THE SIZE OF THE CLASSES OF $H^{(N)}$-SETS

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Abstract. The class of $H^{(N)}$-sets forms an important subclass of the class of sets of uniqueness for trigonometric series. We investigate the size of this class which is reflected by the family of measures (called polar) annihilating all sets from this class. The main aim of this paper is to answer in the negative a question stated by Lyons, whether the polars of the classes of $H^{(N)}$-sets are the same. To prove our result we also present a new description of $H^{(N)}$-sets.

1. Introduction

Let M be a collection of closed subsets of $[0,1]$ and $\mathcal{M}([0,1])$ be the set of all Radon measures on the interval $[0,1]$. Then the *polar* $M^\perp \subset \mathcal{M}([0,1])$ is defined by

$$M^\perp = \{\nu \in \mathcal{M}([0,1]);\ \forall B \in M:\ \nu(B) = 0\}.$$

We say that $\mu \in \mathcal{M}([0,1])$ is *Rajchman* if $\lim_{|n|\to\infty} \widehat{\mu}(n) = 0$. The family of all Rajchman measures is denoted by $\mathcal{R}$. Let us recall that closed sets of *extended uniqueness* (U_0 sets) are those closed sets which are annihilated by every Rajchman measure. Thus by definition we have that $\mathcal{R} \subset U_0^\perp$.

Rajchman investigated classes A with the property $A^\perp = \mathcal{R}$. He introduced in [9] an important subclass of U sets, so called H-sets (or $H^{(1)}$-sets) (see the next section or [4] for the definitions of U and $H^{(1)}$) and investigated whether $H^\perp = \mathcal{R}$. Lyons in [5] showed that $\mathcal{R} = U_0^\perp$. On the other hand Kaufman proved that $U^\perp \neq U_0^\perp = \mathcal{R}$ ([3]). Thus U_0 can be considered much larger than U in the sense of polars. More generally, one can consider two families of closed sets $A \subset B$ and may ask whether $B^\perp \subsetneq A^\perp$. If this is the case then B can be considered much larger than A.

Rajchman conjectured that every set of uniqueness is a countable union of H-sets. This was disproved by Pyatetskii-Shapiro in [7] (see also [8]), where he also introduced the classes of $H^{(N)}$-sets. Further he showed $H^{(N)} \subset H^{(N+1)} \subset U \subset U_0$ and he stated that there is an $H^{(N+1)}$-set which cannot be written as a countable union of $H^{(N)}$-sets. Lyons in [6] showed that $\mathcal{R} \subsetneq \left(\bigcup_{N\in\mathbb{N}} H^{(N)}\right)^\perp$. Thus, classes $H^{(N)}$ are "small" in U_0 in the sense given above. Lyons in [6] posed a question whether $(H^{(N+1)})^\perp = (H^{(N)})^\perp$. The aim of this paper is to prove the next theorem which answers Lyons' question in the negative for every $N \in \mathbb{N}$.

Theorem 1.1. *Let $N \in \mathbb{N}$. Then $(H^{(N+1)})^\perp \neq (H^{(N)})^\perp$.*

2000 *Mathematics Subject Classification.* 43A46, 42A63.
Key words and phrases. Sets of uniqueness, polar, $H^{(N)}$ sets.

We will prove Theorem 1.1 using Theorem 2.5 on a description of $H^{(N)}$-sets. This result can be used to reprove Šleich's result that each $H^{(N)}$-set is σ-porous ([12]).

The case $N = 1$ in Theorem 1.1, which is much simpler, was presented without proofs in [11].

There also arises an open question, whether $\left(\bigcup_{N\in\mathbb{N}} H^{(N)}\right)^{\perp} \supsetneq U^{\perp}$. Zelený and Pelant ([13]) show that there is a non σ-porous closed set of uniqueness. Thus this set is a set of uniqueness, which cannot be written as a countable union of elements of $\bigcup_{N\in\mathbb{N}} H^{(N)}$.

2. Proof of Theorem 1.1

Notation 2.1.

(i) We denote the Lebesgue measure on $\mathbb{R}$ by λ and the number of elements of a finite set A by $\sharp\, A$.

(ii) The symbol $\langle x\rangle$ stands for the fractional part of $x \in \mathbb{R}$, i.e., $\langle x\rangle = x - [x]$, where $[x]$ is the integer part of x. Further, for $B \subset \mathbb{R}$ we denote $\langle B\rangle = \{\langle x\rangle;\ x \in B\}$.

(iii) For $N \in \mathbb{N}$ and $\boldsymbol{a} \in \left(\mathbb{R}^N\right)^{\mathbb{N}}$, we employ the following coordinate notation $\boldsymbol{a} = \{a_j\}_{j\in\mathbb{N}}$ and $a_j = (a_j^1, \dots, a_j^N) \in \mathbb{R}^N$.

(iv) By an *open interval* $J \subset \mathbb{R}^N$ we mean any product of nonempty open intervals $J^i \subset \mathbb{R}$, $i = 1, \dots, N$.

(v) Let $x \in \mathbb{R}$ and $r > 0$. We denote the interval $(x - r, x + r)$ by $B(x, r)$.

Definition 2.2. Let $N \in \mathbb{N}$, $L \in \mathbb{R}$, and $P \subset \mathbb{R}$.

(i) A sequence of vectors $\boldsymbol{a} \in (\mathbb{R}^N)^{\mathbb{N}}$ is *quasi-independent* if for every nonzero $\alpha \in \mathbb{Z}^N$ we have $\lim_j |(\alpha, a_j)| = \infty$, where (u, v) denotes the scalar product of vectors $u, v \in \mathbb{R}^N$. The set of all quasi-independent sequences of vectors from P^N is denoted by $\mathcal{Q}^N(P)$.

(ii) A closed set $A \subset [0, 1]$ is in $H^{(N)}(P)$ if there exist $\boldsymbol{a} \in \mathcal{Q}^N(P)$ and an open interval $J \subset [0, 1]^N$ such that for every $x \in A$ and every $j \in \mathbb{N}$ we have $\langle xa_j\rangle := (\langle xa_j^1\rangle, \dots, \langle xa_j^N\rangle) \notin J$. We will write just $H^{(N)}$ instead of $H^{(N)}(\mathbb{N})$ and $H^{(N)*}$ instead of $H^{(N)}(\mathbb{R} \setminus \{0\})$. The subsets of elements of $H^{(N)}$ are called $H^{(N)}$-sets.

(iii) A closed set $A \subset [0, 1]$ is in $H_L^{(N)*}$ if there exist $\boldsymbol{a} \in \mathcal{Q}^N(\mathbb{R} \setminus \{0\})$ and an open interval $J = \prod_{i=1}^N J^i \subset [0, 1]^N$ witnessing $A \in H^{(N)*}$ and satisfying

$$\left|\frac{a_j^{i+1}\lambda(J^i)}{a_j^i}\right| \geq L$$

for every $i \in \{1, \dots, N-1\}$ and $j \in \mathbb{N}$.

Remark 2.3. (i) Let $N, M \in \mathbb{N}$, $N \leq M$ and $L, K \in \mathbb{R}$, $L \leq K$. Then we clearly have $H_K^{(N)*} \subset H_L^{(M)*}$, $H^{(N)*} = H_0^{(N)*}$ and $H^{(N)} \subset H^{(N)*}$. Further, family $H^{(N)}$ is hereditary, i.e., if $A \in H^{(N)}$, $A \supset B$ and B is closed then $B \in H^{(N)}$. Similarly, families $H^{(N)*}$ and $H_L^{(N)*}$ are also hereditary.

(ii) Bari [1] denotes $H^{(N)*}$ as $H^{(N)}(\mathbb{R})$. We use $\mathbb{R}\setminus\{0\}$ instead of $\mathbb{R}$ to avoid dividing by zero. It is easy to see that $H^{(N)}(\mathbb{R}) = H^{(N)}(\mathbb{R}\setminus\{0\})$. Thus, both of these definitions define the same object. Note that each set from $H^{(N)*}$ is a finite union of elements of $H^{(N)}$ (see [1, pp. 919–921]). Consequently, $(H^{(N)*})^{\perp} = (H^{(N)})^{\perp}$.
(iii) Let $N \in \mathbb{N}$. Then the collection $H^{(N)}$ consists of closed $H^{(N)}$-sets.

The proof of the main result is based on the following two results which will be proved in the next sections.

Lemma 2.4. *Let $N \in \mathbb{N}$. Then $(H^{(N+1)})^{\perp} \subsetneq (H^{(N)*}_{10})^{\perp}$.*

Theorem 2.5. *Let $N, L \in \mathbb{N}$. Then $H^{(N)*}_{L} = H^{(N)*}$.*

Granting these results the proof goes as follows.

Proof of Theorem 1.1. Using Lemma 2.4, Theorem 2.5, and Remark 2.3(ii) we get

$$(H^{(N+1)})^{\perp} \subsetneq (H^{(N)*}_{10})^{\perp} = (H^{(N)*})^{\perp} = (H^{(N)})^{\perp}.$$

□

3. Proof of Lemma 2.4

Throughout this section $N \in \mathbb{N}$ will be fixed. We will construct a measure $\mu \in (H^{(N)*}_{10})^{\perp} \setminus (H^{(N+1)})^{\perp}$.

