obtain the estimate of $D^s(\xi_1 \dots \xi_j)$ with $0 \le s < k$. It follows that

$$\boldsymbol{p}^{(k)}_{x_q,10r_q}(1-\rho_q) \le 10^k A^{-k}$$

as required. □

5. Uniqueness of the integral

In this section, let $\mathscr{P}$ be the family of seminorms (1), (2). where $\boldsymbol{p}_k$ are defined by (2).

Theorem 5.1. *Let $\Omega \subset \mathbb{R}^n$ be an open set and $\mathscr{H} : \Omega \to \mathcal{D}'(\Omega)$ be a distribution-valued function. Then there exists at most one indefinite $\mathscr{P}$-packing integral of $\mathscr{H}$.*

Proof. By linearity, it is enough to prove that 0 is the only indefinite integral of $\mathscr{H} \equiv 0$. Let $\mathcal{F}$ be an indefinite integral of 0 and $\eta \in \mathcal{D}(\mathbb{R}^n)$. We need to show that $\langle \mathcal{F}, \eta \rangle = 0$. Since the support of η is compact, we may assume that $|\Omega| < \infty$. Let $\boldsymbol{p} \in \mathscr{P}$ and τ be as in the definition of integral. Choose $\varepsilon > 0$ and find a gauge δ as in the definition of integral. With each $x \in \operatorname{spt}\eta$ we associate a ball $B(x, r_x)$ such that $\overline{B}(x, 10r_x) \subset \Omega$, $r_x < \delta(x)$ and

$$\boldsymbol{p}^*_{x,10r_x}(\mathcal{F}) \le C\boldsymbol{p}^*_{x,\tau r_x}(\mathcal{F}) + \varepsilon r_x^n \tag{5}$$

(such r_x exists, see [8]). Using Vitali covering technique and compactness of $\operatorname{spt}\eta$, we select a finite pairwise disjoint system $(B_i)_{i=1}^m$, $B_i = B(x_i, r_i)$, such that $r_i = r_{x_i}$ $(i = 1, \dots, m)$, and $B(x_i, 5r_i)$ cover $\operatorname{spt}\eta$. If $(\omega_i)_{i=1}^m$ is a partition of unity as in Theorem 4.1 we obtain

$$|\langle \mathcal{F}, \eta \rangle| = \Big| \sum_{i=1}^{m} \langle \mathcal{F}, \eta\omega_i \rangle \Big| \le \sup_{i=1,\dots,m} \boldsymbol{p}_{x_i,10r_i}(\eta\omega_i) \sum_{i=1}^{m} \boldsymbol{p}^*_{x_i,10r_i}(\mathcal{F}).$$

Now by (5) we have

$$\sum_{i=1}^{m} \boldsymbol{p}^*_{x_i,10r_i}(\mathcal{F}) \le C \sum_{i=1}^{m} \Big(\boldsymbol{p}^*_{x_i,\tau r_i}(\mathcal{F}) + \varepsilon r_i^n\Big) \le C\varepsilon(1 + |\Omega|).$$

By the property (e) in Theorem 4.1 and the special choice of $\boldsymbol{p} = \boldsymbol{p}^{(k)}$, there exists a constant $C(k, \eta)$ such that for each $i = 1, \dots, m$ we have

$$\boldsymbol{p}_{x_i,10r_i}(\eta\omega_i) \le C(k, \eta)\boldsymbol{p}_{x_i,10r_i}(\omega_i) \le C_k C(k, \eta).$$

It follows that $\langle \mathcal{F}, \eta \rangle = 0$. □

6. Absolutely convergent integration

We consider the distribution

$$\langle \mathcal{G}, \varphi \rangle = \int_\Omega \varphi(y)\, d\mu(y).$$

Recall that a.e. point $x \in \mathbb{R}^n$ is a Lebesgue point for $f \in L^1(\Omega, \mu)$, this means

$$\lim_{r\to 0+} \fint_{B(x,r)} |f(y) - f(x)|\, d\mu(y) = 0,$$

see [6, 1.7].

