Implicit relations in fluid mechanics and in abstract parabolic problems

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Implicitly constituted materials: Modeling, analysis, and computing November 24-27, 2013, Chateau Liblice, Czech Republic

Abstract parabolic systems

Let $\mathcal{A}$ be a maximal monotone graph. We look for $u : Q \to \mathbb{R}$ and $A : Q \to \mathbb{R}^d$ such that

$$\begin{aligned} u_t - \mathrm{div}\, A = f \quad & \text{in } Q, \\ (\nabla u, A) \in \mathcal{A}(t,x) \quad & \text{in } Q, \\ u(0,x) = u_0 \quad & \text{in } \Omega, \\ u(t,x) = 0 \quad & \text{on } (0,T) \times \partial\Omega. \end{aligned}$$

where $\Omega \subset \mathbb{R}^d$ is an open, bounded set with a $\mathcal{C}^1$ boundary $\partial\Omega$, $(0,T)$ is the time interval with $T < \infty$, $Q := (0,T) \times \Omega$.

$$(\nabla u, A) \in \mathcal{A} \quad \Longleftrightarrow \quad \mathbf{G}(\nabla u, A) = 0\,,$$

What do we assume about $\mathcal{A}$?

We put the following assumptions on $\mathcal{A}(t,x)$:

(A1) $\mathcal{A}$ comes through the origin.

(A2) $\mathcal{A}$ is a monotone graph, namely

$$(A_1 - A_2)\cdot(\xi_1 - \xi_2) \geq 0 \quad \text{for all } (\xi_1, A_1), (\xi_2, A_2) \in \mathcal{A}(t,x)\,.$$

(A3) $\mathcal{A}$ is a maximal monotone graph. Let $(\xi_2, A_2) \in \mathbb{R}^d \times \mathbb{R}^d$.

$$\text{If } (A_1 - A_2)\cdot(\xi_1 - \xi_2) \geq 0 \quad \text{for all } (\xi_1, A_1) \in \mathcal{A}(t,x)$$
$$\text{then } (\xi_2, A_2) \in \mathcal{A}(t,x).$$

(A4) $\mathcal{A}$ is an $M-$ *graph.* There are non-negative $k \in L^1(Q)$, $c_* > 0$ and N-function M such that

$$A\cdot\xi \geq -k(t,x) + c_*(M(t,x,|\xi|) + M^*(t,x,|A|))$$

for all $(\xi, A) \in \mathcal{A}(t,x)$.

(A5) The existence of a measurable selection. There is $\tilde{A} : Q \times \mathbb{R}^d \to \mathbb{R}^d$ such that $(\xi, \tilde{A}(t,x,\xi)) \in \mathcal{A}(t,x)$ for all $\xi \in \mathbb{R}^d$ and $\tilde{A}$ is measurable.

$N-$ functions

Notation

By an $N-$function we mean that $M : \bar{Q} \times \mathbb{R}_+ \to \mathbb{R}_+$, $M(t,x,a)$ is measurable w.r.t. (t,x) for all $a \in \mathbb{R}_+$ and continuous w.r.t. a for a.a. $(t,x) \in \bar{Q}$, convex in a, has superlinear growth, $M(t,x,a) = 0$ iff $a = 0$ and

$$\lim_{a\to\infty} \inf_{(t,x)\in Q} \frac{M(t,x,a)}{a} = \infty.$$

Moreover the conjugate function M^* is defined as

$$M^*(t,x,b) = \sup_{a\in\mathbb{R}_+} (b \cdot a - M(t,x,a)).$$

Example

Δ_2–condition

We say that M satisfies Δ_2–condition if for some constant $C > 0$ and an integrable function m

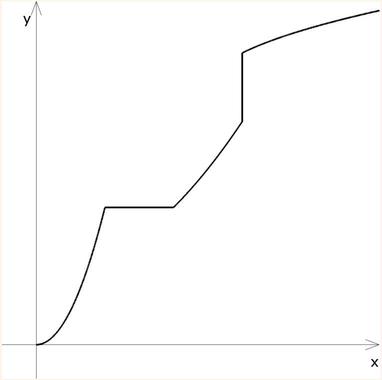
$$M(t,x,2\xi) \leq CM(t,x,\xi) + m(t,x) \text{ for all } \xi \in \mathbb{R}^d \text{ and a.a. } t,x \in Q.$$

None of the functions M and M^* satisfies Δ_2–condition

$$M(\xi) = M(\xi_1,\xi_2) = e^{|\xi_1|} - |\xi_1| - 1 + (1+|\xi_2|)\ln(1+|\xi_2|) - |\xi_2|$$

$$M^*(\eta) = M^*(\eta_1,\eta_2) = (1+|\eta_1|)\ln(1+|\eta_1|) - |\eta_1| + e^{|\xi_2|} - |\eta_2| - 1$$

Implicit constitutive relation



$(t,x)-$dependence of an $N-$function

Another examples

- $M(t,x,\xi) = (e^{\xi})^{p(t,x)} - 1$
- $M(t,x,\xi) = \xi^{p(t,x)} \ln(\xi + 1)$.

log-Hölder continuity

There exists a constant $H > 0$ such that for all $x, y \in \Omega, t, s \in [0,T], |x-y| + |t-s| \leq \frac{1}{2}$

$$|p(t,x) - p(s,y)| \leq \frac{H}{\ln \frac{1}{|t-s|+|x-y|}}$$

holds.

