Nečas Center for Mathematical Modeling

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MINISTRY OF EDUCATION, YOUTH AND SPORTS

On implicit constitutive relations in continuum mechanics and thermodynamics

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Section 1

Introduction

Continuum mechanics

- Matter is a continuously distributed substance.
- Newton's second law, $m\frac{\mathrm{d}^2\mathbf{x}}{\mathrm{d}t^2} = \mathbf{F}$, is reformulated for the continuously distributed substance.

Governing equations – Eulerian description

Eulerian description:

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} + \rho \operatorname{div} \mathbf{v} = 0$$
$$\rho \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \operatorname{div} \mathbb{T} + \rho \mathbf{b}$$
$$\mathbb{T} = \mathbb{T}^{\top}$$

Eulerian description, convective form:

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{v}) = 0$$
$$\frac{\partial \rho \mathbf{v}}{\partial t} + \operatorname{div}(\rho \mathbf{v} \otimes \mathbf{v}) = \operatorname{div} \mathbb{T} + \rho \mathbf{b}$$
$$\mathbb{T} = \mathbb{T}^{\top}$$

All functions are functions of $\mathbf{x}$ and t, $\mathbb{T}$ is the Cauchy stress tensor.

Governing equations – Lagrangian description

Lagrangian description, $\mathbb{F} =_{\mathrm{def}} \frac{\partial \chi(\mathbf{X},t)}{\partial \mathbf{X}}$:

$$\begin{aligned} \rho_R(\mathbf{X}, t_0) &= \rho(\chi(\mathbf{X}, t), t) \det \mathbb{F}(\mathbf{X}, t) \\ \rho_R \frac{\partial^2 \chi}{\partial t^2} &= \operatorname{Div} \mathbb{T}_{\mathrm{R}} \\ \mathbb{T}_{\mathrm{R}} \mathbb{F}^{\top} &= \mathbb{F} \mathbb{T}_{\mathrm{R}}^{\top} \end{aligned}$$

All functions are, if not stated otherwise, functions of $\mathbf{X}$ and t, $\mathbb{T}_{\mathrm{R}}$ denotes the first Piola–Kirchhoff stress tensor

$$\mathbb{T}_{\mathrm{R}} =_{\mathrm{def}} (\det \mathbb{F})\, \mathbb{T} \mathbb{F}^{-\top}.$$

Constitutive relations

- Governing equations are universally valid for all materials. (As long as we believe that it makes sense to describe the material in the continuum mechanics framework.)
- Properties of particular material are specified by a relation between Cauchy stress tensor (or the first Piola–Kirchhoff tensor) and the kinematical variables.
- Boundary conditions can be seen as a specification of properties of material interface.

Dynamic slip model

Dynamic slip model, $u_s =_{\mathrm{def}} \mathbf{v} \bullet \mathbf{t}$, $\sigma_w =_{\mathrm{def}} (\mathbb{T}\mathbf{n}) \bullet \mathbf{t}$:

$$u_s + \lambda_s \frac{\mathrm{d} u_s}{\mathrm{d} t} = a\sigma_w^m$$

Savvas G. Hatzikiriakos. Wall slip of molten polymers. *Prog. Polym. Sci.*, 37(4):624–643, 2012

Constitutive relations

Standard approach: Stress is an function of kinematical variables.

$$\mathbb{T} = \mathfrak{f}(\mathbb{D})$$

Alternative approach: There is a relation between stress and kinematical variables.

$$\mathfrak{f}(\mathbb{T}, \mathbb{D}) = \mathbb{0}$$

Section 2

Implicit constitutive relations

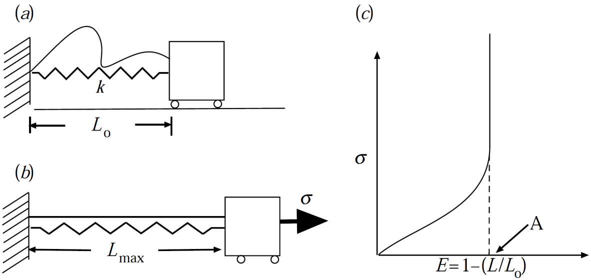


Figure 1. A simple mechanical example of a spring mass system with a stiffening spring in parallel with an inextensible string. (a) Initial state with unstretched spring and slack string. (b) The response of the spring to an applied load demonstrating the maximum extension. (c) Resulting stress versus strain curve.

