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MINISTRY OF EDUCATION, YOUTH AND SPORTS

Mathematical Analysis of Models Describing the Motion of Implicitly Constituted Materials

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Governing equations for fluids and solids

- Incompressible fluids: $\mathbf{v}$-velocity, ϱ-density, $\mathbf{T}$-Cauchy stress, $\mathbf{f}$-body forces

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\begin{aligned}
(\varrho \mathbf{v})_{,t} + \operatorname{div}(\varrho \mathbf{v} \otimes \mathbf{v}) &= \operatorname{div} \mathbf{T} + \varrho \mathbf{f}, \\
\varrho_{,t} + \operatorname{div}(\varrho \mathbf{v}) &= 0, \\
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- Initial and boundary conditions
- Constitutive equation for the Cauchy stress

Implicit character of the constitutive equations

To find a suitable description of the Cauchy stress by the use of the other relevant quantities, i.e., the symmetric part of the velocity gradient $\mathbf{D}(\mathbf{v})$ and the strain tensor $\varepsilon(\mathbf{u})$, where,

$$\mathbf{D}(\mathbf{v}) := \frac{1}{2}(\nabla \mathbf{v} + (\nabla \mathbf{v})^T), \qquad \varepsilon(\mathbf{u}) := \frac{1}{2}(\nabla \mathbf{u} + (\nabla \mathbf{u})^T),$$

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the density ϱ, the pressure p, the temperature θ, etc.

- The typical “old fashion" strategy:

$$\mathbf{T} := \mathbf{T}^*(\mathbf{D}(\mathbf{v}), \varepsilon(\mathbf{u}), \theta, \varrho, ...)$$

unable to justify the dependence of the viscosity on the pressure (incompressible fluids), unable to describe in a reasonable way “discontinuities" (Binghan-like models), unable to justify non-linear models in the linearized elasticity, lost of a huge class of physically reasonable models

Implicit character of the constitutive equations

☞ "New fashion" - still not optimal strategy:

$$\mathbf{D}(\mathbf{v}) := \mathbf{D}^*(\mathbf{T}, \ldots),$$
$$\varepsilon(\mathbf{u}) := \varepsilon^*(\mathbf{T}, \ldots).$$

new class of possible relations, Bingham-like fluids (or whatever with an activation criteria) expressed by a continuous mapping, non-linear linearized elasticity can be easily justified, still lost of a class of models

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✌ Optimal strategy: fully implicit models:

$$\boxed{\mathbf{G}(\mathbf{T}, \mathbf{D}(\mathbf{v}), \varepsilon(\mathbf{u}), \varrho, \theta, ...) = \mathbf{0}.}$$

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$$\mathbf{v} = \mathbf{0} \qquad \text{on } \partial\Omega$$

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pressure exists, non-validity for most materials

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- Threshold slip (κ - the threshold)

$$\left.\begin{aligned} \mathbf{v}\cdot\mathbf{n} &= 0, \\ \frac{(|(\mathbf{Tn})_\tau| - \kappa)_+}{|(\mathbf{Tn})_\tau|}(\mathbf{Tn})_\tau &= -f(\mathbf{v})\mathbf{v} \end{aligned}\right\} \qquad \text{on } \partial\Omega$$

Bingham-like fluids

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- “old fashioned": $\mathbf{T} = \mathbf{S} + p\mathbf{I}$, where

$$\mathbf{S} := 2\mu_0 \frac{\mathbf{D}}{|\mathbf{D}|} + 2\mu_1(|\mathbf{D}|)\mathbf{D}, \qquad \text{if } \mathbf{D} \neq \mathbf{0},$$

$$|\mathbf{S}| \leq 2\mu_0 \qquad \text{if } \mathbf{D} = \mathbf{0}.$$

where $2\mu_1 = \mathit{const}$ - Bingham fluid or $2\mu_1(|\mathbf{D}|) \sim 2\mu_1|\mathbf{D}|^{r-2}$, Herschley-Bulkey fluid

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$$\boxed{2\mu_1(|\mathbf{D}|)\mathbf{D} - \frac{(|\mathbf{T}^d| - 2\mu_0)_+ \mathbf{T}^d}{|\mathbf{T}^d|} = \mathbf{0},}$$

where $\mathbf{T}^d$ is the deviatoric part of $\mathbf{T}$, i.e., $\mathbf{T}^d := \mathbf{T} - \frac{\operatorname{tr}\mathbf{T}}{d}\mathbf{I}$.

