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On BMO estimates for the p-Laplacian and p-fluids

joint work with
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The p-Laplacian (scalar or vectorial)

Laplacian: Minimizer of $\int \frac{1}{2}|\nabla w|^2\,dx + \int \nabla w \cdot F\,dx$ solve

$$-\Delta v = \operatorname{div} F.$$

p-Laplacian: Minimizer of $\int \frac{1}{p}|\nabla w|^p\,dx + \int \nabla w \cdot F\,dx$, $1 < p < \infty$ solve

$$-\operatorname{div}(A(\nabla u)) := -\operatorname{div}(|\nabla u|^{p-2}\nabla u) = \operatorname{div} F = \operatorname{div}(A(G))$$

Theorem

$G \in L^p$ implies $\nabla u \in L^p$.
$F \in L^{p'}$ implies $A(\nabla u) \in L^{p'}$.

The *p*-Laplacian (scalar or vectorial)

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Harmonic vs. p-harmonic

Linear:

If h is harmonic, i.e. $-\Delta h = 0$, then $h \in C^\infty$

Non-linear:

If h is p-harmonic, i.e. for $1 < p < \infty$

$$-\mathrm{div}(A(\nabla h)) = -\mathrm{div}(|\nabla h|^{p-2}\nabla h) = 0,$$

then $\nabla h \in C^{0,\alpha}$ and $A(\nabla h) \in C^{0,\beta}$.

[Uraltseva; Uhlenbeck; Acerbi-Fusco; Tolksdorf; ...]

Decay estimate [e.g. Diening, Stroffolini, Verde '09]: For some $\alpha > 0$

$$\fint_{B_r} |V(\nabla h) - \langle V(\nabla h)\rangle_{B_r}|^2\,dx \le c\left(\frac{r}{R}\right)^\alpha \fint_{B_R} |V(\nabla h) - \langle V(\nabla h)\rangle_{B_R}|^2\,dx$$

for $V(\nabla h) := |\nabla h|^{\frac{p}{2}}\frac{\nabla h}{|\nabla h|}$.

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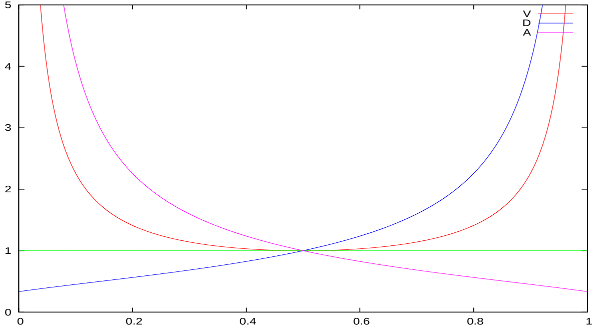
Bad $p-$harmonic functions in the plane

Find positive harmonic function $h \in W_0^{1,p}(\Omega)$ on $\Omega := (0,\infty)^2$.

Copy h with alternating signs to other quadrants to get global solution $h \in C^\alpha(\mathbb{R}^n)$ with $\alpha = \frac{7p-6+\sqrt{p^2+12p-12}}{6p-6}$.

$D := \nabla u$
$A := A(\nabla u)$
$V := |\nabla u|^{\frac{p}{2}} \frac{\nabla u}{|\nabla u|}$

Optimal result
[Aaronson '88]
[Iwaniec, Manfredi '89]



Linear Calderón-Zygmund theory

Let u solve $-\Delta u = \mathrm{div} F$.

Then $F \mapsto \nabla u$ is singular integral operator.

Theorem (Calderón-Zygmund)

Singular integrals map from L^q to L^q for $1 < q < \infty$.

Theorem

$F \in L^q$ implies $\nabla u \in L^q$ for $1 < q < \infty$.

Goal: Extend this principle to non-linear operators!

Non-linear Calderón-Zygmund theory

$$-\mathrm{div}(A(\nabla u)) = -\mathrm{div}(|\nabla u|^{p-2}\nabla u) = \mathrm{div} F = \mathrm{div} A(G).$$

Theorem (weak solution)

$F \in L^{p'}$ implies $A(\nabla u) \in L^{p'}$.
$G \in L^p$ implies $\nabla u \in L^p$.

