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ON THE REGULARITY OF TWO-DIMENSIONAL UNSTEADY FLOWS OF HEAT-CONDUCTING GENERALIZED NEWTONIAN FLUIDS

MIROSLAV BULÍČEK, JOSEF MÁLEK, AND TIMOFEI NIKOLAEVICH SHILKIN

Abstract. We study regularity properties of unsteady flows of an incompressible heat-conducting fluid in a two-dimensional spatially periodic setting. Under certain structural assumptions on the Cauchy stress that include generalizations of the Ladyzhenskaya or power-law like models we establish the existence of a classical solution to such problems.

1. Introduction and Main Results

In this study we investigate the long-time and large-data regularity properties of unsteady flows of homogeneous incompressible heat-conducting fluid in two spatial dimensions. For simplicity, we restrict ourselves to a spatially periodic setting. In other words, we are interested in classical solvability of the following system of equations

$$\begin{aligned} \operatorname{div} \boldsymbol{v} &= 0, \\ \boldsymbol{v}_{,t} + \operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{v}) - \operatorname{div} \mathsf{S} &= -\nabla p + \boldsymbol{f}, \\ E_{,t} + \operatorname{div}\left(\boldsymbol{v}(E+p)\right) + \operatorname{div} \boldsymbol{q} - \operatorname{div}(\mathsf{S}\boldsymbol{v}) &= \boldsymbol{f} \cdot \boldsymbol{v}, \end{aligned} \tag{1.1}$$

that is supposed to be satisfied in the space-time cylinder $Q_T := \Omega \times (0,T)$, where T is the length of time interest and $\Omega := (0,1)^2$ is a unit cube in $\mathbb{R}^2$. The system (1.1) is complemented with the following initial and "boundary" conditions

$$\boldsymbol{v}, E, p, \boldsymbol{q}, \mathsf{S} \text{ are spatially periodic,} \tag{1.2}$$

$$\boldsymbol{v}(t,x) = \boldsymbol{v}_0(x), \quad E(t,x) = E_0(x). \tag{1.3}$$

In (1.1)–(1.3), $\boldsymbol{v} : Q_T \to \mathbb{R}^2$ denotes the velocity field, $p : Q_T \to \mathbb{R}$ is the mean normal stress (the pressure), $\mathsf{S} : Q_T \to \mathbb{R}^{2\times 2}_{sym}$ the constitutively determined part of the Cauchy stress, $\boldsymbol{q} : Q_T :\to \mathbb{R}^2$ denotes the heat flux, $E : Q_T \to \mathbb{R}_+$ is the density of the global energy of the fluid and $\boldsymbol{f} : Q_T \to \mathbb{R}^2$ denotes the density of the external body forces. The equation $(1.1)_1$ represents the incompressibility constraint that

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is for a homogeneous fluid equivalent to the mass balance equation. The second (vectorial) equation in (1.1) is then the balance equation for linear momentum and the last equation $(1.1)_3$ represents the balance of (total) energy.

In order to study a complete system of partial differential equations, we have to add to (1.1) constitutive relationships for the Cauchy stress, the heat flux and the global energy. First, the global energy is given as the sum of the kinetic and the internal energy, i.e., denoting by e the internal energy of the fluid, E takes the form $E := \frac{1}{2}|\boldsymbol{v}|^2 + e$. In addition, we have to specify how the density of the internal energy e depends on other state variables that in our setting means how it depends on the temperature. Thus, denoting θ the temperature of the fluid we will in what follows assume for simplicity that $e = c_v\theta$ and we in addition set $c_v \equiv 1$. Similarly, for the heat flux $\boldsymbol{q}$ we will consider the Fourier law, i.e., we will assume that

$$-\boldsymbol{q} = \kappa_0 \nabla\theta, \tag{1.4}$$

and we also set $\kappa_0 \equiv 1$. Finally, for the part of the Cauchy stress we assume that it takes the form

$$\mathbf{S} = \mathbf{S}(\theta, \mathbf{D}(\boldsymbol{v})) = 2\nu(\theta, |\mathbf{D}(\boldsymbol{v})|^2)\mathbf{D}(\boldsymbol{v}), \quad \mathbf{D}(\boldsymbol{v}) := \frac{1}{2}(\nabla\boldsymbol{v} + (\nabla\boldsymbol{v})^T), \tag{1.5}$$

where ν denotes the kinematic viscosity of the fluid. Substituting all these relations into (1.1) and assuming also for simplicity that the external body forces are zero we get the following system

$$\begin{aligned} \operatorname{div}\boldsymbol{v} &= 0, \\ \boldsymbol{v}_{,t} + \operatorname{div}(\boldsymbol{v}\otimes\boldsymbol{v}) - \operatorname{div}\mathbf{S}(\theta, \mathbf{D}(\boldsymbol{v})) &= -\nabla p, \\ (|\boldsymbol{v}|^2/2 + \theta)_{,t} + \operatorname{div}\big(\boldsymbol{v}(|\boldsymbol{v}|^2/2 + \theta + p)\big) - \triangle\theta - \operatorname{div}(\mathbf{S}(\theta, \mathbf{D}(\boldsymbol{v}))\boldsymbol{v}) &= 0. \end{aligned} \tag{1.6}$$

Finally, if we assume that the solution is smooth enough, we can take the scalar product of $(1.6)_2$ with $\boldsymbol{v}$ and subtract the result from $(1.6)_3$ to get the following final system of partial differential equations

$$\left.\begin{aligned} \operatorname{div}\boldsymbol{v} &= 0 \\ \boldsymbol{v}_{,t} + \operatorname{div}(\boldsymbol{v}\otimes\boldsymbol{v}) - \operatorname{div}\mathbf{S}(\theta, \mathbf{D}(\boldsymbol{v})) &= -\nabla p \\ \theta_{,t} + \operatorname{div}(\boldsymbol{v}\theta) - \Delta\theta &= \mathbf{S}(\theta, \mathbf{D}(\boldsymbol{v})) : \mathbf{D}(\boldsymbol{v}) \end{aligned}\right\} \text{ in } Q_T, \tag{1.7}$$

$$\boldsymbol{v} \text{ and } \theta \text{ are spatially periodic} \tag{1.8}$$

$$\boldsymbol{v}(0,x) = \boldsymbol{v}_0(x), \qquad \theta(0,x) = \theta_0(x)(\geq \theta_{min} > 0). \tag{1.9}$$

Note that the systems (1.6) and (1.7) are equivalent if and only if we can multiply $(1.6)_2$ by (or use as a test function in the weak formulation of $(1.6)_2$) the velocity $\boldsymbol{v}$. If it is not the case, the system (1.6) is preferable since it represents the primar formulation of the balance of energy where all its terms are in the divergence form. As we are interested in regularity of such flows, we will always observe that $\boldsymbol{v}$ is an admissible test function in balance of linear momentum and therefore the "equivallence" described above can be justified. For more details concerning the existence analysis of the system (1.6) we refer to [4, 5, 6]. For physical motivation of such models we refer to [18].

The problem (1.7) is the system of nonlinear PDE's where the nonlinearity is present even in the leading elliptic term in $(1.7)_2$, and in the heat equation, i.e., in $(1.7)_3$, it enters the right-hand side where it represents the term with critical growth. Such a system belongs (at least formally) to the so-called class of systems with strong nonlinearities; see [11]. For these systems, in contrast to the scalar quasilinear equations, there is so far no regularity theory, and various counter-examples show that such a theory is not in general possible even if the leading elliptic term is of the diagonal form, see eg. [7, 10, 12, 24]. Even more, if one assumes that the viscosity depends only on the temperature, the system (1.7) becomes linear w.r.t. velocity $\boldsymbol{v}$ in the highest order term in momentum equation (eq $(1.7)_2$) but is still strongly nonlinear because of the critical term on the right hand side of $(1.7)_3$ and up to our best knowledge there is no regularity theory available even in two-dimensional setting if we merely assume that

$$\nu(\theta, |\mathbf{D}|^2) = \nu(\theta), \qquad 0 < \nu_0 \le \nu(s) \le \nu_1, \qquad |\nu'(s)| \le \nu_2. \tag{1.10}$$

The only known regularity result concerning (1.10), we are aware of, is established in a recent work [3] where the authors proved the maximal L^2 regularity theory for the generalized Stokes system (i.e., the system (1.7) without convective terms) provided that the viscosity in addition to (1.10) satisfies

$$-\frac{1}{15(s-\theta_{\min})} \le \frac{\nu'(s)}{\nu(s)} \le \frac{1}{15(s-\theta_{\min})}. \tag{1.11}$$

Note that (1.11) is satisfied for a large class physically relevant viscosities; see [3].

In the present study we consider a different subclass of viscosities ν, of the form (1.5) satisfying the assumptions (1.12)–(1.13) below, for which the kind of L^2 maximal regularity can be obtained directly from a priori estimates. Having then such estimates, we are able to show that we can continue in the regularity ladder and establish better properties for the velocity and the temperature that finally suffice for an application of the theory[1] developed for the stationary as well as evolutionary generalized Stokes problem by Kaplický et al.; see [13] and [14]. The fact that such a scheme is successful in our setting is based on an observation that the time derivative and the second spatial derivatives of the temperature belong to $L^q(Q_T)$ for some $q > 2$. Saying differently, the purpose of this study is to identify the class of temperature and shear dependent viscosities so that, for large smooth data, the planar flows for the resulting generalized Navier-Stokes-Fourier system are described by the *Hölder continuous* velocity, pressure and temperature fields that have Hölder continuous spatial and time derivatives (it then implies that solution is as smooth as data permit).

Before we formulate the assumption on the viscosity ν and also the main theorem of this paper, we introduce notation that we will use in what follows. The vector-valued functions will be denoted by bold italic, i.e., $\boldsymbol{v}(t,x) : Q_T \to \mathbb{R}^2$, similarly the tensor-valued functions by bold capital letter, e.g. $\mathbf{D}$. Through the whole paper we also use Einstein summation convection over repeated indices. For simplicity, we also denote $\boldsymbol{u} \cdot \boldsymbol{v} = u_i v_i \equiv \sum_{i=1}^2 u_i v_i$ the scalar product of the vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ and similarly we denote $\mathbf{A} : \mathbf{B} = A_{ij} B_{ij} \equiv \sum_{i,j=1}^2 A_{ij} B_{ij}$, for scalar product of any $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{2\times 2}$, and $|\mathbf{A}| := (\mathbf{A} : \mathbf{A})^{1/2}$. In addition, we denote $\mathbb{I} : \mathbb{M}^{2\times 2} \to \mathbb{M}^{2\times 2}$ the identity tensor of the rank 4. In order to simplify the notation of partial derivative, we will frequently use $\boldsymbol{v}_{,k} := \frac{\partial \boldsymbol{v}}{\partial x_k}$, and similarly $\boldsymbol{v}_{,t} := \frac{\partial \boldsymbol{v}}{\partial t}$. Note that using this notation we have $(\nabla \boldsymbol{v})_{ij} = v_{i,j}$.

[1] The origin of this theory goes back to works by Nečas and Stará developed for elliptic problems in two dimensions; see [20] and [23]. The key tool in their approach plays L^p-theory as developed by Bojarski in [2] and Meyers in [19].