Identify $\mathbf{G}(\mathbf{T}^d, \mathbf{D}) = \mathbf{G}(\mathbf{S}, \mathbf{D}) = \mathbf{0}$ with the maximal monotone r-graph $\mathcal{A} \subset \mathbb{R}^{d\times d}_{sym} \times \mathbb{R}^{d\times d}_{sym}$, i.e.

A1 Monotone graph: for all $(\mathbf{S}_{1,2}, \mathbf{D}_{1,2}) \in \mathcal{A}$ we have

$$(\mathbf{S}_1 - \mathbf{S}_2) \cdot (\mathbf{D}_1 - \mathbf{D}_2) \geq 0$$

A2 Maximality (replaces the continuity of $\mathbf{T}^*$): If $(\mathbf{S}, \mathbf{D}) \in \mathbb{R}^{d\times d}_{sym} \times \mathbb{R}^{d\times d}_{sym}$ fulfills

$$(\mathbf{S} - \overline{\mathbf{S}}) \cdot (\mathbf{D} - \overline{\mathbf{D}}) \geq 0 \qquad \text{for all } (\overline{\mathbf{S}}, \overline{\mathbf{D}}) \in \mathcal{A}$$

then $(\mathbf{S}, \mathbf{D}) \in \mathcal{A}$

A3 the r-coercivity: there exists $C_1, C_2 > 0$ such that for all $(\mathbf{S}, \mathbf{D}) \in \mathcal{A}$ we have

$$\mathbf{S} \cdot \mathbf{D} \geq C_1(|\mathbf{D}|^r + |\mathbf{S}|^{r'}) - C_2.$$

Lemma (The key convergence lemma)

Let O be a measurable set and let $\mathcal{A}$ be a maximal monotone r-graph. Assume that $(\mathbf{S}^n, \mathbf{D}^n) \in \mathcal{A}$ a.e. in O and

$$\mathbf{S}^n \rightharpoonup \mathbf{S} \qquad \text{weakly in } L^{r'},$$
$$\mathbf{D}^n \rightharpoonup \mathbf{D} \qquad \text{weakly in } L^{r},$$
$$\limsup_{n\to\infty} \int_O \mathbf{S}^n \cdot \mathbf{D}^n \le \int_O \mathbf{S} \cdot \mathbf{D}.$$

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Then $(\mathbf{S}, \mathbf{D}) \in \mathcal{A}$ a.e. in O.

- If the energy equality holds, i.e., if we can test by the solution, i.e., if the a priori estimates are good enough, the convergence lemma can be easily applied for $O := Q = (0, T) \times \Omega$ to get the existence of a weak solution
- In case of “bad" a priori estimates, we can (but **not** easily) still prove for certain r's that the assumptions of the convergence lemma hold for O fulfilling $|Q \setminus O| \le \varepsilon$ with $\varepsilon > 0$ arbitrary and therefore we can still identify the limit

Results for Bingham-like fluids

Theorem (Constant density)

Let $\mathcal{A}$ be a maximal monotone r-graph. Then for all reasonable data (initial, boundary) and reasonable body forces there exists a weak solution provided that

$$r > \frac{2d}{d+2} \qquad (\Longrightarrow \textit{compactness of the convective term})$$

$r \geq \frac{3d+2}{d+2}$ Minty, Lions, Ladyzhenskaya, Gwiazda, Świerczewska-Gwiazda, MORE

$r > \frac{2d}{d+2}$ Nečas, Frehse, Steinhauer, Wolf, Diening, Ružička, Shelukin, Gwiazda, Świerczewska-Gwiazda, MORE

