Nonperiodic Homogenization of Pradtl-Reuss Plasticity

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Plasticity (Prandtl-Reuss Model)

Reference domain of material Q on time interval $(0, T)$.

Variables

displacement	u	:	$Q \times (0,T) \to \mathbb{R}^n$
Strain	$\nabla^s u$	:	$Q \times (0,T) \to \mathbb{R}^{n\times n}_s$
Cauchy stress	σ	:	$Q \times (0,T) \to \mathbb{R}^{n\times n}_s$

with $\nabla^s u = \frac{1}{2}(\nabla u + (\nabla u)^t)$.

Equations

Conservation of linear Momentum: $-\nabla \cdot \sigma = f$

Additive strain decomposition: $\nabla^s u = \underbrace{e}_{\text{elastic strain}} + \underbrace{p}_{\text{plastic strain}}$

Hooke's law: $C\sigma = e$

Force f and elasticity tensor C are given fields.

The flow rule

Flow rule with kinematic hardening

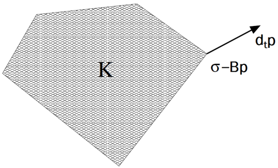
$$\partial_t p \in \partial\Psi(\sigma - \underbrace{B}_{\text{hardening parameter}} p)$$

Example: Let Ψ be the indicator function of the set $K \subset \mathbb{R}_s^{n\times n}$

$$\psi = \begin{cases} 0 & \text{if } \sigma \in K \\ +\infty & \text{if } \sigma \notin K \end{cases}$$

$\partial\Psi$ the subdifferential.
The flow rule demands

$$\sigma - Bp \in K\,.$$



Von-Mises example:
Decompositon of $\mathbb{R}_s^{n\times n}$: Every tensor is decomposed as $\sigma = \sigma^S + \sigma^D$, where

$$\sigma^S = \frac{1}{n}\mathrm{tr}(\sigma)\mathrm{id}$$

Von-Mises yield criterium

$$K := \left\{\sigma \in \mathbb{R}_s^{n\times n} \;:\; \left|\sigma^D\right| \le \gamma\right\}$$

Known results

- H.-D. Alber. Evolving microstructures and homogenization. (2000)
- A. Visintin. On homogenization of elasto-plasticity. (2005)
- A. Visintin. Homogenization of the nonlinear Kelvin-Voigt model of viscoelasticity and of the Prager model of plasticity. (2006)
- S. Nesenenko. Homogenization in viscoplasticity. (2007)
- A. Mielke and A.M. Timofte. Two-scale homogenization for evolutionary variational inequalities via the energetic formulation (2007)
- H.-D. Alber, S. Nesenenko. Justification of homogenization in viscoplasticity: From convergence on two scales to an asymptotic solution. (2009)
- B. Schweizer: Homogenization of the Prager model in one-dimensional plasticity (2009)
- B. Schweizer, M. Veneroni: Homogenization of plasticity equations with two-scale convergence methods. (preprint)

ε-Problem

$$-\nabla \cdot \sigma^\varepsilon = f\,, \qquad \sigma^\varepsilon = C_\varepsilon^{-1} e^\varepsilon\,,$$
$$\nabla^s u^\varepsilon = e^\varepsilon + p^\varepsilon\,, \qquad \partial_t p^\varepsilon \in \partial\Psi_\varepsilon(\sigma^\varepsilon - B_\varepsilon p^\varepsilon)$$

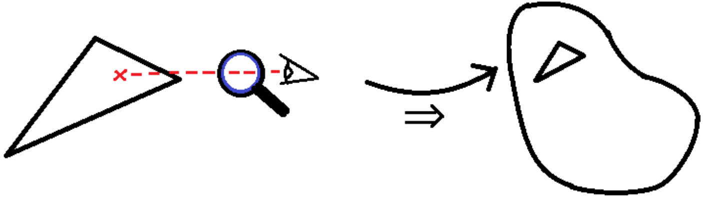
- $C_\varepsilon, B_\varepsilon \in L^\infty(Q; \mathcal{L}(\mathbb{R}_s^{n\times n}, \mathbb{R}_s^{n\times n}))$ are symmetric, uniformly elliptic.
- $\Psi_\varepsilon(\cdot, x):\ \mathbb{R}_s^{n\times n} \to \overline{\mathbb{R}}$ is convex and for each $\varepsilon > 0$ there is $C(\varepsilon) > 0$ such that

$$|\Psi_\varepsilon^*(\sigma; x_1) - \Psi_\varepsilon^*(\sigma; x_2)| \le C(\varepsilon)\,|x_1 - x_2|\,|\sigma|\ .$$

Question

Suppose $u^\varepsilon \to u$ in $L^2(0,T; L^2(Q))$ for $\varepsilon \to 0$.
What is the effective equation for u ?

