# The Effect of Rough Walls on Laminar Flows

works with

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# 1. Setting of the problem

Motivation : Drag reduction in microfluidics.

Issue: to make fluids flow through very small devices.

*Minimizing the drag at the walls is important.*

Many theoretical and experimental works.
[Tabeling, 2004], [Bocquet, 2007 and 2012], [Vinogradova, 2012].
Some of these works claim that the usual no-slip condition is not always satisfied at the micrometer scale:

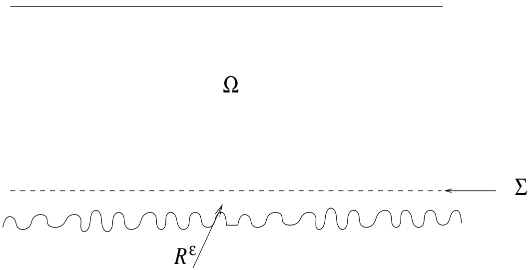
*Some rough surfaces may generate a substantial slip*.

However, these results are still debated.

Maths may help, notably through a homogenization approach.

## 2. A simple model

2D rough channel: $\Omega^\varepsilon = \Omega \cup \Sigma \cup R^\varepsilon$




- $\Omega$ : smooth part: $\mathbb{R} \times (0,1)$.
- $R^\varepsilon$ : rough part, *typical size* $\varepsilon \ll 1$.

$$R^\varepsilon \;=\; \{x = (x_1, x_2), \quad 0 > x_2 > \varepsilon\omega(x_1/\varepsilon)\}$$

$\omega$ with values in $(-1, 0)$, and $K$-Lipschitz.

- $\Sigma$ : interface: $\mathbb{R} \times \{0\}$.

Stationary Navier-Stokes, with given flow rate:

$$\boxed{\begin{cases} u \cdot \nabla u - \Delta u + \nabla p \; = \; 0, \quad x \in \Omega^\varepsilon, \\ \operatorname{div} u = 0, \quad x \in \Omega^\varepsilon, \\ u|_{\partial\Omega^\varepsilon} = 0, \quad \displaystyle\int_\sigma u_1 = \phi, \end{cases}} \qquad (\mathrm{NS}^\varepsilon)$$

with $\phi > 0$, $\sigma$ vertical cross-section.

*Homogenization problem*, as $\varepsilon \to 0$.

Goal: To get rid the oscillations, that is to replace $\Omega^\varepsilon$ by $\Omega$.

Question: What is the effective boundary condition at $\Sigma$ ?

Effective $=$ regular in $\varepsilon$.

# 3. Results

## a) Zeroth order B.C. : Dirichlet (no slip)

Idea: $$u^\varepsilon \approx u_D$$

where $u_D$ is the solution of Navier-Stokes in $\Omega$, with wall law

$$\boxed{u|_\Sigma = 0.}$$

Solution: *Poiseuille Flow* :

$$u_D = u_D(x_2) = (6\phi x_2(1 - x_2), 0).$$

**Theorem 1 :** *For $\phi$ small enough, $(\mathrm{NS}^\varepsilon)$ has a unique solution $u^\varepsilon$ in $H^1_{uloc}(\Omega^\varepsilon)$. Moreover,*

$$\|u^\varepsilon - u_D\|_{H^1_{uloc}(\Omega)} \leq C\sqrt{\varepsilon},$$
$$\|u^\varepsilon - u_D\|_{L^2_{uloc}(\Omega)} \leq C\,\varepsilon.$$

The limit boundary condition is no slip.

Question: Can we do better ?

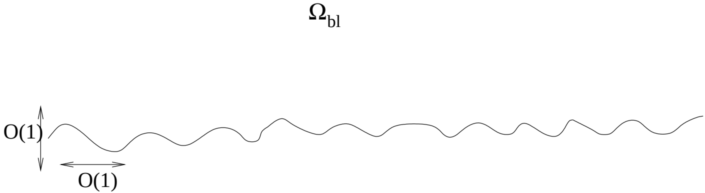
Can we detect slip at first order in $\varepsilon$ ?

