First-Order System Approaches to Hyperelastic Deformation Models

Gerhard Starke

Fakultät für Mathematik
Universität Duisburg - Essen
Essen, Germany

Joint work with Benjamin Müller (Fakultät für Mathematik, UDE)

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Implicitly Constituted Materials: Modeling, Analysis and Computing

Overview

Overview

Variational Formulations for Incompressible Linear Elasticity

Computational Results and Motivation

Hyperelasticity as a First-Order System

Computational Experiments

Conclusions

Incompressible Linear Elasticity

First-Order System Formulation of Linear Elasticity

Displacement field $\mathbf{u} : \Omega \to \mathbb{R}^d$
Stress tensor $\boldsymbol{\sigma} : \Omega \to \mathbb{R}^{d \times d}$

$$\begin{aligned} \operatorname{div} \boldsymbol{\sigma} + \mathbf{f} &= \mathbf{0} \text{ in } \Omega \\ \mathcal{A}\boldsymbol{\sigma} - \boldsymbol{\varepsilon}(\mathbf{u}) &= \mathbf{0} \text{ in } \Omega \\ \mathbf{u} &= \mathbf{0} \text{ on } \Gamma_D \\ \boldsymbol{\sigma} \cdot \mathbf{n} &= \mathbf{t} \text{ on } \Gamma_N \end{aligned}$$

$$\boldsymbol{\varepsilon}(\mathbf{u}) = \frac{\nabla \mathbf{u} + \nabla \mathbf{u}^T}{2}$$

$$\mathcal{A}\boldsymbol{\sigma} = \frac{1}{2\mu}\left(\boldsymbol{\sigma} - \frac{\lambda}{\lambda d + 2\mu}(\operatorname{tr} \sigma)\,\mathbf{I}\right)$$

$$\mathcal{A} = \mathcal{C}^{-1} \text{ if } \lambda < \infty$$

$\mathbf{u} \in H^1_{\Gamma_D}(\Omega)^d$
$\boldsymbol{\sigma} \in \boldsymbol{\sigma}^N + H_{\Gamma_N}(\operatorname{div}, \Omega)^d$ ($\boldsymbol{\sigma}^N \in H(\operatorname{div}, \Omega)^d$ s.t. $\boldsymbol{\sigma}^N \cdot \mathbf{n} = \mathbf{t}$ on Γ_N)

Incompressible Linear Elasticity

Displacement-Pressure (Galerkin) Formulation

Insert new variable p into material (2nd) eqn:

$$\mathcal{A}\boldsymbol{\sigma} = \frac{1}{2\mu}\left(\boldsymbol{\sigma} - \frac{\lambda}{\lambda d + 2\mu}(\mathrm{tr}\,\sigma)\,\mathbf{I}\right) = \frac{1}{2\mu}\left(\boldsymbol{\sigma} - p\,\mathbf{I}\right) = \boldsymbol{\varepsilon}(\mathbf{u})$$

and combine this with momentum balance (1st) equation:

Determine $\mathbf{u}^g \in H^1_{\Gamma_D}(\Omega)^d$, $p^g \in L^2(\Omega)$ such that

$$2\mu\left(\varepsilon(\mathbf{u}^g), \varepsilon(\mathbf{v})\right)_{L^2(\Omega)} + (p, \mathrm{div}\ v)_{L^2(\Omega)} = (\mathbf{f}, \mathbf{v})_{L^2(\Omega)} + \langle \mathbf{t}, \mathbf{v}\rangle_{L^2(\Gamma_N)}$$

$$(\mathrm{div}\,\mathbf{u}, q)_{L^2(\Omega)} = \frac{1}{\lambda}(p, q)_{L^2(\Omega)}$$

holds for all $\mathbf{v} \in H^1_{\Gamma_D}(\Omega)^d$, $q \in L^2(\Omega)$

$$\boldsymbol{\sigma}^g = 2\mu\,\varepsilon(\mathbf{u}^g) + p^g\,\mathbf{I} \in L^2(\Omega)^{d\times d}$$

Incompressible Linear Elasticity

Hellinger-Reissner (Mixed) Formulation of Linear Elasticity

Determine $\boldsymbol{\sigma}^m \in \boldsymbol{\sigma}^N + H_{\Gamma_N}(\mathrm{div}, \Omega)^d$, $\mathbf{u}^m \in L^2(\Omega)^d$ and $\gamma^m \in L^2(\Omega)^{d\times d,\mathrm{skew}}$ such that

$$\begin{aligned}(\mathcal{A}\boldsymbol{\sigma}^m, \boldsymbol{\tau})_{L^2(\Omega)} + (\mathbf{u}^m, \mathrm{div}\,\boldsymbol{\tau})_{L^2(\Omega)} + (\gamma^m, \mathrm{skew}\,\boldsymbol{\tau})_{L^2(\Omega)} &= 0\\ (\mathrm{div}\,\boldsymbol{\sigma}^m + \mathbf{f}, \mathbf{v})_{L^2(\Omega)} &= 0\\ (\mathrm{skew}\,\boldsymbol{\sigma}^m, \boldsymbol{\eta})_{L^2(\Omega)} &= 0\end{aligned}$$

