Nonlinear parabolic equations degenerating on a part of the domain.

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Setting of the problem

Let a domain $\Omega \subset \mathbb{R}^n$ cross the hyperplane $\Sigma = \{(x_1, \dots, x_n) : x_n = 0\}$. In the cylinder $Q_T = \Omega \times [0, T]$ consider the nonlinear parabolic equation

$$\omega_\varepsilon(x) \frac{\partial u}{\partial t} = \mathrm{div}\left(\omega_\varepsilon(x) \mathbf{A}(x, t, u, \nabla u)\right), \qquad (1)$$

where

$$\omega_\varepsilon(x) = \begin{cases} \varepsilon, & x_n > 0, \\ 1, & x_n < 0, \end{cases} \qquad \varepsilon > 0. \qquad (2)$$

and the flow $\mathbf{A}(x, t, u, \xi)$ is a Caratheodory function satisfying the structure conditions

$$|\mathbf{A}(x, t, u, \xi)| \leq \Lambda |\xi|^{p-1}, \quad \mathbf{A}(x, t, u, \xi) \cdot \xi \geq \lambda |\xi|^p, \qquad (3)$$

where $p = const > 2$ and λ, Λ are positive constants, for all $(x, t) \in \mathbb{R}^n \times (0, \infty)$ and $\xi \in \mathbb{R}^n$.

The model equation is

$$\omega_\varepsilon(x)\frac{\partial u}{\partial t} = \mathrm{div}\left(\omega_\varepsilon(x)|\nabla u|^{p-2}\nabla u\right). \tag{4}$$

For any fixed ε this equation (and, of course, (1)) falls within the framework of the theory developed by E. DiBenedetto in 1980's for degenerate parabolic equations of the p-Laplace type. So, for any fixed positive ε we have at our disposal

1. Pointwise estimates of DiGiorgi-Moser (-Ladyzhenskaya-Uraltseva-...) type: It is possible to estimate the L^∞ norm of a solution in a smaller cylinder via its L^p norm in a larger cylinder.
2. (Hölderian) Estimates for the modulus of continuity of solutions: solutions are Hölder continuous in the interior, and up to the boundary if the boundary data is also Hölder continuous.
3. (Intrinsic form of) the Harnack inequality. *(DiBenedetto, Gianazza, Vespri: 2007)*.

The problem:

Is it possible to obtain standard a priori estimates for solutions of (1), independent of the (small) parameter ε?

It is easy that for any r on the set $Q \cap \{|x_n| > r\}$ the model equation (4) is reduced to the standard p-Laplace equation. Thus, the main problem is to glue together estimates in $\{x_n < -r\}$ and $\{x_n > r\}$ at the interface $\{x_n = 0\}$.

The essence of this problem can be easily shown on the example of the linear elliptic equation

$$\operatorname{div}(\omega_\varepsilon \mathcal{A} \nabla u) = 0 \tag{5}$$

with the uniformly elliptic matrix $\mathcal{A}$. For the classical 2nd order elliptic equation

$$\operatorname{div}(\mathcal{A} \nabla u) = 0 \tag{6}$$

the standard methods are based on the energy estimates

$$\int_{B_R} |\nabla (u-k)_\pm|^p \xi^p \, \mathrm{d}x \le C \int_{B_R} (u-k)_\pm^p |\nabla \xi|^p \, \mathrm{d}x,$$

where $k \in \mathbb{R}$ and $\xi \in C_0^\infty(B_R)$.

This estimate is obtained by using the test function $(u-k)_{\pm}\xi^p$ in the definition of a solution

$$\int \mathcal{A}\nabla u \cdot \nabla \varphi \,\mathrm{d}x = 0, \quad \varphi \in W_0^{1,2}(B_R)$$

and the Young inequality.

Following the same way, for equation (5), we obtain

$$\int_{B_R} |\nabla (u-k)_{\pm}|^p \xi^p \omega_\varepsilon \,\mathrm{d}x \leq C \int_{B_R} (u-k)_{\pm}^p |\nabla \xi|^p \omega_\varepsilon \,\mathrm{d}x. \qquad (7)$$

From here, on the lower part of B_R, $B_R^{(2)} = B_R \cap \{x_n < 0\}$, we obtain a "good" estimate

$$\int_{B_R^{(2)}} |\nabla(u-k)_\pm|^p \xi^p \, \mathrm{d}x \le C \int_{B_R} (u-k)_\pm^p |\nabla \xi|^p \omega_\varepsilon \, \mathrm{d}x. \qquad (8)$$

On the upper part of B_R, $B_R^{(1)} = B_R \cap \{x_n < 0\}$, we can easily obtain from here only a "bad" estimate

$$\varepsilon \int_{B_R^{(1)}} |\nabla(u-k)_\pm|^p \xi^p \, \mathrm{d}x \le C \int_{B_R} (u-k)_\pm^p |\nabla \xi|^p \omega_\varepsilon \, \mathrm{d}x \qquad (9)$$

which becomes useless as ε goes to zero.