K. R. Rajagopal. On implicit constitutive theories. *Appl. Math., Praha*, 48(4):279–319, 2003 K. R. Rajagopal and A. R. Srinivasa. On the response of non-dissipative solids. *Proc. R. Soc. Lond., Ser. A, Math. Phys. Eng. Sci.*, 463(2078):357–367, 2007

Section 3

Examples

Examples

- Consistent approach to geometric linearization for non-linear stress–strain relations.
- Non-dissipative solids that are not hyperelastic.
- Implicit constitutive relations for fluid like materials with fading memory.
- Stress power-law fluids.

Nonlinear stress–strain relation and geometric linearization

Governing equations – Lagrangian description

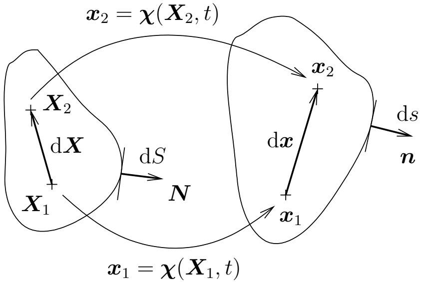
Lagrangian description, $\mathbb{F} =_{\mathrm{def}} \frac{\partial \chi(\mathbf{X},t)}{\partial \mathbf{X}}$:

$$
\begin{aligned}
\rho_R(\mathbf{X}, t_0) &= \rho(\chi(\mathbf{X}, t), t) \det \mathbb{F}(\mathbf{X}, t) \\
\rho_R \frac{\partial^2 \chi}{\partial t^2} &= \operatorname{Div} \mathbb{T}_{\mathrm{R}} \\
\mathbb{T}_{\mathrm{R}} \mathbb{F}^{\top} &= \mathbb{F} \mathbb{T}_{\mathrm{R}}^{\top}
\end{aligned}
$$

All functions are, if not stated otherwise, functions of $\mathbf{X}$ and t, $\mathbb{T}_{\mathrm{R}}$ denotes the first Piola stress tensor

$$
\mathbb{T}_{\mathrm{R}} =_{\mathrm{def}} (\det \mathbb{F}) \mathbb{T} \mathbb{F}^{-\top}.
$$

Geometric nonlinearity



Boundary condition:

$$\mathbb{T}\mathbf{n} = \mathbf{s}(\mathbf{x}, t)$$

Transformed boundary condition:

$$\mathbb{T}\mathbf{n} = \mathbb{T}_{\mathrm{R}}\mathbf{N}$$

$$\mathbb{T}_{\mathrm{R}}\mathbf{N} = (\det \mathbb{F}) \left| \mathbb{F}^{-\top}\mathbf{N} \right| \mathbf{s}(\chi(\mathbf{X}, t), t)$$

Geometric linearization

Formulate the problem in terms of displacement:

$$\mathbf{u} = \chi(\mathbf{X}, t) - \mathbf{X}$$

Deformation gradient:

$$\mathbb{F} = \frac{\partial \chi}{\partial \mathbf{X}} = \mathbb{I} + \frac{\partial \mathbf{u}}{\partial \mathbf{X}}$$

Classical approach, $\mathbb{e} =_{\mathrm{def}} \frac{1}{2}\left(\mathbb{F}\mathbb{F}^{\top} - \mathbb{I}\right)$:

$$\mathbb{T} = \mathbb{T}(\mathbb{e})$$

Declare $\left|\frac{\partial \mathbf{u}}{\partial \mathbf{X}}\right|$ to be a small quantity and neglect nonlinear terms:

$$\mathbb{T}_{\mathrm{R}} =_{\mathrm{def}} (\det \mathbb{F})\, \mathbb{T}\mathbb{F}^{-\top} \approx \mathbb{T}$$

Unfortunately:

$$\mathbb{T} = \mathbb{T}(\mathbb{e}) \approx \lambda \left(\operatorname{div} \mathbf{u}\right) \mathbb{I} + 2\mu \left[\frac{1}{2}\left(\frac{\partial \mathbf{u}}{\partial \mathbf{X}} + \left(\frac{\partial \mathbf{u}}{\partial \mathbf{X}}\right)^{\top}\right)\right] = \lambda \left(\operatorname{div} \varepsilon\right) \mathbb{I} + 2\mu\varepsilon$$

Geomeric linearization – implicit constitutive relation

Recall:

$$\mathbb{F} = \frac{\partial \boldsymbol{\chi}}{\partial \mathbf{X}} = \mathbb{I} + \frac{\partial \mathbf{u}}{\partial \mathbf{X}}$$