$(t, x)-$dependence of an $N-$function

We will assume log-Hölder continuous dependence on t and x of the function M and its conjugate M^*, i.e., it is supposed to satisfy the following:

(M) there exists a constant $H > 0$ such that for all $x, y \in \Omega, t, s \in [0, T], |x - y| + |t - s| \leq \frac{1}{2}$

$$\frac{M(t,x,a)}{M(s,y,a)} \leq a^{\frac{H}{\ln \frac{1}{|t-s|+|x-y|}}}$$

for all $a \in \mathbb{R}_+$

Definition of weak solutions

Definition

Assume that $u_0 \in L^2(\Omega)$, We say that (u, A) is weak solution to our problem if

$$u \in C_{\text{weak}}(0, T; L^2(\Omega)), \nabla u \in L_M(Q),\ \ A \in L_{M^*}(Q),$$

$$\int_Q (-u\varphi_t + A \cdot \nabla\varphi)\, dxdt + \int_\Omega u_0(x)\varphi(0,x)dx = 0$$

$$\text{for all } \varphi \in \mathcal{C}_c^\infty((-\infty, T) \times \Omega)\,,$$

$$(\nabla u((t,x)), A(t,x)) \in \mathcal{A} \text{ for a.a. } (t,x) \in Q.$$

Musielak–Orlicz spaces

$$L_M(Q) := \{v : \exists\ \lambda \text{ s.t. } \int_Q M(t,x,|v|/\lambda)dxdt < \infty\}$$

$$\|v\|_{L_M} = \inf\{\lambda > 0 : \int_Q M(t,x,|v|/\lambda)dxdt \leq 1\}$$

Theorem on existence of solutions

Theorem

Let $\mathcal{A}$ satisfy the assumptions **(A1)–(A5)** *and let an $N-$function satisfy* (**M**). *Then there exists a weak solution to our problem.*

Some properties of Orlicz spaces

The space E_M

By $E_M(Q)$ we denote the closure of $L^\infty(Q)$ in $L_M(Q)$

- $(E_M)^* = L_{M^*}$
- If M does not satisfy Δ_2–condition, then $E_M \subsetneq L_M$
- If M satisfies Δ_2–condition, L_M is separable and $L_M = E_M$.

Definition

A sequence z^j converges modularly to z in $L_M(Q)$ if there exists $\lambda > 0$ such that $\int_Q M\left((z^j - z)/\lambda\right) dxdt \to 0$.

Properties

- Orlicz spaces are separable w.r.t. the modular convergence and smooth functions are dense

Problems/strategies – density arguments

We want to approximate the function vanishing on $\partial\Omega$ having gradients in Musielak–Orlicz space

- Standard procedure (Lipschitz domains): $\Omega = \bigcup \Omega_i, \quad \Omega_i$ are star-shaped domains and proceed with an appropriate partition of unity and scaling the function on star-shaped sets.
- Partition of unity requires using the **Poincaré inequality** or **truncating** the function.
- The option of truncating the function would need the integration by parts formula for truncations, which in the case of time-dependent $N-$functions does not hold.
- For these reasons we use a non-standard approximation method, which consists in constructing a mapping which transfers the area near the boundary of Ω to its interior.

Lemma

Let $S \in \mathcal{C}_c^\infty(\mathbb{R}^{d+1})$*, symmetric and* $\int_{\mathbb{R}^{d+1}} S(\tau, y)\, dy\, d\tau = 1$*. We define* $S_\delta(t,x) := 1/\delta^{d+1} S(t/\delta, x/\delta)$*. Consider the family of operators*

$$\mathcal{S}_\delta z(t,x) := \int_Q S_\delta(t-s, \Psi^\delta(x) - y) z\,(s,y)\; dy\, ds.$$

Let an $N-$*function satisfy condition (M). Then there exist constants* $c, \lambda > 0$ *(independent of* δ*) such that for every* $z \in L_M(Q)$ *the following estimate holds*

$$\int_Q M(t,x,|\mathcal{S}_\delta z(t,x))|)\, dx\, dt \le c \int_Q M(t,x,|z(t,x)|)\, dx\, dt.$$