For clarity we also introduce a notation of function spaces used in the paper. The usual Lebesgue and Sobolev spaces equipped with the standard norm are denoted by $L_q(\Omega)$, $L_q(Q_T)$ and $W^k_q(\Omega)$. The dual space to $W^k_q(\Omega)$ is denoted by $W^{-k}_q(\Omega)$. Moreover, for any Banach space X we use the notation $X^m := \underbrace{X \times \cdots \times X}_{m-\text{times}}$.

We further set $L_{q,s}(Q_T) := L_s(0,T;L_q(\Omega))$ to shorten notation for the Bochner spaces equipped with the standard norm. We also use the abbreviation $W^{k,m}_q(Q_T) := \{v \in L_q(0,T;L_q(\Omega)); \nabla^k v \in L_q(Q_T), \frac{\partial^m v}{\partial t^m} \in L_q(Q_T)\}$. Finally, for any $k,m \in \mathbb{N}$ and any $\alpha, \beta \in [0,1]$ we introduce a notation for space of Hölder continuous functions as $\mathcal{C}^{k+\alpha,m+\beta}(Q_T) := \{v \in \mathcal{C}(Q_T); \nabla^k v \in \mathcal{C}^\alpha(Q_T), \frac{\partial^m v}{\partial t^m} \in \mathcal{C}^\beta(Q_T)\}$, where $\mathcal{C}^\alpha(\Omega)$ or $\mathcal{C}^\alpha(Q_T)$ stand for the usual space of α-Hölder continuous functions defined on Ω or Q_T.

Finally, in what follows we denote by the constant $C_{\boldsymbol{v}_0,\theta_0}$ some universal constant depending only on the structure of the viscosity ν (see (1.5)), the length of time interval T and also on the initial data $\boldsymbol{v}_0, \theta_0$.

Next, we formulate our assumption on the Cauchy stress tensor given by (1.5), give several examples satisfying such assumptions and then we formulate the main theorem of the paper.

1.1. **Assumptions on the Cauchy stress $\mathbf{S}$.** Throughout the paper we assume that the constitutively determined part of the Cauchy stress $\mathbf{S}$ is given by (1.5) and we assume that $\mathbf{S} : \mathbb{R}_+ \times \mathbb{R}^{2\times 2}_{sym} \to \mathbb{R}^{2\times 2}_{sym}$ is at least Lipschitz mapping. In addition we assume that there are three positive constants ν_0, ν_1 and ν_3, there is some $r \in (1,\infty)$ and there is some $\lambda > 0$ such that

$$\left|\frac{\partial \mathbf{S}(\theta,\mathbf{D})}{\partial \theta}\right| \le \nu_3 \frac{(1+|\mathbf{D}|^2)^{\frac{r-2}{4}}}{(1+\theta)^{\lambda+\frac{1}{2}}} \tag{1.12}$$

$$\nu_0(1+|\mathbf{D}|^2)^{\frac{r-2}{2}}|\mathbf{A}|^2 \le \frac{\partial \mathbf{S}(\theta,\mathbf{D})}{\partial \mathbf{D}} : (\mathbf{A}\otimes\mathbf{A}) \le \nu_1(1+|\mathbf{D}|^2)^{\frac{r-2}{2}}|\mathbf{A}|^2 \tag{1.13}$$

for all $\mathbf{D}, \mathbf{A} \in \mathbb{R}^{2\times 2}_{sym}$ and all $\theta \in \mathbb{R}_+$.

At this point, we give two examples of viscosities that satisfy (1.12)–(1.13). Note, that from physical point of view the viscosity should be decreasing function of the temperature. Also note that from the point of view of engineering applications the values of r between $(1,2)$ plays the most important role. We refer to [18] for details. Hence, let us assume that the viscosity takes the form

$$\nu(\theta,\mathbf{D}) = \left(\gamma(\theta) + (1+|\mathbf{D}|^2)^p\right)^{\frac{r-2}{2p}}, \qquad p \ge 1, \tag{1.14}$$

where γ is bounded smooth nonnegative function satisfying

$$|\gamma'(\theta)| \le c(1+\theta)^{-\frac{1}{2}-\lambda} \quad \text{for some } \lambda > 0. \tag{1.15}$$

It is not difficult to observe that the viscosities of the form (1.14) meet the condition (1.13). In order to deduce also the validity of (1.12) we can compute

$$\begin{aligned}\left|\frac{\partial \mathbf{S}(\theta,\mathbf{D})}{\partial\theta}\right| &= |\mathbf{D}|\left|\frac{\partial \nu(\theta,|\mathbf{D}|^2)}{\partial\theta}\right| \\ &= |\mathbf{D}|\frac{|r-2|}{2p}\left(\gamma(\theta)+(1+|\mathbf{D}|^2)^p\right)^{\frac{r-2-2p}{2p}}|\gamma'(\theta)| \\ &\le \nu_3(1+\theta)^{-\frac{1}{2}-\lambda}\left(\gamma(\theta)+(1+|\mathbf{D}|^2)^p\right)^{\frac{r-1-2p}{2p}} \\ &\le \nu_3(1+\theta)^{-\frac{1}{2}-\lambda}\left(1+|\mathbf{D}|^2\right)^{\frac{r-1-2p}{2}}\end{aligned}$$

and we see that if we chose $p \ge \frac{r}{4}$ the inequality (1.12) holds.

The next example,

$$\nu(\theta,|\mathbf{D}|^2) = \arctan(1+\gamma(\theta)+|\mathbf{D}|^2),$$

describes a fluid for which (1.12)–(1.13) hold with $r = 2$, yet the relation between $\mathbf{S}$ and $\mathbf{D}$ is nonlinear.

Finally, we finish this subsection by summarizing the consequences of the assumptions (1.5) and (1.12)–(1.13).

Lemma 1.1. *Let $r \in (1,\infty)$ and assume that $\mathbf{S}$ satisfies (1.5) and (1.12)–(1.13). Let*

$$m(\theta,\mathbf{D}) := \int_0^{|\mathbf{D}|^2} \nu(\theta,\tau)\,d\tau. \tag{1.16}$$

Then m satisfies the following inequalities

$$c_1(1+|\mathbf{D}(\boldsymbol{v})|^2)^{\frac{r}{2}} \le m(\theta,\mathbf{D}(\boldsymbol{v})) \le c_2(1+|\mathbf{D}(\boldsymbol{v})|^2)^{\frac{r}{2}}, \tag{1.17}$$

$$\left|\frac{\partial m(\theta,\mathbf{D}(\boldsymbol{v}))}{\partial\theta}\right| \le c_3(1+|\mathbf{D}(\boldsymbol{v})|^2)^{\frac{r}{4}}(1+\theta)^{-\frac{1}{2}-\lambda}. \tag{1.18}$$

Proof. The estimates (1.17) are direct consequences of (1.12). The second relation, i.e., the inequality (1.18), can be deduced by using (1.5) and (1.13). We refer reader to [17] for details. □

1.2. **The main theorem.** Some results on the existence of the weak solutions to the system (1.7)–(1.9) can be found in [6], [9]. In order to prove the existence of classical solution to (1.7)–(1.9) we need to assume smoothness and compatibility conditions on the initial data. Namely, we assume that

$$\begin{gathered}\exists\,\alpha>0 \;\text{ such that } \boldsymbol{v}_0 \in \mathcal{C}^{2+\alpha}(\mathbb{R}^2)^2, \quad \theta_0 \in \mathcal{C}^{2+\alpha}(\mathbb{R}),\\ \boldsymbol{v}_0,\ \theta_0 \quad \text{are} \quad \Omega\text{-periodic.}\end{gathered} \tag{1.19}$$

The main result of the present paper is the following theorem:

Theorem 1.1. *Let $\mathbf{S}$ be given by (1.5) and satisfy (1.12)–(1.13) for some $r \in (\frac{4}{3},\infty)$. Then for any $\boldsymbol{v}_0$ and θ_0 satisfying (1.19) there exists a triple $(\boldsymbol{v},p,\theta)$ such that*

$$\boldsymbol{v} \in \mathcal{C}^{2+\alpha,1+\frac{\alpha}{2}}(\overline{Q_T})^2, \quad \nabla p \in \mathcal{C}^{\alpha,\frac{\alpha}{2}}(\overline{Q_T})^2, \quad \theta \in \mathcal{C}^{2+\alpha,1+\frac{\alpha}{2}}(\overline{Q_T}), \tag{1.20}$$

which solves (1.7)–(1.9). Moreover, this solution is unique in the class (1.20).

The rest of the paper is organized as follows. Section 2 contains some auxiliary tools and lemmas that are known and will be used to the proof of Theorem 1.1. The key part of the paper is presented in Section 3 where we establish the principal a priori estimates of the solution of the problem (1.7) in Hölder continuous spaces. Finally in Section 4 we sketch an application of the variant of the Leray-Schauder fixed point theorem which is appropriate for our case.

2. Auxiliary Results

First, we discuss the higher integrability results for the generalized Stokes system. We consider both time-independent and evolutionary case referring to [13] and [14] for details.

Proposition 2.1 (Evolutionary problem, see Proposition 2.1 in [14])**.** *Let a fourth order tensor $\mathbb{A} = (A_{ijkl}(x,t))$ possess the following properties:*

$$A_{ijkl} \in L_\infty(Q_T), \qquad A_{ijkl} = A_{klij} = A_{jikl} = A_{ijlk} \tag{2.1}$$
$$\nu_0|\mathbf{D}|^2 \leq \mathbb{A}(x,t) : (\mathbf{D}\otimes\mathbf{D}) \leq \nu_1|\mathbf{D}|^2,$$

for all $\mathbf{D} \in R^{2\times 2}_{sym}$ and for a.e. $(x,t) \in Q_T$. Assume that $\mathbf{F} \in L_{q_0}(Q_T)^{2\times 2}$ for some $q_0 > 2$ and assume that $\boldsymbol{u} \in W^{1,0}_2(Q_T)^2$, $p \in L_2(Q_T)$ satisfy the generalized Stokes problem with periodical boundary conditions:

$$\boldsymbol{u}_{,t} - \operatorname{div}(\mathbb{A}\mathbf{D}(\boldsymbol{u})) + \nabla p = \operatorname{div}\mathbf{F},$$
$$\operatorname{div}\boldsymbol{u} = 0, \quad \boldsymbol{u} \text{ is periodic.}$$

Then for any $q \in (2, q_0]$ such that

$$1 - \frac{2}{q} \leq c_0\frac{\nu_0}{\nu_1}, \tag{2.2}$$

the following estimates hold:

$$\|\nabla\boldsymbol{u}\|_{L_q(Q_T)^{2\times 2}} \leq \frac{c_1}{\nu_0}\|\mathbf{F}\|_{L_q(Q_T)^{2\times 2}}, \tag{2.3}$$

$$\|\boldsymbol{u}\|_{L_\infty(0,T;L_q(\Omega)^2)} \leq c_2\frac{\nu_1^{1/q}}{\nu_0}\|\mathbf{F}\|_{L_q(Q_T)^{2\times 2}}, \tag{2.4}$$

where c_i, $i = 0, 1, 2$, are some absolute constants.