3.1. Construction of the measure μ.

Notation 3.1. Let $\mathcal{A}$ be a collection of subsets of $\mathbb{R}$ and $S \subset \mathbb{R}$. We denote

$$\mathcal{A}^S = \{V \in \mathcal{A};\ V \subset S\}.$$

Notation 3.2. We find and fix some $\boldsymbol{x} \in (\mathbb{N}^{N+1})^{\mathbb{N}}$ such that for every $n \in \mathbb{N}$ and $j = 1, \ldots, N$ we have that $\frac{x_n^{j+1}}{2x_n^j}$ and $\frac{x_{n+1}^1}{2x_n^{N+1}}$ are natural numbers bigger than n^2.
For $n \in \mathbb{N}$ and $j = 1, \ldots, N+1$ we set the following:

$$P_n = \left\{x \in [0,1];\ \langle x \cdot x_i \rangle \notin (1/2, 1)^{N+1}, i = 1, \ldots, n\right\}, \tag{3.1}$$
$$\mathcal{P}_{n,j} = \left\{ \left[\frac{i-1}{2x_n^j}, \frac{i}{2x_n^j}\right] \subset [0,1];\ i \in \mathbb{N}, \left(\frac{i-1}{2x_n^j}, \frac{i}{2x_n^j}\right) \subset P_n \right\},$$
$$\|\mathcal{P}_{n,j}\| = 1/(2x_n^j).$$

Remark 3.3 and Lemma 3.4 will explain some basic facts concerning collections $\mathcal{P}^I_{n,j}$.

Remark 3.3. Let $n \in \mathbb{N}$. Since $\frac{x_n^{j+1}}{2x_n^j}$ and $\frac{x_{n+1}^1}{2x_n^{N+1}}$ are natural numbers we can easily obtain the following three statements:

- $\bigcup \mathcal{P}_{n,N+1} = P_n$.
- $\mathcal{P}_{n+1,j} = \bigcup_{I \in \mathcal{P}_{n,N+1}} \mathcal{P}^I_{n+1,j}$.

- If $j \in \{1, \dots, N+1\}$, $i \in \mathbb{N}$, $I \in \mathcal{P}_{n,N+1}$ and $\left[\frac{i-1}{2x^j_{n+1}}, \frac{i+1}{2x^j_{n+1}}\right] \subset I$ then

$$\left[\frac{i-1}{2x^j_{n+1}}, \frac{i}{2x^j_{n+1}}\right] \in \mathcal{P}_{n+1,j} \text{ or } \left[\frac{i}{2x^j_{n+1}}, \frac{i+1}{2x^j_{n+1}}\right] \in \mathcal{P}_{n+1,j}.$$

Lemma 3.4.

(i) *If $V \in \mathcal{P}_{n,j}$, then $\|\mathcal{P}_{n,j}\| = \lambda(V)$.*

(ii) *Let $k \geq n$ and $i, j \leq N+1$ such that $k > n$ or $j \geq i$. Let $I, J \in \mathcal{P}_{n,i}$. Then we have $\sharp\, \mathcal{P}^I_{k,j} = \sharp\, \mathcal{P}^J_{k,j}$.*

(iii) *Let $n > 1$, $I \in \mathcal{P}_{n-1,N+1}$ and $1 \leq j \leq i \leq N+1$. Then we have*

$$\sharp\, \mathcal{P}^I_{n,i} \leq 2 \sum_{R \in \mathcal{P}^I_{n,j}} \sharp\, \mathcal{P}^R_{n,i}.$$

(iv) *Let $n_1, n_2, n_3 \in \mathbb{N}$, $n_1 < n_2 \leq n_3$, $j_1, j_2, j_3 \in \{1, \dots, N+1\}$ and $I \in \mathcal{P}_{n_1,j_1}$ be such that $n_2 < n_3$ or $j_2 \leq j_3$. Then we have*

$$\sharp\, \mathcal{P}^I_{n_3,j_3} \leq 2 \sum_{R \in \mathcal{P}^I_{n_2,j_2}} \sharp\, \mathcal{P}^R_{n_3,j_3}.$$

(v) *Let $n \in \mathbb{N}$ and $1 \leq j \leq N$. Then $\|\mathcal{P}_{n,j}\| \geq n^2 \|\mathcal{P}_{n,j+1}\|$.*

Proof. (i) Let $V \in \mathcal{P}_{n,j}$. Then there exists $i \in \mathbb{N}$ such that $V = \left[\frac{i-1}{2x^j_n}, \frac{i}{2x^j_n}\right]$. Thus, $\lambda(V) = 1/(2x^j_n) = \|\mathcal{P}_{n,j}\|$.

(ii) Let $x = \min(I)$ and $y = \min(J)$. It is easy to verify that $\mathcal{P}^J_{k,j} = \mathcal{P}^I_{k,j} + y - x$.

(iii) By Remark 3.3 we can easily obtain

$$\sharp\, \mathcal{P}^I_{n,i} \leq 2x^i_n \lambda(I) \leq 2 \sum_{R \in \mathcal{P}^I_{n,j}} \sharp\, \mathcal{P}^R_{n,i}.$$

(iv) Assume $n_2 < n_3$, Then we have

$$\sharp\, \mathcal{P}^I_{n_3,j_3} = \sum_{V \in \mathcal{P}^I_{n_2-1,N+1}} \sum_{W \in \mathcal{P}^V_{n_2,N+1}} \sharp\, \mathcal{P}^W_{n_3,j_3},$$

$$\sum_{R \in \mathcal{P}^I_{n_2,j_2}} \sharp\, \mathcal{P}^R_{n_3,j_3} = \sum_{V \in \mathcal{P}^I_{n_2-1,N+1}} \sum_{R \in \mathcal{P}^V_{n_2,j_2}} \sum_{W \in \mathcal{P}^R_{n_2,N+1}} \sharp\, \mathcal{P}^W_{n_3,j_3}.$$

Using (ii) and (iii) we obtain the desired inequality.

Assume $n_2 = n_3$ and $j_2 \leq j_3$. Then we have

$$\sharp\, \mathcal{P}^I_{n_3,j_3} = \sum_{V \in \mathcal{P}^I_{n_2-1,N+1}} \sharp\, \mathcal{P}^V_{n_3,j_3},$$

$$\sum_{R \in \mathcal{P}^I_{n_2,j_2}} \sharp\, \mathcal{P}^R_{n_3,j_3} = \sum_{V \in \mathcal{P}^I_{n_2-1,N+1}} \sum_{R \in \mathcal{P}^V_{n_2,j_2}} \sharp\, \mathcal{P}^R_{n_3,j_3}.$$

Using (ii) and (iii) we obtain the desired inequality.

(v) Clearly, $\|\mathcal{P}_{n,j}\| = \frac{x_n^{j+1}}{x_n^j}\|\mathcal{P}_{n,j+1}\| \geq 2n^2\|\mathcal{P}_{n,j+1}\|$. □

Notation 3.5. Let $V \subset [0,1]$ and $x \in \mathbb{R} \setminus \{0\}$. We set

$$\mathcal{T}(x,V) = \left\{ \frac{1}{x}(V+n);\ n \in \mathbb{Z} \right\}.$$

Lemma 3.6. *Let $W, S \subset [0,1]$ be intervals, $x \in \mathbb{R}\setminus\{0\}$ and $\lambda(S) \geq \frac{4}{|x|}$. Then $\lambda\left(\bigcup \mathcal{T}(x,W)^S\right) \geq \frac{1}{2}\lambda(S)\lambda(W)$.*

Proof. Clearly, $\sharp\, \mathcal{T}(x,W)^S \geq \lambda(S)\cdot|x| - 2$. Thus,

$$\lambda\left(\bigcup \mathcal{T}(x,W)^S\right) = \frac{\lambda(W)}{|x|}\cdot \sharp\, \mathcal{T}(x,W)^S \geq \lambda(S)\lambda(W) - \frac{2\lambda(W)}{|x|}.$$

Since $\lambda(S) \geq \frac{4}{|x|}$ we have

$$\lambda(S)\lambda(W) - \frac{2\lambda(W)}{|x|} \geq \frac{1}{2}\lambda(S)\lambda(W).$$

□

Lemma 3.7. *Let $n, s, j \in \mathbb{N}$, $n > 1$, $s, j \leq N+1$, $I \in \mathcal{P}_{n-1,s}$ and let $S \subset I$ be an interval with $\lambda(S) \geq 8\|\mathcal{P}_{n,j}\|$. Then $\lambda\left(\bigcup \mathcal{P}_{n,j}^S\right) \geq \frac{1}{4}\lambda(S)$.*

Proof. It is easy to verify that $I = \bigcup \mathcal{P}_{n-1,N+1}^I$ and $\mathcal{P}_{n,j}^V \supset \mathcal{T}(x_n^j, [0,1/2])^V$ for every $V \in \mathcal{P}_{n-1,N+1}^I$. So $\mathcal{P}_{n,j}^I \supset \mathcal{T}(x_n^j,[0,1/2])^I$. Thus $\mathcal{P}_{n,j}^S \supset \mathcal{T}(x_n^j,[0,1/2])^S$. Thus

$$\lambda\left(\bigcup \mathcal{P}_{n,j}^S\right) \geq \lambda\left(\bigcup \mathcal{T}(x_n^j,[0,1/2])^S\right).$$

We know that $\lambda(S) \geq 8\|\mathcal{P}_{n,j}\| = \frac{4}{x_n^j}$. Thus, we can use Lemma 3.6 and obtain

$$\lambda\left(\bigcup \mathcal{T}(x_n^j,[0,1/2])^S\right) \geq \frac{1}{4}\lambda(S).$$

□

Construction 3.8. For $I = \left[\frac{i-1}{2x_n^{N+1}}, \frac{i}{2x_n^{N+1}}\right]$, where $n \in \mathbb{N}$ and $i \in \{1,\dots,2x_n^{N+1}\}$, we define

$$\mu(I) = \begin{cases} \frac{1}{\sharp\, \mathcal{P}_{n,N+1}}, & \text{whenever } I \in \mathcal{P}_{n,N+1}, \\ 0, & \text{whenever } I \notin \mathcal{P}_{n,N+1}. \end{cases} \tag{3.2}$$

Now we use the standard mass distribution principle, see e.g. [2, Proposition 1.7], to extend μ to the desired measure.