A related problem.

The problem described above can be considered as a model case for a more difficult problem. Take the weight

$$\omega(x) = \begin{cases} \omega_1(x), & x_n > 0, \\ \omega_2(x), & x_n < 0, \end{cases} \tag{10}$$

where both weights ω_1 and ω_2 belong to the Muckenhoupt class A_p and are symmetric with respect to Σ. A standard example of such weight is given by $\omega_i(x) = |x|^{\alpha_i}$, $-n < \alpha_i < p$.
Assume, additionally, that

$$\frac{\omega_1(x)}{\omega_1(B_r)} \leq C \frac{\omega_2(x)}{\omega_2(B_r)}, \quad \forall x \in B_r, \tag{11}$$

for balls B_r of radius $r \leq r_0$ centered on Σ. An easy consequence of (11) is that

$$\omega_1(x) \leq C\omega_2(x)$$

in the neighbourhood of Σ.

Consider the parabolic equation

$$\omega(x)u_t = \mathrm{div}\left(\omega(x)|\nabla u|^{p-2}\nabla u\right), \quad x \in \Omega, t \in [0, T],$$

or its elliptic analogue

$$\mathrm{div}\left(\omega(x)|\nabla u|^{p-2}\nabla u\right) = 0, \quad x \in \Omega.$$

If we stay away from the interface Σ, the solutions enjoy all standard properties like local boundedness, Hölder continuity, the Harnack inequality. However, as soon as we touch the interface, the standard scheme breaks.

For instance, it is easy to see that the weighted form of the Sobolev inequality

$$\frac{1}{\omega(B_r)}\int_{B_r}|v|^{p\kappa}\omega\,\mathrm{d}x \le C\left(\frac{r^p}{\omega(B_r)}\int_{B_r}|\nabla v|^p\omega\,\mathrm{d}x\right)^{\kappa},$$

$\kappa > 1$, can not hold if $\omega_2(B_r)/\omega_1(B_r) \to 0$ as $r \to 0$ for balls B_r centered on Σ.

Moreover, in the elliptic case it was established that the standard form of the Harnack inequality does not hold in general, though the Hölder continuity of solutions is present.

Previous results in this direction.

1. Elliptic equations with "partial Muckenhoupt" weights: Yu. Alkhutov, V. Zhikov (2001,2003) - sup estimates and continuity of solutions. A special form of the Harnack inequality was proved by Yu. Alkhutov and E. Hrenova in 2011:

$$\inf_{B_R} u > \gamma \sup_{B_R^-} u, \quad B_R^- = B_R \cap \{-R < x_n < -R/2\}.$$

2. For the linear elliptic case

$$\mathrm{div}\,(\omega_\varepsilon \mathcal{A} \nabla u) = 0. \tag{12}$$

pointwise sup estimates and the Hölder continuity of solutions were established by Yu. Alkhutov and S. Guseinov (2009).

3. The linear parabolic case

$$\omega_\varepsilon u_t = \operatorname{div}\left(\omega_\varepsilon \mathcal{A} \nabla u\right) \tag{13}$$

was due to Yu. Alkhutov and V.A. Liskevich (2012). In the first of their two joint papers they obtained local pointwise estimates

$$\sup_{Q_{R/2}} u \leq C \left(\frac{1}{|Q|} \int_{Q_R} u^2 \,\mathrm{d}x \,\mathrm{d}t \right)^{1/2}, \quad Q_R = B_R \times (t_0 - R^2, t_0),$$

and continuity of solutions, which was followed by the second paper where the uniform Nash-Aronson type upper bound for the heat kernel was established:

$$K_\varepsilon(x, y, t) \leq c_1 t^{-n/2} e^{-c_2 |x-y|^2/t}.$$

The answer to the problem

is **YES**, at least what regards the pointwise estimates for the L^∞ norm of solutions and estimates for the modulus of continuity: it is possible to obtain these estimates uniform with respect to ε.