Alternative approach, $\mathbb{e} =_{\mathrm{def}} \frac{1}{2}\left(\mathbb{F}\mathbb{F}^{\top} - \mathbb{I}\right)$:

$$\mathbb{e} = \mathfrak{f}\left(\mathbb{T}\right)$$

Declare $\left|\frac{\partial \mathbf{u}}{\partial \mathbf{X}}\right|$ to be a small quantity and neglect nonlinear terms:

$$\mathbb{T}_{\mathrm{R}} =_{\mathrm{def}} \left(\det \mathbb{F}\right) \mathbb{T}\mathbb{F}^{-\top} \approx \mathbb{T}$$

But linearization of the constitutive relation is:

$$\varepsilon = \frac{1}{2}\left(\frac{\partial \mathbf{u}}{\partial \mathbf{X}} + \left(\frac{\partial \mathbf{u}}{\partial \mathbf{X}}\right)^{\top}\right) \approx \mathfrak{f}\left(\mathbb{T}\right)$$

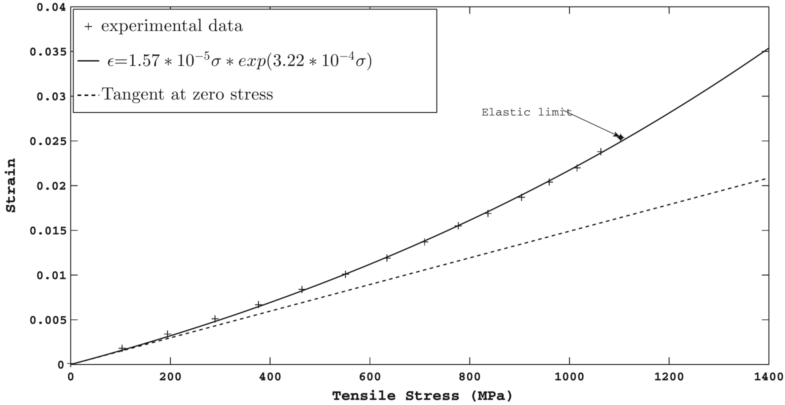


Fig. 1 Strain-stress relation for a Gum metal alloy (Figure is taken from the paper by Saito et al. [29])

K.R. Rajagopal. On the nonlinear elastic response of bodies in the small strain range. *Acta Mech.*, pages 1–9, 2013

Mechanical response of human patellar tendons

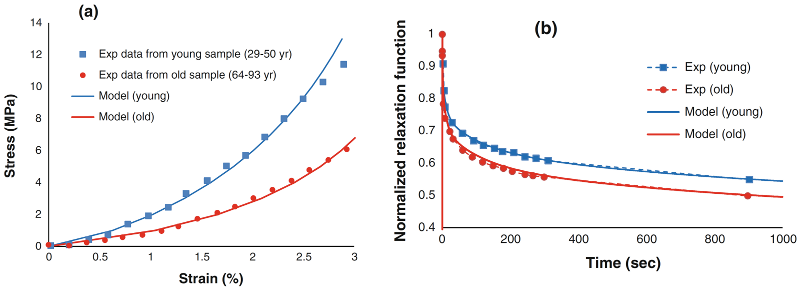


Fig. 5 Response of human patellar tendons under **a** ramp loadings with a constant strain rate and **b** stress relaxation

$$\varepsilon = \mathbf{f}(\mathbb{T}(0), t) + \int_{s=0}^{t} \frac{\partial \mathbf{f}}{\partial \mathbb{T}}(\mathbb{T}(s), t-s)\frac{\mathrm{d}\mathbb{T}}{\mathrm{d}s}\mathrm{d}s$$

A. Muliana, K. R. Rajagopal, and A. S. Wineman. A new class of quasi-linear models for describing the nonlinear viscoelastic response of materials. *Acta Mech.*, 224(9):2169–2183, 2013

Non-dissipative solids that are not hyperelastic

Classical approach, $\mathbb{e} =_{\mathrm{def}} \frac{1}{2}\left(\mathbb{F}\mathbb{F}^{\top} - \mathbb{I}\right)$:

$$\mathbb{T} = \mathbb{T}(\mathbb{e})$$

Alternative approach, $\mathbb{E} =_{\mathrm{def}} \frac{1}{2}\left(\mathbb{F}^{\top}\mathbb{F} - \mathbb{I}\right)$, $\mathbb{S}_{\mathrm{R}} = \mathbb{F}^{-1}\mathbb{T}_{\mathrm{R}}$:

$$\mathfrak{f}(\mathbb{S}_{\mathrm{R}}, \mathbb{E}) = \mathbb{0}$$

Taking time derivative yields:

$$\mathfrak{g}(\mathbb{S}_{\mathrm{R}}, \mathbb{E})\frac{\mathrm{d}\mathbb{S}_{\mathrm{R}}}{\mathrm{d}t} + \mathfrak{h}(\mathbb{S}_{\mathrm{R}}, \mathbb{E})\frac{\mathrm{d}\mathbb{E}}{\mathrm{d}t} = \mathbb{0}$$