We also set

$$P = \left\{ x \in [0,1];\ \forall i \in \mathbb{N}:\ \langle x\cdot x_i\rangle \notin (1/2,1)^{N+1} \right\}. \tag{3.3}$$

We can easily obtain the following properties of the measure μ.

Lemma 3.9. *The measure μ is a continuous Radon probability measure and the support of μ is a subset of P.*

Proof. Let $x \in [0,1]$ and $n \in \mathbb{N}$ be arbitrary. Then there exists $1 \le i \le 2x_n^{N+1}$ such that $x \in [\frac{i-1}{2x_n^{N+1}}, \frac{i}{2x_n^{N+1}}]$. By (3.2) we have

$$\mu(\{x\}) \le \mu\left(\left[\frac{i-1}{2x_n^{N+1}}, \frac{i}{2x_n^{N+1}}\right]\right) \le \frac{1}{\sharp\,\mathcal{P}_{n,N+1}}.$$

Since $\lim_{n\to\infty} \frac{1}{\sharp\,\mathcal{P}_{n,N+1}} = 0$ we have $\mu(\{x\}) = 0$.

By (3.2) and Remark 3.3 the support of μ is a subset of $\bigcup \mathcal{P}_{n,N+1} = P_n$ for every $n \in \mathbb{N}$. By (3.1) $P = \bigcap_{n\in\mathbb{N}} P_n$. Thus the support of μ is a subset of P. □

3.2. Verification of $\mu \notin (H^{(N+1)})^\perp$.

Lemma 3.10. *The set P is a closed $H^{(N+1)}$-set and $\mu(P) = 1$.*

Proof. Let $\alpha = (\alpha_1, \dots, \alpha_{N+1}) \in \mathbb{Z}^{N+1}$ be a nonzero vector. We find the largest $i \le N+1$ such that $\alpha_i \ne 0$. Since $\lim_{n\to\infty} \frac{x_n^j}{x_n^i} = 0$ for every $1 \le j < i$ we have

$$\lim_{n\to\infty} |(x_n, \alpha)| = \lim_{n\to\infty} \Big|\sum_{j=1}^{i} x_n^j \alpha_j\Big| = \lim_{n\to\infty} x_n^i \Big|\sum_{j=1}^{i} \frac{x_n^j \alpha_j}{x_n^i}\Big| = |\alpha_i| \lim_{n\to\infty} x_n^i = \infty.$$

Thus $\{x_n\}_{n\in\mathbb{N}} \in \mathcal{Q}^{N+1}(\mathbb{N})$ and therefore $P \in H^{(N+1)}$. By Lemma 3.9 we have $\mu(P) = 1$. □

3.3. **Verification of $\mu \in (H_{10}^{(N)*})^\perp$.** We fix an arbitrary $X \in H_{10}^{(N)*}$. We find an open interval $W = \prod_{j=1}^N W_j \subset [0,1]^N$ and $\boldsymbol{z} \in \mathcal{Q}^N(\mathbb{R} \setminus \{0\})$ witnessing $X \in H_{10}^{(N)*}$. Thus, we have

$$\left|\frac{z_i^{j+1}\lambda(W_j)}{z_i^j}\right| \ge 10 \qquad \text{for every } i \in \mathbb{N}, j \in \{1, \dots, N-1\}. \tag{3.4}$$

Let $0 \le \sigma \le \rho \le N$ be integers. We set

$$A_{k,\sigma,\rho} = \{x \in [0,1]; \exists j \in \mathbb{N},\ \sigma < j \le \rho : \langle x \cdot z_k^j\rangle \notin W_j\},$$
$$A_k = \{x \in [0,1]; \forall i \le k : \langle x \cdot z_i\rangle \notin W\} = \bigcap_{i \le k} A_{i,0,N},$$
$$A = \bigcap_{k\in\mathbb{N}} A_k = \bigcap_{k\in\mathbb{N}} A_{k,0,N}.$$

We have $X \subset A$. We want to show that $\mu(X) = 0$, so it is sufficient to prove $\mu(A) = 0$.

Further in this section fix a constant $l \in \mathbb{N}$ such that

$$l > 100 \quad \text{and} \quad l > 1/\lambda(W_j), \quad j = 1, \dots, N. \tag{3.5}$$

Notation 3.11. Let $n, k \in \mathbb{N}$, $S, T \subset [0,1]$ and $\mathcal{D}$ be a collection of subsets of $[0,1]$. We define

$$\mathcal{V}(\mathcal{D}, T) = \{V \in \mathcal{D};\ V \cap T = \emptyset\}$$

and if $\mathcal{P}^S_{n,N+1} \neq \emptyset$, then we set

$$\mu^S_{n,k} = 1 - \frac{\sharp\, \mathcal{V}\left(\mathcal{P}_{n,N+1}, A_k\right)^S}{\sharp\, \mathcal{P}^S_{n,N+1}} \qquad \text{and} \qquad \mu_{n,k} = \mu^{[0,1]}_{n,k}.$$

Lemma 3.12.

(i) *Let $n, k \in \mathbb{N}$. Then $\mu(A) \leq \mu_{n,k}$.*

(ii) *If $n, s, k \in \mathbb{N}$ and $n \geq s$ then $\mu_{n,k} \leq \sup\{\mu^V_{n,k};\ V \in \mathcal{P}_{s,N+1}\} \cdot \mu_{s,k}$.*

Proof. (i) We have

$$A \cap P \subset A_k \cap P \subset A_k \cap P_n \subset \bigcup (\mathcal{P}_{n,N+1} \setminus \mathcal{V}\left(\mathcal{P}_{n,N+1}, A_k\right)). \tag{3.6}$$

Using Lemma 3.9, (3.6) and (3.2) we can conclude

$$\begin{aligned}
\mu(A) &= \mu(A \cap P) \leq \mu\left(\bigcup (\mathcal{P}_{n,N+1} \setminus \mathcal{V}\left(\mathcal{P}_{n,N+1}, A_k\right)\right) \\
&= \sum_{J \in \left(\mathcal{P}_{n,N+1} \setminus \mathcal{V}\left(\mathcal{P}_{n,N+1}, A_k\right)\right)} \mu(J) = \frac{\sharp(\mathcal{P}_{n,N+1} \setminus \mathcal{V}\left(\mathcal{P}_{n,N+1}, A_k\right))}{\sharp\, \mathcal{P}_{n,N+1}} = \mu_{n,k}.
\end{aligned}$$

(ii) It is easy to verify that

$$\begin{aligned}
\mu_{n,k} &= 1 - \frac{\sharp\, \mathcal{V}\left(\mathcal{P}_{n,N+1}, A_k\right)}{\sharp\, \mathcal{P}_{n,N+1}} = 1 - \sum_{V \in \mathcal{P}_{s,N+1}} \frac{\sharp\, \mathcal{V}\left(\mathcal{P}_{n,N+1}, A_k\right)^V}{\sharp\, \mathcal{P}_{s,N+1} \cdot \sharp\, \mathcal{P}^V_{n,N+1}} \\
&= \frac{1}{\sharp\, \mathcal{P}_{s,N+1}} \sum_{V \in \mathcal{P}_{s,N+1}} 1 - \frac{\sharp\, \mathcal{V}\left(\mathcal{P}_{n,N+1}, A_k\right)^V}{\sharp\, \mathcal{P}^V_{n,N+1}} = \frac{\sum_{V \in \mathcal{P}_{s,N+1}} \mu^V_{n,k}}{\sharp\, \mathcal{P}_{s,N+1}} \\
&= \frac{\sum_{V \in \mathcal{P}_{s,N+1} \setminus \mathcal{V}\left(\mathcal{P}_{s,N+1}, A_k\right)} \mu^V_{n,k}}{\sharp\, \mathcal{P}_{s,N+1}},
\end{aligned}$$

where the last equality follows from the fact that $\mu^V_{n,k} = 0$ for all $V \in \mathcal{V}\left(\mathcal{P}_{s,N+1}, A_k\right)$. Thus, we have

$$\begin{aligned}
\mu_{n,k} &\leq \sup\{\mu^V_{n,k};\ V \in \mathcal{P}_{s,N+1}\} \cdot \frac{\sharp(\mathcal{P}_{s,N+1} \setminus \mathcal{V}\left(\mathcal{P}_{s,N+1}, A_k\right))}{\sharp\, \mathcal{P}_{s,N+1}} \\
&= \sup\{\mu^V_{n,k};\ V \in \mathcal{P}_{s,N+1}\} \cdot \mu_{s,k}.
\end{aligned}$$

□

Let us assume that $k \in \mathbb{N}$ is fixed in the following definition, Lemma 3.14, Lemma 3.15 and Lemma 3.16.

Definition 3.13. Let $S \subset [0,1]$ be an interval and $j \in \{0,\dots,N-1\}$. We inductively define

$$\begin{aligned}\mathcal{K}_{j,j+1}(S) &= \mathcal{T}(z_k^{j+1}, W_{j+1})^S,\\ \mathcal{K}_{j,t}(S) &= \bigcup_{L \in \mathcal{K}_{j,t-1}(S)} \mathcal{T}(z_k^t, W_t)^L, \qquad t = j+2,\dots,N.\end{aligned}$$

Lemma 3.14. *We can easily obtain the following five properties.*

(i) *For every $Z \in \mathcal{K}_{j,t}(S)$ we have $\lambda(Z) = \frac{\lambda(W_t)}{|z_k^t|} \geq \frac{1}{l|z_k^t|}$.*

(ii) *Let $K, L \subset [0,1]$ and $K \cap L = \emptyset$. Then $\mathcal{K}_{j,t}(K) \cap \mathcal{K}_{j,t}(L) = \emptyset$.*

(iii) *Let $K, L \in \mathcal{K}_{j,t}(S)$. Then $K = L$ or $K \cap L = \emptyset$.*

(iv) *$\bigcup \mathcal{K}_{j,t}(S) \cap A_{k,j,t} = \emptyset$.*

Proof. Statements (i)–(iii) are easy to verify.