Theorem 6.1. *Suppose that $\mathscr{P}$ contains a seminorm $\boldsymbol{p}$ with $\boldsymbol{p} \geq \boldsymbol{p}^{(0)}$. Let $\Omega \subset \mathbb{R}^n$ be an open set, μ be a Radon measure on Ω and $f \in L^1_{\mathrm{loc}}(\Omega, \mu)$. Then the distribution*

$$\langle \mathcal{F}, \varphi \rangle = \int_\Omega f\varphi\, dy, \qquad \varphi \in \mathcal{D}(\Omega)$$

is an indefinite $\mathscr{P}$-packing integral of f with respect to $\mathcal{G}$ with $\tau = 1$.

Proof. Given $\varepsilon > 0$, we define a gauge δ according to the following rules: Let L be the set of all μ-Lebesgue points for f and $E_k = \{x \in \Omega \setminus L : \ |f(x)| \leq k\}$.

Let $(U_k)_k$ be a sequence of open sets of finite μ-measure such that $\mu(U_k) > 0$ and $\Omega = \bigcup_k U_k$. Let V be a open set containing the μ-null set $\Omega \setminus L$ with the property that

$$\int_V |f(y)|\, d\mu(y) < \varepsilon$$

and W_k be open sets containing E_k such that $\mu(W_k) < \frac{\varepsilon}{k2^k}$, $k = 1, 2, \dots$.

If $x \in L \cap U_k$, we choose $\delta(x) > 0$ such that $B(x, \delta(x)) \subset U_k$ and

$$0 < r < \delta \implies \fint_{B(x,r)} |f(y) - f(x)|\, d\mu(y) \leq \frac{\varepsilon}{2^k \mu(U_k)}.$$

Then, if $0 < r < \delta(x)$ and $\varphi \in \mathcal{D}(\Omega)$ with $\operatorname{spt}\varphi \in B(x,r)$ and $\boldsymbol{p}^{(0)}(\varphi) \leq 1$, we have

$$|\langle \mathcal{F} - f(x)\mathcal{G}, \varphi \rangle| = \Big| \int_{B(x,r)} (f(y) - f(x))\, \varphi(y)\, d\mu(y) \Big| \leq \varepsilon \frac{\mu(B(x,r))}{2^k \mu(U_k)}.$$

If $x \in E_k$, we choose $\delta(x) > 0$ such that $B(x, \delta(x)) \subset V \cap W_k$. Then, if $0 < r < \delta(x)$ and $\varphi \in \mathcal{D}(\Omega)$ with $\operatorname{spt}\varphi \in B(x,r)$ and $\boldsymbol{p}^{(0)}(\varphi) \leq 1$, we have

$$\begin{aligned} |\langle \mathcal{F} - f(x)\mathcal{G}, \varphi \rangle| &= \Big| \int_{B(x,r)} (f(y) - f(x))\, \varphi(y)\, d\mu(y) \Big| \\ &\leq \varepsilon \int_{B(x,r)} |f(y)|\, d\mu(y) + k\mu(B(x,r)). \end{aligned}$$

Now, it is clear to observe that for each δ-fine packing $(B(x_i, r_i))_i$ of Ω we have

$$\sum_i \boldsymbol{p}^*_{x_i, r_i}(\mathcal{F} - f(x)\mathcal{G}) \leq \int_V |f(y)|\, d\mu(y) + \sum_{k=1}^{\infty} \frac{\varepsilon \mu(U_k)}{2^k \mu(U_k)} + \sum_{k=1}^{\infty} k|W_k| \leq 3\varepsilon.$$

□

7. Singular integral

We demonstrate the power of the integral we just defined on the classical example of the singular integral with Calderón–Zygmund kernel. We do not pursue the maximal possible generality, and therefore we settle for the smooth homegeneous kernel while noting that many other cases may be handled in analogous fashion. Let us take a C^1 function Φ defined on the unit sphere S^{n-1} of mean value 0 with respect to the surface measure.

We consider the kernel

$$K(x) = |x|^{-n}\Phi\Big(\frac{x}{|x|}\Big).$$

We imagine the singular integral operator T as the convolution $Tf = K * f$. However, this convolution usually does not make sense as Lebesgue integral. We write

$$\begin{aligned} T^x_\varepsilon f &= \int_{B^c(0,\epsilon)} f(x-y)K(y)\,dy, \\ T^x f &= (\text{p.v.})\int_{\mathbb{R}^n} f(x-y)K(y)\,dy = \lim_{\epsilon\to 0} T^x_\varepsilon f. \end{aligned} \tag{6}$$

The definition

$$Tf : x \mapsto T^x f$$

gives a sense to the singular integral at least if $f \in \mathcal{D}(\mathbb{R}^n)$. In this case the principal value integral converges for each $x \in \mathbb{R}^n$. The operator is then shown to be apriori L^p bounded for $1 < p < \infty$ and of the weak type $1-1$. This boundedness allows to extend the operator T to all L^p functions. The extension, however, does not guarantee the convergence of the principal value. To avoid a misunderstanding, we use the symbol $T^x f$ exclusively for the principal value integral (6).

