Structure/regularity of solutions to sudden directional diffusion systems

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based joint results with Piotr Rybka, Monika Muszkieta

The subject of our analysis will be

$$u_t - \partial_x L(u_x) = f$$

with suitable boundary conditions and initial datum
we prefer Dirichlet boundary conditions

Function L is assumed to be monotone (increasing), in other words:

$$(L(u_x), u_x) \in \mathcal{A} \qquad \text{with } \mathcal{A} \qquad \text{maximal monotone}$$

Whenever our key goal is to study multidimensional case like

$$u_t - \operatorname{div}(\operatorname{sgn} u_{x_1}, \operatorname{sgn} u_{x_2}) = 0$$

$$u_t - \operatorname{div}(\operatorname{sgn} u_{x_1}, \operatorname{sgn} u_{x_2}) - \Delta u = 0$$

What we can observe are special feature which are not present in qualitative picture of monodimensional solutions.

A way to find a connection between 1d and multi-d case are inhomogeneous systems.

In 1d we'd like to study model case:

$$L(w) = w + \operatorname{sgn} w = \begin{cases} w+1 & \text{for} \quad w>0 \\ [-1,1] & \text{for} \quad w=0 \\ w-1 & \text{for} \quad w<0 \end{cases}$$

goal: the behavior/structure of solutions
we are interested in facets, flat regions of the solutions
ruled surfaces

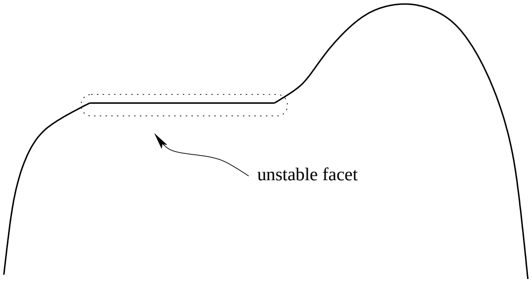

Figure: A facet

The results are based on:

PBM Stagnation, creation, breaking, in preparation ...

PBM, M Muszkieta, P Rybka, Two cases of squares evolving by anisotropic diffusion, arXiv:1303.1655

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PBM Regular solutions to a model monodimensional system with discontinuous elliptic operator, *Interfaces Free Boundaries*, 2012.

PBM, P Rybka, A note on a model system with sudden directional diffusion, *Journal of Statistical Physics 146*, 5, 975-988 (2012).

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PBM, P Rybka, A caricature of a singular flow in the plane, *Nonlinearity 21*, 2281-2316 (2008).

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Y Giga, R Kobayashi, Equations with singular diffusivity, J STATISTICAL PHYSICS 95, 1187-1220 (1999) .

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G Bellettini, V Caselles, M Novaga, The total variation flow in R-N, J DIFF EQS 184, 475-525 (2002).

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A Briani, A Chambolle, M Novaga, O Giandomenico, On the gradient flow of a one-homogeneous functional, arXiv:1109.6765

A Chambolle, M Novaga, Existence and uniqueness for planar anisotropic and crystalline curvature flow, arXiv:1302.2216

We consider

$$u_t - \partial_x(u_x + \operatorname{sgn} u_x) = f(t)$$

with boundary data

$$u(0,t) = 0, \qquad u(1,t) = 0$$

and initial datum

$$u\big|_{t=0} = u_0$$

We are allowed to determine typical shapes of solutions

Typical front of solution (flanc)

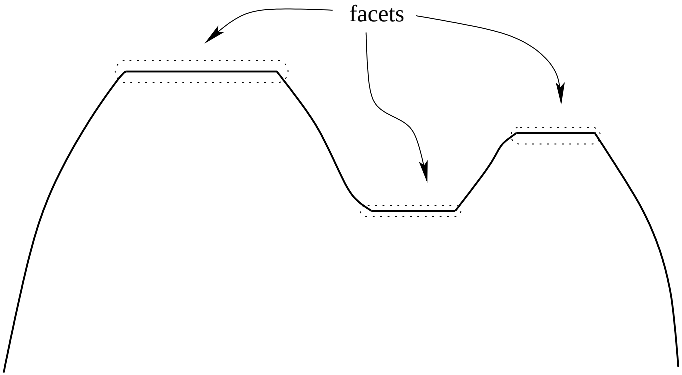

Figure: Addmissible profile

an example of unstable shape
(it may happen just in isolated time point)