(iv) It is straightforward to verify that

$$\bigcup \mathcal{K}_{j,t}(S) \subset \bigcap_{i=j+1}^{t} \bigcup \mathcal{T}(z_k^i, W_i)^S,$$

$$A_{k,j,t} = \bigcup_{i=j+1}^{t} ([0,1] \setminus \bigcup \mathcal{T}(z_k^i, W_i)^{\mathbb{R}}).$$

Since $\mathcal{T}(z_k^i, W_i)^S \subset \mathcal{T}(z_k^i, W_i)^{\mathbb{R}}$ for every $1 \leq i \leq N$ we have

$$\bigcap_{i=j+1}^{t} \bigcup \mathcal{T}(z_k^i, W_i)^S \cap \bigcup_{i=j+1}^{t} ([0,1] \setminus \bigcup \mathcal{T}(z_k^i, W_i)^{\mathbb{R}}) = \emptyset.$$

So, we are done. □

Lemma 3.15. *Let $0 \leq j < t \leq N$ and $S \subset [0,1]$ be an interval with $\lambda(S) \geq 4/|z_k^{j+1}|$. Then $\lambda\left(\bigcup \mathcal{K}_{j,t}(S)\right) \geq \lambda(S) \cdot (2l)^{j-t}$.*

Proof. We will prove this lemma by induction. First, we assume that $t = j+1$. Then $\mathcal{K}_{j,t}(S) = \mathcal{T}(z_k^t, W_t)^S$ and $\lambda(S) \geq 4/|z_k^t|$. By Lemma 3.6 and (3.5) we have

$$\begin{aligned}\lambda\left(\bigcup \mathcal{K}_{j,t}(S)\right) &= \lambda\left(\bigcup \mathcal{T}(z_k^t, W_t)^S\right) \overset{L.3.6}{\geq} \frac{1}{2}\lambda(S)\lambda(W_t)\\ &\overset{(3.5)}{\geq} \lambda(S) \cdot (2l)^{-1} = \lambda(S) \cdot (2l)^{j-t}.\end{aligned}$$

Now, we assume that $t > j+1$ and we already proved

$$\lambda\left(\bigcup \mathcal{K}_{j,t-1}(S)\right) \geq \lambda(S) \cdot (2l)^{j-t+1}. \tag{3.7}$$

Let $L \in \mathcal{K}_{j,t-1}(S)$ be arbitrary. Then $\lambda(L) = \frac{\lambda(W_{t-1})}{|z_k^{t-1}|}$. By (3.4) we have $\frac{\lambda(W_{t-1})}{|z_k^{t-1}|} \geq \frac{10}{|z_k^t|}$. Thus we have

$$\lambda(L) \geq \frac{4}{|z_k^t|}. \tag{3.8}$$

By Lemma 3.14 (ii)-(iii), Lemma 3.6, (3.8), (3.5) and (3.7) we have

$$\begin{aligned}
\lambda\left(\bigcup \mathcal{K}_{j,t}(S)\right) &= \lambda\Big(\bigcup \bigcup_{L\in\mathcal{K}_{j,t-1}(S)} \mathcal{T}(z_k^t, W_t)^L\Big)\\
&\overset{L.3.14(ii),(iii)}{=} \sum_{L\in\mathcal{K}_{j,t-1}(S)} \lambda\Big(\bigcup \mathcal{T}(z_k^t, W_t)^L\Big)\\
&\overset{L.3.6}{\geq} \sum_{L\in\mathcal{K}_{j,t-1}(S)} \frac{1}{2}\lambda(L)\lambda(W_t)\\
&\overset{(3.5)}{\geq} \sum_{L\in\mathcal{K}_{j,t-1}(S)} \lambda(L)(2l)^{-1}\\
&\overset{L.3.14(iii)}{=} (2l)^{-1}\lambda\left(\bigcup \mathcal{K}_{j,t-1}(S)\right)\\
&\overset{(3.7)}{\geq} \lambda(S)\cdot(2l)^{j-t},
\end{aligned}$$

where the formula (3.8) was used to verify the condition of Lemma 3.6.

□

Lemma 3.16. *Let $0 \leq \sigma < \rho \leq N$, $1 \leq s \leq N$, $1 \leq j \leq N+1$, n, k be natural numbers and $I \in \mathcal{P}_{n,s}$. Suppose that the following conditions are satisfied*

$$n \geq l^2, \tag{3.9}$$

$$n\|\mathcal{P}_{n,s+1}\| \geq \frac{1}{|z_k^i|} \geq (n+1)\|\mathcal{P}_{n+1,j}\|, \qquad \sigma < i \leq \rho. \tag{3.10}$$

Then

$$\frac{\sharp\,\mathcal{V}(\mathcal{P}^I_{n+1,j}, A_{k,\sigma,\rho})}{\sharp\,\mathcal{P}^I_{n+1,j}} \geq \frac{1}{4}\cdot(2l)^{\sigma-\rho}. \tag{3.11}$$

Proof. By Lemma 3.14(iii),(iv) we have

$$\mathcal{V}(\mathcal{P}^I_{n+1,j}, A_{k,\sigma,\rho}) \supset \mathcal{P}^{\bigcup\mathcal{K}_{\sigma,\rho}(I)}_{n+1,j} \supset \bigcup_{K\in\mathcal{K}_{\sigma,\rho}(I)} \mathcal{P}^K_{n+1,j}. \tag{3.12}$$

By (3.12), Lemma 3.14(iii) and Lemma 3.4(i) we have

$$\begin{aligned}
\frac{\sharp\,\mathcal{V}(\mathcal{P}^I_{n+1,j}, A_{k,\sigma,\rho})}{\sharp\,\mathcal{P}^I_{n+1,j}} &\geq \sum_{K\in\mathcal{K}_{\sigma,\rho}(I)} \frac{\sharp\,\mathcal{P}^K_{n+1,j}}{\sharp\,\mathcal{P}^I_{n+1,j}} = \sum_{K\in\mathcal{K}_{\sigma,\rho}(I)} \frac{\lambda(\bigcup\mathcal{P}^K_{n+1,j})}{\lambda(\bigcup\mathcal{P}^I_{n+1,j})}\\
&\geq \sum_{K\in\mathcal{K}_{\sigma,\rho}(I)} \frac{\lambda(\bigcup\mathcal{P}^K_{n+1,j})}{\lambda(I)}.
\end{aligned}$$

Thus, it is enough to verify that

$$\sum_{K\in\mathcal{K}_{\sigma,\rho}(I)} \lambda\big(\bigcup \mathcal{P}^K_{n+1,j}\big) \geq \frac{1}{4}\lambda(I)(2l)^{\sigma-\rho}. \tag{3.13}$$

By (3.9) and (3.5) we have $n \geq l^2$ and $l > 4$. By Lemma 3.4(v) and (3.10) we have

$$\lambda(I) = \|\mathcal{P}_{n,s}\| \geq n^2\|\mathcal{P}_{n,s+1}\| \geq \frac{n}{z_k^{\sigma+1}} \geq \frac{4}{z_k^{\sigma+1}}.$$

Thus by Lemma 3.15 we have

$$\lambda\left(\bigcup \mathcal{K}_{\sigma,\rho}(I)\right) \geq \lambda(I)(2l)^{\sigma-\rho}. \tag{3.14}$$

Let $K \in \mathcal{K}_{\sigma,\rho}(I)$ be arbitrary. By Lemma 3.14(i), (3.10) and $n+1 > 8l$ we have

$$\lambda(K) \geq \frac{1}{lz_k^{\rho}} \geq 8\|\mathcal{P}_{n+1,j}\|.$$

Thus by Lemma 3.7 we have

$$\lambda\big(\bigcup \mathcal{P}^K_{n+1,j}\big) \geq \frac{1}{4}\lambda(K). \tag{3.15}$$

By (3.15), Lemma 3.14(iii) and (3.14) we have

$$\sum_{K\in\mathcal{K}_{\sigma,\rho}(I)} \lambda\big(\bigcup \mathcal{P}^K_{n+1,j}\big) \geq \frac{1}{4}\sum_{K\in\mathcal{K}_{\sigma,\rho}(I)} \lambda(K) = \frac{1}{4}\lambda\left(\bigcup \mathcal{K}_{\sigma,\rho}(I)\right) \geq \frac{1}{4}\lambda(I)(2l)^{\sigma-\rho}.$$

So, we verify (3.13). □

Lemma 3.17. *Let $n_0 \leq n_1 < n_2 \in \mathbb{N}$, $1 \leq j_1 < j_2 < j_3 \leq N+1$ and $T_1, T_2 \subset [0,1]$. If there exist $\alpha_1 > 0$, $\alpha_2 > 0$ such that*

$$\frac{\sharp\,\mathcal{V}(\mathcal{P}^{I_1}_{n_1+1,j_2}, T_1)}{\sharp\,\mathcal{P}^{I_1}_{n_1+1,j_2}} \geq \alpha_1, \tag{3.16}$$

$$\frac{\sharp\,\mathcal{V}(\mathcal{P}^{I_2}_{n_2+1,j_3}, T_2)}{\sharp\,\mathcal{P}^{I_2}_{n_2+1,j_3}} \geq \alpha_2, \tag{3.17}$$

for every $I_1 \in \mathcal{P}_{n_0,j_1}$ and $I_2 \in \mathcal{P}_{n_2,j_2}$ then

$$\frac{\sharp\,\mathcal{V}(\mathcal{P}^{I}_{n_2+1,j_3}, T_1\cup T_2)}{\sharp\,\mathcal{P}^{I}_{n_2+1,j_3}} \geq \frac{1}{4}\alpha_1\alpha_2$$

for every $I \in \mathcal{P}_{n_0,j_1}$.