Theorem (Inhomogeneous)

Let $\mathcal{A}$ be a maximal monotone r-graph. Then for all reasonable data (initial, boundary) and reasonable body forces there exists a weak solution provided that

$$r > \frac{2d}{dr+2r-2d} \qquad (\Longrightarrow \textit{we can test by } u \in L^{\infty}(0,T;W^{1,r}(\Omega)))$$

- Frehse, Ružička, Zhikov, MORE

Pressure dependent viscosities

Many experiments shows that even in incompressible fluids the viscosity should depend on the pressure

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- ✖ “old fashion" approach

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- ✌ implicit approach

$$\mathbf{T}^d - 2\mu\left(\frac{\operatorname{tr}\mathbf{T}}{d}, |\mathbf{D}|\right)\mathbf{D} = \mathbf{0},$$

and then we denote

$$dp := \operatorname{tr}\mathbf{T}.$$

“

Pressure dependent viscosities - results

The prototype model

$$\mathbf{T}^d = (1 + \gamma(\operatorname{tr}\mathbf{T}) + |\mathbf{D}|^2)^{\frac{r-2}{2}}\mathbf{D}.$$

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Theorem (Only homogeneous case)

Let $r \in (\frac{2d}{d+2}, 2]$ and γ be nonnegative function fulfilling

$$(2-r)\gamma'(s) \ll 1 \qquad \textit{for all } s \in \mathbb{R}.$$

Then for all reasonable initial data and for all boundary data (Navier's slip or threshold slip) there exists a weak solution.

- Nečas, MORE and more

Kelvin-Voigt-like model for viscoelasticity

The key splitting condition

$$\mathbf{T} = \mathbf{T}_{el} + \mathbf{T}_{dis},$$

where

$$\mathbf{G}_1(\mathbf{T}_{el}, \varepsilon) = \mathbf{0}, \qquad \mathbf{G}_2(\mathbf{T}_{dis}, \mathbf{D}) = \mathbf{0}$$

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3. Linear elastic part

$$\mathbf{G}_2(\mathbf{T}_{dis}, \mathbf{D}) = \mathbf{0}, \qquad \mathbf{T}_{el} \sim \varepsilon.$$

4. Fully nonlinear

$$\mathbf{G}_2(\mathbf{T}_{dis}, \mathbf{D}) = \mathbf{0}, \qquad \mathbf{G}_1(\mathbf{T}_{el}, \varepsilon) = \mathbf{0}.$$

In all cases we identify $\mathbf{G}_1$ with the graph $\mathcal{A}_{el}$ and $\mathbf{G}_2$ with the graph $\mathcal{A}_{dis}$.

Kelvin-Voigt-like model - easy results

Theorem (Fully linear)

Let

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Then for all reasonable initial and boundary data there exists unique weak solution. Moreover, the solution is as smooth as data allows.

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Theorem (Linear viscous part)

Let

$$\mathbf{T}_{dis} \sim \mathbf{D}$$

and $\mathcal{A}_{el}$ be a maximal monotone q-graph. Then for all reasonable data there exists a weak solution provided that either $q \le 2$ or $\mathcal{A}_{el}$ has the potential structure, i.e., for all $(\mathbf{T}_{el}, \varepsilon) \in \mathcal{A}_{el}$

$$\mathbf{T}_{el} \cdot \varepsilon_{,t} = \frac{d}{dt}\varphi(\mathbf{T}_{el}, \varepsilon)$$

with nonnegative φ fulfilling $\varphi(\mathbf{T}_{el}, \varepsilon) \ge C_1(|\mathbf{T}_{el}|^{q'} + |\varepsilon|^q)$.

Theorem (Linear elastic part)

Let

$$\mathbf{T}_{el} \sim \varepsilon$$

with the potential structure and $\mathcal{A}_{dis}$ be a maximal monotone r-graph. Then for all reasonable initial and boundary data there exists unique weak solution.