holds for all $\boldsymbol{\tau} \in H_{\Gamma_N}(\mathrm{div}, \Omega)^d$, $\mathbf{v} \in L^2(\Omega)^d$ and $\boldsymbol{\eta} \in L^2(\Omega)^{d\times d,\mathrm{skew}}$

$$\mathrm{skew}\boldsymbol{\tau} = \frac{1}{2}(\boldsymbol{\tau} - \boldsymbol{\tau}^T)$$

$$L^2(\Omega)^{d\times d,\mathrm{skew}} = \{\boldsymbol{\tau} \in L^2(\Omega)^{d\times d} : \boldsymbol{\tau} + \boldsymbol{\tau}^T = \mathbf{0}\}$$

Incompressible Linear Elasticity

First-Order System Least Squares

Determine $\boldsymbol{\sigma}^{ls} \in \boldsymbol{\sigma}^N + H_{\Gamma_N}(\mathrm{div},\Omega)^d$ and $\mathbf{u}^{ls} \in H^1_{\Gamma_D}(\Omega)^d$ such that

$$\|\mathrm{div}\,\boldsymbol{\sigma} + \mathbf{f}\|^2_{L^2(\Omega)} + \|\mathcal{A}\boldsymbol{\sigma} - \boldsymbol{\varepsilon}(\mathbf{u})\|^2_{L^2(\Omega)}$$

is minimized

Equivalently: $\boldsymbol{\sigma}^{ls} \in \boldsymbol{\sigma}^N + H_{\Gamma_N}(\mathrm{div},\Omega)^d$ and $\mathbf{u}^{ls} \in H^1_{\Gamma_D}(\Omega)^d$ s.t.

$$\begin{aligned}(\mathrm{div}\,\boldsymbol{\sigma}^{ls} + \mathbf{f}, \mathrm{div}\,\boldsymbol{\tau})_{L^2(\Omega)} + (\mathcal{A}\boldsymbol{\sigma}^{ls} - \boldsymbol{\varepsilon}(\mathbf{u}^{ls}), \mathcal{A}\boldsymbol{\tau})_{L^2(\Omega)} &= 0 \\ (\mathcal{A}\boldsymbol{\sigma}^{ls} - \boldsymbol{\varepsilon}(\mathbf{u}^{ls}), \boldsymbol{\varepsilon}(\mathbf{v}))_{L^2(\Omega)} &= 0\end{aligned}$$

holds for all $\boldsymbol{\tau} \in H_{\Gamma_N}(\mathrm{div},\Omega)^d$ and $\mathbf{v} \in H^1_{\Gamma_D}(\Omega)^d$

From now on: $(\ ,\ )$ instead of $(\ ,\ )_{L^2(\Omega)}$, $\|\ \ \|$ instead of $\|\ \ \|_{L^2(\Omega)}$

Incompressible Linear Elasticity

Approximation Properties for First-Order System Least Squares

Coercivity of the first-order system least squares bilinear form

$$\mathcal{B}(\mathbf{u},\boldsymbol{\sigma};\mathbf{v},\boldsymbol{\tau}) = (\operatorname{div}\boldsymbol{\sigma},\operatorname{div}\boldsymbol{\tau}) + (\mathcal{A}\boldsymbol{\sigma} - \varepsilon(\mathbf{u}), \mathcal{A}\boldsymbol{\tau} - \varepsilon(\mathbf{v}))$$

in $H^1_{\Gamma_D}(\Omega)^d \times H_{\Gamma_N}(\mathrm{div},\Omega)^d$ with respect to

$$|||(\mathbf{v},\boldsymbol{\tau})||| = \left(\|\varepsilon(\mathbf{v})\|^2 + \|\operatorname{div}\boldsymbol{\tau}\|^2 + \|\boldsymbol{\tau}\|^2\right)^{1/2}$$

holds uniformly for $\lambda \to \infty$ (Cai/St., 2004)

$\Longrightarrow$ Optimal order convergence:

$$|||(\mathbf{u} - \mathbf{u}_h^{ls}, \boldsymbol{\sigma} - \boldsymbol{\sigma}_h^{ls})||| \eqsim \inf_{\mathbf{v}_h,\boldsymbol{\tau}_h} |||(\mathbf{u} - \mathbf{v}_h, \boldsymbol{\sigma} - \boldsymbol{\tau}_h)|||$$

for subspaces $\mathbf{V}_h \subset H^1_{\Gamma_D}(\Omega)^d$, $\boldsymbol{\Sigma}_h \subset H_{\Gamma_N}(\mathrm{div},\Omega)^d$

Incompressible Linear Elasticity

Finite Element Spaces and Approximation Properties

In comparison, for the displacement-pressure formulation:

$$|||(\mathbf{u} - \mathbf{u}_h^g, \mathbf{0})||| = \|\boldsymbol{\varepsilon}(\mathbf{u} - \mathbf{u}_h^g)\| \eqsim \inf_{\mathbf{v}_h \in \mathbf{V}_h} \|\boldsymbol{\varepsilon}(\mathbf{u} - \mathbf{v}_h)\|$$