Averaging Property



Averaging Property

Definition

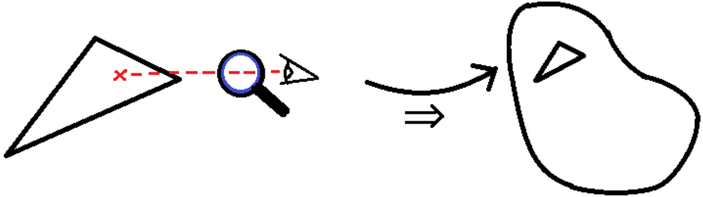
The coefficients C_ε, B_ε and Ψ_ε allow averaging, if there exists an operator

$$\Sigma \ : \ H^1(0,T;\mathbb{R}^{n\times n}_s) \to H^1(0,T;\mathbb{R}^{n\times n}_s)$$

such that the following property holds:

For a simplex $\mathcal{T} \subset \mathbb{R}^n$, $\xi \in H^1(0,T;\mathbb{R}^{n\times n}_s)$ let u^ε, e^ε, p^ε and σ^ε be solutions to the ε-problem on $\mathcal{T}$ with $f=0$ and $u^\varepsilon|_{\partial\mathcal{T}} = \xi x$. Then, as $\varepsilon \to 0$, there holds

$$⨍_{\mathcal{T}} \sigma^\varepsilon(t) \to \Sigma(\xi)(t)\,.$$



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$$⨍_{\mathcal{T}} \sigma^\varepsilon(t) \to \Sigma(\xi)(t)\,.$$

Theorem

Under some assumptions on Σ, the problem

$$-\nabla \cdot \Sigma(\nabla^s u) = f$$

has a solution u and $u^\varepsilon \rightharpoonup u$ weakly in $H^1(0,T;H^1(Q))$ as $\varepsilon \to 0$.

Needle Problem for plasticity

- Use a discretization Q_h of Q via simplices $\mathcal{T}_k$ and call the grid $\mathbb{T}_h = \{\mathcal{T}_k\}_{k\in\Lambda_h}$
- $\mathcal{N}_h := \{\phi \in H_0^1(Q) \ : \ \phi|_{\partial\mathcal{T}_k}$ is affine $\forall\ \mathcal{T}_k \in \mathbb{T}_h,\ \phi \equiv 0$ on $Q \setminus Q_h\}$
- Construct discretization g_h of RHS such that

$$\int_{\bigcup \partial\mathcal{T}_k} g_h \cdot \varphi = \int_Q f \cdot \varphi \quad \text{for piecewise affine test functions } \varphi$$

Definition (Needle problem in plasticity)

Find $u_h^\varepsilon \in H^1(0,T;\mathcal{N}_h)$, $e_h^\varepsilon, p_h^\varepsilon, \sigma_h^\varepsilon \in H^1(0,T;L^2(Q_h;\mathbb{R}_s^{n\times n}))$, s.t.

$$\int_0^T \int_{Q_h} \sigma_h^\varepsilon : \nabla\varphi = \int_0^T \int_{\bigcup \partial\mathcal{T}_k} g_h \cdot \varphi \qquad \forall\varphi \in L^2(0,T;\mathcal{N}_h)\,,$$

and almost everywhere in Q_h holds

$$\nabla^s u_h^\varepsilon = e_h^\varepsilon + p_h^\varepsilon\ , \qquad C_\varepsilon\sigma_h^\varepsilon = e_h^\varepsilon\ , \qquad \partial_t p_h^\varepsilon \in \partial\Psi_\varepsilon\left(\sigma_h^\varepsilon - B_\varepsilon p_h^\varepsilon\right)\,.$$

Homogenization proof

1. $\lim_{h\to 0}\lim_{\varepsilon\to 0}\|u_h^\varepsilon - u^\varepsilon\|_{L^\infty(0,T;H^1(Q))} \to 0$