- for $r \geq 2$ the proof is relatively easy - we can test by $\mathbf{v}$ - energy equality
- for $r < 2$ the proof is more delicate but we can still "test" by $\mathbf{v}$ - energy equality

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Theorem (Fully nonlinear)

Let $\mathcal{A}_{dis}$ be maximal monotone r-graph with $r \geq 2$ fulfilling for all $(\mathbf{T}_{1,2}, \mathbf{D}_{1,2})$

$$(\mathbf{T}_1 - \mathbf{T}_2) \cdot (\mathbf{D}_1 - \mathbf{D}_2) \geq C_1 |\mathbf{D}_1 - \mathbf{D}_2|^2$$

and $\mathcal{A}_{el}$ be Lipschitz continuous graph, i.e., for all $(\mathbf{T}_{1,2}, \varepsilon_{1,2}) \in \mathcal{A}_{el}$

$$|\mathbf{T}_1 - \mathbf{T}_2| \leq C_2 |\varepsilon_1 - \varepsilon_2|.$$

Then for all reasonable data there exists unique weak solution. Moreover, for smoother data we have $\nabla^2 \mathbf{v} \in L^2$ and further in 2D setting the solution is classical provided that data are smooth enough.

Limiting strain model

Consider only elastic deformation and "forget" evolution, i.e.,

$$\begin{aligned} -\operatorname{div}\mathbf{T} &= \mathbf{f} && \text{in } \Omega, \\ \mathbf{G}(\mathbf{T},\varepsilon) &= \mathbf{0} && \text{in } \Omega, \\ \mathbf{u} &= \mathbf{0} && \text{on } \Gamma_1, \\ \mathbf{T}\mathbf{n} &= \mathbf{g} && \text{on } \Gamma_2. \end{aligned} \tag{El}$$

☞ The key assumption in linearized elasticity

$$\boxed{|\varepsilon| \ll 1}. \tag{A}$$

✖ Consider Ω a domain with non-convex corner at x_0, $\Gamma = \partial\Omega$, $\mathbf{u}_0 = \mathbf{0}$ and $\mathbf{G}$ of the form

$$\mathbf{T} = \varepsilon.$$

Then there exists a smooth $\mathbf{f}$ such that the solution $(\mathbf{T}, \varepsilon)$ fulfils

$$|\mathbf{T}(x)| = |\varepsilon(x)| \overset{x \to x_0}{\to} \infty.$$

$\Longrightarrow$ contradicts the assumption of the model (A) $\Longrightarrow$ not valid model at least in the neighborhood of x_0.

Limiting strain model

- Consider implicit models which a priori guarantees $|\varepsilon| \leq K$:

$$\boxed{\varepsilon = \lambda_1(|\operatorname{tr}\mathbf{T}|)(\operatorname{tr}\mathbf{T})\mathbf{I} + \lambda_2(|\mathbf{T}|)\mathbf{T} + \lambda_3(|\mathbf{T}^d|)\mathbf{T}^d}, \qquad \text{(L-S)}$$

where

$$|\lambda_{1,2,3}(s)| \leq \frac{K}{3(s+1)}.$$

- A priori estimates: from (L-S)

$$|\varepsilon| \leq K.$$

From the equation, we may hope that

$$\int_\Omega \lambda_1(|\operatorname{tr}\mathbf{T}|)|\operatorname{tr}\mathbf{T}|^2 + \lambda_2(|\mathbf{T}|)|\mathbf{T}|^2 + \lambda_3(|\mathbf{T}^d|)|\mathbf{T}^d|^2 = \int_\Omega \mathbf{T}\cdot\varepsilon \leq C.$$

- The reasonable assumptions (∞-Laplacian-like problem):

$$\left.\begin{aligned} \lambda_{1,2,3}(s) &\geq 0, \\ \lambda_3(s) &\geq \frac{\alpha}{s+1}. \end{aligned}\right\} \implies \int_\Omega |\mathbf{T}^d| \leq C.$$

Limiting strain model & monotonicity

- Apriori estimates for $\mathbf{T}^d$ in L^1
- For the convergence at least some monotonicity needed, the minimal assumption:

$$0 \leq \frac{d}{ds}(\lambda_{1,2,3}(s)s). \tag{M}$$

- If we would have a sequence fulfilling

$$\int_{\Omega_0} |(\mathbf{T}^d)^n|^{1+\delta} \leq C(\Omega_0) \qquad \text{for all } \Omega_0 \subset\subset \Omega,$$
$$\implies \mathbf{T}^n \rightharpoonup \mathbf{T} \qquad \text{weakly in } L^1_{loc}.$$

then using (M) we can identify the limit.