The Harnack inequality is the subject of my future work: first, it is necessary to understand the form it takes here: the standard form fails even in the elliptic case $+$ add the interplay with the intrinsic character of Harnack for nonlinear equations.

The main estimate

The main tool in the regularity theory is the classical energy estimate (see, for instance, the book of DiBenedetto)

$$\int_\Omega (u-k)_\pm^2 \xi^p \,\mathrm{d}x \Big|_{t=a}^{t=b} + \int_{\Omega\times(a,b)} |\nabla (u-k)_\pm|^p \xi^p \,\mathrm{d}x\,\mathrm{d}t$$
$$\leq p \int_{\Omega\times(a,b)} (u-k)_\pm^2 \xi^{p-1} \xi_t \,\mathrm{d}x\,\mathrm{d}t$$
$$+ C \int_{\Omega\times(a,b)} (u-k)_\pm^p |\nabla \xi|^p \,\mathrm{d}x\,\mathrm{d}t$$

for any $k \in \mathbb{R}$ and any smooth cut-off function ξ vanishing on $\partial\Omega \times (a,b)$. To obtain this estimate one multiplies the equation formally by $(u-k)_\pm \xi^p$, integrates by parts and uses the Young inequality. Formal justification is by the Steklov averaging.

In our case, this estimate takes the form

$$\int_{\Omega}(u-k)_{\pm}^{2}\xi^{p}\omega_{\varepsilon}\,\mathrm{d}x\bigg|_{t=a}^{t=b}+\int_{\Omega\times(a,b)}|\nabla(u-k)_{\pm}|^{p}\xi^{p}\omega_{\varepsilon}\,\mathrm{d}x\,\mathrm{d}t$$

$$\leq p\int_{\Omega\times(a,b)}(u-k)_{\pm}^{2}\xi^{p-1}\xi_{t}\omega_{\varepsilon}\,\mathrm{d}x\,\mathrm{d}t$$

$$+C\int_{\Omega\times(a,b)}(u-k)_{\pm}^{p}|\nabla\xi|^{p}\omega_{\varepsilon}\,\mathrm{d}x\,\mathrm{d}t.$$

While in the "lower" part $Q^{(2)} = Q \times \{x_n < 0\}$ it yields a good estimate

$$\int_{\Omega^{(2)}} (u-k)_\pm^2 \xi^p \,\mathrm{d}x \Big|_{t=b} + \int_{\Omega^{(2)} \times (a,b)} |\nabla (u-k)_\pm|^p \xi^p \,\mathrm{d}x\,\mathrm{d}t$$
$$\leq p \int_{\Omega \times (a,b)} (u-k)_\pm^2 \xi^{p-1} \xi_t \omega_\varepsilon \,\mathrm{d}x\,\mathrm{d}t$$
$$+ C \int_{\Omega \times (a,b)} (u-k)_\pm^p |\nabla \xi|^p \omega_\varepsilon \,\mathrm{d}x\,\mathrm{d}t$$
$$+ \int_{\Omega} (u-k)_\pm^2 \xi^p \,\mathrm{d}x \Big|_{t=a},$$

on the "upper" part of the cylinder we have from here only

$$\varepsilon \int_{\Omega^{(1)}} (u-k)_\pm^2 \xi^p \,\mathrm{d}x \Big|_{t=b} + \varepsilon \int_{\Omega^{(1)}\times(a,b)} |\nabla(u-k)_\pm|^p \xi^p \,\mathrm{d}x\,\mathrm{d}t$$

$$\leq p \int_{\Omega\times(a,b)} (u-k)_\pm^2 \xi^{p-1} \xi_t \omega_\varepsilon \,\mathrm{d}x\,\mathrm{d}t$$

$$+ C \int_{\Omega\times(a,b)} (u-k)_\pm^p |\nabla\xi|^p \omega_\varepsilon \,\mathrm{d}x\,\mathrm{d}t$$

$$+ \int_\Omega (u-k)_\pm^2 \xi^p \,\mathrm{d}x \Big|_{t=b},$$

which degenerates at small values of ε.

Instead of this estimate we use another one. Denote the even continuation of u from $\mathcal{Q}^{(2)}$ across Σ to $Q^{(1)}$ by $\tilde{u}$. An additional assumption here is that the symmetric reflection of $Q^{(2)}$ across Σ covers $Q^{(1)}$. Let

$$v = \begin{cases} \max(u, \tilde{u}) \quad \text{in} \quad Q^{(1)}, \\ u \quad \text{in} \quad Q^{((2)}. \end{cases}$$

Denote the even continuation of the cut-off function ξ from $Q^{(2)}$ to $Q^{(1)}$ across Σ by $\hat{\xi}$.
Moreover, let the cut-off function ξ vanish on the parabolic boundary of Q or $u < k$ on the lower base of the cylinder Q.