Proof. Let $I \in \mathcal{P}_{n_0,j_1}$ be arbitrary. Clearly,

$$\sharp\,\mathcal{V}(\mathcal{P}^{I}_{n_2,j_2}, T_1) \geq \sum_{V\in\mathcal{V}(\mathcal{P}^{I}_{n_1+1,j_2},T_1)} \sharp\,\mathcal{P}^{V}_{n_2,j_2}. \tag{3.18}$$

By (3.18), Lemma 3.4(ii),(iv) and (3.16) we have

$$\begin{aligned}
\text{(3.19)} \qquad \frac{\sharp\,\mathcal{V}(\mathcal{P}^I_{n_2,j_2},T_1)}{\sharp\,\mathcal{P}^I_{n_2,j_2}} &\overset{(3.18),L.3.4(iv)}{\geq} \frac{\sum_{V\in\mathcal{V}(\mathcal{P}^I_{n_1+1,j_2},T_1)}\sharp\,\mathcal{P}^V_{n_2,j_2}}{2\sum_{W\in\mathcal{P}^I_{n_1+1,j_2}}\sharp\,\mathcal{P}^W_{n_2,j_2}} \\
&\overset{L.3.4(ii)}{\geq} \frac{\sharp\,\mathcal{V}(\mathcal{P}^I_{n_1+1,j_2},T_1)}{2\,\sharp\,\mathcal{P}^I_{n_1+1,j_2}} \\
&\overset{(3.16)}{\geq} \frac{1}{2}\alpha_1.
\end{aligned}$$

Clearly,

$$\text{(3.20)} \qquad \sharp\,\mathcal{V}(\mathcal{P}^I_{n_2+1,j_3},T_1\cup T_2) \geq \sum_{V\in\mathcal{V}(\mathcal{P}^I_{n_2,j_2},T_1)} \sharp\,\mathcal{V}(\mathcal{P}^V_{n_2+1,j_3},T_2).$$

By (3.20), (3.19), (3.17) and Lemma 3.4(iv) we have

$$\begin{aligned}
\frac{\sharp\,\mathcal{V}(\mathcal{P}^I_{n_2+1,j_3},T_1\cup T_2)}{\sharp\,\mathcal{P}^I_{n_2+1,j_3}} &\overset{(3.20),L.3.4(iv)}{\geq} \frac{\sum_{V\in\mathcal{V}(\mathcal{P}^I_{n_2,j_2},T_1)}\sharp\,\mathcal{V}(\mathcal{P}^V_{n_2+1,j_3},T_2)}{2\sum_{W\in\mathcal{P}^I_{n_2,j_2}}\sharp\,\mathcal{P}^W_{n_2+1,j_3}} \\
&\overset{(3.17)}{\geq} \alpha_2\frac{\sharp\,\mathcal{V}(\mathcal{P}^I_{n_2,j_2},T_1)}{2\,\sharp\,\mathcal{P}^I_{n_2,j_2}} \\
&\overset{(3.19)}{\geq} \frac{1}{4}\alpha_2\alpha_1.
\end{aligned}$$

□

Lemma 3.18. *There exists $\varepsilon>0$ such that for every $n,k\in\mathbb{N}$ there exist $\tilde{n}\in\mathbb{N}$ and $\tilde{k}\in\mathbb{N}$ such that $\tilde{n}>n$, $\tilde{k}>k$ and*

$$\frac{\sharp\,\mathcal{V}(\mathcal{P}^I_{\tilde{n},N+1},A_{\tilde{k},0,N})}{\sharp\,\mathcal{P}^I_{\tilde{n},N+1}} \geq \varepsilon$$

for every $I\in\mathcal{P}_{n,N+1}$.

Proof. Set $\varepsilon=2(32l)^{-N}$. Let $n,k\in\mathbb{N}$ be arbitrary. We set $n_0=\max\{n+1,l^2\}$. We will construct $\tilde{k}>k$, $s\leq N$ and sequences $n_0<n_1<\dots<n_s$ and $0=v_0<v_1<\dots<v_s=N$ such that

$$\forall 0<i\leq s\ \forall v_{i-1}<j\leq v_i:\ n_i\|\mathcal{P}_{n_i,v_{i-1}+2}\| \geq \frac{1}{\left|z^j_{\tilde{k}}\right|} > (n_i+1)\|\mathcal{P}_{n_i+1,v_i+1}\|.$$

Since $\boldsymbol{z}\in\mathcal{Q}^N(\mathbb{R}\setminus\{0\})$ and (3.4) holds we have $\lim|z^1_i|=\infty$ and $\left|z^{j+1}_i\right|\geq 10\left|z^j_i\right|$ for every $i\in\mathbb{N}, j<N$. Thus, we can find $\tilde{k}>k$ such that $\frac{1}{\left|z^1_{\tilde{k}}\right|}\leq\|\mathcal{P}_{n_0+1,2}\|(n_0+1)$. We set $v_0=0$. Assume that we have already constructed $n_0,\dots,n_i$ and $v_0,\dots,v_i$ for some $i\geq 0$.

If $v_i = N$ we set $s = i$ and we are done. If $v_i < N$ we find $n_{i+1} \in \mathbb{N}$ such that

$$n_{i+1}\|\mathcal{P}_{n_{i+1},v_i+2}\| \geq \frac{1}{\left|z_{\tilde{k}}^{v_i+1}\right|} > (n_{i+1}+1)\|\mathcal{P}_{n_{i+1}+1,v_i+2}\|.$$

Further we find the largest $v_{i+1} \in \{v_i+1, \dots, N\}$ such that

$$\frac{1}{\left|z_{\tilde{k}}^{v_{i+1}}\right|} > (n_{i+1}+1)\|\mathcal{P}_{n_{i+1}+1,v_{i+1}+1}\|$$

and we are done. We set $\tilde{n} = n_s + 1$.

We use Lemma 3.16 replacing σ, ρ, s, j, n, k by v_{i-1}, v_i, $v_{i-1}+1$, v_i+1, n_i, $\tilde{k}$ respectively to obtain

$$\frac{\sharp\,\mathcal{V}(\mathcal{P}^V_{n_i+1,v_i+1}, A_{\tilde{k},v_{i-1},v_i})}{\sharp\,\mathcal{P}^V_{n_i+1,v_i+1}} \geq \frac{1}{4}\cdot(2l)^{v_{i-1}-v_i} \tag{3.21}$$

for every $V \in \mathcal{P}_{n_i,v_{i-1}+1}$ and $1 \leq i \leq s$.

We prove by induction that

$$\frac{\sharp\,\mathcal{V}(\mathcal{P}^V_{n_j+1,v_j+1}, A_{\tilde{k},v_0,v_j})}{\sharp\,\mathcal{P}^V_{n_j+1,v_j+1}} \geq 4^{-j}(2l)^{-v_j}.4^{-j+1} \tag{3.22}$$

for every $V \in \mathcal{P}_{n_1,1}$ and $1 \leq j \leq s$.

By (3.21) we have (3.22) for $j = 1$.

Suppose that $1 < j \leq s$ and (3.22) holds for $j-1$. Thus, by (3.21) and Lemma 3.17 replacing n_0, n_1, n_2, j_1, j_2, j_3, T_1, T_2 by n_1, n_{j-1}, n_j, 1, $v_{j-1}+1$, v_j+1, $A_{\tilde{k},v_0,v_{j-1}}$, $A_{\tilde{k},v_{j-1},v_j}$ respectively we have

$$\begin{aligned}
\frac{\sharp\,\mathcal{V}(\mathcal{P}^V_{n_j+1,v_j+1}, A_{\tilde{k},v_0,v_j})}{\sharp\,\mathcal{P}^V_{n_j+1,v_j+1}} &= \frac{\sharp\,\mathcal{V}(\mathcal{P}^V_{n_j+1,v_j+1}, A_{\tilde{k},v_0,v_{j-1}} \cup A_{\tilde{k},v_{j-1},v_j})}{\sharp\,\mathcal{P}^V_{n_j+1,v_j+1}} \\
&\geq \frac{1}{4}\big(4^{-j+1}(2l)^{-v_{j-1}}.4^{-j+2}\big)\big(\frac{1}{4}(2l)^{v_{j-1}-v_j}\big) \\
&= 4^{-j}(2l)^{-v_j}.4^{-j+1}.
\end{aligned}$$

Thus we have (3.22).