- Assume kind of uniform monotonicity, i.e., for some $\alpha, a, K > 0$

$$\frac{\alpha}{(K+s)^a} \leq \frac{d}{dt}(\lambda_3(s)s) \tag{UM}$$

for example

$$\boxed{\lambda_3(s) := \frac{1}{(1+s^a)^{\frac{1}{a}}}}.$$

Limiting strain model & Theorems

Theorem (Dirichlet data)

Let $\Omega \subset \mathbb{R}^d$, $\lambda_{1,2}$ fulfil (M) *and λ_3 satisfy* (UM) *with $a < \frac{1}{d}$. Then there exists a weak solution $(\mathbf{T}, \mathbf{u})$. Moreover, $\mathbf{u}$ is unique. Further, if either λ_1 or λ_2 are strictly monotone then also $\mathbf{T}$ is unique.*

- ☞ Proper approximation (p-Laplacian)
- ☞ Uniform L^1 estimates
- ☞ Uniform $L^{1+\delta}_{loc}$ estimates by showing that $\mathbf{T} \in \mathcal{N}^{\alpha,1}$ for some $\alpha \in (0,1)$.

Theorem (Periodic data)

Let $\lambda_{1,2}$ fulfil (M) *and λ_3 satisfy* (UM) *with $a < \frac{2}{d}$. Then there exists a weak solution $(\mathbf{T}, \mathbf{u})$. Moreover, $\mathbf{u}$ is unique. Further, if either λ_1 or λ_2 are strictly monotone then also $\mathbf{T}$ is unique.*

- ☞ The same as before but no problem with localization $\Longrightarrow$ better bound for a

Theorem (Periodic data II)

Let $\lambda_{1,2}$ fulfil (M) *and λ_3 satisfy* (UM) *with $a > 0$. Then there exists a $(\mathbf{T}, \mathbf{u})$ fulfilling the implicit relation a.e. such that*

$$\mathbf{T} \in L^1, \qquad \varepsilon \in L^\infty, \qquad \frac{\nabla \mathbf{T}}{(1+|\mathbf{T}|)^{\frac{a+1}{2}}} \in L^2$$

the energy inequality holds, i.e.,

$$\int_\Omega \mathbf{T} \cdot \varepsilon(\mathbf{u}) \leq \int_\Omega \mathbf{f} \cdot \mathbf{u},$$

and fulfill the renormalized equation, i.e., for all smooth periodic $\mathbf{v}$ and all $g \in \mathcal{D}(\mathbb{R})$ there holds

$$\int_\Omega \mathbf{T} \cdot (g(|\mathbf{T}|)\nabla \mathbf{v} + \mathbf{v} \otimes \nabla g(|\mathbf{T}|)) = \int_\Omega g(|\mathbf{T}|)\mathbf{f} \cdot \mathbf{v}.$$

Moreover, if $\mathbf{T} \in L^{a+1}$ the the solution is weak.