(if an inf-sup stable Stokes finite element pair is used)

And, for the Hellinger-Reissner (mixed) formulation:

$$\begin{aligned} |||(\mathbf{0}, \boldsymbol{\sigma} - \boldsymbol{\sigma}_h^m)||| &= \left(\|\operatorname{div}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h^m)\|^2 + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h^m\|\right)^{1/2} \\ &\eqsim \inf_{\boldsymbol{\tau}_h \in \boldsymbol{\Sigma}_h} \left(\|\operatorname{div}(\boldsymbol{\sigma} - \boldsymbol{\tau}_h)\|^2 + \|\boldsymbol{\sigma} - \boldsymbol{\tau}_h\|\right)^{1/2} \end{aligned}$$

(if Σ_h is part of an inf-sup stable finite element combination)

$$\Longrightarrow \quad |||(\mathbf{u} - \mathbf{u}_h^{ls}, \boldsymbol{\sigma} - \boldsymbol{\sigma}_h^{ls})||| \eqsim |||(\mathbf{u} - \mathbf{u}_h^g, \boldsymbol{\sigma} - \boldsymbol{\sigma}_h^m)|||$$

Incompressible Linear Elasticity

Finite Element Spaces and Approximation Properties

Advantages and Disadvantages

	Galerkin $(\mathbf{u}^g, p^g)$ $(H^1)^d/L^2$	Mixed $(\boldsymbol{\sigma}^m, \mathbf{u}^m)$ $H(\mathrm{div})^d/(L^2)^d$	FOSLS $(\boldsymbol{\sigma}^{ls}, \mathbf{u}^{ls})$ $H(\mathrm{div})^d/(H^1)^d$
# unknowns	+	-	-
comp. condition	-	-	+
momentum bal.	-	+	$\circ$
nonlinear form.	+	-	+
error estimation	$\circ$	$\circ$	+
scaling issue	+	+	-

Proper scaling of the individual terms in the ls functional

$$\|\mathrm{div}\,\boldsymbol{\sigma} + \mathbf{f}\|^2_{L^2(\Omega)} + \|\mathcal{A}\boldsymbol{\sigma} - \boldsymbol{\varepsilon}(\mathbf{u})\|^2_{L^2(\Omega)}$$

needed: depending on μ (physical units used), size of domain Ω

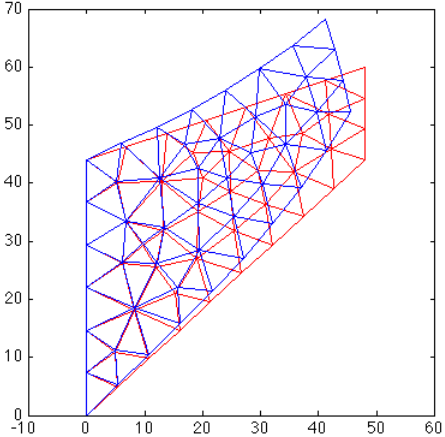
Computational Results and Motivation

Cook's Membrane

Finite element spaces based on a triangulation $\mathcal{T}_h$

$\mathbf{V}_h$: H^1-conforming $\mathcal{P}_2$ elements

$\boldsymbol{\Sigma}_h$: $H(\mathrm{div})$-conforming $\mathcal{RT}_1$ elements



incompressible case

$\lambda = \infty$

Boundary conditions:

$\mathbf{u} = \mathbf{0}$ at left

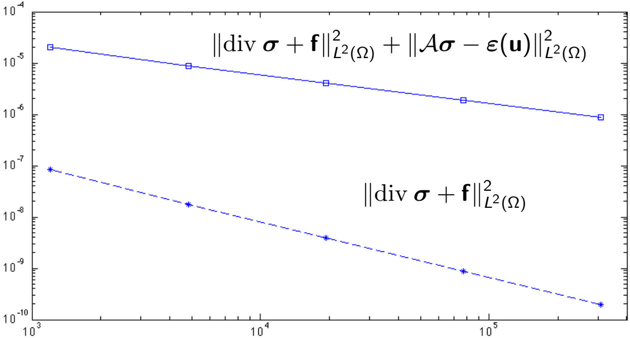
$\boldsymbol{\sigma} \cdot \mathbf{n} = \mathbf{0}$ at top/bottom

$\boldsymbol{\sigma} \cdot \mathbf{n} = (0, \gamma)$ at right

Computational Results and Motivation

Cook's Membrane

First-Order System Least Squares does not satisfy momentum balance exactly but ...