Simple one dimensional model $\psi(\sigma, \varepsilon) = 0$

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{\sigma - \frac{\partial \psi(\sigma,\varepsilon)}{\partial \varepsilon}}{\frac{\partial \psi}{\partial \sigma}(\sigma, \varepsilon)} \frac{\mathrm{d}\varepsilon}{\mathrm{d}t}$$

$$\psi(\sigma, \varepsilon) =_{\mathrm{def}} a_1 \sigma^2 + a_2 \varepsilon^2 + a_3 \sigma \varepsilon + a_4 \sigma + a_5 \varepsilon$$

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{(1 - a_3)\sigma - 2a_2\varepsilon - a_5}{2a_1\sigma + a_3\varepsilon + a_4} \frac{\mathrm{d}\varepsilon}{\mathrm{d}t}$$

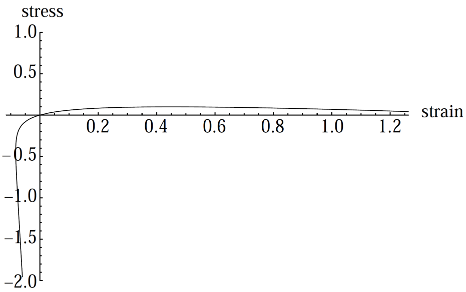


Figure 2. Stress–strain response of the non-dissipative material, whose implicit constitutive equation is given by equation (3.11) with constants given by $a_1 = 0.01$, $a_2 = -0.1$, $a_3 = 1.1$, $a_4 = 0.1$ and $a_5 = -0.1$. Note the fact that the compressive response is multivalued.

K. R. Rajagopal and A. R. Srinivasa. On the response of non-dissipative solids. *Proc. R. Soc. Lond., Ser. A, Math. Phys. Eng. Sci.*, 463(2078):357–367, 2007

An implicit elastic theory for lung parenchyma

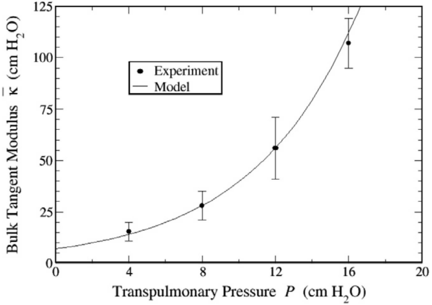


Fig. 2. Bulk tangent modulus $\bar{\kappa}$ for canine lung. Data are from Lai-Fook et al. (1976) and Lai-Fook (1979) and are tabulated in Table 2. Correlations are from Eqs. (25) and (31) where $\alpha = 4$ cm H_2O and $\kappa = 7$ cm H_2O.

Alan D. Freed and Daniel R. Einstein. An implicit elastic theory for lung parenchyma. *Int. J. Eng. Sci.*, 62(0):31–47, 2013

An implicit elastic theory for lung parenchyma

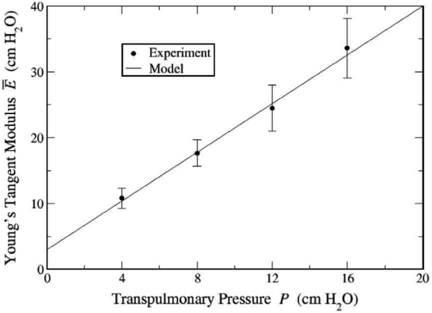


Fig. 4. Elastic tangent modulus $\overline{E}$ of Young for canine lung. Data are from Lai-Fook et al. (1976) and are tabulated in Table 2. A correlation of these data to the model of Eq. (47) are drawn for $\mu = 1.5$ cm H_2O and $\beta = 3.85$.