Non-linear Calderón-Zygmund theory goes back to T. Iwaniec '82:

Theorem (Iwaniec; Manfredi, Di Benedetto)

$F \in L^s$ implies $A(\nabla u) \in L^s$ for all $s \in [p', \infty)$.
$G \in L^q$ implies $\nabla u \in L^q$ for all $q \in [p, \infty)$.

Whole space and local result are possible.

Remark: Riesz potential estimate for $\mathrm{div} F \mapsto A(\nabla u)$ [Mingione, Kuusi,..]

Limit case L^∞

Let u solve $-\Delta u = \mathrm{div} F$, then $\|\nabla u\|_\infty \not\lesssim \|F\|_\infty$.

Example: Let $u := x_1 \ln|x|$, then $\nabla u \notin L^\infty$ but $\Delta u \in \mathrm{div} L^\infty$.

$$-\Delta u = -\mathrm{div}\left(|x|^{-2} \begin{pmatrix} -2x_1 x_2 \\ x_2^2 - x_1^2 \end{pmatrix} \right)$$

Singular integrals: Not bounded on L^∞, but bounded on BMO

Definition (bounded mean oscillation, BMO)

$$w \in \mathrm{BMO} \quad :\Leftrightarrow \quad \|w\|_{\mathrm{BMO},\Omega} = \sup_{Q \text{ is cube}} ⨍_Q |w - \langle w \rangle_Q| \, dx < \infty.$$

Theorem

$F \in \mathrm{BMO}$ *implies* $\nabla u \in \mathrm{BMO}$.

Non-linear limit case: Main result

Let $-\mathrm{div}(A(\nabla u)) = \mathrm{div}(F)$.

Theorem (Manfredi, Di Benedetto 93')

If $p \geq 2$, then

$$\|\nabla u\|_{\mathrm{BMO},Q}^{p-1} \leq c\,\|F\|_{\mathrm{BMO},2Q} + c\left(\fint_{2Q}\left|\frac{u-\langle u\rangle_{2Q}}{R_Q}\right|^p dx\right)^{\frac{p-1}{p}}$$

Theorem (Diening, Kaplický, Schwarzacher '12)

For any $p > 1$,

$$\|A(\nabla u)\|_{\mathrm{BMO},Q} \leq c\,\|F\|_{\mathrm{BMO},2Q} + c\fint_{2Q}|A(\nabla u) - \langle A(\nabla u)\rangle_{2Q}|\,dx.$$

Explain proof below in several steps.

Comparison to *p*-harmonic functions:

Let $-\mathrm{div}(A(\nabla u)) = \mathrm{div} F$. Recall $|V(Q)|^2 = A(Q) : Q$.

Let h be p-harmonic on B with $h = u$ on ∂B. Then

$$\underbrace{\langle A(\nabla u) - A(\nabla h), \nabla u - \nabla h\rangle}_{\|V(\nabla u) - V(\nabla h)\|_2^2} = \langle F - \langle F\rangle_B, \nabla u - \nabla h\rangle.$$

Comparison help to transfer decay estimate of $V(\nabla h)$ to $V(\nabla u)$.

Problem: Different growth of $|A|$ and $|V|^2$, namely $p-1$

Tools: John-Nirenberg for BMO, reverse Hölder estimate

Beyond BMO

Refined BMO: $\|w\|_{\mathrm{BMO},\omega,\Omega} = \sup_{Q \text{ is cube}} \frac{1}{\omega(r_Q)} \fint_Q |w - \langle w \rangle_Q| \, dx.$

Theorem (Diening, Kaplický, Schwarzacher '12)

$$\|A(\nabla u)\|_{\mathrm{BMO},\omega,Q} \leq c\,\|F\|_{\mathrm{BMO},\omega,2Q} + \frac{c}{\omega(R_Q)} \fint_{2Q} |A(\nabla u) - \langle A(\nabla u)\rangle_{2Q}| \, dx$$

with $\omega(r) r^{-\frac{\alpha}{p'}}$ decreasing. (α from decay estimates of $V(\nabla u)$.)