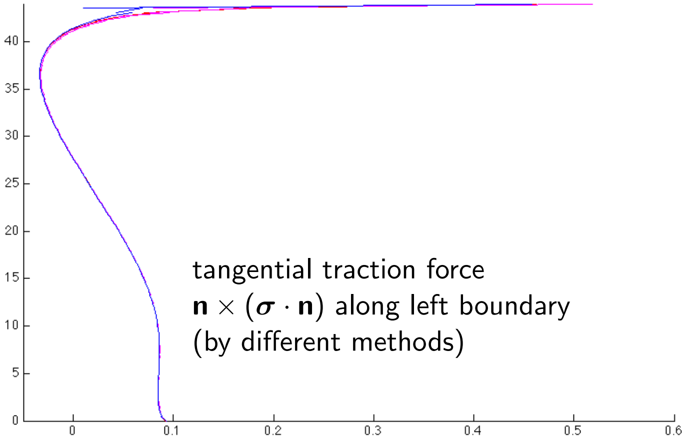
... approximates momentum balance at a higher rate!

St./Schröder/Schwarz (2012): Theory for a slight modification

Computational Results and Motivation

Cook's Membrane

When is accurate momentum balance important?



tangential traction force
$\mathbf{n} \times (\boldsymbol{\sigma} \cdot \mathbf{n})$ along left boundary
(by different methods)

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Hyperelasticity as a First-Order System

Hyperelastic Material Models

Displacement field $\mathbf{u} : \Omega \to \mathbb{R}^3$

Deformation gradient $\mathbf{F}(\mathbf{u}) = \mathbf{I} + \nabla \mathbf{u}$

Right Cauchy-Green strain tensor $\mathbf{C}(\mathbf{u}) = \mathbf{F}(\mathbf{u})^T \mathbf{F}(\mathbf{u})$

Left Cauchy-Green strain tensor $\mathbf{B}(\mathbf{u}) = \mathbf{F}(\mathbf{u}) \mathbf{F}(\mathbf{u})^T$

Stored energy function $\psi : \mathbb{R}^{3\times 3}_{\mathrm{sym}} \to \mathbb{R}$

Minimize the total energy

$$W(\mathbf{u}) = \int_\Omega \psi(\mathbf{C}(\mathbf{u}))\, dx - \int_\Omega \mathbf{f} \cdot \mathbf{u}\, dx$$

among all admissible $\mathbf{u} : \Omega \to \mathbb{R}^3$

Hyperelasticity as a First-Order System

Hyperelastic Material Models

1st Piola-Kirchhoff stress tensor $\mathbf{P} = \partial_{\mathbf{F}} \psi(\mathbf{C}(\mathbf{u}))$

First-order system:
Determine $\mathbf{u} : \Omega \to \mathbb{R}^3$, $\mathbf{P} : \Omega \to \mathbb{R}^{3\times 3}$ such that

$$\begin{aligned} \operatorname{div} \mathbf{P} + \mathbf{f} &= \mathbf{0} \\ \mathbf{P} - \partial_{\mathbf{F}} \psi(\mathbf{C}(\mathbf{u})) &= \mathbf{0} \end{aligned}$$

Hyperelasticity as a First-Order System

Hyperelastic Material Models

1st Piola-Kirchhoff stress tensor $\mathbf{P} = \partial_{\mathbf{F}} \psi(\mathbf{C}(\mathbf{u}))$

First-order system:
Determine $\mathbf{u} : \Omega \to \mathbb{R}^3$, $\mathbf{P} : \Omega \to \mathbb{R}^{3\times 3}$ such that

$$\begin{aligned} \operatorname{div} \mathbf{P} + \mathbf{f} &= \mathbf{0} \\ \mathbf{P}\mathbf{F}(\mathbf{u})^T - \partial_{\mathbf{F}} \psi(\mathbf{C}(\mathbf{u}))\mathbf{F}(\mathbf{u})^T &= \mathbf{0} \end{aligned}$$

For example: Neo-Hooke material (with $J = \det \mathbf{F}$):

$$\psi_{NH}(\mathbf{C}) = \frac{\mu}{2} \operatorname{tr} \mathbf{C} + \frac{\lambda}{4} J^2 - \left(\frac{\lambda}{2} + \mu \right) \ln J$$

$$\partial_{\mathbf{F}} \psi_{NH}(\mathbf{C}) = \mu \, \mathbf{F} + \left(\frac{\lambda}{2} (J^2 - 1) - \mu \right) \mathbf{F}^{-T}$$

Hyperelasticity as a First-Order System

Neo-Hooke model

$$\partial_{\mathbf{F}} \psi_{NH}(\mathbf{C}) \mathbf{F}^T = \mu \, \mathbf{F} \mathbf{F}^T + \left(\frac{\lambda}{2} (J^2 - 1) - \mu \right) \mathbf{I}$$
$$= \mu \, \mathbf{B} + \left(\frac{\lambda}{2} (J^2 - 1) - \mu \right) \mathbf{I} =: \mathcal{G}_{NH}(\mathbf{B})$$

with $J^2 = \det(\mathbf{B})$

Determine $\mathbf{u} : \Omega \to \mathbb{R}^3$, $\mathbf{P} : \Omega \to \mathbb{R}^{3\times 3}$ such that

$$\operatorname{div} \mathbf{P} + \mathbf{f} = \mathbf{0}$$
$$\mathbf{P} \mathbf{F}(\mathbf{u})^T - \mathcal{G}_{NH}(\mathbf{B}(\mathbf{u})) = \mathbf{0}$$