Approximation theorem

Theorem

If $u \in L^2(Q)$, $\nabla u \in L_M(Q)$ and $u_t \in V_{M^*}(Q) + L^2(Q)$, then there exists a sequence $v^\delta \in \mathcal{C}_c^\infty([0,T] \times \Omega))$ satisfying

$$\nabla v^\delta \xrightarrow{M} \nabla u \quad \text{modularly in } L_M(Q) \text{ and } v^\delta \to u \text{ strongly in } L^2(Q).$$

Moreover we can write

$$\frac{\partial v^\delta}{\partial t} = \mathrm{div}\ v_A^\delta + v_f^\delta \quad \text{and} \quad \frac{\partial u}{\partial t} = \mathrm{div}\ v_A + v_f$$

with

$$v_A^\delta \xrightarrow{M^*} v_A \quad \text{modularly in } L_{M^*}(Q) \quad \text{and} \quad v_f^\delta \to v_f \text{ strongly in } L^2(Q).$$

$$V_{M^*}(Q) = \{\phi = \mathrm{div}\ \phi_i : \phi_i \in L_{M^*}(Q)\}.$$

Monotonicity method

Lemma

Let $\mathcal{A}$ be maximal monotone M-graph. Assume that there are sequences $\{A^n\}_{n=1}^{\infty}$ and $\{\nabla u^n\}_{n=1}^{\infty}$ defined on G such that the following conditions hold:

$$
\begin{aligned}
(\nabla u^n, A^n) &\in \mathcal{A} && \textit{a.e. in } G,\\
\nabla u^n &\overset{*}{\rightharpoonup} \nabla u && \textit{weakly}^*\textit{in } L_M(G),\\
A^n &\overset{*}{\rightharpoonup} A && \textit{weakly}^*\textit{in } L_{M^*}(G),
\end{aligned}
$$

$$\limsup_{n\to\infty} \int_G A^n \cdot \nabla u^n \, \mathrm{d}z \le \int_G A \cdot \nabla u \, \mathrm{d}z.$$

Then

$$(\nabla u, A) \in \mathcal{A} \quad \textit{a.e. in } G,$$

Flow of fluid described by an implicit constitutive relation

System

Let $\Omega \subseteq \mathbb{R}^n$ be a bounded, open set and $Q = (0, T) \times \Omega$. We say that the velocity field v and the pressure p describe the unsteady flow of the incompressible fluid if

$$\begin{aligned} v_t + v \cdot \nabla v - \mathrm{div}\, S + \nabla p = f \\ \mathrm{div}\, v = 0 \\ (D, S) \in \mathcal{A} \end{aligned}$$

where $\mathbf{S}$ is the stress tensor and f are given body forces.

- There exist a positive constant c and an N-function satisfying $M(x, \xi) \geq C|\xi|^q$ for some q and M^* satisfies Δ_2 condition.

Implicit formulation

Assume that the shear stress and the shear rate are related

$$\mathbf{G}(\mathbf{S},\mathbf{D}) = \mathbf{0}\,.$$

Examples

1. Fluids of a Bingham type

$$\mathbf{D} = \frac{(|\mathbf{S}| - \tau^*)_+}{\nu(|\mathbf{D}|)|\mathbf{S}|}\mathbf{S} \quad \text{with } \tau^* > 0$$

which is equivalent to the traditional description

$$|\mathbf{S}| \le \tau^* \Leftrightarrow \mathbf{D} = \mathbf{0}, \quad \text{and} \quad |\mathbf{S}| > \tau^* \Leftrightarrow \mathbf{S} = \frac{\tau^*\mathbf{D}}{|\mathbf{D}|} + 2\nu(|\mathbf{D}|^2)\mathbf{D}.$$

2. Fluids with activation criteria

$$\mathbf{S} = \nu(|\mathbf{D}|)\mathbf{D}$$

with ν being discontinuous at some d^*

Implicit constitutive relation

We are interested in investigating fluids in which $\mathbf{S}$ and $\mathbf{D}(v)$ are related implicitly, i.e., $\mathbf{D}(v)$ and $\mathbf{S}$ fulfill the constitutive equation

$$\mathbf{G}(\mathbf{D}(v), \mathbf{S}) = \mathbf{0},$$

whereas we restrict ourselves to those responses that are described by *maximal monotone graphs*. Introducing the graph $\mathcal{A}$ through the natural characterization

$$(\mathbf{D}, \mathbf{S}) \in \mathcal{A} \quad \Longleftrightarrow \quad \mathbf{G}(\mathbf{D}, \mathbf{S}) = \mathbf{0}\,,$$

we can specify precisely the class of admissible responses $\mathbf{G}$ by specifying the assumptions for $\mathcal{A}$

References

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