Then

$$\sup_{a\le t\le b}\int_\Omega (v-k)_+^2\xi^p\,\mathrm{d}x+\int_Q|\nabla(v-k)_+|^p\xi^p\,\mathrm{d}x\,\mathrm{d}t$$
$$\le C\int_Q (u-k)_+^2(|\xi|+|\hat{\xi}_t|)\,\mathrm{d}x\,\mathrm{d}t \tag{14}$$
$$+C\int_Q (u-k)_+^p(|\nabla\xi|^p+|\nabla\hat{\xi}|^p)\,\mathrm{d}x\,\mathrm{d}t$$

with the constant C independent of ε and the solution.
A similar estimate holds for $()_+$ replaced by $()_-$ with max replaced by min in the definition of v.

The proof follows *Alkhutov, Liskevich 2012* and is by a choice of the test-function (modulo the Steklov averaging process)

$$\varphi = \left((u-k)_+ - (\min(u,\tilde{u})-k)_+^3 (u-k)_+^{-2}\right) \xi^p.$$

It is easy that φ is not zero only on the set where $\{u > \tilde{u}\}$, which is contained in the upper part of the cylinder $Q^{(1)} = Q \cap \{x_n > 0\}$. Thus, one gets rid of ε. Further manipulations rely on the fact that $\tilde{u}$ is a solution to an equation of the same type

$$\tilde{u}_t = \operatorname{div} \mathbf{B}(x,t,\tilde{u},\nabla\tilde{u}),$$

where

$$(\mathbf{B}(\ldots,\xi_1,\ldots,\xi_{n-1},\xi_n))_i = (\mathbf{A}(\ldots,\xi_1,\ldots,\xi_{n-1},-\xi_n))_i\,,$$
$$i = 1,\ldots,n-1,$$
$$(\mathbf{B}(\ldots,\xi_1,\ldots,\xi_{n-1},\xi_n))_n = -(\mathbf{A}(\ldots,\xi_1,\ldots,\xi_{n-1},-\xi_n))_n\,.$$

In the model case $\tilde{u}$ satisfies just the same equation with ω_ε reflected across Σ.

In the end we obtain

$$\begin{aligned}
&\sup \int_{\Omega\cap\{u>\tilde{u}\}} (u-k)_+^2 \xi^p \, \mathrm{d}x + \int_{Q\cap\{u>\tilde{u}\}} |\nabla (u-k)_+|^p \xi^p \, \mathrm{d}x \, \mathrm{d}t \\
&\quad \leq C \int_{Q\cap\{u>\tilde{u}\}} \left[(u-k)_+^2 \xi^{p-1} |\xi_t| + (u-k)_+^p |\nabla \xi|^p \right] \mathrm{d}x \, \mathrm{d}t \\
&\quad + C \int_{Q\cap\{u>\tilde{u}\}} |\nabla (\tilde{u}-k)_+|^p \xi^p + C \int_{\Omega\cap\{u>\tilde{u}\}} (\tilde{u}-k)_+^2 \xi^p \, \mathrm{d}x.
\end{aligned} \tag{15}$$

It remains to use the “good” estimate for u in $Q^{(2)}$ to complete the proof.
Having obtained estimate (15), the rest of the proof follows the standard DiBenedetto's scheme.

The only difference is that we have to use only such cut-off functions ξ that vanish on the lower base of the cylinder. It leads to the cylinders we used in the proof being asymmetric with respect to Σ. They have the structure $B_R \times (a, b)$, where $|B_R \cap \{x_n > 0\}| = \alpha |B_R|$ with some small positive constant α.

In the standard case on of the instruments used in the proof is the following asserion:

Let u be a solution in the cylinder $B_R \times [0, T]$ and $|\{u(\cdot, 0) > k\} \cap B_R| > \alpha |B_R|$. There exist constants $\beta, \delta \in (0, 1)$ and a positive constant θ such that $|\{u(\cdot, t) > \delta k\} \cap B_R| > \beta |B_R|$ for any $0 \leq t \leq \theta k^{2-p} R^p$.

In our case, this estimate is replaced by the following one.