Alan D. Freed and Daniel R. Einstein. An implicit elastic theory for lung parenchyma. *Int. J. Eng. Sci.*, 62(0):31–47, 2013

Implicit constitutive relations for fluids with fading memory

Fading memory

Explicit formula for Cauchy stress:

$$\mathbb{T} = -p\mathbb{I} + \mathfrak{F}_{s=0}^{+\infty}(\mathbb{C}_t\,(t-s))$$

Bernard D. Coleman and Walter Noll. An approximation theorem for functionals, with applications in continuum mechanics. *Arch. Ration. Mech. Anal.*, 6:355–370, 1960

Implicit relation between the histories:

$$\mathfrak{H}_{s=0}^{+\infty}\,(\mathbb{T}(t-s), \mathbb{C}_t\,(t-s)) = \mathbb{0}$$

V. Průša and K. R. Rajagopal. On implicit constitutive relations for materials with fading memory. *J. Non-Newton. Fluid Mech.*, 181–182:22–29, 2012

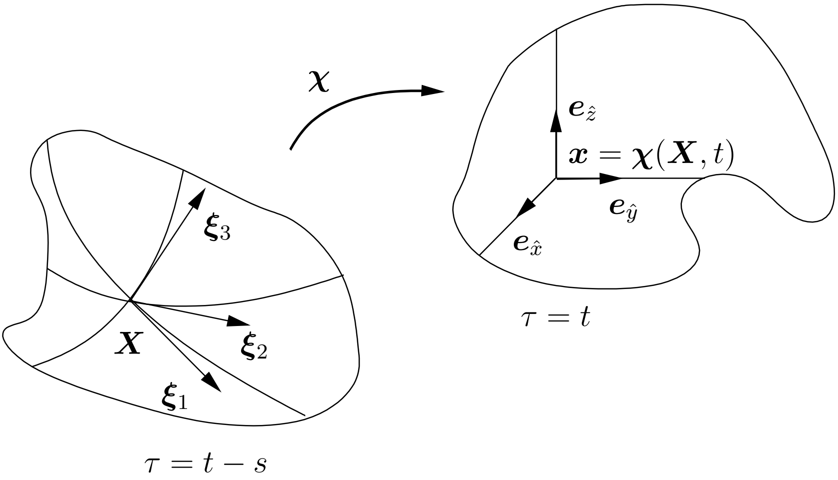
Independent variables

[. . .] the properties of a material element may depend upon the previous rheological states through which that element has passed, but not in any way on the states of neighbouring elements and not on the motion of the element as a whole in the space.

[. . .] only those tensor quantities need to be considered which have a significance for the material element independent of its motion as a whole in space.

J. G. Oldroyd. On the formulation of rheological equations of state. *Proc. R. Soc. A-Math. Phys. Eng. Sci.*, 200(1063):523–541, 1950

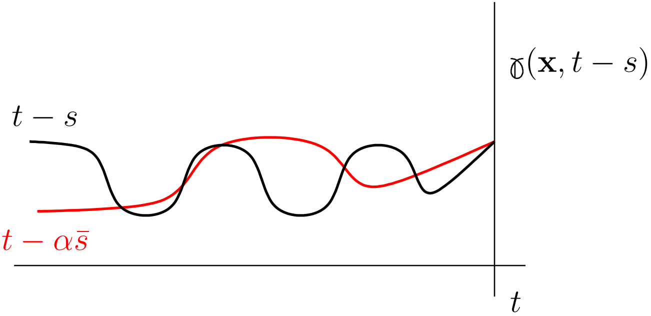
Convected coordinate system



Special form of the general implicit constitutive relation

$$
\begin{aligned}
\mathbb{T} &= -\pi \mathbb{I} + \mathbb{S} \\
\mathbb{0} &= \mathfrak{G}_{s=0}^{+\infty} \left(\mathbb{F}(\xi, t-s) - \mathbb{I}, \mathbb{S}(\xi, t-s) \right)
\end{aligned}
$$

Slow history/fading memory



Plan

- Formulate the constitutive relation in the *convected coordinate system*.

  H. Hencky. Die Bewegungsgleichungen beim nichtstationären Fließen plastischer massen. *Z. Angew. Math. Mech.*, 5:144–146, 1925

- Formulate the constitutive relation in an implicit form.

  K. R. Rajagopal. On implicit constitutive theories. *Appl. Math., Praha*, 48(4):279–319, 2003

- Expand the functional using an analogue of the *retardation theorem*.

  Bernard D. Coleman and Walter Noll. An approximation theorem for functionals, with applications in continuum mechanics. *Arch. Ration. Mech. Anal.*, 6:355–370, 1960

- Use *representation theorems* for isotropic linear and bilinear functions.

  C. Truesdell and W. Noll. The non-linear field theories of mechanics. In S. Flüge, editor, *Handbuch der Physik*, volume III/3. Springer, Berlin, 1965

- Transform the constitutive relation to a *fixed-in-space coordinate system*.