The standard method to prove the convergence of the principal value is to consider the maximal singular operator

$$T^* f(x) = \sup_{\varepsilon>0} |T^x_\varepsilon f|.$$

This operator is well defined for any function f in L^1 and is of the weak type $1-1$. From this fact it is then possible to obtain the convergence of the principal value $T^x f$ almost everywhere by a simple argument. The seminal papers in the theory are [3] and [4], for overview see [13].

The principal value integral acts a distribution

$$\langle \mathcal{T}^x, \varphi\rangle = T^x\varphi.$$

Moreover, given $f \in L^1(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$ such that the singular integral $T^x f$ exists, we define the distribution $\mathcal{F}^x$ as

$$\langle \mathcal{F}^x, \varphi\rangle = T^x(f\varphi).$$

This definition is correct according to the following lemma.

Lemma 7.1. *Let $f \in L^1(\mathbb{R}^n)$, $x \in \mathbb{R}^n$ and $r > 0$. Suppose that the principal value integral $T^x f$ converges and that x is a Lebesgue point for f. Let $\varphi \in \mathcal{D}(\mathbb{R}^n)$, $\operatorname{spt}\varphi \subset \overline{B}(x,r)$. Then $T^x(f\varphi)$ exists and*

$$\begin{aligned} |T^x(f\varphi) - f(x)T^x\varphi| \le Cr\|\nabla\varphi\|_\infty &\sup_{0<\rho\le r} \fint_{B(x,\rho)} |f(y)-f(x)|\,dy \\ &+ C\|\varphi\|_\infty |T^x f - T^x_r f|. \end{aligned} \tag{7}$$

Proof. We start from the estimate

$$\int_{B(x,r)} |y-x|^{1-n}|f(y)-f(x)|\,dy \le (n-1)\int_0^r \Big(\rho^{-n}\int_{B(x,\rho)} |f(y)-f(x)|\,dy\Big)\,d\rho \tag{8}$$
$$+ r^{1-n}\int_{B(x,r)} |f(y)-f(x)|\,dy \le Cr \sup_{0<\rho\le r} \fint_{B(x,\rho)} |f(y)-f(x)|\,dy,$$

see [1, Lemma 3.1.1]. The right part of (8) is finite as x is a Lebesgue point for f. Hence

$$\int_{\mathbb{R}^n} |x-y|^{1-n}|f(y)|\,dy \le |f(x)|\int_{B(x,r)} |x-y|^{1-n}\,dy \tag{9}$$
$$+ Cr \sup_{0<\rho\le r} \fint_{B(x,\rho)} |f(y)-f(x)|\,dy + r^{1-n}\int_{\mathbb{R}^n} |f(y)|\,dy.$$

Since

$$|K(x-y)(\varphi(y)-\varphi(x))| \le C\|\nabla\varphi\|_\infty |y-x|\,|K(x-y)| \le C\|\nabla\varphi\|_\infty |x-y|^{1-n}, \tag{10}$$

the integral $T^x((\varphi-\varphi(x))f)$ has a sense even as the Lebesgue integral. Thus, $f\mapsto T^x(\varphi f)$ is a sum of distributions $\varphi\mapsto T^x((\varphi-\varphi(x))f)$ and $\varphi\mapsto\varphi(x)T^x f$.