Hyperelasticity as a First-Order System

Neo-Hooke model

$$\mathcal{G}_{NH}(\mathbf{B}) = \mu\,(\mathbf{B} - \mathbf{I}) + \frac{\lambda}{2}\,(J^2 - 1)\mathbf{I}$$

with $J^2 = \det(\mathbf{B})$

$$\mathcal{G}'_{NH}(\mathbf{B})[\mathbf{E}] = \mu\,\mathbf{E} + \frac{\lambda}{2} J^2 (\mathbf{B}^{-T} : \mathbf{E})\mathbf{I}$$

In particular: $$\mathcal{G}'_{NH}(\mathbf{I})[\mathbf{E}] = \mu\mathbf{E} + \frac{\lambda}{2}(\mathrm{tr}\mathbf{E})\mathbf{I}$$

Small strain limit: Linear elasticity system

$$\mathrm{div}\,\mathbf{P} + \mathbf{f} = \mathbf{0}$$

$$\mathbf{P} - \underbrace{\mathcal{G}_{NH}(\mathbf{I})}_{=\mathbf{0}} + \mathcal{G}'_{NH}(\mathbf{I})[\nabla\mathbf{u} + (\nabla\mathbf{u})^T] = \mathbf{0}$$

Hyperelasticity as a First-Order System

Scaling the Stress-Strain Relation

$$\mathcal{G}_{NH}(\mathbf{B}) = \mu\,(\mathbf{B} - \mathbf{I}) + \frac{\lambda}{2}\,(J^2 - 1)\mathbf{I}$$

Existence of an inverse function $\mathbf{B} = \mathcal{G}_{NH}^{-1}(\boldsymbol{\Sigma})$ leads to a first-order system which remains valid in the incompressible limit

$$\lambda \to \infty : \quad \mathcal{G}_{NH}'(\mathbf{B})^{-1}[\boldsymbol{\Theta}] \to \frac{1}{\mu}\left(\boldsymbol{\Theta} - \frac{1}{\mathrm{tr}(\mathbf{B}^{-1})}(\mathbf{B}^{-T} : \boldsymbol{\Theta})\mathbf{I}\right)$$

Determine $\mathbf{u} : \Omega \to \mathbb{R}^3$, $\mathbf{P} : \Omega \to \mathbb{R}^{3\times 3}$ such that

$$\begin{aligned} \mathrm{div}\,\mathbf{P} + \mathbf{f} &= 0 \\ \mathcal{A}(\mathbf{P}\mathbf{F}(\mathbf{u})^T) - \mathbf{B}(\mathbf{u}) &= \mathbf{0} \end{aligned}$$

where $\mathcal{A} = \mathcal{G}_{NH}^{-1}$ for $\lambda < \infty$ and $\mathcal{A}$ is also well-defined for $\lambda = \infty$

Wriggers/Nonlinear FE Methods: Inversion based on $\widetilde{\mathcal{G}}(\mathbf{C}(\mathbf{u}))$

Hyperelasticity as a First-Order System

Scaling the Stress-Strain Relation

B. Müller/St./Schwarz/Schröder (2013):

For $\mathbf{u} \in W^{1,4}_{\Gamma_D}(\Omega)^3$, $\mathbf{P} \in W^4_{\Gamma_N}(\mathrm{div}, \Omega)^3$ and $\mathbf{f} \in L^2(\Omega)$,

$$\mathcal{R}(\mathbf{P}, \mathbf{u}) = \begin{pmatrix} \mathrm{div}\, \mathbf{P} + \mathbf{f} \\ \mathcal{A}(\mathbf{P}\mathbf{F}(\mathbf{u})^T) - \mathbf{B}(\mathbf{u}) \end{pmatrix} \in L^2(\Omega)^3 \times L^2(\Omega)^{3\times 3}$$

Determine $\mathbf{u} \in W^{1,4}_{\Gamma_D}(\Omega)^3$, $\mathbf{P} \in W^4_{\Gamma_N}(\mathrm{div}, \Omega)^3$ such that

$$\mathcal{F}(\mathbf{P}, \mathbf{u}) = \|\mathrm{div}\, \mathbf{P} + \mathbf{f}\|^2_{L^2(\Omega)} + \|\mathcal{G}^{-1}_{NH}(\mathbf{P}\mathbf{F}(\mathbf{u})^T) - \mathbf{B}(\mathbf{u})\|^2_{L^2(\Omega)}$$

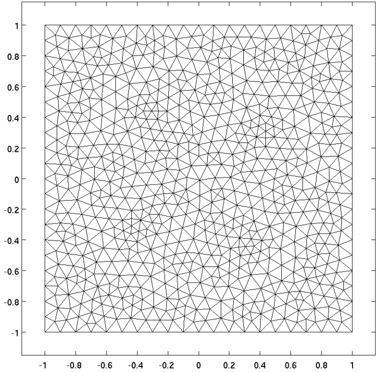
is minimized.