Let u be a solution in the cylinder $B_R \times [0, T]$ and $|\{u(\cdot, 0) > k\} \cap B_R| > \alpha |B_R|$. Assume that $|B_R \cap \{x_n > 0\}| \leq \frac{\alpha}{8} |B_R|$. There exist constants $\delta \in (0, 1)$ and $\theta > 0$ such that $|\{u(\cdot, t) > \delta k\} \cap B_R| > \frac{\alpha}{2} |B_R|$.

To prove the Hölder continuity it is sufficient to obtain the reduction of oscillation in the following basic situation.

Lemma

Let u be a solution to (1) in the cylinder $Q = B_R \times [-T, 0]$, satisfying $0 \leq u \leq 1$ in Q. Then there exist $N \in \mathbb{N}$, $\alpha \in (0, 1)$ and $\delta \in (0, 1)$ such that

$$\operatorname*{ess\,osc}_{B_{\alpha R} \times [-1,0]} u \leq 1 - \delta$$

provided that $T > 2N$. The constants N, α, δ are independent of u and ε.

The remaining instruments in the proof are the following:

Lemma (De Giorgi type lemma)

Let u be a solution to (1) in the cylinder $Q = B_R \times [-T, 0]$. Let $\mu_+ \geq \operatorname{ess\,sup}_Q u$, $\mu_- \leq \operatorname{ess\,inf}_Q u$, $\omega \in \mu_+ - \mu_-$. For any $a \in (0,1)$ and $\sigma \in (0,1)$ there exists a positive number $\varepsilon = \varepsilon(a, \sigma, TR^{-p}\omega^{p-2})$ such that $u < \mu_+ - a\omega$ in $B_{\sigma R} \times [-T/2, 0]$ provided that $|\{u > \mu_+ - \omega\} \cap Q| < \varepsilon |Q|$. Analogously, $u > \mu_- + a\omega$ in $B_{\sigma R} \times [-T/2, 0]$ provided that $|\{u < \mu_- + \omega\} \cap Q| < \varepsilon |Q|$ with the same ε.

Lemma

Let u be a solution to (1) in the cylinder $Q = B_R \times [t_0, t_1]$. Assume that $u(\cdot, t_0) > \mu_- + \omega$ a.e. on B_R, where

$$\mu_+ \geq \operatorname*{ess\,sup}_Q u, \quad \mu_- \leq \operatorname*{ess\,inf}_Q u, \omega \in \mu_+ - \mu_-.$$

For any $\sigma, a \in (0,1)$ there exists $\theta = \theta(a, \sigma, data)$ such that $u > \mu_- + a\omega$ a.e. in the cylinder $Q' = B_{\sigma R} \times [t_0, t_0 + \theta\omega^{2-p}R^p]$. If $u(\cdot, t_0) < \mu_+ - \omega$ on B_R, then $u < \mu_+ - a\omega$ on Q'.

Lemma (Telescopic argument)

Let u be a solution to (1) in the cylinder $Q = B_R \times [t_0, t_0 + 2\theta\omega^{2-p}R^p]$, $\omega > 0$. Assume that $|\{u(\cdot,t) > \mu_- + \omega\} \cap B_{\sigma R}| > \alpha |B_{\sigma R}|$, $\alpha, \sigma \in (0,1)$. For any $j \in \mathbb{N}$ the following estimate holds:

$$|\{u < \mu_- + 2^{-j}\omega\} \cap Q'| \leq C(\alpha,\sigma) j^{1/p-1} |Q'|,$$
$$Q' = B_{\sigma R} \times [t_0 + \theta\omega^{2-p}R^p, t_0 + 2\theta\omega^{2-p}R^p].$$

provided that $\theta > 2^{(p-2)j}$.

Directions of further work.

1. Complete the proof for the singular case $p < 2$.
2. Obtain the Harnack inequality.
3. Consider the "partial Muckenhoupt weight" scenario.
4. $p = p(x)$?
5. L^1 estimates?
6. The "cross" interface (chessboard type structure).

An example of a “cross”-type interface on the plane was considered by Alkhutov and Zhikov: the equation

$$\left(a_{ij}(x)\omega(x)u_{x_j}\right)_{x_j} = 0$$

where

$$\omega(x) = \begin{cases} |x|^{-\alpha}, & x_1x_2 > 0, \\ |x|^{\alpha}, & x_1x_2 < 0, \end{cases} \qquad 0 < \alpha < 2.$$

The feature of this weight is that it is irregular, i.e. smooth functions are **not** dense in the Soboloev space associated with this equation. Correspondingly, one can speak about two types of solutions: H-solutions and W-solutions. While the former are continuous, the latter are not if they are not H-solutions.

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