Proposition 2.2 (Stationary problem, see Proposition 2.2 in [13])**.** *Let a fourth order tensor $\mathbb{A} = (A_{ijkl}(x))$ possess the following properties:*

$$A_{ijkl} \in L_\infty(\Omega), \qquad A_{ijkl} = A_{klij} = A_{jikl} = A_{ijlk} \tag{2.5}$$
$$\nu_0|\mathbf{D}|^2 \leq \mathbb{A}(x) : (\mathbf{D}\otimes\mathbf{D}) \leq \nu_1|\mathbf{D}|^2,$$

for all $\mathbf{D} \in R^{2\times 2}_{sym}$ and for a.e. $x \in \Omega$. Assume that $\mathbf{F} \in L_{q_0}(\Omega)^{2\times 2}$ for some $q_0 > 2$ and assume that $\boldsymbol{u} \in W^1_2(\Omega)^2$, $p \in L_2(\Omega)$ satisfy the generalized Stokes problem with periodical boundary conditions:

$$-\operatorname{div}(\mathbb{A}\mathbf{D}(\boldsymbol{u})) + \nabla p = \operatorname{div}\mathbf{F},$$
$$\operatorname{div}\boldsymbol{u} = 0, \quad \boldsymbol{u} \text{ is periodic.}$$

Then for any $q \in (2, q_0]$ satisfying (2.2) the following estimates hold

$$\|\nabla\boldsymbol{u}\|_{L_q(\Omega)^{2\times 2}} \leq \frac{C}{\nu_0}\|\mathbf{F}\|_{L_q(\Omega)^{2\times 2}}. \tag{2.6}$$

In conclusion we recall some known facts of the theory of functions.

Proposition 2.3 (see [16]). *For any $w \in W^1_q(\Omega)$ with $q > 2$ there holds*

$$\sup_{x\in\Omega} |w(x)| \le \left(\frac{C}{1-\frac{2}{q}}\right)^{(1-\frac{1}{q})} \|\nabla w\|_{L_q(\Omega)^2}.$$

The following proposition is proved in [21]:

Proposition 2.4 (see [21]). *Assume that $\nabla v \in L_\infty(0,T;\mathcal{C}^\alpha(\overline{\Omega})^2)$ and that $\nabla v_{,t} \in L_q(Q_T)^2$ for some $q > 1$. Then there is $\alpha_0 \in (0,1)$ such that*

$$\nabla v \in \mathcal{C}^{\alpha_0,\frac{\alpha_0}{2}}(\overline{Q_T})^2.$$

Here chosen scheme of the proof of the classical solvability of the system (1.7) is based on the two general statements[2]. The first one is the theorem on existence of the classical solution to the generalized Navier-Stokes fluid (sometimes called the Ladyzhenskaya system) (see Kaplický et al. [14]), and the second is some variant of the Leray-Schauder principle (see [16]). We reproduce their statements here for the completeness.

Proposition 2.5 (generalized Navier-Stokes (or Ladyzhenskaya) system, see Theorem 1.1 in [14]). *Assume that $\frac{4}{3} < r$ and that S is independent of the temperature θ and satisfy* (1.5) *and* (1.13). *Assume that $\boldsymbol{v}_0 \in \mathcal{C}^{2+\alpha}(\overline{\Omega})^2$ is spatially periodic. Then there exists the unique couple $(\boldsymbol{v}, p)$ satisfying $\boldsymbol{v} \in \mathcal{C}^{2+\alpha,1+\frac{\alpha}{2}}(\overline{Q_T})^2$, $\nabla p \in \mathcal{C}^{\alpha,\frac{\alpha}{2}}(\overline{Q_T})$ that represents the classical solution of the Ladyzhenskaya system*

$$\begin{aligned} \boldsymbol{v}_{,t} + \operatorname{div}(\boldsymbol{v}\otimes\boldsymbol{v}) - \operatorname{div}\mathsf{S}(\mathsf{D}(\boldsymbol{v})) + \nabla p &= 0, \\ \operatorname{div}\boldsymbol{v} &= 0, \\ \boldsymbol{v}(0) = \boldsymbol{v}_0,\ \boldsymbol{v} &\text{ is spatially periodic.} \end{aligned} \tag{2.7}$$

Moreover, this solution is unique in the class of all strong solutions to the system (2.7).

Proposition 2.6 (Leray-Schauder, see [16]). *Let X be a Banach space, $B_R := \{u \in X : \|u\|_X < R\}$, and assume*

$$\Phi_\tau : \bar{B}_R \subset X \to X, \qquad \tau \in [0,1]$$

is a family of non-linear operators such that

(1) *for any $\tau \in [0,1]$ the map $\Phi_\tau(\cdot)$ is continuous on $\bar{B}_R$,*
(2) $\sup_{u\in\bar{B}_R} \|\Phi_\tau(u) - \Phi_{\tau_0}(u)\|_X \to 0$ *as* $\tau \to \tau_0$,
(3) *for any $\tau \in [0,1]$ the set $\Phi_\tau(\bar{B}_R)$ is precompact in X,*
(4) *for any $\tau \in [0,1]$ and for any $u \in \bar{B}_R$ such that $u = \Phi_\tau(u)$ we have*

$$\|u\|_X < R \tag{2.8}$$

(5) *there is a unique $u_0 \in B_R$ such that $u_0 = \Phi_0(u_0)$.*

Then there is at least one $u \in B_R$ such that $u = \Phi_1(u)$.

[2]Note that it is possible to built the solution via the Galerkin approximations and uniform apriori estimates established below. These estimates, at an appropriate point, allow one to take the limit and obtained the strong solution for which one can proved the $C^{1,\alpha}$-regularity results similarly as it is performed here

Finally, to verify the validity of all conditions of Proposition 2.6 for the solution operator corresponding to the problem (1.7), we need the following result on the continuous dependence of solutions to the generalized Stokes system on the coefficients.

Proposition 2.7 (Continuous dependence on the coefficients, see [22])**.** *Let for any* $m \in \mathbb{N}$, $\boldsymbol{w}^m$ *denotes solutions to the following problem*

$$\begin{aligned} \boldsymbol{w}^m_{,t} - \mathbb{A}^m \nabla^2 \boldsymbol{w}^m + \nabla q^m &= \boldsymbol{f}^m, \\ \operatorname{div} \boldsymbol{w}^m &= 0 \\ \boldsymbol{w}^m(0) = \boldsymbol{v}_0, \ \boldsymbol{w}^m & \textit{ is spatially periodic}, \end{aligned} \tag{2.9}$$

with the tensor $\mathbb{A}^m := A^m_{ijkl}(x,t)$ *satisfying* (2.1)*. Assume* $\|\mathbb{A}^m\|_{\mathcal{C}^{0,\alpha_0}(\overline{Q_T})^{2^4}} < \infty$ *for some* $\alpha_0 > 0$*, and* $\mathbb{A}^m(x,t) \to \mathbb{A}(x,t)$ *a.e. in* Q_T *as* $m \to \infty$*. Assume also that* $\boldsymbol{f}^m \to \boldsymbol{f}$ *in* $L_s(Q_T)$ *for some* $s > 1$*. Then*

$$\boldsymbol{w}^m \to \boldsymbol{w} \quad \textit{strongly in} \quad W^{2,1}_s(Q_T). \tag{2.10}$$

In particular, if $s > 4$*, then for some* $\alpha > 0$

$$\boldsymbol{w}^m \to \boldsymbol{w} \quad \textit{in} \quad \mathcal{C}^{1+\alpha, \frac{1+\alpha}{2}}(\overline{Q_T}). \tag{2.11}$$

3. A Priori Estimates

In this section we derive formally (under assumption that the solution is smooth) some a priori estimates. These estimates can be however justified by using the semi-Galerkin procedure (see [5] for details) since in the periodic setting it is allowed to use $-\triangle \boldsymbol{v}$ as a test function in balance of linear momentum. Therefore in principle we establish the existence of weak solution to (1.7)–(1.9) and we show which kind of regularity such constructed solution has.

In the rest of this section we will always assume that $\mathbf{S}$ is given by (1.5) and satisfies (1.12)–(1.13) with some fixed $r \in (1,\infty)$. Moreover we will always assume that the initial data satisfies (1.19). To simplify the notation we will usually omit writing of dependence on (x,t) whenever it is clear from the context.

We begin with the first a priori estimates.

Lemma 3.1 (First a priori estimate)**.** *Let* $(\boldsymbol{v}, p, \theta)$ *be smooth solution to* (1.7)–(1.9)*. Then* $\theta \geq \theta_{min}$ *a.e. in* Q_T *and the following estimates hold*

$$\|\boldsymbol{v}\|^2_{L_{2,\infty}(Q_T)^2} + \|\boldsymbol{v}\|^r_{W^{1,0}_r(Q_T)^2} \leq C\|\boldsymbol{v}_0\|_2^2 \leq C_{\boldsymbol{v}_0,\theta_0}, \tag{3.1}$$

$$\|\theta\|_{L_{1,\infty}(Q_T)} + \int_{Q_T} \frac{|\nabla\theta|^2}{\theta^{1+\lambda}} \, dx \, dt \leq C(\lambda^{-1})(\|\boldsymbol{v}_0\|_2^2 + \|\theta_0\|_1) \leq C_{\boldsymbol{v}_0,\theta_0} \tag{3.2}$$

for all $\lambda > 0$.

Proof. The proof of (3.1)–(3.2) is standard. To obtain (3.1) it is enough to multiply balance of linear momentum $(1.7)_2$ by $\boldsymbol{v}$ and integrate over Q_T. Since $\boldsymbol{v}$ is divergence free field, the convective term and the pressure term vanish after integration by parts. Finally, it is enough to use the structural assumption (1.5) and the growth assumption (1.12) together with the Korn inequality. Since the right hand side of the heat equation $(1.7)_3$ is non-negative (which follows from (1.5) and (1.13), see e.g. [5]) it is easy to deduce that $\theta(t,x) \geq \theta_0(x) \geq \theta_{min}$ a.e. in Q_T. Then integration over Ω of the heat equation, and use (3.1) leads to the first estimate in

(3.2). The second estimate in (3.2) can be obtained by using $\theta^{-\lambda}$ as a test function for the heat equation. We refer to [1] or [5] for details. □

Next, we derive the estimates for the second spatial derivatives.