2. $u_h^\varepsilon \to u_h$ as $\varepsilon \to 0$ and u_h is piecewise affine and solves

$$\int_0^T \int_Q \Sigma(\nabla^s u_h) : \nabla\varphi = \int_0^T \int_\Gamma g_h \cdot \varphi \qquad \forall \varphi \in L^2(0,T;Y_h)\,.$$

   - $Y_h := \left\{\phi \in H_0^1(Q) \,:\, \phi|_{\mathcal{T}_k} \text{ is affine } \forall\, \mathcal{T}_k \in \mathbb{T}_h,\ \phi \equiv 0 \text{ on } Q \setminus Q_h\right\}$

3. $u_h \to u$

Homogenization proof

1. $\lim_{h\to 0}\lim_{\varepsilon\to 0}\|u_h^\varepsilon - u^\varepsilon\|_{L^\infty(0,T;H^1(Q))} \to 0$
   - Testing with $\partial_t\left(u_h^\varepsilon - u^\varepsilon\right)$
2. $u_h^\varepsilon \to u_h$ as $\varepsilon \to 0$ and u_h is piecewise affine and solves

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   - Follows from averaging property
3. $u_h \to u$

One term from testing

$$\lim_{h\to 0}\lim_{\varepsilon\to 0}\int_0^{t_0}\int_Q \sigma_h^\varepsilon : \partial_t \nabla^s(\mathcal{P}_h(u^\varepsilon) - u^\varepsilon)$$

- $\mathcal{P}_h \,:\, L^2(Q) \to Y_h$ orthogonal projection

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- $\mathcal{P}_h \;:\; L^2(Q) \to Y_h$ orthogonal projection
- Compensated compactness result:

Theorem (Schweizer and Veneroni)

Let $\{a^\varepsilon\}_{\varepsilon>0} \subset H^1(Q)$ a sequence with weak limit $a^\varepsilon \rightharpoonup a \in H^1(Q)$,
$\Rightarrow\ \exists$ a sequence of grids $\mathbb{T}_h$ as $h \to 0$ and a subsequence a^ε with

$$\lim_{\varepsilon\to 0}\int_{Q_h} q^\varepsilon : \nabla^s a^\varepsilon = \int_{Q_h} q : \nabla a \qquad \forall h > 0$$

for all $q^\varepsilon \rightharpoonup q$ weakly in $L^2(Q)$, with $\nabla \cdot q^\varepsilon = 0$.

One term from testing

$$\int_0^{t_0}\int_Q \sigma_h^\varepsilon : \partial_t \nabla^s(\mathcal{P}_h(u^\varepsilon) - u^\varepsilon)$$

$\downarrow$ time-discretization

$$\sum_{i=0}^{k_0-1}\int_{Q_h} \overline{\sigma}_{h,K}^\varepsilon\left(\tau_K^i,\cdot\right) : \left(\nabla^s\left(\mathcal{P}_h(u^\varepsilon) - u^\varepsilon\right)\left(\tau_K^{i+1},\cdot\right) - \nabla^s\left(\mathcal{P}_h(u^\varepsilon) - u^\varepsilon\right)\left(\tau_K^i,\cdot\right)\right)$$

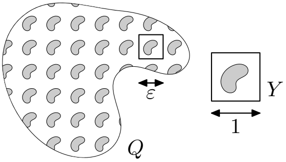
compensated compactness $\downarrow$ $\varepsilon \to 0$

$$\sum_{i=0}^{k_0-1}\int_{Q_h} \overline{\sigma}_{h,K}\left(\tau_K^i,\cdot\right) : \left(\nabla^s\left(\mathcal{P}_h(u) - u\right)\left(\tau_K^{i+1},\cdot\right) - \nabla^s\left(\mathcal{P}_h(u) - u\right)\left(\tau_K^i,\cdot\right)\right)$$

$\downarrow$ $h \to 0$

$$0$$

Periodic case



$$C_\varepsilon(x) = C\left(\frac{x}{\varepsilon}\right), \qquad B_\varepsilon(x) := B\left(\frac{x}{\varepsilon}\right),$$

$$\Psi_\varepsilon(\cdot, x) := \Psi\left(\cdot, \frac{x}{\varepsilon}\right)$$

With the two-scale limit problem

$$\begin{aligned} -\nabla \cdot \left(\int_Y z(x,y)\,dy\right) &= f(x), \\ C(y)z(x,y) &= \nabla_x^s u(x) + \nabla_y^s v(x,y) - w(x,y), \\ \partial_t w(x,y) &\in \partial\Psi(z(x,y) - B(y)w(x,y)\,;y), \\ \nabla_y \cdot z(x,y) &= 0, \end{aligned}$$

where $z = z(x,y,t)$, $w = w(x,y,t)$, $v = v(x,y,t)$.

