Half-homogeneous chainable continua with end points

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Abstract

A point of a chainable continuum is called an end point if for every positive epsilon there is an epsilon-chain such that only the first link contains the point. We prove that up to homeomorphism there are only two half-homogeneous chainable continua with two end points. One of them is an arc and the second one is the quotient of an arc of pseudo-arcs, where the two terminal continua are pushed to points. This answers a question of the second and third author.

Moreover we prove that the two above mentioned continua are the only half-homogeneous chainable continua with a nonempty finite set of end points.

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1 Introduction

A *continuum* is a non-empty compact connected metrizable space. A continuum is said to be *indecomposable* if it can not be written as a union of

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two proper subcontinua. Otherwise a continuum is said to be *decomposable*. A continuum is called *hereditarily unicoherent* if the intersection of any two subcontinua is empty or connected. A point of a continuum is said to be a *cut point* if its complement is not connected. An *orbit* of a topological space X containing a point $x \in X$ is the set of all points $h(x)$, where $h\colon X \to X$ is a homeomorphism. A continuum X is called $\frac{1}{n}$*-homogeneous* if it consists of n orbits exactly, where $n \in \mathbb{N}$. For $n = 1$ we just write *homogeneous*.

A *chain* is a finite sequence $\mathcal{C} = \{C_1, C_2, \ldots, C_n\}$ of open sets in a metric space such that $C_i \cap C_j \neq \emptyset$ if and only if $|i - j| \leq 1$. The elements of a chain are called its *links*. If $\varepsilon > 0$ and the diameter of each link is less than ε, then the chain is called an ε*-chain*. A continuum is *chainable* if for each $\varepsilon > 0$ it can be covered by an ε-chain.

A continuous mapping $f\colon X \to Y$ between metric spaces is called an ε*-mapping* if f is continuous and for every $x \in X$ the diameter of $f^{-1}(f(x))$ is less than ε. A continuum X is called *arc-like* if for every $\varepsilon > 0$ there is an ε-mapping of X onto an arc. It is a well known result that a nondegenerate continuum is arc-like if and only if it is chainable.

Let X be a continuum and $p, q \in X$. The continuum X is called *irreducible between* p *and* q if any subcontinuum containing p and q is equal to X. A continuum is said to be *irreducible* if it is irreducible between some two points.

Fact 1 (Section 5 in [Bi51, p. 660] or [Do08, p. 32])**.** For a point p of a nondegenerate chainable continuum X the following conditions are equivalent.

a) Each nondegenerate subcontinuum of X containing p is irreducible between p and some other point.

b) If there are two subcontinua of X containing p, one of them contains the other.

c) For each positive number ε, there is an ε-chain covering X such that only the first link of the chain contains p.

d) For every $\varepsilon > 0$ there is a continuous mapping f of X onto $[0, 1]$ such that preimages of points have diameter less than ε and $f(p) = 0$.

Definition 2. A point p of a chainable continuum X is called an *end point* if it satisfies one condition (or all conditions) from Fact 1.

In Fact 1 we would like to emphasize especially the condition b), because we will use it quite often. A classical example is an arc which contains two end points, but it may happen that a chainable continuum contains more than two end points. For example the $\sin(1/x)$-continuum contains three end points. In the pseudo-arc, every point is an end point by Theorem 16 from [Bi51]. On the other hand there are continua with no end points. In [Do94] it is even shown that an arbitrary nonnegative integer can be the number of end points of a chainable continuum.

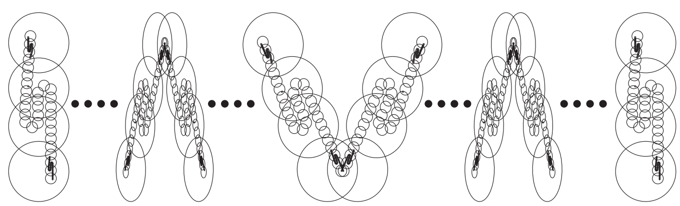

Figure 1: An arc of pseudo-arcs.

An *arc of pseudo-arcs* (see Figure 1) is any chainable continuum A for which there exists a continuous mapping $g\colon A \to [0,1]$ such that preimage of each point is a pseudo-arc. It is known that up to homeomorphism there is only one continuum with these properties. For more details see [BJ59] and [Le85]. It is known that A is $\frac{1}{2}$-homogeneous and the two orbits in X are $g^{-1}(\{0,1\})$ and $g^{-1}((0,1))$ by Example 4.8 from [NPP06]. Moreover the first orbit consists of precisely the end points of A.