Lemma 3.2 (Estimate of the second spatial derivatives). *Let* $(\boldsymbol{v}, p, \theta)$ *be a smooth solution to* (1.7)–(1.9). *Then the following estimates hold*

$$\|\nabla \boldsymbol{v}\|^2_{L_{2,\infty}(Q_T)^{2\times 2}} + \int_{Q_T} (1+|\mathbf{D}(\boldsymbol{v})|^2)^{\frac{r-2}{2}} |\nabla^2 \boldsymbol{v}|^2 \, dx\, dt \le C\|\nabla \boldsymbol{v}_0\|_2^2 \le C_{\boldsymbol{v}_0,\theta_0}. \tag{3.3}$$

In particular, we have

$$\||\mathbf{D}(\boldsymbol{v})|^{\frac{r}{2}}\|^2_{W_2^{1,0}(Q_T)} \le C_{\boldsymbol{v}_0,\theta_0}. \tag{3.4}$$

Proof. Multiplying the momentum equation (i.e., equation $(1.7)_2$) by $-\Delta v$, integrating the result over Ω and using integration by parts, we obtain (note that the pressure term again vanishes)

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\nabla \boldsymbol{v}\|_2^2 &+ \int_\Omega \frac{\partial \mathbf{S}(\theta, \mathbf{D}(\boldsymbol{v}))}{\partial \mathbf{D}} : (\mathbf{D}(\nabla \boldsymbol{v}) \otimes \mathbf{D}(\nabla \boldsymbol{v}))\, dx \\ &= -\int_\Omega \left(\frac{\partial \mathbf{S}(\theta, \mathbf{D}(\boldsymbol{v}))}{\partial \theta} \otimes \nabla\theta\right) : \mathbf{D}(\nabla \boldsymbol{v})\, dx - \int_\Omega \operatorname{div}(\boldsymbol{v}\otimes\boldsymbol{v}) \cdot \Delta \boldsymbol{v}\, dx \\ &=: I_1 + I_2.\end{aligned}$$

Next, using (1.13) we obtain that the second term on the left hand side can be estimated as

$$\int_\Omega \frac{\partial \mathbf{S}(\theta, \mathbf{D}(\boldsymbol{v}))}{\partial \mathbf{D}} : (\mathbf{D}(\nabla \boldsymbol{v}) \otimes \mathbf{D}(\nabla \boldsymbol{v}))\, dx \ge \nu_0 \int_\Omega (1+|\mathbf{D}(\boldsymbol{v})|^2)^{\frac{r-2}{2}} |\mathbf{D}(\nabla \boldsymbol{v})|^2\, dx := J(t). \tag{3.5}$$

For the term I_2 we get

$$I_2 = \int_\Omega \frac{\partial v_i}{\partial x_j}\frac{\partial v_j}{\partial x_k}\frac{\partial v_k}{\partial x_i}\, dx = 0,$$

where for the first identity we used integration by parts formula, and in the second inequality we used the fact that we study the two-dimensional problem and therefore the integrand is identically zero (see [17] for details).

To estimate I_1 we use the assumption (1.13) and the Young inequality to get

$$I_1 \le \nu_3 \int_\Omega \frac{|\nabla\theta|}{\theta^{\lambda+\frac{1}{2}}}(1+|\mathbf{D}(\boldsymbol{v})|^2)^{\frac{r-2}{4}} |\mathbf{D}(\nabla \boldsymbol{v})|\, dx \le \frac{1}{2}J(t) + \frac{\nu_3^2}{2\nu_0}\int_\Omega \frac{|\nabla\theta|^2}{\theta^{2\lambda+1}}\, dx.$$

Therefore, combining all together we end up with the estimate

$$\frac{d}{dt}\|\nabla \boldsymbol{v}\|_2^2 + \nu_0 \int_\Omega (1+|\mathbf{D}(\boldsymbol{v})|^2)^{\frac{r-2}{2}} |\mathbf{D}(\nabla \boldsymbol{v})|^2\, dx \le C\int_\Omega \frac{|\nabla\theta|^2}{\theta^{2\lambda+1}}\, dx. \tag{3.6}$$

Owing to the point-wise relation $|\nabla^2 \boldsymbol{v}| \le C|\mathbf{D}(\nabla \boldsymbol{v})|$ we can easily replace $\mathbf{D}(\nabla \boldsymbol{v})$ in (3.6). Moreover, having a priori estimate (3.2) we see that the right hand side is integrable w.r.t. time t and therefore we can deduce the relation (3.3).

To complete the proof we need to show (3.4). It however suffices to combine (3.3) with the following point-wise estimate

$$|\nabla(1+|\mathbf{D}(\boldsymbol{v})|^2)^{\frac{r}{4}}|^2 \le C(1+|\mathbf{D}(\boldsymbol{v})|^2)^{\frac{r-2}{2}}|\mathbf{D}(\nabla \boldsymbol{v})|^2.$$

Thus the proof of Lemma 3.2 is complete. □

The next step is focused on improving the regularity of the time derivative of the solution.

Lemma 3.3 (Estimates on time derivative)**.** *Let $(\boldsymbol{v}, p, \theta)$ be a smooth solution to* (1.7)–(1.9)*. Then the following estimates hold*

$$\begin{aligned}\|\boldsymbol{v}_{,t}\|^2_{L_{2,\infty}(Q_T)} + \int_{Q_T} (1+|\mathbf{D}(\boldsymbol{v})|^2)^{\frac{r-2}{2}}|\mathbf{D}(\boldsymbol{v}_{,t})|^2 \, dxdt \\ \le C(\|\boldsymbol{v}_0\|_{W^2_{2+\varepsilon}(\Omega)^2}, \|\theta_0\|_{W^1_2(\Omega)}) \le C_{\boldsymbol{v}_0,\theta_0}.\end{aligned} \tag{3.7}$$

Proof. **Step 1:** *Derivation of the estimates that follow from* (3.3)–(3.4) *by standard interpolation techniques.* First, for $r \ge 2$, we get by using the embedding $W_2^{\frac{2}{r+2}}(\Omega) \hookrightarrow L_{\frac{2(r+2)}{r}}(\Omega)$ and the standard interpolation inequality that

$$\begin{aligned}\|\mathbf{D}(\boldsymbol{v})\|^{r+2}_{L_{r+2}(\Omega)^{2\times 2}} &= \||\mathbf{D}(\boldsymbol{v})|^{\frac{r}{2}}\|^{\frac{2(r+2)}{r}}_{L_{\frac{2(r+2)}{r}}(\Omega)} \le C\||\mathbf{D}(\boldsymbol{v})|^{\frac{r}{2}}\|^{\frac{2(r+2)}{r}}_{W_2^{\frac{2}{r+2}}(\Omega)} \\ &\le C\||\mathbf{D}(\boldsymbol{v})|^{\frac{r}{2}}\|^2_{L_2(\Omega)}\||\mathbf{D}(\boldsymbol{v})|^{\frac{r}{2}}\|^{\frac{4}{r}}_{W^1_2(\Omega)} \\ &\le C\|\mathbf{D}(\boldsymbol{v})\|^{\frac{4}{r}}_{L_2(\Omega)^{2\times 2}}\|\mathbf{D}(\boldsymbol{v})\|^{r-\frac{4}{r}}_{L_{r+2}(\Omega)^{2\times 2}}\||\mathbf{D}(\boldsymbol{v})|^{\frac{r}{2}}\|^{\frac{4}{r}}_{W^1_2(\Omega)}\end{aligned}$$

and consequently we observe that

$$\|\mathbf{D}(\boldsymbol{v})\|^{r+2}_{L_{r+2}(\Omega)^{2\times 2}} \le C\|\mathbf{D}(\boldsymbol{v})\|^2_{L_2(\Omega)^{2\times 2}}\||\mathbf{D}(\boldsymbol{v})|^{\frac{r}{2}}\|^2_{W^1_2(\Omega)}. \tag{3.8}$$

Next, for $r \le 2$ we use the embedding $W_{\frac{4}{r}}^{\frac{r^2}{2(r+2)}}(\Omega) \hookrightarrow L^{\frac{2(r+2)}{r}}(\Omega)$, the standard interpolation inequality and the embedding $W^1_2(\Omega) \hookrightarrow W^{\frac{r}{2}}_{\frac{4}{r}}(\Omega)$ to deduce

$$\begin{aligned}\|\mathbf{D}(\boldsymbol{v})\|^{r+2}_{L_{r+2}(\Omega)^{2\times 2}} &= \||\mathbf{D}(\boldsymbol{v})|^{\frac{r}{2}}\|^{\frac{2(r+2)}{r}}_{L_{\frac{2(r+2)}{r}}(\Omega)} \le C\||\mathbf{D}(\boldsymbol{v})|^{\frac{r}{2}}\|^{\frac{2(r+2)}{r}}_{W_{\frac{4}{r}}^{\frac{r^2}{2(r+2)}}(\Omega)} \\ &\le C\||\mathbf{D}(\boldsymbol{v})|^{\frac{r}{2}}\|^{\frac{4}{r}}_{L_{\frac{4}{r}}(\Omega)}\||\mathbf{D}(\boldsymbol{v})|^{\frac{r}{2}}\|^2_{W^{\frac{r}{2}}_{\frac{4}{r}}(\Omega)},\end{aligned}$$

which again leads to (3.8).

Hence, integrating (3.8) with respect to time and using the a priori estimates (3.3) and the Korn inequality we conclude that

$$\|\nabla \boldsymbol{v}\|_{L_{r+2}(Q_T)^{2\times 2}} \le C_{v_0,\theta_0}. \tag{3.9}$$

Step 2: *We show that*

$$\|\operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{v})\|_{L_2(Q_T)^2} \le C_{v_0,\theta_0}. \tag{3.10}$$

Indeed, in the case $r > 2$ thanks to the energy estimate (3.1) we have $\boldsymbol{v} \in L_r(0,T;W^1_r(\Omega)^2) \hookrightarrow L_r(0,T;L_\infty(\Omega)^2)$ and hence after using (3.9) we obtain

$$\|\operatorname{div}(\boldsymbol{v}\otimes\boldsymbol{v})\|^2_{L_2(Q_T)^2} \le \|\nabla\boldsymbol{v}\|^2_{L_{2,\infty}(Q_T)^2}\int_0^T \|\boldsymbol{v}\|^2_{L_\infty(\Omega)^2}\,dt \le C_{\boldsymbol{v}_0,\theta_0}. \tag{3.11}$$

For $1 < r \le 2$ we use the following inequality (see also [17] for details)

$$\|\nabla^2\boldsymbol{v}\|^2_{L_2\left(0,T;L_{\frac{4}{4-r}}(\Omega)^{2\times 2}\right)} \le \left(1+\|\mathbf{D}(\boldsymbol{v})\|^{2-r}_{L_\infty(0,T;L_2(\Omega))^{2\times 2}}\right)\int_{Q_T}\frac{|\nabla^2\boldsymbol{v}|^2}{(1+|\mathbf{D}(\boldsymbol{v})|^2)^{\frac{2-r}{2}}}\,dxdt.$$

and obtain

$$\|\boldsymbol{v}\|_{L_2(0,T;W^2_{\frac{4}{4-r}}(\Omega)^2)} \le C_{\boldsymbol{v}_0,\theta_0}. \tag{3.12}$$

Consequently, using the imbedding $W^2_{\frac{4}{4-r}}(\Omega) \hookrightarrow W^1_{\frac{4}{2-r}}(\Omega) \hookrightarrow L_\infty(\Omega)$ and also (3.12) we conclude that

$$\|\boldsymbol{v}\|_{L_2(0,T;L_\infty(\Omega)^2)} \le C_{\boldsymbol{v}_0,\theta_0}, \tag{3.13}$$

and we see that integral in (3.11) is finite, and (3.10) holds.