Includes VMO.

By characterization of Campanato: $C^{0,\alpha} = \mathrm{BMO}_\beta$.

Theorem (Diening, Kaplický, Schwarzacher '12)

If $F \in C^{0,\beta}$ and $\beta < \frac{\alpha}{p'}$, then $A(\nabla u) \in C^{0,\beta}$

Summary p-Laplacian

$$-\mathrm{div}(A(\nabla u)) = \mathrm{div}(F)$$

Theorem (Non-linear Calderón-Zygmund theory)

$$\|A(\nabla u)\|_q \leq \|F\|_q \qquad \text{for } q \geq p' \text{ [Iwaniec '82; Kinnunen-Zhou '01]},$$

$$\|A(\nabla u)\|_{\mathrm{BMO}} \leq \|F\|_{\mathrm{BMO}}, \qquad [DKS'12]$$

$$\|A(\nabla u)\|_{C^{0,\beta}} \leq \|F\|_{C^{0,\beta}} \qquad \text{for some } \beta \in (0, \tfrac{\alpha}{p'})$$

$$\|A(\nabla u)\|_{\mathrm{VMO}} \leq \|F\|_{\mathrm{VMO}}.$$

Remark: Everything holds for more general growth $\varphi(|\nabla u|)$

Theorem (Breit, Diening, Chianci, Schwarzacher '13/14)

Everything holds up to zero Dirichlet or Neumann boundary.

p-Stokes

Consider
$$-\mathrm{div}\mathbf{A}(\varepsilon(\mathbf{u})) + \nabla q = 0,$$
$$\mathrm{div}\mathbf{u} = 0$$

Model for honey, ketchup, polymers, blood ($p < 2$).

Theorem (Diening, Kaplicky, Schwarzacher '12) Let $\mathbf{V} := \mathbf{V}(\varepsilon(\mathbf{u}))$.

$$\left(\fint_B |\nabla \mathbf{V}|^{2+\delta}\, dx \right)^{\frac{1}{2+\delta}} \leq c \left(\fint_{2B} |\mathbf{V} - \langle \mathbf{V} \rangle_{2B}|^2\, dx \right)^{\frac{1}{2}}$$

Decay estimates: (Diening, Kaplicky, Schwarzacher '12)

$$n = 2: \quad \left(\fint_{B_r} |\mathbf{V} - \langle \mathbf{V} \rangle_{B_r}|^2\, dx \right)^{\frac{1}{2}} \leq c \left(\frac{r}{R} \right)^{\alpha} \left(\fint_{B_R} |\mathbf{V} - \langle \mathbf{V} \rangle_{B_R}|^2\, dx \right)^{\frac{1}{2}}$$

Summary p-Stokes

Consider $-\mathrm{div}(A(\varepsilon(\mathbf{u}))) + \nabla q = \mathrm{div}(\mathbf{F})$.

Lemma

$$\fint_B |q - \langle q\rangle|^s \, dx \leq c \fint_B |\mathbf{A}(\varepsilon(\mathbf{u})) - \langle \mathbf{A}(\varepsilon(\mathbf{u}))\rangle|^s \, dx + c \fint_B |\mathbf{F} - \langle \mathbf{F}\rangle|^s \, dx.$$

Theorem (Diening, Kaplický, Schwarzacher '12)

If $\mathbf{F} \in X$, *then* $\mathbf{A}(\varepsilon(\mathbf{v})) \in X$ *and* $q \in X$.

Case $n = 3$, $X = L^q_{\mathrm{loc}}$ *with* $p' \leq q \leq 3p'$.

Case $n = 2$, $X = L^q_{\mathrm{loc}}$ *with* $p' \leq q$.

Case $n = 2$: $X = \mathrm{BMO}, \mathrm{VMO}, C^{0,\alpha}$ *(for all* $1 < p < \infty$*!).*