Half-homogeneous indecomposable circle-like continuum

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Abstract

A continuum is said to be $\frac{1}{2}$-homogeneous if there are exactly two types of points. We give an example of a $\frac{1}{2}$-homogeneous indecomposable circle-like continuum. This answers a question of V. Neumann-Lara, P. Pellicer-Covarrubias and I. Puga.

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1 Introduction

A *continuum* is a non-empty compact connected metrizable space. A continuum X is said to be *connected im kleinen* at a point x if any neighbourhood of x contains a connected subset whose interior contains the point x. An *orbit* of a space X containing a point $x \in X$ is a set of all points of the form $h(x)$, where $h\colon X \to X$ is a homeomorphism. A continuum X is called $\frac{1}{2}$*-homogeneous* if it consists of two orbits exactly. A continuum is said to be *indecomposable* if it can not be written as a union of two proper subcontinua. A continuum X is said to be *arc-like* (resp. *circle-like*) if for any $\varepsilon > 0$ there

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is a continuous mapping of X onto an arc (resp. a circle) such that preimages of points have diameter less than ε.

We recall a characterization of an arc-like continuum via chains. A *chain* is a finite sequence $\mathcal{C} = \{C_1, C_2, \ldots, C_n\}$ of open sets in a metric space such that $C_i \cap C_j \neq \emptyset$ if and only if $|i - j| \leq 1$. The elements of a chain are called its *links*. If $\varepsilon > 0$ and the diameter of each link is less than ε, then the chain is called an ε-chain. A continuum is chainable if for each $\varepsilon > 0$, the continuum can be covered by an ε-chain. It is well known that a continuum is arc-like if and only if it is chainable.

We give a positive answer to Problem 4.10 from [6], i.e. we give an example of a $\frac{1}{2}$-homogeneous continuum which is indecomposable and circle-like.[1]

2 Example

Let A be an arc of pseudoarcs i.e. an arc-like continuum for which there exists a continuous mapping $g\colon A \to [0,1]$ such that preimage of each point is a pseudoarc. For more details see [2] and [5]. It is argued in [6, Example 4.8], that A is a $\frac{1}{2}$-homogeneous and the two orbits in X are $g^{-1}(\{0,1\})$ and $g^{-1}((0,1))$.

Let B be a quotient of A, where the sets $g^{-1}(0)$ and $g^{-1}(1)$ are degenerated to points. Let us call these two points p and q. We note that B is an arc-like continuum. Moreover it is $\frac{1}{2}$-homogeneous with orbits $\{p,q\}$ and $B \setminus \{p,q\}$.

For any natural number n we define a space X_n as a quotient of the product $\{1, 2, \ldots, 2^n\} \times B$ where the points (k,q) and $(k+1,p)$ are identified for every $k < 2^n$ and $(2^n, q)$ is identified with $(1,p)$. Thus X_n is a "circle" made of 2^n copies of B.

We define continuous maps $f_n\colon X_{n+1} \to X_n$ as

$$f\big((k,x)\big) = \begin{cases} (k,x), & \text{if } k \leq 2^n \\ (k-2^n, x), & \text{if } 2^n < k \leq 2^{n+1} \end{cases}$$

for $k \leq 2^{n+1}$ and $x \in B$. Let X be the inverse limit of the inverse sequence $(X_n, f_n)_{n=1}^{\infty}$. Thus X is a subspace of the product $\prod X_n$ which consists of those points $(x_n)_{n=1}^{\infty}$ such that $f_n(x_{n+1}) = x_n$ for any $n \in \mathbb{N}$. Since X is an

[1]After submitting this paper, we were made aware of a paper by Jan Boronski on the same problem. That paper appears in the same issue as this paper.

inverse limit of continua with continuous bonding mappings, it is a continuum by [8, Theorem 2.4, p. 19].

Claim 1. X is circle-like.

Proof. Any continuum X_n is circle-like, since the continuum B is arc-like. Moreover, an inverse limit of an inverse sequence of circle-like continua with onto bonding mappings is again a circle-like continuum (see [7, Lemma 1, p. 147]). Thus X is a circle-like continuum. □

Claim 2. X is indecomposable.