Step 3: *Improvement of regularity for θ following from* (3.3). First, thanks to (3.9) and the growth assumption on $\mathbf{S}$ (1.13), wee see that the right-hand side of the heat equation $(1.7)_3$ is bounded in $L_{\frac{r+2}{r}}(Q_T)$. Due to the first part of the estimate (3.3) we also have that $\|\boldsymbol{v}\|_{L_\infty(0,T;L_q(\Omega)^2)} \le C_{\boldsymbol{v}_0,\theta_0}$ for any $q \in (1,\infty)$. Next, we easily deduce from (3.2) (see e.g. [5] for details) that $\|\theta\|_{L_q(0,T;W^1_q(\Omega))} \le C_{\boldsymbol{v}_0,\theta_0}$ for all $q \in (1,\frac{4}{3})$. Consequently, using the fact that $\operatorname{div}\boldsymbol{v} = 0$ we obtain that

$$\|\operatorname{div}(\boldsymbol{v}\theta)\|_{L_q(Q_T)} = \|\boldsymbol{v}\cdot\nabla\theta\|_{L_q(Q_T)} \le C_{\boldsymbol{v}_0,\theta} \qquad \text{for all } q \in (1,\frac{4}{3}).$$

Hence, moving the convective term onto the right hand side of $(1.7)_3$ and referring to L^q maximal regularity of the heat equation (see [16, Chapter IV, Theorem 9.1]), we deduce that

$$\|\theta\|_{W^{2,1}_{\tilde q}(Q_T)} \le C_{\boldsymbol{v}_0,\theta_0} \tag{3.14}$$

for all $\tilde q < \frac{4}{3}$ and $\tilde q \le \frac{r+2}{2}$. Hence, in case $\frac{r+2}{r} < \frac{4}{3}$ we directly obtain

$$\|\theta\|_{W^{2,1}_{\frac{r+2}{r}}(Q_T)} \le C_{\boldsymbol{v}_0,\theta_0}. \tag{3.15}$$

On the other hand, in case that $\frac{r+2}{r} \ge \frac{4}{3}$ we can bootstrap our estimate to still conclude (3.15). Indeed, it follows from the standard parabolic interpolation (see for example [8]) and (3.14) that

$$\||\nabla\theta|\|_{L^{\frac{4\tilde q}{4-\tilde q}}(Q_T)} \le C_{\boldsymbol{v}_0,\theta_0}.$$

Using (3.3) and the fact that $\tilde q \in [1,\frac{4}{3})$ is arbitrary, we conclude that

$$\|\operatorname{div}(\boldsymbol{v}\theta)\|_{L_q(Q_T)} = \|\boldsymbol{v}\cdot\nabla\theta\|_{L_q(Q_T)} \le C_{\boldsymbol{v}_0,\theta} \qquad \text{for all } q \in (1,2).$$

If $\frac{r+2}{r} < 2$ we move the convective term onto the right hand side and conclude (3.15). Otherwise, we repeat the bootstrap argument again. It is evident that after finite number of steps we always deduce (3.15).

Step 4: *Derivation of*

$$\|\boldsymbol{v}_{,t}\|_{L_2(Q_T)^2} + \|\theta_{,t}\|_{L_2(Q_T)} + \|\nabla \boldsymbol{v}\|_{L_{r,\infty}(Q_T)^{2\times 2}} + \|\nabla\theta\|_{L_{2,\infty}(Q_T)^2} \le C_{\boldsymbol{v}_0,\theta_0}. \tag{3.16}$$

We start with multiplying the momentum equation $(1.7)_1$ by $\boldsymbol{v}_{,t}$, integrating over Ω and using the integration by parts (the term with pressure vanishes due to the divergence free constraint). It leads to the identity

$$\|\boldsymbol{v}_{,t}\|^2_{L_2(\Omega)} + \int_\Omega \mathbf{S}(\theta, \mathbf{D}(\boldsymbol{v})) : \mathbf{D}(\boldsymbol{v}_{,t})\, dx = -\int_\Omega \boldsymbol{v}_{,t} \cdot \operatorname{div}(\boldsymbol{v}\otimes\boldsymbol{v})\, dx.$$

Applying the Young inequality to the term on the right hand side and absorbing the term with time derivative into the first term on the left-hand side, we observe that

$$\frac{1}{2}\|\boldsymbol{v}_{,t}\|^2_{L_2(\Omega)} + \int_\Omega \mathbf{S}(\theta, \mathbf{D}(\boldsymbol{v})) : \mathbf{D}(\boldsymbol{v}_{,t})\, dx \le \frac{1}{2}\|\operatorname{div}(\boldsymbol{v}\otimes\boldsymbol{v})\|^2_{L_2(\Omega)^2}. \tag{3.17}$$

Taking further (1.5) and the definition of m (see (1.16)) into account, we can rewrite the second term on the left-hand side of (3.17) as

$$\begin{aligned}\int_\Omega \mathbf{S}(\theta, \mathbf{D}(\boldsymbol{v})) : \mathbf{D}(\boldsymbol{v}_{,t})\, dx &= \frac{1}{2}\int_\Omega \nu(\theta, |\mathbf{D}(\boldsymbol{v})|^2)|\mathbf{D}(\boldsymbol{v})|^2_{,t}\, dx \\ &= \frac{1}{2}\frac{d}{dt}\int_\Omega m(\theta, \mathbf{D}(\boldsymbol{v}))\, dx - \frac{1}{2}\int_\Omega \frac{\partial m(\theta, \mathbf{D})}{\partial\theta}\theta_{,t}\, dx\end{aligned}$$

Thus, inserting this relation into (3.17) and using the Young inequality together with (1.18) we obtain

$$\begin{aligned}\|\boldsymbol{v}_{,t}\|^2_{L_2(\Omega)} + \frac{d}{dt}\int_\Omega m(\theta, \mathbf{D}(\boldsymbol{v}))\, dx &\le \|\operatorname{div}(\boldsymbol{v}\otimes\boldsymbol{v})\|^2_{L_2(\Omega)^2} + \frac{1}{2}\|\theta_{,t}\|_{L_2(\Omega)} \\ &\quad + C(\|\mathbf{D}(\boldsymbol{v})\|^r_{L_r(\Omega)^{2\times 2}} + 1).\end{aligned} \tag{3.18}$$

Next, multiplying $(1.7)_3$ by $\theta_{,t}$ and integrating the result over Ω, one can easily observe, after using (1.12) and the Young inequality, that

$$\|\theta_{,t}\|^2_{L_2(\Omega)} + \frac{d}{dt}\|\nabla\theta\|^2_{L_2(\Omega)^2} \le C(1 + \|\mathbf{D}(\boldsymbol{v})\|^{2r}_{L_{2r}(\Omega)} + \|\boldsymbol{v}\cdot\nabla\theta\|^2_2). \tag{3.19}$$

In order to be able to control the right hand side of (3.19) by means of the Gronwall lemma, we proceed as follows. First, for sufficiently small $\varepsilon > 0$, we get by the Hölder inequality and the interpolation inequality that

$$\begin{aligned}\|\boldsymbol{v}\cdot\nabla\theta\|^2_{L_2(\Omega)} &\le C\|\boldsymbol{v}\|^2_{W^1_2(\Omega)^2}\|\nabla\theta\|^2_{L_{2+\varepsilon}(\Omega)^2} \\ &\le C\|\boldsymbol{v}\|^2_{W^1_2(\Omega)^2}(1 + \|\nabla\theta\|^2_{L_2(\Omega)^2}\|\theta\|_{W^2_{\frac{r+2}{r}}(\Omega)}).\end{aligned} \tag{3.20}$$

Similarly, using the Ladyzhenskaya inequality (see [15]) and the estimate (1.17) we get

$$\begin{aligned}\|\mathbf{D}(\boldsymbol{v})\|^{2r}_{L_{2r}(\Omega)^{2\times 2}} = \||\mathbf{D}(\boldsymbol{v})|^{\frac{r}{2}}\|^4_{L_4(\Omega)} &\le C\||\mathbf{D}(\boldsymbol{v})|^{\frac{r}{2}}\|^2_{L_2(\Omega)}\||\mathbf{D}(\boldsymbol{v})|^{\frac{r}{2}}\|^2_{W^1_2(\Omega)} \\ &\le C\||\mathbf{D}(\boldsymbol{v})|^{\frac{r}{2}}\|^2_{W^1_2(\Omega)}\int_\Omega m(\theta, \mathbf{D}(\boldsymbol{v}))\, dx.\end{aligned} \tag{3.21}$$

Finally, summing (3.18) and (3.19) and using (3.1), (3.4), (3.11), (3.15), (3.20) and (3.21), we obtain

$$
\begin{aligned}
&\|\boldsymbol{v}_{,t}\|^2_{L_2(\Omega)^2} + \|\theta_{,t}\|^2_{L_2(\Omega)} + \frac{d}{dt}\left(\|\nabla\theta\|^2_{L_2(\Omega)^2} + \int_\Omega m(\theta, \mathbf{D}(\boldsymbol{v}))\,dx\right) \\
&\qquad \le Cg(t)(\|\nabla\theta\|^2_{L_2(\Omega)^2} + \int_\Omega m(\theta,\mathbf{D}(\boldsymbol{v}))\,dx + 1),
\end{aligned} \tag{3.22}
$$

where $g := C(\|\,|\mathbf{D}(\boldsymbol{v})|^{\frac{r}{2}}\,\|^2_{W^1_2(\Omega)} + \|\boldsymbol{v}\|^2_{W^1_2(\Omega)^2} + \|\theta\|_{W^2_{\frac{r+2}{r}}(\Omega)} + \|\operatorname{div}(\boldsymbol{v}\otimes\boldsymbol{v})\|_{L_2(Q_T)^2})$ fulfills

$$
\int_0^T g(t) \le C_{\boldsymbol{v}_0,\theta_0}.
$$

Consequently, using (1.17) and the Gronwall inequality, we arrive at (3.16).

Step 5: *Consequences following from* (3.16). First, referring to (3.21), we observe that (3.16) implies that

$$
\|\nabla\boldsymbol{v}\|_{L_{2r}(Q_T)^2} \le C_{v_0,\theta_0}. \tag{3.23}
$$

Next, using the properties of the heat equation solution operator, it is not difficult to see that

$$
\|\theta\|_{W^{2,1}_2(Q_T)} \le C_{v_0,\theta_0}. \tag{3.24}
$$

Step 6: *Proof of the estimate* (3.7). In fact, we use the procedure similar to the one that led to the estimate (3.3). Indeed, applying the time derivative to the the momentum equation $(1.7)_2$, multiplying the result by $\boldsymbol{v}_{,t}$ and integrating over Ω, we obtain (after integration by parts and using $\operatorname{div}\boldsymbol{v} = 0$)

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\boldsymbol{v}_{,t}\|^2_{L_2(\Omega)^2} + \int_\Omega \frac{\partial\mathbf{S}(\theta,\mathbf{D}(\boldsymbol{v}))}{\partial\mathbf{D}} : (\mathbf{D}(\boldsymbol{v}_{,t})\otimes\mathbf{D}(\boldsymbol{v}_{,t}))\,dx \\
&= -\int_\Omega \theta_{,t}\frac{\partial\mathbf{S}(\theta,\mathbf{D}(\boldsymbol{v}))}{\partial\theta} : \mathbf{D}(\boldsymbol{v}_{,t})\,dx + \int_\Omega (\boldsymbol{v}\otimes\boldsymbol{v})_{,t} : \mathbf{D}(\boldsymbol{v}_{,t})\,dx.
\end{aligned}
$$

First, using the fact that $\operatorname{div}\boldsymbol{v} = 0$ and integration by parts, we get

$$
\int_\Omega (\boldsymbol{v}\otimes\boldsymbol{v})_{,t} : \mathbf{D}(\boldsymbol{v}_{,t})\,dx = -\int_\Omega (\boldsymbol{v}_{,t}\otimes\boldsymbol{v}_{,t}) : \mathbf{D}(\boldsymbol{v})\,dx.
$$