Since $v_s = N$, $s \leq N$ and (3.22) holds we have

$$\frac{\sharp\,\mathcal{V}(\mathcal{P}^V_{\tilde{n},N+1}, A_{\tilde{k},0,N})}{\sharp\,\mathcal{P}^V_{\tilde{n},N+1}} \geq 4^{-s}(2l)^{-N}.4^{-s+1} \geq 2\varepsilon \tag{3.23}$$

for every $V \in \mathcal{P}_{n_1,1}$. Fix arbitrary $I \in \mathcal{P}_{n,N+1}$. Clearly,

$$\sharp\,\mathcal{V}(\mathcal{P}^I_{\tilde{n},N+1}, A_{\tilde{k},0,N}) \geq \sum_{V\in\mathcal{P}^I_{n_1,1}} \sharp\,\mathcal{V}(\mathcal{P}^V_{\tilde{n},N+1}, A_{\tilde{k},0,N}). \tag{3.24}$$

By (3.24), (3.23) and Lemma 3.4(iv),(ii) we have

$$\frac{\sharp\,\mathcal{V}(\mathcal{P}^I_{\tilde{n},N+1},A_{\tilde{k},0,N})}{\sharp\,\mathcal{P}^I_{\tilde{n},N+1}} \geq \frac{\sum_{V\in\mathcal{P}^I_{n_1,1}}\sharp\,\mathcal{V}(\mathcal{P}^V_{\tilde{n},N+1},A_{\tilde{k},0,N})}{2\sum_{W\in\mathcal{P}^I_{n_1,1}}\sharp\,\mathcal{P}^W_{\tilde{n},N+1}} \geq \varepsilon.$$

So, we are done. □

Proof of Lemma 2.4. We need to show that $\mu(A)=0$. Set $\varepsilon = 2(32l)^{-N}$. Let $n,k\in\mathbb{N}$ be arbitrary. By Lemma 3.18 there exist $\tilde{n},\tilde{k}\in\mathbb{N}$ such that

$$\frac{\sharp\,\mathcal{V}(\mathcal{P}^I_{\tilde{n},N+1},A_{\tilde{k},0,N})}{\sharp\,\mathcal{P}^I_{\tilde{n},N+1}} \geq \varepsilon$$

for every $I\in\mathcal{P}_{n,N+1}$. Since $A_{\tilde{k}}\subset A_{\tilde{k},0,N}$ we have

$$\mu^I_{\tilde{n},\tilde{k}} = \frac{\sharp\,\mathcal{P}^I_{\tilde{n},N+1}-\sharp\,\mathcal{V}(\mathcal{P}^I_{\tilde{n},N+1},A_{\tilde{k}})}{\sharp\,\mathcal{P}^I_{\tilde{n},N+1}} \leq \frac{\sharp\,\mathcal{P}^I_{\tilde{n},N+1}-\sharp\,\mathcal{V}(\mathcal{P}^I_{\tilde{n},N+1},A_{\tilde{k},0,N})}{\sharp\,\mathcal{P}^I_{\tilde{n},N+1}} \leq 1-\varepsilon$$

for every $I\in\mathcal{P}_{n,N+1}$. Thus by Lemma 3.12(ii) we have $\mu_{\tilde{n},\tilde{k}}\leq(1-\varepsilon)\mu_{n,k}$. Thus $\inf\{\mu_{n,k};\ n,k\in\mathbb{N}\}=0$. By Lemma 3.12(i) we have

$$0\leq\mu(A)\leq\inf\{\mu_{n,k};\ n,k\in\mathbb{N}\}=0.$$

So, $\mu(A)=0$. □

4. Proof of Theorem 2.5

Notation 4.1. Let $N,n\in\mathbb{N}$, $\boldsymbol{a}\in\mathcal{Q}^N(\mathbb{R}\setminus\{0\})$, $y\in\mathbb{R}$ and $J\subset\mathbb{R}$, $\mathcal{J}=\prod_{j=1}^N J^j\subset[0,1]^N$ be open intervals. We set

$$T(y,J)=\{x\in[0,1];\ \langle xy\rangle\in\langle J\rangle\},$$

$$H^n(\boldsymbol{a},\mathcal{J})=[0,1]\setminus\bigcap_{p=1}^N T(a^p_n,J^p),$$

$$H(\boldsymbol{a},\mathcal{J})=\bigcap_{n\in\mathbb{N}}H^n(\boldsymbol{a},\mathcal{J}).$$

Notation 4.2. Let $m\in\mathbb{N}$, $I\subset[0,1]^m$ be an interval and $\boldsymbol{z}\in\mathcal{Q}^m(\mathbb{R}\setminus\{0\})$. Then we define

$$H(\boldsymbol{z},I)=\{x\in[0,1];\ \forall k\in\mathbb{N}:\ \langle x\cdot z_k\rangle\notin I\}.$$

Remark 4.3. Let $m\in\mathbb{N}$.

(i) If $A\in H^{(m)*}$ then there exist $\boldsymbol{z}\in\mathcal{Q}^m(\mathbb{R}\setminus\{0\})$ and an open interval $W\subset[0,1]^m$ such that $A\subset H(\boldsymbol{z},W)$.

(ii) If $I\subset J\subset[0,1]^m$ are open intervals and $\boldsymbol{r}\in\mathcal{Q}^m(\mathbb{R}\setminus\{0\})$, then $H(\boldsymbol{r},J)\subset H(\boldsymbol{r},I)$.

(iii) Let $m\in\mathbb{N}$, $I\subset[0,1]^m$ be an interval and $\boldsymbol{z}\in\mathcal{Q}^m(\mathbb{R}\setminus\{0\})$. Then

$$H(\boldsymbol{z},I)=\{x\in[0,1];\ \forall k\in\mathbb{N}:\ \langle x\cdot z_k\rangle\notin I\}.$$

Lemma 4.4. *Let $N\in\mathbb{N},\boldsymbol{a}=\{a_j\}\in\mathcal{Q}^N(\mathbb{R}\setminus\{0\})$, $\{j_k\}$ be an increasing sequence of integers and $\mathcal{J}\subset\mathcal{U}\subset[0,1]^N$ be open intervals. Then the following assertions hold.*

(i) $\{a_{j_k}\} \in \mathcal{Q}^N(\mathbb{R} \setminus \{0\})$
(ii) $H(\boldsymbol{a},\mathcal{U}) \subset H(\{a_{j_k}\},\mathcal{U})$
(iii) $H(\boldsymbol{a},\mathcal{U}) = \bigcap_{n\in\mathbb{N}} H^n(\boldsymbol{a},\mathcal{U})$
(iv) *Let $L \in \mathbb{R}^{N\times N}$ be a regular matrix. Then there exists a finite set $M \subset \mathbb{N}$ such that for every increasing sequence $\{v_k\}$ of elements from $\mathbb{N} \setminus M$ we have $\{L(a_{v_k})\} \in \mathcal{Q}^N(\mathbb{R} \setminus \{0\})$.*
(v) *Let $y \in \mathbb{R}\setminus\{0\}$ and $J \subset [0,1]$ be an open interval. Then $T(y,J) = \bigcup_{n\in\mathbb{Z}} \frac{1}{y}(J+ n)\cap [0,1] = \bigcup \mathcal{T}(y,J) \cap [0,1]$.*
(vi) *Let $m \in \mathbb{Z} \setminus \{0\}$, $y \in \mathbb{R} \setminus \{0\}$ and $u,r \in \mathbb{R}$. Then we have $T(y,B(u,r)) \supset T(\frac{y}{m}, B(\frac{u}{m}, \frac{r}{|m|}))$, where the symbol $B(x,s) = (x-s,x+s)$ for $s > 0$.*
(vii) *Let $y \in \mathbb{R} \setminus \{0\}$, $J \subset \mathbb{R}$ and $V \subset \langle J\rangle$ be open intervals. Then $T(y,J) \supset T(y,V)$.*

Proof. (i)-(iii),(v),(vii) are trivial.

(iv) We set $M = \{i \in \mathbb{N};\ \exists s \le N :\ (L(a_i))^s = 0\}$. Let $\{v_k\}$ be arbitrary increasing sequence of elements from $\mathbb{N} \setminus M$. Then $\{L(a_{v_k})\} \in ((\mathbb{R} \setminus \{0\})^N)^{\mathbb{N}}$. Let $\alpha \in \mathbb{Z}^N$ be nonzero vector. Then $L^T(\alpha)$ be nonzero vector, where L^T be transpose of matrix L. Thus we have

$$\lim_{n\to\infty} |(L(a_{v_k}),\alpha)| = \lim_{n\to\infty} |(a_{v_k}, L^T(\alpha))| = \infty.$$

Thus, $L(a_{v_k}) \in \mathcal{Q}^N(\mathbb{R} \setminus \{0\})$.

(vi) Clearly, $\mathcal{T}(y,B(u,r)) \supset \mathcal{T}(\frac{y}{m}, B(\frac{u}{m}, \frac{r}{|m|}))$. Thus (vi) follows from (v). □

We will use the following well known approximation theorem.

Lemma 4.5. [10, Dirichlet's Theorem on Simultaneous Approximations] *Let $\alpha_1,\dots,\alpha_n$ be real numbers and $Q > 1$ be an integer. Then there exist integers $q,p_1,\dots,p_n$ with $1 \le q < Q^n$ and $|\alpha_i q - p_i| \le 1/Q$ for all $1 \le i \le n$.*

Lemma 4.6. *Let $N \in \mathbb{N}, \boldsymbol{a} \in \mathcal{Q}^N(\mathbb{R}\setminus\{0\})$ and $\mathcal{U}_n = U^1\times\cdots\times U^{N-1}\times U^N_n \subset [0,1]^N$, $n \in \mathbb{N}$ be open intervals. If there exists $\alpha > 0$ such that $\lambda(U^N_n) \ge \alpha$ for all $n \in \mathbb{N}$ then there exist an increasing sequence $\{j_n\}$ of positive integers and an open interval $\mathcal{J} = U^1\times\cdots\times U^{N-1}\times J^N \subset [0,1]^N$ such that for every $n \in \mathbb{N}$ we have*
(i) $4\lambda(J^N) \ge \lambda(U^N_{j_n})$,
(ii) $H^n(\{a_{j_n}\},\mathcal{U}_{j_n}) \subset H^n(\{a_{j_n}\},\mathcal{J})$.