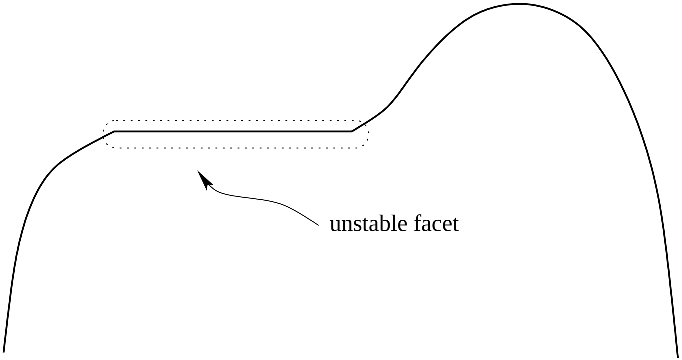

Figure: Nonadmissible shape

Qualitative analysis for 1d problems

We want to analyze the possible behavior of facets:

– stagnation

– creation

– breaking

STAGNATION

Observation

Let $u_0 \equiv 0$, if

$$\|f\|_{L_\infty} \text{ is sufficiently small}$$

then

$$u \equiv 0$$

The proof of this fact is an immediate consequence of the result for the steady case

More precisely we have

THEOREM

If $u_0 = 0$ and

$$0 \leq \int_0^x f(s,t)ds \leq 2 \quad \text{for} \quad x \in (0,1) \quad \text{and} \quad t \in (0,T)$$

Then

$$u \equiv 0$$

The weak formulation

$$(u_t, \phi)_{L_2(0,1)} + (\sigma, \phi_x)_{L_2(0,1)} = (f, \phi)_{L_2(0,1)}$$

where $\sigma(x,t) \in L \circ u_x$

PROOF.

we have

$u \equiv 0$ and

$$\sigma(x,t) = -1 + \int_0^x f(s,t)ds \in [-1,1] = L(0)$$

So the proof of the theorem is done.

ps. we see that there is a room to generalize the assumptions on f

More complex case

Let u_0 be a steady solution to the problem

$$-\partial_x(u_x + \operatorname{sgn} u_x) = f(0)$$

say $f(0) \geq 0$

Then there exist points $\xi_- < \xi_+$ such that $0 < \xi_- < \xi_+ < 1$ and

$$\int_{\xi_-}^{\xi^+} f(s, 0) ds = -2$$

We have a picture

Figure: Shape of u_0

u the solution to the evolutionary system

$$u_t - \partial_x(u_x + \operatorname{sgn} u_x) = f(t)$$

is <u>static</u>

$$ie. \qquad u(x,t) = u_0(x)$$

if $f(t)$ fulfills

supp $f(t) - f(0) \subset [\xi_-, \xi_+]$ for all $t \in (0, T)$

$\displaystyle\int_{\xi_-}^{\xi_+} f(x,t)dx = -2$

$\displaystyle|\int_a^b f(x,t)dx| < 2$ for all a, b s.t. $\xi_- < a < b < \xi_+$.

A proof of the above fact follows directly from the considerations for the stationary case,

just for all t, the function $f(t)$ generates the same stationary solution,

so the uniqueness of the evolutionary system yields the stagnation of the initial state.

CREATION

The first observation is the following:

facets are typical

fixing time, the minima, maxima are realized over facets (non degenerated)

Let u be sufficiently smooth solution to

$$u_t - \partial_x(u_x + \operatorname{sgn} u_x) = f(t)$$

with $u(0,t) = u(1,t) = 0$

Fix $t > 0$, we look at the function $u(\cdot, t)$ over $(0, 1)$.

Assume that at $\xi_0 \in (0, 1)$ the function u has a minimum

Take two sequences: $\{a_n\}, \{b_n\}$ such that

$$u(a_n), u(b_n) > u(\xi_0)$$

and

$$a_n \to \xi_0^- \quad \text{and} \quad b_n \to \xi_0^+$$

Integrate the equation over (a_n, b_n)

$$-(u_x + \operatorname{sgn} u_x)|_{a_n}^{b_n} = \int_{a_n}^{b_n} (f(t) - u_t)dx$$

Passing to the limit we get

$$-2 = \int_a^b (f(t) - u_t)dx$$

As f, u_t is integrable then we conclude that $a \neq b$, so

$$u(\xi_0) = u(a) = u(b)$$

the minimum is obtained on the interval $[a, b]$

Observation:

The regularity of solutions seems to be very important here

BREAKING

If a facet can be broken?