Then, applying (1.12)–(1.13) to the corresponding terms above we get

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\boldsymbol{v}_{,t}\|^2_{L_2(\Omega)^2} + \nu_0\int_\Omega (1+|\mathbf{D}(\boldsymbol{v})|^2)^{\frac{r-2}{2}}|\mathbf{D}(\boldsymbol{v}_{,t})|^2\,dx \\
&\le C\int_\Omega |\theta_{,t}|(1+|\mathbf{D}(\boldsymbol{v})|^2)^{\frac{r-2}{4}}|\mathbf{D}(\boldsymbol{v}_{,t})|\,dx + C\int_\Omega |\boldsymbol{v}_{,t}|^2|\mathbf{D}(\boldsymbol{v})|\,dx.
\end{aligned} \tag{3.25}
$$

The first integral at the right-hand side can be handled by using the Young inequality:

$$
C\int_\Omega |\theta_{,t}|(1+|\mathbf{D}(\boldsymbol{v})|^2)^{\frac{r-2}{4}}|\mathbf{D}(\boldsymbol{v}_{,t})|\,dx \le \frac{\nu_0}{4}\int_\Omega (1+|\mathbf{D}(\boldsymbol{v})|^2)^{\frac{r-2}{2}}|\mathbf{D}(\boldsymbol{v}_{,t})|^2\,dx + C\|\theta_{,t}\|^2_2.
$$

Next, we estimate the second term on the right-hand side of (3.25). First, for $r \geq 2$ we have after using the Hölder and Ladyzhenskaya inequalities

$$\int_\Omega |\boldsymbol{v}_{,t}|^2 |\mathsf{D}(\boldsymbol{v})| \, dx \leq \|\boldsymbol{v}_{,t}\|^2_{L_4(\Omega)^2} \|\mathsf{D}(\boldsymbol{v})\|_{L_2(\Omega)^{2\times 2}}$$
$$\leq \frac{\nu_0}{4} \|\mathsf{D}(\boldsymbol{v}_{,t})\|^2_{L_2(\Omega)^{2\times 2}} + \|\boldsymbol{v}_{,t}\|^2_{L_2(\Omega)^2} \|\mathsf{D}(\boldsymbol{v})\|^2_{L_2(\Omega)^{2\times 2}}$$

For $r \in (1,2)$ we proceed as follows. By means of the Hölder inequality and (3.3) we have

$$\|\mathsf{D}(\boldsymbol{v}_{,t})\|_{L_{\frac{4}{4-r}}(\Omega)^{2\times 2}} \leq \left(1 + \|\mathsf{D}(\boldsymbol{v})\|^{2-r}_{L_2(\Omega)}\right)^{\frac{1}{2}} \left(\int_\Omega |\mathsf{D}(\boldsymbol{v}_{,t})|^2 (1 + |\mathsf{D}(\boldsymbol{v})|^2)^{\frac{r-2}{2}} \, dx\right)^{\frac{1}{2}}$$
$$\leq C_{\boldsymbol{v}_0,\theta_0} \left(\int_\Omega |\mathsf{D}(\boldsymbol{v}_{,t})|^2 (1 + |\mathsf{D}(\boldsymbol{v})|^2)^{\frac{r-2}{2}} \, dx\right)^{\frac{1}{2}}.$$

Having this estimate we then get for $r \in (1,2)$ that (after using also (3.3))

$$\int_\Omega |\boldsymbol{v}_{,t}|^2 |\mathsf{D}(\boldsymbol{v})| \, dx \leq \|\boldsymbol{v}_{,t}\|^2_{L_4(\Omega)^2} \|\mathsf{D}(\boldsymbol{v})\|_{L_2(\Omega)^{2\times 2}}$$
$$\leq C_{\boldsymbol{v}_0,\theta_0} \|\boldsymbol{v}_{,t}\|^{\frac{2(r-1)}{r}}_{L_2(\Omega)^2} \|\mathsf{D}(\boldsymbol{v}_{,t})\|^{\frac{2}{r}}_{L_{\frac{4}{4-r}}(\Omega)^{2\times 2}}$$
$$\leq \frac{\nu_0}{4} \int_\Omega |\mathsf{D}(\boldsymbol{v}_{,t})|^2 (1 + |\mathsf{D}(\boldsymbol{v})|^2)^{\frac{r-2}{2}} \, dx + C_{\boldsymbol{v}_0,\theta_0} \|\boldsymbol{v}_{,t}\|^2_{L_2(\Omega)^2}.$$

Thus, combining above estimates with (3.25) we obtain

$$\frac{d}{dt}\|\boldsymbol{v}_{,t}\|^2_{L_2(\Omega)^2} + \int_\Omega (1 + |\mathsf{D}(\boldsymbol{v})|^2)^{\frac{r-2}{2}} |\mathsf{D}(\boldsymbol{v}_{,t})|^2 \, dx \leq C_{\boldsymbol{v}_0,\theta_0} (\|\boldsymbol{v}_{,t}\|^2_{L_2(\Omega)^2} + \|\theta_{,t}\|^2_{L_2(\Omega)}). \tag{3.26}$$

Finally using (3.24) and the Gronwall inequality we deduce[3] the relation (3.7). □

Lemma 3.4 (Further estimates for $\boldsymbol{v}$)**.** *Let $(\boldsymbol{v}, p, \theta)$ be a smooth solution to (1.7)–(1.9). Then the following estimate holds*

$$\|\boldsymbol{v}\|_{L_\infty(Q_T)^2} \leq C_{v_0,\theta_0}. \tag{3.27}$$

Moreover, for $r \geq 2$ we have

$$\sup_{t\in(0,T)} \int_\Omega (1 + |\mathsf{D}(\boldsymbol{v})|^2)^{\frac{r-2}{2}} |\nabla^2 \boldsymbol{v}|^2 \, dx \leq C_{v_0,\theta_0} \tag{3.28}$$

and consequently

$$\|\nabla^2 \boldsymbol{v}\|_{L_\infty(0,T;L_2(\Omega)^{2\times 2\times 2})} \leq C_{v_0,\theta_0}. \tag{3.29}$$

For $r \in (1,2)$ there holds

$$\|\nabla \boldsymbol{v}\|_{L_\infty(0,T;L_{r+1}(\Omega)^{2\times 2})} \leq C_{v_0,\theta_0}. \tag{3.30}$$

[3]For using the Gronwall lemma we need that $\|\boldsymbol{v}_{,t}(0)\|_{L_2(\Omega)^2}$ is finite. However, this can be deduce directly from equation $(1.7)_2$ provided that $\boldsymbol{v}_0 \in W^2_{2+\varepsilon}(\Omega)^2$ and $\theta_0 \in W^1_2(\Omega)$.

In addition, we have

$$\|\boldsymbol{v}_{,t}\|_{L_{q_0}(Q_T)^2} \le C_{v_0,\theta_0}, \qquad q_0 = \begin{cases} 4 & 2 \le r, \\ r+2 & 1 < r < 2. \end{cases} \tag{3.31}$$

Proof. For $r > 2$, the relation (3.27) is a direct consequence of (3.16). For $r \in (1,2]$, we have

$$\frac{d}{dt}\int_\Omega (1+|\mathsf{D}(\boldsymbol{v})|^2)^{\frac{r+1}{2}}\,dx = (r+1)\int_\Omega (1+|\mathsf{D}(\boldsymbol{v})|^2)^{\frac{r-1}{2}}\mathsf{D}(\boldsymbol{v}) : \mathsf{D}(\boldsymbol{v}_{,t})\,dx$$

$$\le C\left(\int_\Omega (1+|\mathsf{D}(\boldsymbol{v})|^2)^{\frac{r-2}{2}}|\mathsf{D}(\boldsymbol{v}_{,t})|^2\,dx\right)^{1/2}\left(\int_\Omega (1+|\mathsf{D}(\boldsymbol{v})|^2)^{\frac{r+2}{2}}\,dx\right)^{1/2}$$

$$\le C\int_\Omega (1+|\mathsf{D}(\boldsymbol{v})|^2)^{\frac{r-2}{2}}|\mathsf{D}(\boldsymbol{v}_{,t})|^2\,dx + C\left(1+\|\mathsf{D}(\boldsymbol{v})\|^{r+2}_{L_{r+2}(\Omega)^{2\times 2}}\right).$$

Upon the integration by parts and use of (3.9) and (3.7), we obtain (3.30). Consequently, the embedding theorem $W^1_{r+1}(\Omega) \hookrightarrow L_\infty(\Omega)$ implies (3.27).

To observe (3.28), we use (3.6) but move the time derivative on the right hand side. Since

$$-\frac{d}{dt}\|\nabla\boldsymbol{v}\|^2_{L_2(\Omega)^{2\times 2}} = \int_\Omega \boldsymbol{v}_{,t}\cdot\triangle\boldsymbol{v}\,dx \le \frac{\nu_0}{2}\|\mathsf{D}(\nabla\boldsymbol{v})\|^2_{L_2(\Omega)^{2\times 2\times 2}} + C\|\boldsymbol{v}_{,t}\|^2_{L_2(\Omega)^2}.$$

we obtain (note that here we need $r \ge 2$)

$$\int_\Omega (1+\mathsf{D}(\boldsymbol{v})|^2)^{\frac{r-2}{2}}|\nabla^2\boldsymbol{v}|^2\,dx \le C(\|\nabla\theta\|^2_{L_2(\Omega)^2} + \|\boldsymbol{v}_{,t}\|^2_{L_2(\Omega)^2}).$$

Finally the estimates (3.7) and (3.16) imply (3.28).

The relation (3.31) can be deduced from (3.7) and (3.9) by using standard interpolation (see e.g. [14] for details). □

Lemma 3.5 (Further estimates for θ)**.** *Let $(\boldsymbol{v}, p, \theta)$ be a smooth solution to* (1.7)–(1.9)*. Then the following estimate holds*

$$\|\theta\|_{W^{2,1}_s(Q_T)^2} \le C_{v_0,\theta_0}, \qquad \begin{cases} \text{for all } s < +\infty, & 2 \le r \\ s = 3, & 1 < r < 2 \end{cases} \tag{3.32}$$

In particular, we obtain

$$\|\nabla\theta\|_{L_\infty(0,T;L_s(\Omega)^2)} \le C_{v_0,\theta_0}, \qquad \begin{cases} \text{for all } s < +\infty, & 2 \le r \\ s = 3, & 1 < r < 2 \end{cases} \tag{3.33}$$

Proof. In the case of $2 \le r$ the estimate (3.32) follows from (3.29) and the standard properties of the heat operator. In the case of $1 < r < 2$ we use the the embedding $W^{1,1}_2(Q_T) \hookrightarrow L_6(Q_T)$ and obtain the estimate

$$\|\nabla v\|_{L_{3r}(Q_T)} \le C_{v_0,\theta_0}.$$

Using estimates of the heat equation we get the inequality

$$\|\theta\|_{W^{2,1}_3(Q_T)} \le C_{v_0,\theta_0}.$$

Lemma 3.32 is proved. □

Let us mention that the above estimates suffice to establish the existence of strong solution. The rest of this section is devoted to a priori estimate for a classical solution. We note that this is the part where the restriction to the case $r > \frac{4}{3}$ appears.