Proof. We can easily observe that the system $(X_n, f_n)_{n=1}^{\infty}$ forms an indecomposable inverse sequence, i.e. whenever there are two continua A and B such that $X_{n+1} = A \cup B$, then $f_n(A) = X_n$ or $f_n(B) = X_n$. Since an inverse limit of an indecomposable inverse sequence is an indecomposable continuum by [8, Theorem 2.7], we conclude that X is an indecomposable continuum. □

Claim 3. X is $\frac{1}{2}$-homogeneous.

Proof. Let us consider any two points $x = (x_n)_{n=1}^{\infty} \in X$ and $y = (y_n)_{n=1}^{\infty} \in X$ such that $x_1, y_1 \in \{p, q\}$ or $x_1, y_1 \notin \{p, q\}$. There is a homeomorphism $h_1 \colon X_1 \to X_1$ such that $h_1(x_1) = y_1$, because B is $\frac{1}{2}$-homogeneous. By induction we find a homeomorphism $h_n \colon X_n \to X_n$ for any $n \geq 2$ such that $h_n(x_n) = y_n$ and $h_n \circ f_{n+1} = f_{n+1} \circ h_{n+1}$. We define a homeomorphism $h \colon X \to X$ which is given by $h((x_n)_{n=1}^{\infty}) = (h_n(x_n))_{n=1}^{\infty}$. Clearly $h(x) = y$. Thus there are at most two orbits in X, namely the set $\{x \in X \colon x_1 = p \text{ or } x_1 = q\}$ and its complement $\{x \in X \colon p \neq x_1 \neq q\}$.

It remains to show that X is not homogeneous. Let $x = (x_n)_{n=1}^{\infty} \in X$ be the point for which $x_n = (1, p)$ for every $n \in \mathbb{N}$. Every proper subcontinuum of X which contains the point x is connected im kleinen at x.

Let $r \in B$ be an arbitrary point distinct from p and q. We consider the point $y = (y_n)_{n=1}^{\infty}$ where $y_n = (1, r)$ for every $n \in \mathbb{N}$. We define a continuum K which consists of points $z = ((1, s))_{n=1}^{\infty} \in X$ for some $s \in B$. We observe that K and B are homeomorphic. Since B is not connected im kleinen at r, we get that K is not connected im kleinen at y.

Thus the points x and y are not in the same orbit and hence X is not homogeneous. □

3 Remarks

Remark 4. Our construction is a modification of the dyadic solenoid (see [4] and [9]). We could easily modify the construction to obtain an example from each p-adic solenoid.

Remark 5. The continuum B is a special quotient of the arc of pseudoarcs. If we try to replace the arc of pseudoarcs just with the arc or with the pseudoarc, we do not succeed. In the first case we obtain a solenoid which is a homogeneous continuum. In the second case we get a continuum with at least three orbits.

There is a natural question whether there is another possible choice of the continuum B formulated in Problem 8. We just need to define a notion of an end-point in an arc-like continuum.

Definition 6. A point x of an arc-like continuum X is called an *end-point* if for every $\varepsilon > 0$ there is a continuous mapping f of X onto the segment $[0, 1]$ such that preimages of points have diameter less than ε and $f(x) = 0$.

There is a characterization of end-points in an arc-like continuum (see [1, Section 5, p. 660] and [3, p. 32]):

Proposition 7. For a point p of a chainable continuum X the following conditions are equivalent.

a) Each nondegenerate subcontinuum of X containing p is irreducible from p to some other point.

b) If each of two subcontinua of X contains p, one of the subcontinua contains the other.

c) For each positive number ε, there is an ε-chain covering M such that only the first link of the chain contains p.

d) p is an end-point of X.

Problem 8. Does there exist a $\frac{1}{2}$-homogeneous arc-like continuum with exactly two end-points which is neither homeomorphic to the arc nor to the continuum B?

Regarding to the result obtained in this paper we also recall Problem 4.9 from [6].

Problem 9. Does there exist an indecomposable, $\frac{1}{2}$-homogeneous, arc-like continuum?

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