Let χ_B be the characteristic function of $B=B(x,r)$. Then

$$T^x(f\varphi) - f(x)T^x\varphi = T^x((f-f(x))\varphi\chi_B) \tag{11}$$
$$= T^x((f-f(x))(\varphi-\varphi(x))\chi_B) + \varphi(x)T^x((f-f(x))\chi_B).$$

By (10), the integral $T^x((f-f(x))(\varphi-\varphi(x))\chi_B)$ converges as the Lebesgue integral and

$$|T^x((f-f(x))(\varphi-\varphi(x))\chi_B)| \le Cr\|\nabla\varphi\|_\infty \sup_{0<\rho\le r}\fint_{B(x,\rho)} |f(y)-f(x)|\,dy. \tag{12}$$

For the second term in (11) we first observe that $T^x\chi_B=0$ by definition of the kernel (the mean value requirement on Φ). Hence

$$T^x((f-f(x))\chi_B) = T^x(f\chi_B) = T^x f - T^x(f\chi_{B^c}) = T^x f - T^x_r f. \tag{13}$$

From (11), (12) and (13) we obtain the required estimate (7). □

Theorem 7.2. *Let $\mathscr{P}$ be a system of seminorms on $\mathcal{D}(B)$ containing a seminorm $\boldsymbol{p}$ with $\boldsymbol{p}\ge\boldsymbol{p}^{(1)}$ (see (1) for the notation). Let $f\in L^1$. Then for almost every $z\in\mathbb{R}^n$, the distribution $\mathcal{F}^z$ is the indefinite $\mathscr{P}$-packing integral of f with respect to $\mathcal{T}^z$ with $\tau=1$.*

Proof. We choose z such that z is a Lebesgue point for f and the principal value integral $T^z f$ converges. Given $\varepsilon>0$, by Theorem 6.1 there exists a gauge $\delta:\mathbb{R}^n\setminus\{z\}\to(0,\infty)$ such that for each δ-fine packing $(B(x_i,r_i))_{i=1}^m$ in $\mathbb{R}^n\setminus\{z\}$ we have

$$\sum_{i=1}^m \boldsymbol{p}^*_{x_i,r_i}(\mathcal{F}^z - f(x_i)\mathcal{T}^z) < \varepsilon.$$

Moreover, we may assume that $\delta(x)<|x-z|$ for each $x\in\mathbb{R}^n\setminus\{z\}$. We define $\delta(z)$ such that

$$\sup_{0<r\le\delta(z)}\fint_{B(x,r)} |f(y)-f(x)|\,dy < \varepsilon$$

and

$$\sup_{0<r\le\delta} |T^z_r f - T^z f| < \varepsilon.$$

Consider a δ-fine packing $(B(x_i,r_i))_{i=1}^m$ in $\mathbb{R}^n$. Then at most one of the balls $\overline{B}(x_i,r_i)$ contains z and in this case this is a ball $\overline{B}(z,r)$ with $r<\delta(z)$. For such a ball $\overline{B}(z,r)$ and $\varphi\in\mathcal{D}(\mathbb{R}^n)$ with $\operatorname{spt}\varphi\subset\overline{B}(z,r)$ we have

$$\langle \mathcal{F}^z - f(z)\mathcal{T}^z,\ \varphi\rangle \le C\varepsilon r\|\nabla\varphi\|_\infty + C\varepsilon\|\varphi\|_\infty$$

by the definition of $\delta(z)$ and Lemma 7.1. Since

$$\|\varphi\|_\infty + r\|\nabla\varphi\|_\infty \le C\boldsymbol{p}_{z,r}(\varphi),$$

we have

$$\boldsymbol{p}^*_{z,r}(\mathcal{F}^z - f(z)\mathcal{T}^z) \le C\varepsilon.$$

Hence for any δ-fine packing as above we obtain

$$\sum_{i=1}^m \boldsymbol{p}^*_{x_i,r_i}(\mathcal{F}^z - f(x_i)\mathcal{T}^z) < (1+C)\varepsilon.$$

□

Remark 7.3. We have just proved that the singular integral defined by $z\mapsto\mathcal{F}^z$ makes sense for all L^1 functions. However, it is clear that there is much more than L^1 functions which are integrable in this sense. For example, if

$$f(x)=\frac{\sin\frac{1}{x^2}}{x}$$

and the singular kernel is just $1/x$, then the integral $\mathcal{F}^z$ has sense for $z\neq 0$, and thus for a.e. $z\in\mathbb{R}$. Indeed, the integrand

$$\frac{\sin\frac{1}{(z-x)^2}}{x(z-x)}$$

has two singularities; the singularity at $x=0$ can be handled via Theorem 7.2, whereas the singularity at $x=z$ does not bother for non-absolutely convergent integrals of Denjoy-Perron type and their generalizations.

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