We go back to the total variation flow
(the simples case)

$$u_t - \partial_x \operatorname{sgn} u_x = f$$

Let us consider the case

$u(0, t) = u(1, t) = 0$ and $u_0 = 0$.

Take $f(x, t) = tg(x)$ and supp $g \subset [1/4, 3/4]$.

However one can consider the problem over the interval $[1/4, 3/4]$

$$u_t^R - \partial_x \operatorname{sgn} u_x^R = f$$

and assume

$u^R(1/4, t) = u^R(3/4, t) = 0$ and $u_0^R = 0$.

The existence of weak solutions is clear in both cases.

In addition it is a simple exercise that an extension of u^R on the interval $(0, 1)$ by zero defines a weak solution to the first problem,

so the uniqueness of solutions implies the following shape of the solution

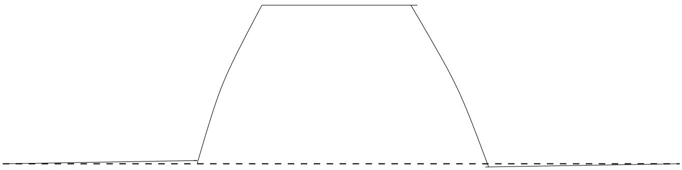

Figure: Breaking of facet

What about multidimensional case ...

We put our attention on the 2d mentioned systems:

$$u_t - \operatorname{div}\left(\operatorname{sgn} u_{x_1}, \operatorname{sgn} u_{x_2}\right) - \gamma \Delta u = 0$$

with suitable boundary and initial conditions.

we keep in mind

$$\gamma > 0 \quad \text{or} \quad \gamma = 0$$

THEOREM

Let $u_0 \in H^1(K)$, then the solutions fulfill the following estimate

$$\|u_t\|_{L_2(0,T\times K)} + \sup_{t\in[\delta,T]} \|u_t, \gamma\nabla^2 u\|_{L_2(K)}(t) \leq DATA(\delta).$$

Ruled surface and convexity

The first phenomenon we study are features of minimizers and maximizers of the solution.

We ask about possible structure of sets where the function u, for fixed time t, admits extrema.

Since the issue of regularity is not well studied here we prove only the following result.

LEMMA

Let u be a solution to equation with $\gamma > 0$ or $\gamma = 0$.

Let $t > 0$ and for x_0 in the domain $u(\cdot, t)$ has a minimum at x_0
$u(\cdot, t)$ is a convex function in a neighborhood N
of set $u(\cdot, t) = u(x_0, t)$,

then the set

$$M = \{x : u(x, t) = u(x_0, t)\} \cap N$$

is a closed set with nonempty interior.

Proof.

There is a sequence m_n converging to $m := u|_M$ from above and such that each

level sets $\{u(\cdot, t) = m_n\}$ are convex closed curves.

Moreover, the sets $M_n = \{u(\cdot, t) \le m_n\}$ are convex too.

Consider the equation over these sets

$$\int_{M_n} \big(u_t - \gamma \Delta u - \operatorname{div}\big(\operatorname{sgn} u_{x_1}, \operatorname{sgn} u_{x_2}\big)\big)\, dx_1 dx_2 = 0.$$

Integration by parts and convexity lead to

$$\int_{\{u=m_n\}}(\gamma\frac{\partial u}{\partial n}+n_1\,\mathrm{sgn}\,u_{x_1}+n_2\,\mathrm{sgn}\,n_{x_2})dH^1=\int_{M_n}u_t\,dx_1dx_2.$$

But convexity implies that

$\frac{\partial u}{\partial n}\geq 0$ at ∂M_n.