Proof. Since $\inf\{\lambda(U^N_n);\ n \in \mathbb{N}\} \ge \alpha > 0$ there exists an increasing sequence v_n of positive integers such that

$$4\inf\{\lambda(U^N_{v_n});\ n \in \mathbb{N}\} > 3\sup\{\lambda(U^N_{v_n});\ n \in \mathbb{N}\}.$$

We find $l \in \mathbb{N}$ such that

$$\frac{2}{l} \le \inf\{\lambda(U^N_{v_n});\ n \in \mathbb{N}\} < \frac{3}{l}.$$

For all $j \in \mathbb{N}$ we find $b_j \in \mathbb{N}_0$ and an open interval $J^N_j = (\frac{b_j}{l}, \frac{b_j+1}{l})$ such that $J^N_j \subset U^N_{v_j}$. Since the set $\{J^N_j;\ j \in \mathbb{N}\}$ is finite there exists an increasing sequence $\{p_n\}$ of positive integers and an open interval J^N such that $J^N_{p_n} = J^N$ for all $n \in \mathbb{N}$. We set $\mathcal{J} = U^1 \times \cdots \times U^{N-1} \times J^N$ and $j_n = v_{p_n}$. Thus,

$$H^n(\{a_{j_n}\}, \mathcal{U}_{j_n}) \subset H^n(\{a_{j_n}\}, \mathcal{J})$$

for every $n \in \mathbb{N}$. Clearly,

$$4\lambda(J^N) = \frac{4}{l} \geq \frac{4}{3} \inf\{\lambda(U^N_{v_n});\ n \in \mathbb{N}\} > \sup\{\lambda(U^N_{v_n});\ n \in \mathbb{N}\} \geq \lambda(U^N_{j_m})$$

for all $m \in \mathbb{N}$.

□

The following lemma was inspired by Zajíček [12].

Lemma 4.7. *Let $y, z \in \mathbb{R} \setminus \{0\}$, $y \neq z$, $U = B(u, r_1), V = B(v, r_2)$ be subsets of $[0,1]$ and $\delta \leq \min\left\{\frac{\lambda(V)}{|y|}, \frac{\lambda(U)}{|z|}\right\}$. If $4|y| > 3|z|$ then*

$$T(y, V) \cap T(z, U) \supset T\left(z, B\left(u, \frac{|z|\delta}{4}\right)\right) \cap T\left(y - z, B\left(v - u, \frac{r_2}{4}\right)\right).$$

Proof. Since $|z|\delta/4 \leq r_1$ we have $B(u, |z|\delta/4) \subset U$. Thus

$$T(z, U) \supset T\left(z, B\big(u, |z|\delta/4\big)\right).$$

Let $x \in T\left(z, B\big(u, |z|\delta/4\big)\right) \cap T\left(y - z, B\big(v - u, r_2/4\big)\right)$. Then there exist $\xi \in B(0, r_2/4)$, $\mu \in B(0, |z|\delta/4)$ and $m, n \in \mathbb{Z}$ such that

$$x = (\xi + v - u + n)\frac{1}{y - z},$$
$$x = (\mu + u + m)\frac{1}{z}.$$

Thus, $x = (\xi + \mu + v + m + n)\frac{1}{y}$. Since $|\xi + \mu| \leq \frac{r_2}{4} + \frac{|z|\delta}{4} < \frac{r_2}{4} + \frac{|y|\delta}{3} < \frac{r_2}{4} + \frac{2r_2}{3} < r_2$ we have $\xi + \mu + v \in V$. Thus, $x \in T(y, V)$.

□

Lemma 4.8. *Let $N \in \mathbb{N}, \boldsymbol{a} \in \mathcal{Q}^N(\mathbb{R} \setminus \{0\})$, $\mathcal{U} = \prod_{i=1}^N U^i \subset [0,1]^N$ be an open interval, $L \in \mathbb{N}$ and $\delta_j = \min\left\{\frac{\lambda(U^i)}{|a^i_j|};\ i = 1, \ldots, N\right\}$ for every $j \in \mathbb{N}$. Then there exist a regular matrix $\mathcal{L} \in \mathbb{Q}^{N \times N}$, an increasing sequence $\{v_n\}$ of positive integers and an open interval $\mathcal{J} = \prod_{i=1}^N J^i \subset [0,1]^N$ such that*

(a) $\boldsymbol{x} := \{\mathcal{L}(a_{v_n})\} \in \mathcal{Q}^N(\mathbb{R} \setminus \{0\})$,
(b) $\forall n \in \mathbb{N} : H^n(\{a_{v_n}\}, \mathcal{U}) \subset H^n(\boldsymbol{x}, \mathcal{J})$,
(c) $\forall n \in \mathbb{N}\ \forall i < N : \left|\frac{x^N_n \lambda(J^i)}{x^i_n}\right| \geq L$,
(d) $\frac{\lambda(J^N)}{|x^N_n|} \geq \delta_{v_n}/16$.

Proof. Going to a subsequence and permuting indexes if necessary, we can assume that $|a_n^i| < |a_n^{i+1}|$ for all $n \in \mathbb{N}$ and $i < N$. We find $Q \in \mathbb{N}$ such that $\frac{1}{Q} < \frac{\min\{\lambda(U^i);\ i=1,\dots,N\}}{8L}$. By Lemma 4.5 for every $j \in \mathbb{N}$ there exist $q_j, p_j^1, \dots, p_j^{N-1} \in \mathbb{Z}$ such that

$$\begin{aligned} 1 \le q_j &\le Q^{N-1}, \\ \left| q_j \frac{a_j^i}{a_j^N} - p_j^i \right| &\le \frac{1}{Q}, \qquad i = 1, \dots, N-1. \end{aligned} \tag{4.1}$$

Since $\frac{|a_j^i|}{|a_j^N|} < 1$, we have $|p_j^i| \le Q^{N-1}$ for every $j \in \mathbb{N}$ and $i = 1, \dots, N-1$. Going to a subsequence if necessary, we can assume that there exist $q, p^1, \dots, p^{N-1}$ such that $q = q_j, p^i = p_j^i$ for every $j \in \mathbb{N}$. Clearly, there exists $0 \le s < N$ such that $p^i = 0$ if and only if $i \le s$. Denote by u^i the center of the interval U^i and set

$$y_j^i = \begin{cases} a_j^i & \text{for } i \le s, \\ \frac{a_j^i}{p^i} - \frac{a_j^N}{q} & \text{for } s < i < N, \\ \frac{a_j^N}{q} & \text{for } i = N, \end{cases} \qquad j \in \mathbb{N}.$$

Further we define

- $J^i = U^i$ for $i \le s$,
- $\widetilde{J}^i = B\left(\frac{u^i}{p^i} - \frac{u^N}{q}, \frac{\lambda(U^i)}{8|p^i|}\right)$ for $s < i < N$,
- $\widetilde{J}_j^N = B\left(\frac{u^N}{q}, \frac{\delta_j |y_j^N|}{4}\right)$ for $j \in \mathbb{N}$,
- $J_j^N = \widetilde{J}_j^N \cap (0,1)$.

Since $\frac{u^N}{q} \in (0,1)$ we have $\lambda(J_j^N) \ge \frac{1}{2}\lambda(\widetilde{J}_j^N)$. Going to a subsequences if necessary and using Lemma 4.4(iv) we have that $\boldsymbol{y} := \left\{(y_j^1, \dots, y_j^N)\right\}_j$ is in $\mathcal{Q}^N(\mathbb{R} \setminus \{0\})$. For every $s < i < N$ we find an open interval $J^i \subset [0,1]$ such that $\lambda(J^i) \ge \frac{\lambda(\widetilde{J}^i)}{2}$ and $J^i \subset \langle \widetilde{J}^i \rangle$. By Lemma 4.4(vi) we have

$$\begin{aligned} T(a_j^i, U^i) &\supset T\left(\frac{a_j^i}{p^i}, B\left(\frac{u^i}{p^i}, \frac{\lambda(U^i)}{2|p^i|}\right)\right), \\ T(a_j^N, U^N) &\supset T\left(y_j^N, B\left(\frac{u^N}{q}, \frac{\lambda(U^N)}{2q}\right)\right). \end{aligned} \tag{4.2}$$

Since

$$\left| q \frac{a_j^i}{a_j^N} - p^i \right| \le \frac{1}{Q} < \frac{\min\{\lambda(U^i);\ i = 1, \dots, N\}}{8L} \le \frac{1}{8},$$

we have $4\left|\frac{a_j^i}{p^i}\right| > 3\left|y_j^N\right|$. Since $\frac{a_j^i}{p^i} - y_j^N = y_j^i$ and $\boldsymbol{y} \in \mathcal{Q}^N(\mathbb{R} \setminus \{0\})$, we have $\frac{a_j^i}{p^i} \ne y_j^N$. We use Lemma 4.7 replacing y, v, r_2, δ, z, u, r_1 by $\frac{a_j^i}{p^i}$, $\frac{u^i}{p^i}$, $\frac{\lambda(U^i)}{2|p^i|}$, δ_j, y_j^N, $\frac{u^N}{q}$, $\frac{\lambda(U^N)}{2q}$ respectively to obtain

$$(4.3)\qquad T\left(\frac{a^i_j}{p^i}, B\left(\frac{u^i}{p^i}, \frac{\lambda(U^i)}{2|p^i|}\right)\right) \cap T\left(y^N_j, B\left(\frac{u^N}{q}, \frac{\lambda(U^N)}{2q}\right)\right) \supset T(y^N_j, \widetilde{J}^N_j) \cap T(y^i_j, \widetilde{J}^i).$$

Recall that $y - z$ is replaced by $\frac{a^i_j}{p^i} - y^N_j = y^i_j$.