Hyperelasticity as a First-Order System

Example

Auricchio/Beirão da Veiga/Lovadina/Reali (2010)

Uniform volume force $\mathbf{f} = (0, \gamma)$, $\gamma \in \mathbb{R}$, plane strain condition



Bdy conditions 1:
$\mathbf{u} = \mathbf{0}$ left, right and below,
$\mathbf{P} \cdot \mathbf{n} = \mathbf{0}$ on top

Bdy conditions 2:
$\mathbf{u} \cdot \mathbf{n} = 0$ und $(\mathbf{P} \cdot \mathbf{n}) \cdot \mathbf{t} = 0$
left, right and below,
$\mathbf{P} \cdot \mathbf{n} = \mathbf{0}$ on top

Exact solution (for $\lambda \to \infty$):
$\mathbf{P}(x_1, x_2) = \gamma(1 - x_2)\mathbf{I}$, $\mathbf{u} \equiv \mathbf{0}$

Singularities occur at critical load values $\gamma_k > 0, k = 1, 2, \ldots$
Numerical results: See poster by Benjamin Müller

Hyperelasticity as a First-Order System

Linearization

$\mathcal{R}(\mathbf{P}+\mathbf{Q},\mathbf{u}+\mathbf{v}) \approx \mathcal{R}(\mathbf{P},\mathbf{u}) + \mathcal{J}(\mathbf{P},\mathbf{u})[\mathbf{Q},\mathbf{v}]$ where the derivative in direction $(\mathbf{Q},\mathbf{v}) \in W^4_{\Gamma_N}(\mathrm{div},\Omega)^3 \times W^{1,4}_{\Gamma_D}(\Omega)^3$ is given by

$$\mathcal{J}(\mathbf{P},\mathbf{Q})[\mathbf{Q},\mathbf{v}] = \begin{pmatrix} \mathrm{div}\,\mathbf{Q} \\ D\mathbf{G}^{-1}(\mathbf{P}\mathbf{F}(\mathbf{u})^T)[\mathbf{Q},\mathbf{v}] - (\mathbf{I}+\nabla\mathbf{u})\nabla\mathbf{v}^T - \nabla\mathbf{v}(\mathbf{I}+\nabla\mathbf{u})^T \end{pmatrix}$$

with

$$D\mathbf{G}^{-1}(\mathbf{P}\mathbf{F}(\mathbf{u})^T)[\mathbf{Q},\mathbf{v}] = \mathbf{G}'(\mathbf{G}^{-1}(\mathbf{P}\mathbf{F}(\mathbf{u})^T))^{-1}[\mathbf{Q}\mathbf{F}(\mathbf{u})^T + \mathbf{P}\nabla\mathbf{v}^T]$$

Variational formulation:

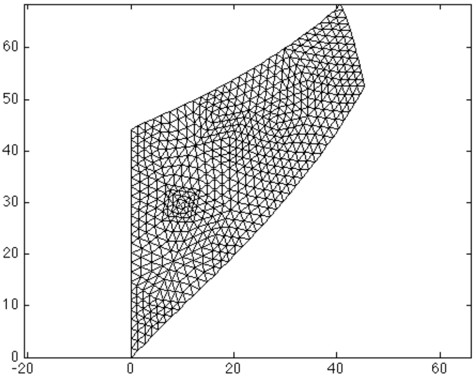
Find $(\mathbf{P},\mathbf{u}) \in W^4_{\Gamma_N}(\mathrm{div},\Omega)^3 \times W^{1,4}_{\Gamma_D}(\Omega)^3$ such that
$(\mathcal{R}(\mathbf{P},\mathbf{u}),\mathcal{J}(\mathbf{P},\mathbf{u})[\mathbf{Q},\mathbf{v}]) = 0$ for all
$(\mathbf{Q},\mathbf{v}) \in W^4_{\Gamma_N}(\mathrm{div},\Omega)^3 \times W^{1,4}_{\Gamma_D}(\Omega)^3$

Overview

Computational Experiments

Cook's membrane (plane strain): incompressible case ($\lambda = \infty$)

Traction force at right boundary: $\mathbf{P} \cdot \mathbf{n} = (0, \gamma)^T$



Computational Experiments

Cook's membrane (plane strain): incompressible case ($\lambda = \infty$)
Traction force at right boundary: $\mathbf{P} \cdot \mathbf{n} = (0, \mu\gamma)^T$

Reduction of least squares functional for $\gamma = 0.1$:

	dim $\mathbf{\Pi}_h$	dim $\mathbf{V}_h$	$\mathcal{F}(\mathbf{P}_h, \mathbf{u}_h)$ (order)	$\|\mathrm{div}\mathbf{P}_h\|^2$
$l = 0$	897	310	1.688e-1	6.598e-4
$l = 1$	3640	1188	7.414e-2 (1.187)	1.416e-4 (2.220)
$l = 2$	14664	4648	3.454e-2 (1.102)	3.200e-5 (2.146)
$l = 3$	58864	18384	1.625e-2 (1.088)	7.021e-6 (2.188)
$l = 4$	235872	73120	7.547e-3 (1.106)	1.415e-6 (2.311)