At the same time for almost all y functions

$x_1\mapsto u(x_1,y,t)$ and $x_2\mapsto u(y,x_2,t)$

are monotone,

hence in a neighborhood of M_n

$$n_1 \operatorname{sgn} u_{x_1} + n_2 \operatorname{sgn} u_{x_2} = |n_1| + |n_2| \geq |n| = 1.$$

So we conclude

$$\int_{\{u=m_n\}} dH^1 \leq |M_n|^{1/2} \left(\int_{M_n} u_t^2 dx_1 dx_2 \right)^{1/2}.$$

Moreover, since u is not constant, then the sets M_n must have positive two-dimensional Lebesgue measure.

On the other hand, due to the isoperimetric inequality we have

$$H^1(\partial M_n) \geq \frac{1}{2\sqrt{\pi}}|M_n|^{1/2},$$

the identity holds for the disc. Hence

$$C \leq (\int_{M_n} u_t^2 dx_1 dx_2)^{1/2}.$$

Since u_t is square integrable, so the RHS above cannot go to zero when $n \to \infty$.
Thus, M is a convex set of positive two-dimensional measure, hence it must have nonempty interior.

□

The next feature concerns the shape of graph of solutions.

To be more precise, we will show that is the regular level set of u is convex, then the graph contains ruled surfaces parts of positive two-dimensional measure.

The tangent is orthogonal to vectors $(0,1)$ or $(1,0)$.

LEMMA

Let u be a sufficiently regular solution to for γ equal to 0 or 1. Furthermore, we assume that for given $c \in \mathbb{R}$, the level set

$$S(c) = \{x \in K : u(t,x) = c\}$$

for a fixed time t is convex and it is regular, everywhere $\nabla u|_{S(c)} \neq 0$, then sets

$$M_1^+ = \{x : x = (m_1^+, x_2) \in S(c)\},$$

where $m_1^+ = \max\{x_1 : (x_1, x_2) \in S(c)\}$;

$$M_1^- = \{x : x = (m_1^-, x_2) \in S(c)\}, ...$$

$$M_2^+ = \{x : x = (x_1, m_2^+) \in S(c)\}, ...$$

$$M_2^- = \{x : x = (x_1, m_2^-) \in S(c)\}, ...$$

do not contain isolated points.

For the level set we expect the following structure in $(0, 1)$ direction:

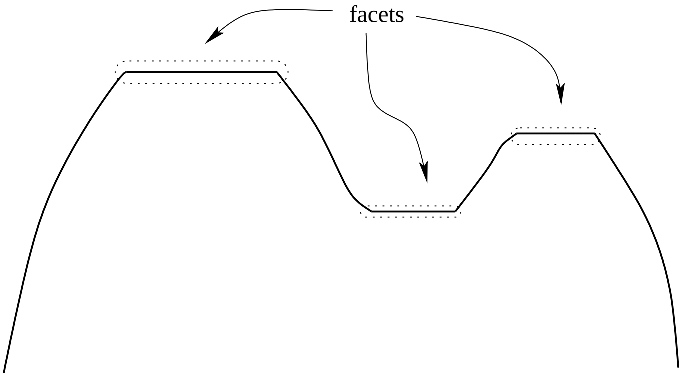

Figure: shape of the level set

Proof.

At the studied point we assume that $x_0 \in M_1^+$ is regular,

$$i.e. \nabla u(x_0) \neq 0,$$

it means $\partial_{x_1} u$ do not change the sign at this point,
so $\partial_{x_1} \operatorname{sgn} \partial_{x_1} u$ is zero at the neighborhood of the set M_1^+.
Just in the considered region the function must be monotone in x_1.

Take a connected component of M_1^+ it is a point or an interval, say $I = [(m_1^+, n^*), (m_1^+, n_*)]$ with $n^* \geq n_*$.

Take a sequence of points $\{a_k\}_{k\in\mathbb{N}}, \{b_k\}_{k\in\mathbb{N}}$ such that

$$(a_k^1, a_k^2) \to ((m_1^+)^-, (n_*)^-) \quad \text{and} \quad (b_k^1, b_k^2) \to ((m_1^+)^-, (n^*)^+)$$

and

$$u(t, a_k), u(t, b_k) = c \quad \text{and} \quad u_{x_2}(t, a_k) < 0, u_{x_2}(t, b_k) > 0.$$

In addition, we require that $a_k^1 = b_k^1$ and
points a_k and b_k are points of continuity of ∇u.

Take a rectangle R_k such that:

– longer edges $L^{\pm}$ are parallel to M_1^+ and shorter edges $S^{\pm}$ contains point a_k, b_k.