  J. G. Oldroyd. On the formulation of rheological equations of state. *Proc. R. Soc. A-Math. Phys. Eng. Sci.*, 200(1063):523–541, 1950

Approximation formulae

First order:

$$\mathfrak{G}_{s=0}^{+\infty}\left(\begin{bmatrix}\mathbb{F}(\xi, t-\alpha\bar{s})-\mathbb{I}\\ \mathbb{\sigma}(\xi, t-\alpha\bar{s})-\mathbb{\sigma}(\xi, t)\end{bmatrix}\right)$$

$$\mapsto b_0\left(\operatorname{Tr}\mathbb{S}\right)\mathbb{I}+b_1\mathbb{S}+2b_3\mathbb{D}+b_4\left(\operatorname{Tr}\overset{\triangledown}{\mathbb{S}}\right)\mathbb{I}+b_5\overset{\triangledown}{\mathbb{S}}+o\left(\alpha\right)$$

Approximation formulae

Second order:

$$
\begin{aligned}
&\mathfrak{G}_{s=0}^{+\infty}\left(\begin{bmatrix}\mathbb{b}(\xi, t-\alpha\bar{s}) - \mathbb{I} \\ \mathbb{\sigma}(\xi, t-\alpha\bar{s}) - \mathbb{\sigma}(\xi, t)\end{bmatrix}\right) \\
&\mapsto \Bigg[b_0\left(\operatorname{Tr}\mathbb{S}\right) + b_4\left(\operatorname{Tr}\overset{\triangledown}{\mathbb{S}}\right) + b_6\left(\operatorname{Tr}\overset{\triangledown\triangledown}{\mathbb{S}}\right) + \left(b_{15} - 2b_8\right)\operatorname{Tr}\left(\mathbb{D}\right)^2 \\
&+ \left(b_{10}\left(\operatorname{Tr}\mathbb{S}\right)^2 + b_{11}\operatorname{Tr}\left(\mathbb{S}\right)^2\right) + \left(b_{18}\left(\operatorname{Tr}\overset{\triangledown}{\mathbb{S}}\right)^2 + b_{19}\operatorname{Tr}\left(\overset{\triangledown}{\mathbb{S}}\right)^2\right) \\
&+ b_{23}\operatorname{Tr}\left(\mathbb{D}\overset{\triangledown}{\mathbb{S}}\right) + \left(b_{27}\operatorname{Tr}\mathbb{S}\operatorname{Tr}\overset{\triangledown}{\mathbb{S}} + b_{28}\operatorname{Tr}\left(\mathbb{S}\overset{\triangledown}{\mathbb{S}}\right)\right) + b_{33}\operatorname{Tr}\left(\mathbb{S}\mathbb{D}\right)\Bigg]\mathbb{I} \\
&+ \left[b_1 + b_{12}\left(\operatorname{Tr}\mathbb{S}\right) + b_{30}\left(\operatorname{Tr}\overset{\triangledown}{\mathbb{S}}\right)\right]\mathbb{S} + \left[b_3 + b_{25}\left(\operatorname{Tr}\overset{\triangledown}{\mathbb{S}}\right) + b_{34}\left(\operatorname{Tr}\mathbb{S}\right)\right]\mathbb{D} \\
&+ b_{13}\left(\mathbb{S}\right)^2 + b_{17}\left(\mathbb{D}\right)^2 + b_{36}\left(\mathbb{S}\mathbb{D} + \mathbb{D}\mathbb{S}\right) \\
&+ b_9\overset{\triangledown}{\mathbb{D}} + \left[b_5 + b_{20}\left(\operatorname{Tr}\overset{\triangledown}{\mathbb{S}}\right) + b_{29}\left(\operatorname{Tr}\mathbb{S}\right)\right]\overset{\triangledown}{\mathbb{S}} + b_{21}\left(\overset{\triangledown}{\mathbb{S}}\right)^2 \\
&+ b_{26}\left(\mathbb{D}\overset{\triangledown}{\mathbb{S}} + \overset{\triangledown}{\mathbb{S}}\mathbb{D}\right) + b_{31}\left(\mathbb{S}\overset{\triangledown}{\mathbb{S}} + \overset{\triangledown}{\mathbb{S}}\mathbb{S}\right) + b_7\overset{\triangledown\triangledown}{\mathbb{S}} + o\left(\alpha^2\right)
\end{aligned}
$$