Computational Experiments

Auricchio/Beirão da Veiga/Lovadina/Reali (2010)

Minimize $W(\mathbf{u}_h)$ with respect to $\mathbf{u}_h \in \mathbf{V}_h$ and handle incompressibility by introducing a pressure-like variable $p = \lambda(J^2 - 1)$ approximated by $p_h \in \Pi_h$

Used combinations:
$\mathbf{V}_h =$ conforming $\mathcal{P}_2$, $\mathbf{\Pi}_h =$ discontinuous $\mathcal{P}_0$
$\mathbf{V}_h =$ conforming $\mathcal{P}_2$, $\mathbf{\Pi}_h =$ discontinuous $\mathcal{P}_1$ (unstable)
$\mathbf{V}_h =$ conforming $\mathcal{P}_2$, $\mathbf{\Pi}_h =$ continuous $\mathcal{P}_1$ (Taylor-Hood)

and

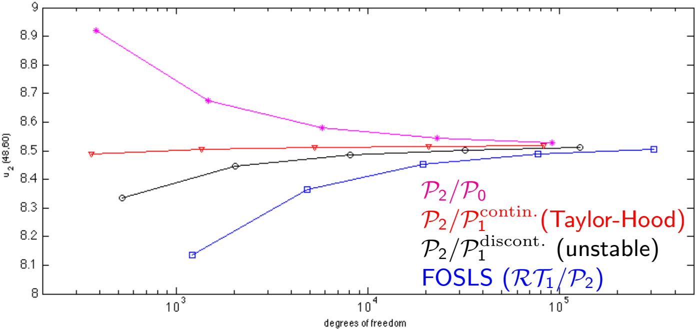
$\mathbf{V}_h =$ conforming $\mathcal{P}_2$, $\mathbf{\Sigma}_h = RT_1$ (First-order system least squares)

Computational Experiments

Cook's membrane (plane strain): incompressible case

Traction force at right boundary: $\mathbf{P} \cdot \mathbf{n} = (0, \mu\gamma)^T$

Behavior of approximation to u_2 at right upper tip for $\gamma = 0.1$:

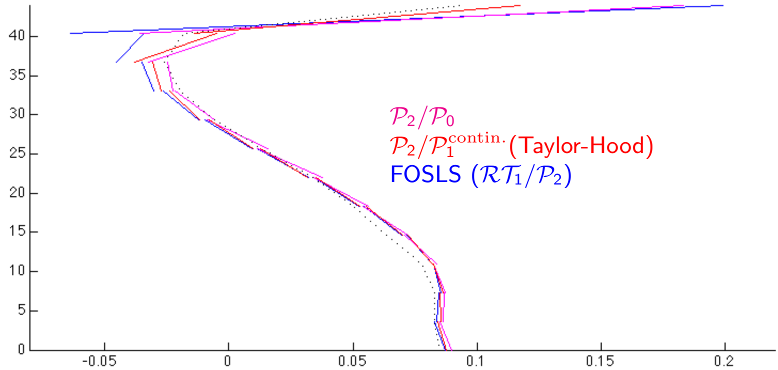


Computational Experiments

Cook's membrane (plane strain): incompressible case

Traction force at right boundary: $\mathbf{P} \cdot \mathbf{n} = (0, \mu\gamma)^T$

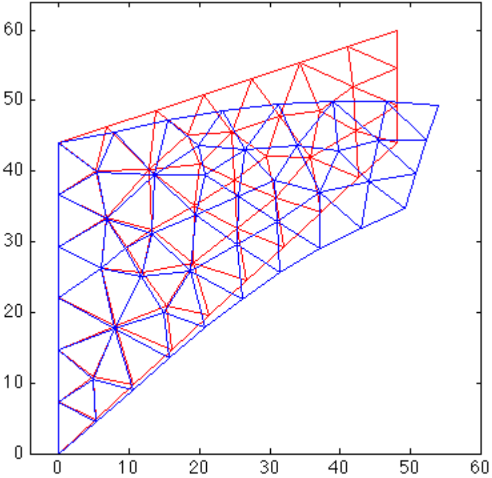
Tangential traction at left boundary at refinement level 1 ($\gamma = 0.1$)



Computational Experiments

Reverse Cook's membrane: incompressible case ($\lambda = \infty$)

Traction force at right boundary: $\mathbf{P} \cdot \mathbf{n} = (0, \gamma)^T$, $\gamma < 0$



Computational Experiments

Cook's membrane (plane strain): incompressible case ($\lambda = \infty$)
Traction force at right boundary: $\mathbf{P} \cdot \mathbf{n} = (0, \mu\gamma)^T$

Reduction of least squares functional for $\gamma = -0.1$:

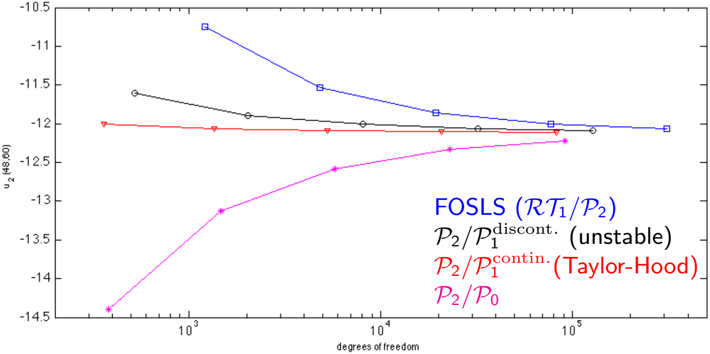
	dim $\mathbf{\Pi}_h$	dim $\mathbf{V}_h$	$\mathcal{F}(\mathbf{P}_h, \mathbf{u}_h)$ (order)	$\|\mathrm{div}\mathbf{P}_h\|^2$
$l = 0$	897	310	2.326e-1	8.943e-4
$l = 1$	3640	1188	9.542e-2 (1.285)	1.599e-4 (2.484)
$l = 2$	14664	4648	4.042e-2 (1.239)	2.790e-5 (2.519)
$l = 3$	58864	18384	1.694e-2 (1.255)	4.325e-6 (2.689)
$l = 4$	235872	73120	6.796e-3 (1.318)	5.335e-7 (3.019)

Computational Experiments

Cook's membrane (plane strain): incompressible case

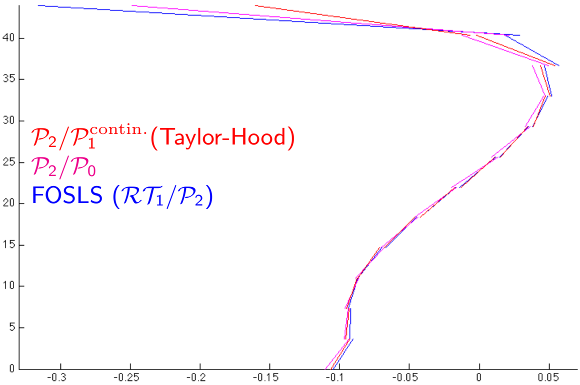
Traction force at right boundary: $\mathbf{P} \cdot \mathbf{n} = (0, \mu\gamma)^T$

Behavior of approximation to u_2 at right upper tip for $\gamma = -0.1$:



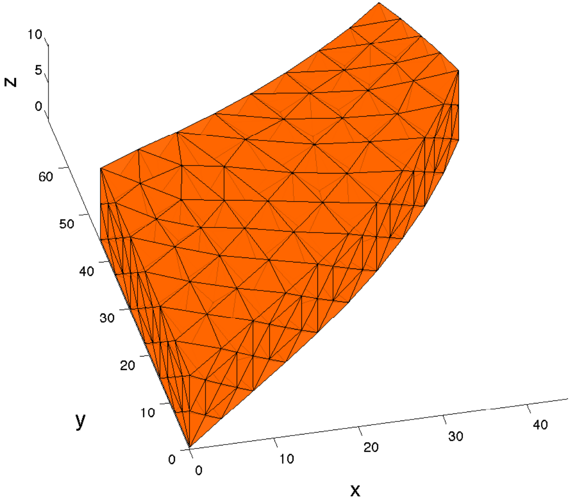
Computational Experiments

Cook's membrane (plane strain): incompressible case
Traction force at right boundary: $\mathbf{P} \cdot \mathbf{n} = (0, \mu\gamma)^T$
Tangential traction at left boundary at refinem. level $(\gamma = -0.1)$



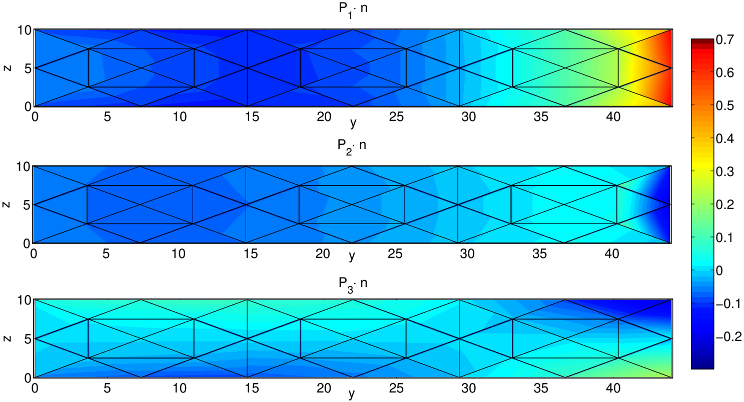
Computational Experiments

3D Cook's Membrane



Computational Experiments

3D Cook's Membrane



Plot of the normal components of the stress tensor on the left clamped boundary

(see poster by Benjamin Müller)

Overview

Conclusions

- First-order system least squares methods in solid mechanics provide simultaneous approximation of displacements and stresses

- Produces accurate results for local evaluations of stress and traction forces important in connection to damage simulations

- Generalizable to nonlinear solid-mechanical models in a natural way as well as, in principle, to implicit constitutive laws

- Local evaluation of least squares functional may be used as an a posteriori error estimator (see poster by Benjamin Müller)