## b) First order BC: Navier condition (slip condition)

Two ideas behind this Navier condition.

Idea 1:
$$u^\varepsilon \approx u_D + 6\phi\varepsilon v\left(\frac{x}{\varepsilon}\right),$$

$v = v(y)$: *Boundary layer corrector*. Cancels the trace of $u_D$ at $\Gamma^\varepsilon$.




Defined on $\Omega^{bl} := \{y_2 > \omega(y_1)\}$. Formally,

$$\boxed{\begin{cases} -\Delta v + \nabla p = 0, & y \in \Omega^{bl}, \\ \operatorname{div} v = 0, & y \in \Omega^{bl}, \\ v(y) = (-\omega(y_1), 0), & y \in \partial\Omega^{bl}. \end{cases}} \qquad \text{(BL)}$$

Idea 2: The boundary layer generates a non-zero mean flow

$$v \to v^\infty = (\alpha, 0), \quad \text{as } y_2 \to +\infty, \text{ for some } \alpha > 0.$$

Consequence: Formal expansion yields

$$u^\varepsilon \approx u_D + 6\phi\varepsilon(\alpha, 0) + o(\varepsilon) \quad \text{in } L^2$$

A better approximation should be the solution $u_N$ of NS in $\Omega$ with *Navier boundary condition*:

$$\boxed{u_2|_\Sigma = 0, \quad u_1|_\Sigma = \varepsilon\,\alpha\,\partial_2 u_1|_\Sigma.}$$

Pb: To make these formal ideas rigorous !

*The analysis of system* (BL) *is difficult.*

- *Well-posedness*:
  No tangential decay at infinity. Requires local bounds.
  No Poincaré's inequality.
  No maximum principle, no Harnack's inequality.

- *Behaviour as $y_2 \to +\infty$ ?*

One easier setting: periodic roughness. [Achdou et al, Jäger et al]

- Solvability: Variational formulation in a space of functions periodic with respect to $y_1$.
- $y_2 \to +\infty$ : Fourier series in $y_1$. *Convergence at exponential rate* of $v$ to some $(\alpha, 0)$.

General setting: much harder.

- Well-posedness holds for general $\omega$.
- Convergence of the boundary layer flow is false in general. Requires some *ergodicity properties*. No speed of convergence in general.

Example: $\omega$ stationary ergodic process, with values in $(-1,0)$, uniformly lipschitz.

Theorem 2: *There exists some $\alpha > 0$ such that:*

$$\|u^\varepsilon - u_N\|_{L^2_{uloc}(P\times\Omega)} = o(\varepsilon)$$

with

$$\|f\|^2_{L^2_{uloc}(P\times\Omega)} := \sup_t \mathbb{E}\int_{\Omega\cap\{|x_1-t|<1\}} |f|^2 dx\, d\mu$$

Remarks:

- Almost sure estimates also available (weaker than $L^2_{uloc}$).
- $o(\varepsilon)$ for Navier, instead of $O(\varepsilon)$ for Dirichlet.
- No rate in general. Rate in special cases:

  - periodic : $O(\varepsilon^{3/2})$

  - stationary with strong decay of correlations: $O(\varepsilon^{3/2}|\ln\varepsilon|)$.

# 4. Real or apparent slip ?

Summary: Rigorous derivation of a Navier condition at $\Sigma$.

Question: Does it prove that roughness enhances slip ?

Not clear ! The positivity of $\alpha$ is linked to the position of our artificial boundary (namely *above the humps*).

If we keep the artificial boundary at $x_2 = 0$ and shift the roughness, things change.

Example: periodic roughness. One can show [Achdou et al, Jäger et al]

$$\begin{aligned} \alpha(\omega + h) &= \alpha(\omega) - h, \quad \forall h, \\ \sup -\omega &\le \alpha(\omega). \end{aligned}$$

In our setting : $\omega < 0$, so $\alpha > 0$.