Rate type models

$$\mathbb{T} = -\pi\mathbb{I} + \mathbb{S}$$

J. G. Oldroyd. Non-newtonian effects in steady motion of some idealized elastico-viscous liquids. *Proc. R. Soc. A-Math. Phys. Eng. Sci.*, 245(1241):278–297, 1958

$$\mathbb{S} + \lambda_1 \overset{\triangledown}{\mathbb{S}} + \frac{\lambda_3}{2}\left(\mathbb{D}\mathbb{S} + \mathbb{S}\mathbb{D}\right) + \frac{\lambda_5}{2}\left(\operatorname{Tr}\mathbb{S}\right)\mathbb{D} + \frac{\lambda_6}{2}\left(\mathbb{S} : \mathbb{D}\right)\mathbb{I} = -\mu\left(\mathbb{D} + \lambda_2 \overset{\triangledown}{\mathbb{D}} + \lambda_4 \mathbb{D}^2 + \frac{\lambda_7}{2}\left(\mathbb{D} : \mathbb{D}\right)\mathbb{I}\right)$$

N. Phan Thien. Non-linear network viscoelastic model. *J. Rheol.*, 22(3):259–283, 1978

$$Y\mathbb{S} + \lambda\overset{\triangledown}{\mathbb{S}} + \frac{\lambda\xi}{2}\left(\mathbb{D}\mathbb{S} + \mathbb{S}\mathbb{D}\right) = -\mu\mathbb{D}$$

$$Y = \mathrm{e}^{-\varepsilon\frac{\lambda}{\mu}\operatorname{Tr}\mathbb{S}}$$

Notation:

$$\overset{\triangledown}{\mathbb{b}^\flat} =_{\mathrm{def}} \frac{\mathrm{d}\mathbb{b}^\flat}{\mathrm{d}t} - [\nabla \mathbf{v}]\,\mathbb{b}^\flat - \mathbb{b}^\flat\,[\nabla \mathbf{v}]^\top$$

Subsection 4

Incompressible stress power-law fluids

Incompressible stress power law fluids

Classical power-law fluids:

$$\mathbb{T} = -p\mathbb{I} + 2\mu_0 \left(1 + \mu_1 \left|\mathbb{D}\right|^2\right)^m \mathbb{D}$$

Stress power-law fluids:

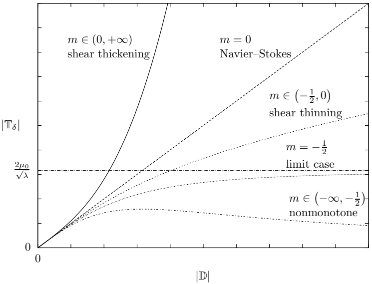
$$\mathbb{D} = \alpha \left(1 + \beta \left|\mathbb{T}_\delta\right|^2\right)^n \mathbb{T}_\delta$$

Stress power-law fluids, linear response at infinity:

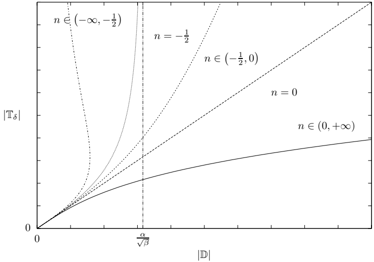
$$\mathbb{D} = \alpha \left(1 + \beta \left|\mathbb{T}_\delta\right|^2\right)^n \mathbb{T}_\delta + \gamma \mathbb{T}_\delta$$

J. Málek, V. Průša, and K. R. Rajagopal. Generalizations of the Navier–Stokes fluid from a new perspective. *Int. J. Eng. Sci.*, 48(12):1907–1924, 2010 Christiaan Roux and K. R. Rajagopal. Shear flows of a new class of power-law fluids. *Applications of Mathematics*, 58(2):153–177, 2013

Qualitative behaviour



(a) Classical power-law type model, $\mathbb{T} = -p\mathbb{I} + 2\mu_0 \left(1 + \mu_1 |\mathbb{D}|^2\right)^m \mathbb{D}$.



(b) Stress power-law type model, $\mathbb{D} = \alpha \left(1 + \beta |\mathbb{T}_\delta|^2\right)^n \mathbb{T}_\delta$.