Limiting strain model - anti-plane stress

We consider the following special geometry

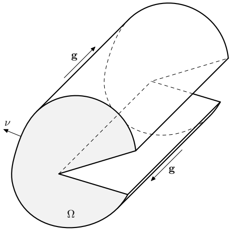

Figure: Anti-plane stress geometry.

and we look for the solution in the following from:

$$\mathbf{u} = \mathbf{u}(x_1, x_2) = (0, 0, u(x_1, x_2)), \quad \mathbf{g}(x) = (0, 0, g(x_1, x_2)),$$

and

$$\mathbf{T}(x) = \begin{pmatrix} 0 & 0 & T_{13}(x_1, x_2) \\ 0 & 0 & T_{23}(x_1, x_2) \\ T_{13}(x_1, x_2) & T_{23}(x_1, x_2) & 0 \end{pmatrix}. \tag{1}$$

Consequences

☞ The problem under consideration then simplifies to

$$
\begin{aligned}
-\frac{\partial T_{13}}{\partial x_1} - \frac{\partial T_{23}}{\partial x_2} &= 0 && \text{in } \Omega,\\
\varepsilon_{13} = \frac{T_{13}}{(1+|\mathbf{T}|^a)^{\frac{1}{a}}} \text{ and } \varepsilon_{23} &= \frac{T_{23}}{(1+|\mathbf{T}|^a)^{\frac{1}{a}}} && \text{in } \Omega,\\
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T_{13}\mathbf{n}_1 + T_{23}\mathbf{n}_2 &= g && \text{on } \partial\Omega.
\end{aligned}
$$

$\Rightarrow$ (and in a simple connected domain also $\Leftarrow$)

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☞ Find $U : \Omega \to \mathbb{R}$ - the Airy stress function such that

$$
T_{13} = \frac{1}{\sqrt{2}}U_{x_2} \quad \text{and} \quad T_{23} = -\frac{1}{\sqrt{2}}U_{x_1}.
$$

Consequences for U

☞ U must satisfy

$$\operatorname{div}\left(\frac{\nabla U}{(1+|\nabla U|^a)^{\frac{1}{a}}}\right) = 0 \qquad \text{in}\Omega,$$

$$U_{x_2}\mathbf{n}_1 - U_{x_1}\mathbf{n}_2 = \sqrt{2}g \qquad \text{on } \partial\Omega.$$

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- Dirichlet problem, indeed assume that $\partial\Omega$ is parametrized by $\gamma(s)=(\gamma_1(s),\gamma_2(s))$. Then

$$\mathbf{n}=\frac{1}{\sqrt{(\gamma_1'(s))^2+(\gamma_2'(s))^2}}(\gamma_2'(s),-\gamma_1'(s)).$$

$$\begin{aligned}
U(\gamma(s_0)) &= a_0 + \int_0^{s_0} \frac{d}{ds}U(\gamma(s))\,ds \\
&= a_0 + \int_0^{s_0} U(\gamma(s))_{x_2}\gamma_2'(s) + U(\gamma(s))_{x_1}\gamma_1'(s)\,ds \\
&= a_0 + \int_0^{s_0} \sqrt{(\gamma_1'(s))^2+(\gamma_2'(s))^2}(U(\gamma(s))_{x_2}\mathbf{n}_1 - U(\gamma(s))_{x_1}\mathbf{n}_2)\,ds \\
&= a_0 + \sqrt{2}\int_0^{s_0} g(\gamma(s))\sqrt{(\gamma_1'(s))^2+(\gamma_2'(s))^2}ds =: U_0(x).
\end{aligned}$$

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$$\operatorname{div}\left(\frac{\nabla U}{(1+|\nabla U|^{a})^{\frac{1}{a}}}\right)=0 \quad \text{in}\Omega, \qquad U=U_0 \quad \text{on } \partial\Omega.$$

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- $a\in(1,2)$ the weak solution may not exists eg. for $\Omega = B_2\setminus B_1$
- on the flat part of the boundary, you can extend the solution outside such that it is still the solution - estimates near the boundary

Limiting strain - Results II

Theorem (MORE)

Let U_0 be arbitrary. Then there exists unique weak solution U provided that one of the following holds.

- Ω *is uniformly convex,* $a > 0$ *is arbitrary and* U_0 *smooth.*
- $a \in (0,2)$ *and* $\partial\Omega = \bigcup_{i=1}^{N} \Gamma_i$ *such that either* Γ_i *is uniformly convex and* U_0 *is smooth on* Γ_i *or* Γ_i *is flat and* U_0 *is constant there.*