Only meaningful case: $<\omega>=0$: same averaged flow rate in the rough and smooth channels.

Problem: Find the maximizer and maximum of

$$\tilde{\alpha}(\omega) := \alpha(\omega) - <\omega>$$

among all rough profiles $\omega \in W^{1,\infty}(\mathbb{T})$ ($W^{1,\infty}(\mathbb{T}^2)$ in 3d ).

Proposition: Maximum slip coefficient is achieved for flat surfaces:

$$\max_{\omega} \tilde{\alpha}(\omega) = \tilde{\alpha}(0) = 0.$$

Conclusion: apparent slip, not real.

# 5. Hydrophobic rough surfaces

Back to our microfluidics problem:

May rough walls generate substantial slip ?

Previous study: suggests the answer is no, starting from a Dirichlet condition.

Closer look at some papers: *cavitation phenomenon*.

- Rough hydrophobic surfaces generate bubbles in their hollows.
- The fluid slips above hollows, sticks at bumps.

Suggestion: To consider a model with a *flat boundary, alternating zones of slip and no-slip*, with arbitrary relative areas.

Example: $\Omega = \mathbb{T}^2 \times \mathbb{R}_+$ (3d model).

- Stokes in $\Omega$, with some forcing.
- Boundary $\mathbb{T}^2 \times \{0\}$ divided in $\sim \varepsilon^{-2}$ square cells of side $\varepsilon$:

$$C_k^\varepsilon \;:=\; \varepsilon\,(k \;+\; C), \quad C = [0,1[^2, \quad k \in [[0, \varepsilon^{-1} - 1]]^2$$

  with patches

$$P_k^\varepsilon \;=\; \varepsilon(k \;+\; P^\varepsilon), \quad P^\varepsilon \subset C.$$

- B.C. is *pure slip at* $\cup (C_k^\varepsilon \setminus P_k^\varepsilon)$, *no-slip at* $\cup P_k^\varepsilon$,

Question : Averaged boundary condition as $\varepsilon \to 0$ ?

Key: Volume fraction of no-slip: $\phi^\varepsilon = |P^\varepsilon| \in [0,1]$.

Two main results:

1. One for *patches*: broadly, $P^\varepsilon \Subset C$ smooth open set.
2. One for *riblets*: $P^\varepsilon = [0,1] \times I^\varepsilon$, $I^\varepsilon$ subinterval.

"Theorem for patches"

- *If* $\phi^\varepsilon >> \varepsilon^2$*, the limit condition is Dirichlet.*
- *If* $\phi^\varepsilon << \varepsilon^2$*, the limit condition is pure slip.*
- *If* $\phi^\varepsilon \sim \varepsilon^2$*, the limit condition is Navier.*

"Theorem for riblets": $C > 0$ arbitrary.

- *If* $\phi^\varepsilon >> \exp(-C/\varepsilon)$*, the limit condition is Dirichlet.*
- *If* $\phi^\varepsilon << \exp(-C\varepsilon)$*, the limit condition is pure slip.*
- *If* $\phi^\varepsilon \sim \exp(-C/\varepsilon)$*, the limit condition is Navier.*

Remarks:

- Significant slip is possible. But the relative area of the no-slip zone needs to be very small (unrealistic ?).
- The riblet geometry is less efficient in improving slip.

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- Significant slip is possible. But the relative area of the no-slip zone needs to be very small (unrealistic ?).
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Proof: More or less already done ! Think of the simpler problem:

$$\boxed{\Delta u^\varepsilon = 0 \text{ in } \Omega, \quad \partial_\nu u^\varepsilon = 1 \text{ in } \cup (C_k \setminus P_k^\varepsilon), \quad u^\varepsilon = 0 \text{ in } \cup P_k^\varepsilon.}$$

*Homogenization of the fractional Laplacian in domains with holes*.

Allows to connect to the existing litterature [Cioranescu et al, 82], [Allaire, 91], [Caffarelli-Mellet, 08].