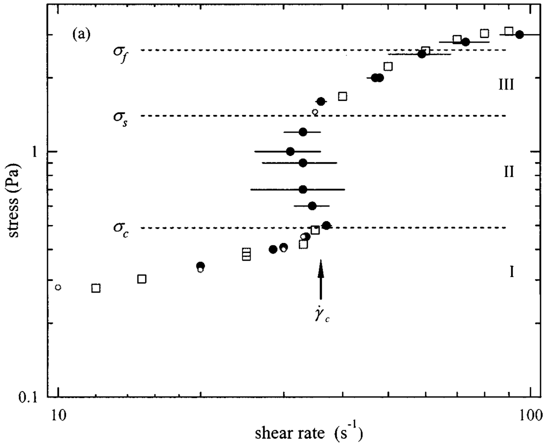
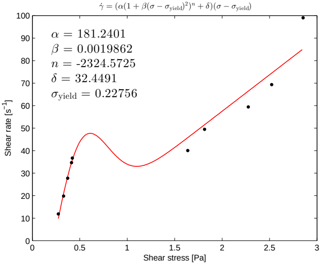
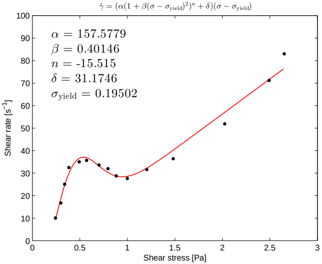


FIG. 2. Summary of the steady state rheology for constant applied stress (solid circles) and shear rate (open squares). The dashed lines indicate the different rheological regimes of behavior. The small open circles locate the final stress/shear-rate pairs for the quenches plotted in Fig. 16.

Y. T. Hu, P. Boltenhagen, and D. J. Pine. Shear thickening in low-concentration solutions of wormlike micelles. I. Direct visualization of transient behavior and phase transitions. *J. Rheol.*, 42:1185–1208, 1998



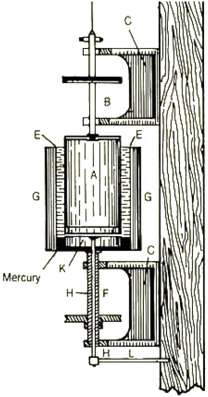
(c) Constant "rate of strain".



(d) Constant shear stress.

(Data fitted by Adam Janečka and Tereza Perlácová.) Y. T. Hu, P. Boltenhagen, and D. J. Pine. Shear thickening in low-concentration solutions of wormlike micelles. I. Direct visualization of transient behavior and phase transitions. *J. Rheol.*, 42:1185–1208, 1998

Constant rate of strain versus constant shear stress experiment



Fully implicit models

Algebraic type relations:

$$\mathfrak{f}(\mathbb{T}, \mathbb{D}) = \mathbb{0}$$

One dimensional data sets can be fitted by the following function:

$$\dot{\gamma} = \frac{p_1 \sigma^3 + p_2 \sigma^2 + p_3 \sigma + p_4}{\sigma^2 + q_1 \sigma + q_2}$$

General relation for isotropic tensor function of $\mathbb{T}$ and $\mathbb{D}$:

$$\alpha_0 \mathbb{I} + \alpha_1 \mathbb{T} + \alpha_2 \mathbb{D} + \alpha_3 \mathbb{T}^2 + \alpha_4 \mathbb{D}^2 + \alpha_5 \left(\mathbb{T}\mathbb{D} + \mathbb{D}\mathbb{T}\right) \\ + \alpha_6 \left(\mathbb{T}^2\mathbb{D} + \mathbb{D}\mathbb{T}^2\right) + \alpha_7 \left(\mathbb{T}\mathbb{D}^2 + \mathbb{D}^2\mathbb{T}\right) + \alpha_8 \left(\mathbb{T}^2\mathbb{D}^2 + \mathbb{D}^2\mathbb{T}^2\right) = \mathbb{0}$$

Invariants:

$$\alpha_i = \alpha_i \left(\operatorname{Tr}\mathbb{D}, \operatorname{Tr}\mathbb{T}, \operatorname{Tr}\mathbb{D}^2, \operatorname{Tr}\mathbb{T}^2, \operatorname{Tr}\mathbb{T}^3, \operatorname{Tr}\mathbb{D}^2, \\ \operatorname{Tr}\left(\mathbb{T}\mathbb{D}\right), \operatorname{Tr}\left(\mathbb{T}^2\mathbb{D}\right), \operatorname{Tr}\left(\mathbb{T}\mathbb{D}^2\right), \operatorname{Tr}\left(\mathbb{T}^2\mathbb{D}^2\right)\right)$$

Section 4

Conclusion

Conclusion

- There exist a unifying approach to the theory of constitutive relations in continuum mechanics.
- The idea of implicit constitutive relations is useful in articulating several theoretical concepts in continuum mechanics.
- There exist materials that can be described *only* using implicit type constitutive relations.
- There exists a solid thermodynamical background for implicit type constitutive relations.