Lemma 3.6. *Let $r > \frac{4}{3}$ and S satisfy (1.5) and (1.12)–(1.13). Then for any initial data $(\boldsymbol{v}_0, \theta_0)$ satisfying (1.19) there is a positive constant C_{v_0,θ_0} such that for any smooth solution $(\boldsymbol{v}, p, \theta)$ of the problem (1.7) the following estimate holds*

$$\|\boldsymbol{v}\|_{\mathcal{C}^{2+\alpha,1+\frac{\alpha}{2}}(\overline{Q_T})^2} + \|\theta\|_{\mathcal{C}^{2+\alpha,1+\frac{\alpha}{2}}(\overline{Q_T})} \leq C_{\boldsymbol{v}_0,\theta_0}. \tag{3.34}$$

Proof. We define

$$V := \sup_{Q_T} (1 + |\mathsf{D}(\boldsymbol{v})|^2)^{\frac{1}{2}} \tag{3.35}$$

and denote by $\boldsymbol{u} = \boldsymbol{v}_{,t}$, $\tilde{\pi} = p_{,t}$. Next, we differentiate $(1.7)_2$ w.r.t. time and obtain

$$\begin{aligned} \boldsymbol{u}_{,t} - \operatorname{div}(\mathbb{A}\mathsf{D}(\boldsymbol{u})) + \nabla\tilde{\pi} &= \operatorname{div}\mathsf{F}, \\ \operatorname{div}\boldsymbol{u} &= 0, \end{aligned} \tag{3.36}$$

where we set

$$\mathbb{A} = \frac{\partial \mathsf{S}(\theta, \mathsf{D}(\boldsymbol{v}))}{\partial \mathsf{D}},$$
$$\mathsf{F} = \frac{\partial \mathsf{S}(\theta, \mathsf{D}(\boldsymbol{v}))}{\partial \theta}\theta_{,t} - (\boldsymbol{v} \otimes \boldsymbol{v})_{,t}.$$

Using (1.12) we get that

$$|\mathsf{F}| \leq C\left((1 + |\mathsf{D}(\boldsymbol{v})|^2)^{\frac{r-2}{4}}|\theta_{,t}| + |\boldsymbol{v}||\boldsymbol{v}_{,t}|\right)$$

and owing to (3.27), (3.31) and (3.32), we obtain

$$\|\mathsf{F}\|_{L_{q_0}(Q_T)^{2\times 2}} \leq C_{\boldsymbol{v}_0,\theta_0} \tag{3.37}$$

for some $q_0 > 2$.

Now we split the proof in two parts: the case $2 \leq r$ and the case $\frac{4}{3} < r < 2$.

(i) The case $2 \leq r$.

In this case, it follows from (1.13) and (3.35) that the tensor $\mathbb{A}$ satisfies

$$\nu_0|\mathsf{D}|^2 \leq \mathbb{A} : (\mathsf{D} \otimes \mathsf{D}) \leq \nu_1 V^{r-2}|\mathsf{D}|^2, \qquad \text{for all } \mathsf{D} \in \mathbb{R}^{2\times 2}_{sym}.$$

Hence by Proposition 2.1 we get for any $q \in (2, q_0)$ satisfying

$$1 - \frac{2}{q} \leq c_0 \frac{\nu_0}{\nu_1}\ V^{2-r} \tag{3.38}$$

that

$$\begin{aligned} \|\nabla\boldsymbol{u}\|_{L_q(Q_T)^{2\times 2}} &\leq C_{\boldsymbol{v}_0,\theta_0}, \\ \|\boldsymbol{u}\|_{L_\infty(0,T;L_q(\Omega)^2)} &\leq C_{\boldsymbol{v}_0,\theta_0}\left(1 + V^{\frac{r-2}{q}}\right). \end{aligned} \tag{3.39}$$

Recalling (3.29) we have

$$\|\boldsymbol{u}\|_{L_\infty(0,T;L_2(\Omega)^2)} \leq C_{\boldsymbol{v}_0,\theta_0}.$$

Interpolating this estimate with the second inequality in (3.39) we obtain for any $a \in [0, 1]$

$$\|\boldsymbol{u}\|_{L_\infty(0,T;L_{q_1}(\Omega)^2)} \leq C_{\boldsymbol{v}_0,\theta_0}\|\boldsymbol{u}\|^{1-a}_{L_\infty(0,T;L_2(\Omega)^2)}\|\boldsymbol{u}\|^a_{L_\infty(0,T;L_q(\Omega)^2)},$$

where q_1 is given by

$$\frac{1}{q_1} = \frac{1-a}{2} + \frac{a}{q}. \tag{3.40}$$

Consequently, we have

$$\|\boldsymbol{u}\|_{L_\infty(0,T;L_{q_1}(\Omega)^2)} \le C_{\boldsymbol{v}_0,\theta_0}\left(1 + V^{a\frac{r-2}{q}}\right) \tag{3.41}$$

and we conclude that

$$\|\boldsymbol{v}_{,t}\|_{L_\infty(0,T;L_{q_1}(\Omega)^2)} \le C_{\boldsymbol{v}_0,\theta_0}\left(1 + V^{a\frac{r-2}{q}}\right). \tag{3.42}$$

Next, we focus on the spatial derivatives, and we set $\boldsymbol{w} := \boldsymbol{v}_{,k} = \frac{\partial \boldsymbol{v}}{\partial x_k}$, $\pi := p_{,k} = \frac{\partial p}{\partial x_k}$ for this purpose. Thus, applying $\frac{\partial}{\partial x_k}$ to $(1.7)_2$ we obtain the identities

$$\begin{aligned} -\operatorname{div}(\mathbb{A}\mathbf{D}(\boldsymbol{w})) + \nabla\pi &= \operatorname{div}\mathbf{G} - \boldsymbol{w}_{,t}, \\ \operatorname{div}\boldsymbol{w} &= 0, \end{aligned} \tag{3.43}$$

where

$$\mathbf{G} := \frac{\mathbf{S}(\theta, \mathbf{D}(\boldsymbol{v}))}{\partial\theta}\theta_{,k} - (\boldsymbol{v}\otimes\boldsymbol{v})_{,k}. \tag{3.44}$$

Since

$$|\mathbf{G}| \le C\left((1+|\mathbf{D}(\boldsymbol{v})|^2)^{\frac{r-2}{4}}|\nabla\theta| + |\boldsymbol{v}|\,|\nabla\boldsymbol{v}|\right)$$

we get by using (3.29), (3.27) and (3.33) that for some $q_0 > 2$ we have the estimate

$$\|\mathbf{G}\|_{L_\infty(0,T;L_{q_0}(\Omega)^{2\times 2})} \le C_{\boldsymbol{v}_0,\theta_0}. \tag{3.45}$$

Thus, applying Proposition 2.2 for $q_1 > 2$ that certainly satisfies (3.38) as $q_1 < q$ we obtain the estimate

$$\|\nabla\boldsymbol{w}\|_{L_\infty(0,T;L_{q_1}(\Omega)^{2\times 2})} \le C_{\boldsymbol{v}_0,\theta_0}(1 + \|\boldsymbol{v}_{,t}\|_{L_\infty(0,T;L_{q_1}(\Omega)^2)}).$$

Using (3.42) and the definition of $\boldsymbol{w}$ we finally end up with

$$\|\boldsymbol{v}\|_{L_\infty(0,T;W^2_{q_1}(\Omega)^2)} \le C_{\boldsymbol{v}_0,\theta_0}\left(1 + V^{a\frac{r-2}{q}}\right). \tag{3.46}$$

Then for any $b \in [0,1]$ we define q_2 as

$$\frac{1}{q_2} := \frac{1-b}{2} + \frac{b}{q_1}$$

and by using Proposition 2.3 and standard interpolation we obtain

$$\begin{aligned} V^{\frac{r}{2}} &\le \frac{C}{(1-\frac{2}{q_2})^{1-\frac{1}{q_2}}}\||\mathbf{D}(\boldsymbol{v})|^{\frac{r}{2}}\|_{L_\infty(0,T;W^1_{q_2}(\Omega)^2)} \\ &\le \frac{C}{(1-\frac{2}{q_2})^{1-\frac{1}{q_2}}}\||\mathbf{D}(\boldsymbol{v})|^{\frac{r}{2}}\|^{1-b}_{L_\infty(0,T;W^1_2(\Omega)^2)}\||\mathbf{D}(\boldsymbol{v})|^{\frac{r}{2}}\|^{b}_{L_\infty(0,T;W^1_{q_1}(\Omega)^2)} \end{aligned} \tag{3.47}$$

Thus, using (3.28), the definition of V and q_2, and (3.46) we observe

$$V^{\frac{r}{2}} \le C_{\boldsymbol{v}_0,\theta_0}\frac{C}{\sqrt{b}(1-\frac{2}{q_1})^{\frac{1}{2}}}(1 + V^{ab\frac{r-2}{q} + b\frac{r-2}{2}}) \tag{3.48}$$

Next, let q_1 be such that (3.38) holds with equality sign, i.e. $1 - \frac{2}{q_1} = c_0\frac{\nu_0}{\nu_1}V^{2-r}$. Then using (3.40) we conclude from (3.48) that

$$V^{\frac{r}{2}} \le C_{\boldsymbol{v}_0,\theta_0}C(a,b)V^{\frac{r-2}{2}}(1 + V^{ab\frac{r-2}{q} + b\frac{r-2}{2}}) \tag{3.49}$$

Finally, we chose a, b such

$$ab\frac{r-2}{q} + b\frac{r-2}{2} < \frac{1}{2},$$

and deduce from (3.49) that

$$V \le C_{\boldsymbol{v}_0,\theta_0}.$$

From (3.46) we obtain $\nabla \boldsymbol{v} \in L_\infty(0,T;\mathcal{C}^\alpha(\Omega)^{2\times 2})$ and consequently using (3.39) and applying Proposition 2.4 we obtain

$$\|\nabla \boldsymbol{v}\|_{\mathcal{C}^{\alpha,\frac{\alpha}{2}}(\overline{Q_T})^{2\times 2}} \le C_{\boldsymbol{v}_0,\theta_0}$$

which implies the first part of (3.34). The second part then follows from Schauder-type estimates in Hölder spaces for the heat equation and for the generalized Stokes problem, see [22]. The case $2 \le r$ is complete.

(ii) The case $\frac{4}{3} < r < 2$.