By Lemma 4.4(vii) and our choice of sets J^i, J^N_j we have

$$(4.4)\qquad T(y^N_j, \widetilde{J}^N_j) \cap T(y^i_j, \widetilde{J}^i) \supset T(y^N_j, J^N_j) \cap T(y^i_j, J^i).$$

By (4.2), (4.3) and (4.4) we have

$$(4.5)\qquad H^n(\boldsymbol{a},\mathcal{U}) \subset H^n(\boldsymbol{y}, J^1 \times \cdots \times J^{N-1} \times J^N_n).$$

Observe that we have

$$\lambda(J^N_j) \geq \frac{1}{2}\lambda(\tilde{J}^N_j) = \frac{1}{4}\delta_j|y^N_j| = \frac{1}{4}\delta_j\frac{|a^N_j|}{q} \geq \frac{1}{4}\frac{\min\{\lambda(U^i); i = 1, \ldots, N\}}{|a^N_j|}\frac{|a^N_j|}{q}$$
$$= \frac{1}{4}\frac{\min\{\lambda(U^i); i = 1, \ldots, N\}}{q}.$$

Thus we can use Lemma 4.6 to get an open interval J^N and an increasing sequence v_n of positive integers such that for every $n \in \mathbb{N}$ we obtain

$$(4.6)\qquad H^n(\{y_{v_n}\}, J^1 \times \cdots \times J^{N-1} \times J^N_n) \subset H^n(\{y_{v_n}\}, J^1 \times \cdots \times J^N),$$
$$4\lambda(J^N) \geq \lambda(J^N_{v_n}).$$

We set $x^i_n := y^i_{v_n}$ and $\mathcal{J} = J^1 \times \cdots \times J^N$. By the definition of $\boldsymbol{y}$ we simply get that $\mathcal{L}$ is triangular matrix without any zero element on diagonal. Thus we have (a). By (4.5) and (4.6) we get (b). Assume $i \leq s$. Since

$$\left|\frac{x^i_j}{x^N_j}\right| = \left|q\frac{a^i_{v_j}}{a^N_{v_j}} - p^i\right| \leq \frac{1}{Q} < \frac{\min\{\lambda(U^i);\ i = 1, \ldots, N\}}{8L}$$

we have

$$\left|\frac{x^N_j\lambda(J^i)}{x^i_j}\right| = \left|\frac{x^N_j\lambda(U^i)}{x^i_j}\right| \geq \left|\frac{x^N_j 8L}{x^i_j Q}\right| \geq 8L.$$

Let $s < i < N$. Since

$$\left|\frac{x^i_j p^i}{x^N_j}\right| = \left|q\frac{a^i_{v_j}}{a^N_{v_j}} - p^i\right| \leq \frac{1}{Q} < \frac{\min\{\lambda(U^i);\ i = 1, \ldots, N\}}{8L}$$

we have

$$\left|\frac{x^N_j\lambda(J^i)}{x^i_j}\right| \geq \left|\frac{x^N_j\lambda(\widetilde{J}^i)}{2x^i_j}\right| = \left|\frac{x^N_j\lambda(U^i)}{8x^i_j p^i}\right| \geq \left|\frac{x^N_j L}{x^i_j Q p^i}\right| \geq L.$$

Thus we have (c). Clearly,

$$16\lambda(J^N) \geq 4\lambda(J^N_{v_n}) \geq 2\lambda(\widetilde{J}^N_{v_n}) = \delta_{v_n}|x^N_n|$$

for all $n \in \mathbb{N}$. Thus we have (d).

$\square$

Lemma 4.9. *Let $N \in \mathbb{N}, \boldsymbol{a} \in \mathcal{Q}^N(\mathbb{R} \setminus \{0\})$, $\mathcal{U} = \prod_{i=1}^N U^i \subset [0,1]^N$ be an open interval, $L \in \mathbb{N}$ and $\delta_j = \min\left\{\frac{\lambda(U^i)}{|a_j^i|};\ i = 1, \ldots, N\right\}$ for every $j \in \mathbb{N}$. Then there exist $\boldsymbol{x} \in \left(\mathbb{R}^N\right)^{\mathbb{N}}$, a regular matrix $M \in \mathbb{Q}^{N\times N}$, an increasing sequence $\{v_n\}$ of positive integers and an open interval $\mathcal{J} = \prod_{i=1}^N J^i \subset [0,1]^N$ such that*

(a) $\boldsymbol{x} := \{M(a_{v_n})\} \in \mathcal{Q}^N(\mathbb{R} \setminus \{0\})$,
(b) $\forall n \in \mathbb{N} : H^n((a_{v_n}),\mathcal{U}) \subset H^n(\boldsymbol{x}, \mathcal{J})$,
(c) $\forall n \in \mathbb{N}\ \forall i < N : \left|\frac{x_n^{i+1}\lambda(J^i)}{x_n^i}\right| \geq L$,
(d) $\frac{\lambda(J^N)}{|x_n^N|} \geq \delta_{v_n}/16$.

Proof. We will proceed by induction over N. The case $N = 1$ is trivial. Assume that our statement holds for some $N - 1 \in \mathbb{N}$, we show that it also holds for N. By Lemma 4.8 there exist a regular matrix $\mathcal{L} \in \mathbb{Q}^{N\times N}$, an increasing sequence $\{p_n\}$ of positive integers and an open interval $\mathcal{V} = \prod_{i=1}^N V^i \subset [0,1]^N$ such that

(i) $\boldsymbol{y} := \{\mathcal{L}(a_{p_n})\} \in \mathcal{Q}^N(\mathbb{R} \setminus \{0\})$,
(ii) $\forall n \in \mathbb{N} : H^n((a_{p_n}),\mathcal{U}) \subset H^n(\boldsymbol{y}, \mathcal{V})$,
(iii) $\forall n \in \mathbb{N}\ \forall i < N : \left|\frac{y_n^N\lambda(V^i)}{y_n^i}\right| \geq 16L$,
(iv) $\frac{\lambda(V^N)}{|y_n^N|} \geq \delta_{p_n}/16$.

Clearly, $\{y^1, \ldots, y^{N-1}\} \in \mathcal{Q}^{N-1}(\mathbb{R} \setminus \{0\})$. By induction hypothesis there exist $\{x_n\} \in \left(\mathbb{Q}^{N-1}\right)^{\mathbb{N}}$, a regular matrix $\mathcal{Z} \in \mathbb{Q}^{(N-1)\times(N-1)}$, an increasing sequence $\{j_n\}$ of positive integers and open intervals $J^i \subset [0,1]$, $0 < i < N$, such that

(1) $\{x_n^1, \ldots, x_n^{N-1}\} := \{\mathcal{Z}(y_{j_n}^1, \ldots, y_{j_n}^{N-1})\} \in \mathcal{Q}^{N-1}(\mathbb{R} \setminus \{0\})$,
(2) $\forall n \in \mathbb{N} :$
$H^n(\{y_{j_n}^1, \ldots, y_{j_n}^{N-1}\}, \prod_{i=1}^{N-1} V^i) \subset H^n(\{x_n^1, \ldots, x_n^{N-1}\}, \prod_{i=1}^{N-1} J^i)$,
(3) $\forall n \in \mathbb{N}\ \forall i < N-1 : \left|\frac{x_n^{i+1}\lambda(J^i)}{x_n^i}\right| \geq L$,
(4) $\frac{\lambda(J^{N-1})}{|x_n^{N-1}|} \geq \frac{1}{16}\min\left\{\frac{\lambda(V^i)}{|y_{j_n}^i|};\ i = 1, \ldots, N-1\right\}$.

We set $v_n = p_{j_n}$, $x_n^N = y_{j_n}^N$ and $J^N = V^N$. We define $\widetilde{\mathcal{Z}} \in \mathbb{Q}^{N\times N}$ by

$$\widetilde{\mathcal{Z}}_{i,j} = \begin{cases} \mathcal{Z}_{i,j} & \text{for } 0 < i, j < N, \\ 1 & \text{for } i = j = N+1, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $\widetilde{\mathcal{Z}}$ is regular. We set $M = \widetilde{\mathcal{Z}} \cdot \mathcal{L}$. Thus M is also regular. Using (i) and (1) we easily obtain (a). By (2) we have

$$\forall n \in \mathbb{N} : H^n\big(\{y_{j_n}^1, \ldots, y_{j_n}^N\}, \mathcal{V}\big) \subset H^n(\boldsymbol{x}, \mathcal{J}) \tag{4.7}$$

Using (4.7) and (ii) we get (b). Using (3) we obtain (c) for $i < N-1$. From (iii) we have $\min\left\{\frac{\lambda(V^i)}{|y^i_{j_n}|};\ i = 1,\ldots,N-1\right\} = \frac{\lambda(V^{N-1})}{|y^{N-1}_{j_n}|}$. Using this, (4) and (iii) again we get the case $i = N-1$. The formula (iv) easily gives (d). □

Proof of Theorem 2.5. The inclusion $H^{(N)*} \supset H_L^{(N)*}$ is trivial.

Let $A \in H^{(N)*}$ be arbitrary. Then there exists $\boldsymbol{a} \in \mathcal{Q}^N(\mathbb{R} \setminus \{0\})$ and an open interval $\mathcal{U} \subset [0,1]^N$ such that $A \subset H(\boldsymbol{a},\mathcal{U})$. By Lemma 4.9 there exists $\boldsymbol{x} \in \left(\mathbb{Q}^N\right)^{\mathbb{N}}$ and an open interval $\mathcal{J} \subset [0,1]^N$ such that $H(\boldsymbol{a},\mathcal{U}) \subset H(\boldsymbol{x},\mathcal{J}) \in H_L^{(N)*}$. So, $A \in H_L^{(N)*}$. □

Acknowledgments. The research was supported by the Grant No. 22308/
B-MAT/MFF of the Grant Agency of the Charles University in Prague and by the grant GAČR 201/09/0067. The author is a (junior) researcher in the University Centre for Mathematical Modelling, Applied Analysis and Computational Mathematics (Math MAC).

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