Periodic case

$$
\begin{aligned}
-\nabla \cdot \left(\int_Y z \, dy \right) &= f, \\
Cz &= \nabla_x^s u + \nabla_y^s v - w, \qquad \leftrightarrow \qquad -\nabla \cdot \Sigma(\nabla^s u) = f \\
\partial_t w &\in \partial \Psi(z - Bw \, ; y), \\
\nabla_y \cdot z &= 0,
\end{aligned}
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$$
\sigma^\varepsilon \rightharpoonup \int_Y z\,dy \qquad \overset{?}{\leftrightarrow} \qquad \sigma^\varepsilon \rightharpoonup \Sigma(\nabla^s u)
$$

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Periodic case

$$-\nabla \cdot \left(\int_Y z \, dy \right) = f, \qquad \partial_t w \in \partial \Psi(z - Bw \, ; y),$$
$$Cz = \nabla_x^s u + \nabla_y^s v - w, \qquad \nabla_y \cdot z = 0,$$

For $\xi \in H^1(0,T;\mathbb{R}^{n\times n}_s)$, we can formally consider the ODE

$$\partial_t w_\xi = \partial \Psi(z_\xi - Bw_\xi; y)\,,$$
$$\xi = Cz_\xi + w_\xi - \nabla_y^s v_\xi \qquad \nabla_y \cdot z_\xi = 0$$

where $z_\xi, w_\xi \in L^2(Y;\mathbb{R}^{n\times n}_s)$ $v_\xi \in H^1_{per}(Y;\mathbb{R}^n)$.

$$\Sigma(\xi) := \int_Y z_\xi \, dy$$

Periodic case

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For $\xi \in H^1(0,T;\mathbb{R}_s^{n\times n})$, we can formally consider the ODE

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$$\Sigma(\xi) := \int_Y z_\xi \, dy$$

Theorem

The operator $\Sigma : H^1(0,T;\mathbb{R}_s^{n\times n}) \to H^1(0,T;\mathbb{R}_s^{n\times n})$ has all properties that are necessary to apply our Main Theorem.

Stochastic case

On a probability space $(\Omega, \Sigma_\Omega, \mathcal{P})$ with an ergodic dynamical system $(\tau_x)_{x\in\mathbb{R}^n}$, we consider $C, B \in L^\infty(\Omega)$ and

$$C_\varepsilon(x) := C(\tau_{\frac{x}{\varepsilon}}\omega)\,, \quad B_\varepsilon(x) := B(\tau_{\frac{x}{\varepsilon}}\omega)\,, \quad \Psi_\varepsilon(\sigma; x) := \Psi(\sigma; \tau_{\frac{x}{\varepsilon}}\omega)\,.$$

Consider the ODE

$$\partial_t p(t,\omega) \in \partial\Psi\left(z(t,\omega) - B(\omega)\,p(t,\omega)\,;\,\omega\right)\,, \qquad C\,z = \xi + v^s - p\,,$$

where $z \in L^2(0,T; L^2_{sol}(\Omega))$, $v \in L^2(0,T; L^2_{pot}(\Omega))$ and define

$$\Sigma(\xi) := \int_\Omega z\,d\mathcal{P}\,.$$

Theorem

The operator $\Sigma\,:\; H^1(0,T;\mathbb{R}^{n\times n}_s) \to H^1(0,T;\mathbb{R}^{n\times n}_s)$ has all properties that are necessary to apply our Main Theorem.

Conclusions

- Non periodic homogenization of the Prandtl-Reuss plasticity model
- Averaging Property
- Needle-problem approach
- Adapted grids
- Special case: Periodic geometry
- Special case: Stochastic homogenization

– THANK YOU ! –