Although $r \in (\frac{4}{3}, 2)$, we still use the same notation introduced in the previous part ($r \ge 2$). Hence, starting with (3.36) we have for the tensor $\mathbb{A}$ that

$$\nu_0 V^{r-2}|\mathbf{D}|^2 \le \mathbb{A} : (\mathbf{D}\otimes\mathbf{D}) \le \nu_1|\mathbf{D}|^2, \qquad \text{for all } \mathbf{D} \in \mathbb{R}^{2\times 2}_{sym}. \tag{3.50}$$

Then by Proposition 2.1 we see that for any $q > 2$ such that

$$1 - \frac{2}{q} \le c_0\frac{\nu_0}{\nu_1}\, V^{r-2} \tag{3.51}$$

the following estimates holds:

$$\|\nabla \boldsymbol{u}\|_{L_q(Q_T)^{2\times 2}} \le CV^{2-r},\ \|\boldsymbol{u}\|_{L_\infty(0,T;L_q(\Omega)^2)} \le C\left(1 + V^{2-r}\right). \tag{3.52}$$

Interpolating the second inequality in (3.52) with the estimate (3.16), i.e, with the estimate

$$\|\boldsymbol{u}\|_{L_\infty(0,T;L_2(\Omega)^2)} \le C_{\boldsymbol{v}_0,\theta_0},$$

for small $a \in [0,1]$ and $q_1 > 2$ such that $\frac{1}{q_1} = \frac{1-a}{2} + \frac{a}{q}$ we obtain the inequality

$$\|\boldsymbol{u}\|_{L_\infty(0,T;L_{q_1}(\Omega)^2)} \le C_{\boldsymbol{v}_0,\theta_0}\left(1 + V^{a(2-r)}\right). \tag{3.53}$$

So, we conclude that

$$\|\boldsymbol{v}_{,t}\|_{L_\infty(0,T;L_{q_1}(\Omega)^2)} \le C\left(1 + V^{a(2-r)}\right). \tag{3.54}$$

Next, we focus on the problem (3.43) and for $\mathbf{G}$ given in (3.44) we conclude from (3.33), (3.27) and (3.30) that for some $q_0 > 2$ (3.45) holds. By Proposition 2.2 we thus derive the estimate

$$\|\nabla \boldsymbol{w}\|_{L_\infty(0,T;L_{q_1}(\Omega)^{2\times 2})} \le CV^{2-r}(1 + \|\boldsymbol{v}_{,t}\|_{L_\infty(0,T;L_{q_1}(\Omega)^2)})$$

and using (3.54) we get

$$\|\boldsymbol{v}\|_{L_\infty(0,T;W^2_{q_1}(\Omega)^2)} \le C_{\boldsymbol{v}_0,\theta_0}V^{2-r}\left(1 + V^{a(2-r)}\right). \tag{3.55}$$

Similarly as in preceding part, we apply Proposition 2.3 and setting q_1 such that (3.38) holds with equality sign, we obtain

$$\begin{aligned} V &\le C(a)C_{\boldsymbol{v}_0,\theta_0}V^{(2-r)(1-\frac{1}{q_1})}\|\boldsymbol{v}\|_{L_\infty(0,T;W^2_{q_1}(\Omega)^2)} \\ &\le C(a)C_{\boldsymbol{v}_0,\theta_0}V^{(2-r)(2-\frac{1}{q_1})}\left(1 + V^{a(2-r)}\right). \end{aligned} \tag{3.56}$$

Finally, it is evident that for any $\frac{4}{3} < r < 2$ we can find $a \in (0,1)$ and $q_1 > 2$ such that

$$(2-r)\left(2 - \frac{1}{q_1} + a\right) < 1$$

and we finally obtain the estimate

$$V \le C_{\boldsymbol{v}_0,\theta_0}.$$

Then the arguments similar to the case $r \ge 2$ leads to the estimate (3.34) for the case $\frac{4}{3} < r < 2$. Thus, Lemma 3.6 is proved. ☐

4. Proof of Theorem 1.1

To prove Theorem 1.1 we use known results on the solvability of Ladyzhenskaya's system (2.7) (see Proposition 2.5) and a version of the Leray-Schauder fixed point theorem (see Proposition 2.6). Thus, for any $\tau \in [0,1]$ we define a tensorial function $\mathbf{S}^\tau(\theta, \mathbf{D})$ by the formula

$$\mathbf{S}^\tau(\theta, \mathbf{D}) := \mathbf{S}(\tau\theta, \mathbf{D}).$$

Note that $\mathbf{S}^\tau$ also satisfies (1.12)–(1.13) with constants independent of τ. We consider the system

$$\begin{aligned} \boldsymbol{v}_{,t} - \mathbb{A}^\tau(\theta, \mathbf{D}(\boldsymbol{v}))\nabla^2 \boldsymbol{v} + \nabla p &= \boldsymbol{f}^\tau(\theta, \boldsymbol{v}, \mathbf{D}(\boldsymbol{v})), \\ \operatorname{div} \boldsymbol{v} &= 0 \\ \theta_{,t} - \Delta\theta &= g^\tau(\theta, \boldsymbol{v}, \mathbf{D}(\boldsymbol{v})), \end{aligned} \tag{4.1}$$

where

$$\begin{aligned} \mathbb{A}^\tau(\theta, \mathbf{D}) &:= \frac{\partial \mathbf{S}^\tau(\theta, \mathbf{D})}{\partial \mathbf{D}}, \\ \boldsymbol{f}^\tau(\theta, \boldsymbol{u}, \mathbf{D}) &:= \frac{\partial \mathbf{S}^\tau(\theta, \mathbf{D})}{\partial \theta}\nabla\theta - \operatorname{div}(\boldsymbol{u} \otimes \boldsymbol{u}), \\ g^\tau(\theta, \boldsymbol{u}, \mathbf{D}) &:= \mathbf{S}^\tau(\theta, \mathbf{D}) : \mathbf{D} - \operatorname{div}(\boldsymbol{u}\theta). \end{aligned}$$

Note that for smooth $\boldsymbol{v}$ and θ the system (4.1) is equivalent to the system (1.7) where we replace $\mathbf{S}$ by $\mathbf{S}^\tau$.

Next, for any $\tau \in [0,1]$ and any $\boldsymbol{u} \in \mathcal{C}^{1+\alpha, \frac{1+\alpha}{2}}(\overline{Q_T})^2$, $\psi \in \mathcal{C}^{1+\alpha, \frac{1+\alpha}{2}}(\overline{Q_T})$ we define the functions $\boldsymbol{w}^\tau \in \mathcal{C}^{2+\alpha, 1+\frac{\alpha}{2}}(\overline{Q_T})^2$, $\nabla q^\tau \in \mathcal{C}^{\alpha, \frac{\alpha}{2}}(\overline{Q_T})$ and $\theta^\tau \in C^{2+\alpha, 1+\frac{\alpha}{2}}(\overline{Q_T})$ as a solution to the following linear system

$$\begin{aligned} \boldsymbol{w}_{,t} - \mathbb{A}^\tau(\psi, \mathbf{D}(\boldsymbol{u}))\nabla^2 \boldsymbol{w}^\tau + \nabla q^\tau &= \boldsymbol{f}^\tau(\psi, \boldsymbol{u}, \mathbf{D}(\boldsymbol{u})), \\ \operatorname{div} \boldsymbol{w} &= 0, \\ \theta^\tau_{,t} - \Delta\theta^\tau &= g^\tau(\psi, \boldsymbol{u}, \mathbf{D}(\boldsymbol{u})), \\ \boldsymbol{w}^\tau(0) = \boldsymbol{v}_0, &\qquad \theta^\tau(0) = \theta_0, \\ \boldsymbol{w}^\tau \text{ and } \theta^\tau &\text{ are spatially periodic.} \end{aligned} \tag{4.2}$$

The existence of the solution to (4.2) as well as the following estimate

$$\begin{aligned} &\|\boldsymbol{w}^\tau\|_{\mathcal{C}^{2+\alpha,1+\frac{\alpha}{2}}(\overline{Q_T})^2} + \|\theta^\tau\|_{\mathcal{C}^{2+\alpha,1+\frac{\alpha}{2}}(\overline{Q_T})} + \|\nabla q^\tau\|_{\mathcal{C}^{\alpha,\frac{\alpha}{2}}(\overline{Q_T})^2} \\ &\qquad \le \mathcal{F}\left(\|\psi\|_{\mathcal{C}^{1+\alpha,\frac{1+\alpha}{2}}(\overline{Q_T})}, \|\boldsymbol{u}\|_{\mathcal{C}^{1+\alpha,\frac{1+\alpha}{2}}(\overline{Q_T})^2}\right) \end{aligned} \tag{4.3}$$

that holds for some nondecreasing continuous function $\mathcal{F}$, follow from the results in [22].

Next, we denote $X := \mathcal{C}^{1+\alpha,\frac{1+\alpha}{2}}(\overline{Q_T})^2 \times \mathcal{C}^{1+\alpha,\frac{1+\alpha}{2}}(\overline{Q_T})$ and define the family of nonlinear operators depending on the parameter $\tau \in [0,1]$ as

$$\Phi_\tau : X \to X, \qquad \Phi_\tau(\boldsymbol{u},\psi) := (\boldsymbol{w}^\tau, \theta^\tau),$$

where $(\boldsymbol{w}^\tau, \theta^\tau)$ are solutions of (4.2) with data $(\boldsymbol{u},\psi)$. We also denote $\mathcal{B}_R$ the ball of radius R in the space X centered at origin. Let us verify that the operators Φ_τ satisfy all conditions of Proposition 2.6. Indeed, for any given $\tau \in [0,1]$ the continuity of the operator $\Phi_\tau : X \to X$ follows directly from Proposition 2.7. To prove convergence

$$\sup_{(\boldsymbol{u},\psi)\in\bar{\mathcal{B}}_R} \|\Phi_{\tau_m}(\boldsymbol{u},\psi) - \Phi_{\tau_0}(\boldsymbol{u},\psi)\|_X \to 0, \qquad \text{as} \quad \tau_m \to \tau_0, \tag{4.4}$$

we derive from (4.2) the identities

$$\begin{aligned}
&(\boldsymbol{w}^{\tau_m} - \boldsymbol{w}^{\tau_0})_{,t} - \mathbb{A}^{\tau_0}(\psi, \mathbf{D}(\boldsymbol{u}))\nabla^2(\boldsymbol{w}^{\tau_m} - \boldsymbol{w}^{\tau_0}) + \nabla(q^{\tau_m} - q^{\tau_0})\\
&\qquad = \boldsymbol{f}^{\tau_m}(\psi,\boldsymbol{u},\mathbf{D}(\boldsymbol{u})) - \boldsymbol{f}^{\tau_0}(\psi,\boldsymbol{u},\mathbf{D}(\boldsymbol{u}))\\
&\qquad + [\mathbb{A}^{\tau_m}(\psi,\mathbf{D}(\boldsymbol{u})) - \mathbb{A}^{\tau_0}(\psi,\mathbf{D}(\boldsymbol{u}))]\nabla^2\boldsymbol{w}^{\tau_m},\\
&\operatorname{div}(\boldsymbol{w}^{\tau_m} - \boldsymbol{w}^{\tau_0}) = 0\\
&(\theta^{\tau_m} - \theta^{\tau_0})_{,t} - \Delta(\theta^{\tau_m} - \theta^{\tau_0}) = g^{\tau_m}(\psi,\boldsymbol{u},\mathbf{D}(\boldsymbol{u})) - g^{\tau_0}(\psi,\boldsymbol{u},\mathbf{D}(\boldsymbol{u})).
\end{aligned}$$

Applying Solonnikov's coercive estimate to the above system (see Proposition 2.7), we obtain convergence $\boldsymbol{w}^{\tau_m} \to \boldsymbol{w}^{\tau_0}$ in $W^{2,1}_s(Q_T)$ with any $s > 1$, from which the convergence (4.4) follows. Furthermore, the set $\Phi(\bar{\mathcal{B}}_R)$ is precompact in X due to (4.3). Moreover, the condition (4) of Proposition 2.6 is satisfied if we set $R > C_{\boldsymbol{v}_0,\theta_0}$, where the constant $C_{\boldsymbol{v}_0,\theta_0}$ is fixed in Lemma 3.6. Finally, the condition (5) of Proposition 2.6 holds thanks to Proposition 2.5. Hence, by Proposition 2.6 there exists at least one classical solution to the problem (1.7)–(1.9). So, the existence result in Theorem 1.1 is proved.

The uniqueness of the classical solution in the class (1.20) follows from the general theory of nonlinear evolutionary equations, see e.g. [16]. The proof of Theorem 1.1 is complete.

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