– $u_{x_2}(t,s) < 0$ for all $s \in S^-$ and $u_{x_2}(t,s) > 0$ for all $s \in S^+$.

– the length of shorter edges is $\epsilon > 0$.

Then we are allowed to integrate the equation over R_k. Indeed, we shall proceed via an approximation of the characteristic function of the set R_k. So we get

$$\int_{R_k} \partial_{x_1}(\operatorname{sgn} u_{x_1}) + \partial_{x_2}(\operatorname{sgn} u_{x_2}) dx_1 dx_2 = \int_{R_k} (u_t - \gamma \Delta u) dx_1 dx_2.$$

So

$$\int_{\partial R_k} \operatorname{sgn} u_{x_2} n_2 d\sigma = \int_{R_k} (u_t - \gamma \Delta u) dx_1 dx_2.$$

The term with $\operatorname{sgn} u_{x_1}$ vanishes because of the monotonicity in x_1, so by the chosen properties of points a_k, b_k we find

$$2\epsilon \leq |\int_{R_k}(u_t - \gamma\Delta u)dx_1dx_2| \leq \epsilon \sup_{\lambda\in(0,1)} \int_{\lambda L^- + (1-\lambda)L^+}(u_t - \gamma\Delta u)dx_2.$$

ie

$$2 \leq \sup_{\lambda\in(0,1)} \int_{\lambda L^- + (1-\lambda)L^+}(u_t - \gamma\Delta u)dx_2.$$

as $k \to \infty$, so the rhs must go to zero
it is a consequence of integrability of $u_t - \gamma\Delta u$,
$2 \leq 0$. Thus, M_1^+ must not be a point. □

THEOREM
Assume that for $t > 0$ and region A the solution $u(\cdot, t)$ restricted to a region A is convex, then
sets

$$S_1 = \{(x, u(x,t)) : x \in A, u_{x_1}(x) = 0\}$$

and

$$S_2 = \{(x, u(x,t)) : x \in A, u_{x_2}(x) = 0\}$$

are ruled surfaces.

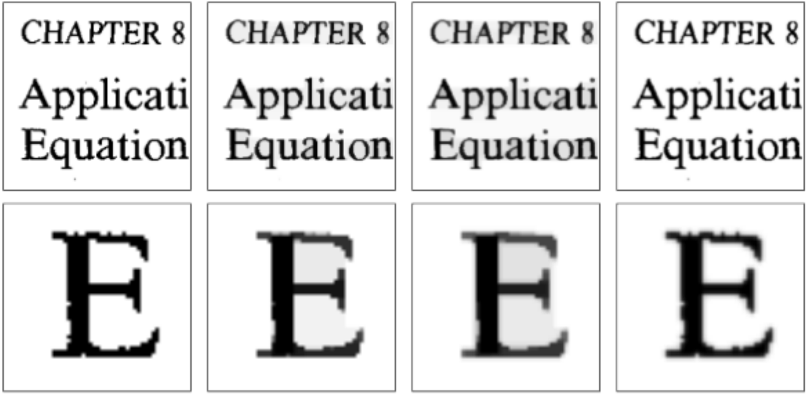

Figure: reconstruction

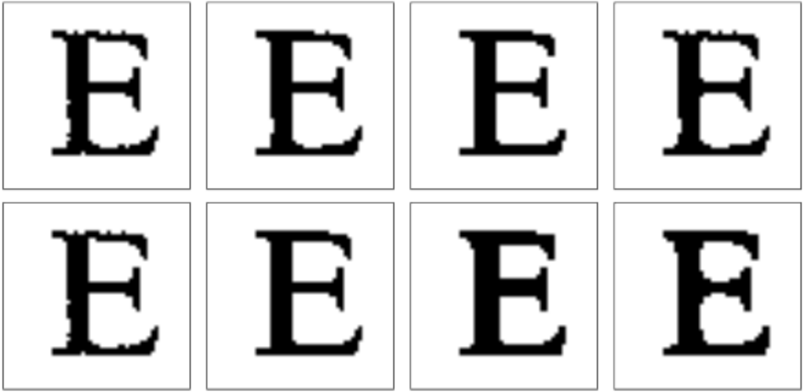

Figure: reconstruction

THANK YOU!

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