Let us consider the quotient of A which is obtained using an upper semi-continuous decomposition $\{g^{-1}(0), g^{-1}(1)\} \cup \{\{x\}\colon x \in g^{-1}((0,1))\}$. Any continuum homeomorphic to this quotient will be called an *arcless-arc*. It follows that an arcless-arc is a $\frac{1}{2}$-homogeneous chainable continuum with two end points.

In Problem 8 of [PV13] the second and third author of this paper settled the following problem.

Problem 3. Does there exist a $\frac{1}{2}$-homogeneous chainable continuum with exactly two end points which is neither an arc nor an arcless-arc?

We prove that there is no such continuum. Moreover we show that any $\frac{1}{2}$-homogeneous chainable continuum with a finite nonempty set of end points contains just two end points and thus it is either an arc or an arcless-arc.

2 Tools

In this section we cite several known results that will be used in proofs in the Main results section. Most of the facts are given without proof, but there is always a reference to the source.

Fact 4 (Boundary Bumping Theorem 5.4 from [Na92, p. 73])**.** *Let X be a continuum, G an open proper subset of G and p a point in G. Then the closure of the component of p in U is a continuum intersecting the boundary of G.*

Fact 5 (Theorem 12.1 from [Na92, p. 230])**.** *Every nondegenerate subcontinuum of a chainable continuum is chainable.*

Fact 6 (Theorem 12.2 from [Na92, p. 230])**.** *Every chainable continuum is hereditarily unicoherent.*

Definition 7. A continuum T is said to be a *weak triod* provided that there exist three subcontinua of T whose intersection is nonempty, whose union is T and none of which is contained in the union of the two remaining.

Fact 8 (Corollary 12.7 from [Na92, p. 233])**.** *A chainable continuum does not contain a weak triod.*

Fact 9 (Theorem 6.17 from [Na92, p. 96])**.** *A continuum containing exactly two non cut points is an arc.*

Fact 10 ([Bi59])**.** *A nondegenerate homogeneous chainable continuum is a pseudo-arc.*

Fact 11 (Theorem 16 from [Bi51] or Theorem 3.13 from [Le99, p. 44])**.** Let X be a chainable continuum each point of which is an end point. Then X is a pseudo-arc.

Fact 12 (Effros theorem [Ef65], [vM04])**.** Suppose that a Polish group acts transitively on a Polish space. Then the group acts microtransitively.

Fact 13. Let X be a continuum with a compatible metric ρ and let G be an open subset of X. Suppose that for every pair of points c and d of G there is a homeomorphism $h\colon X \to X$ such that $h(c) = d$. Then for every $\varepsilon > 0$ and for every $c \in G$ there is $\delta > 0$ such that whenever $\rho(c, d) < \delta$ then there is a homeomorphism $h\colon X \to X$ such that $h(c) = d$ and $\rho(e, h(e)) < \varepsilon$ for every $e \in X$.

Proof. The group of homeomorphisms of a compact space with the topology of uniform convergence is completely metrizable by Corollary 1.3.11 from [vM01, p. 35]. The open set G is a Polish space. Thus by Fact 12 we get the desired result. □

Fact 14 (Theorem 3.4 from [Bo13])**.** Let X be a $\frac{1}{2}$-homogeneous continuum. If X is indecomposable, then each of the two orbits is uncountable.

Lemma 15. Let X be a chainable continuum and let E be a finite subset of the set of end points of X. Then the space $X \setminus E$ is connected.

Proof. Suppose for contradiction that $X \setminus E$ is not connected. Since $X \setminus E$ is nonempty there are two disjoint nonempty open sets U and V in $X \setminus E$ whose union is $X \setminus E$. Clearly U as well as V are open in X. We see that the closure of $U \cup V$ is the whole continuum X and thus the union of closures of U and V is X. Since X is connected we get that there is a point $e \in E$ which lies in the closure of U and also in the closure of V. There exist sequences $\{u_n\}_{n=1}^{\infty} \subseteq U$ and $\{v_n\}_{n=1}^{\infty} \subseteq V$ which converge to the point e. For any $n \in \mathbb{N}$ we denote by K_n (resp. L_n) the closure of the component of the point u_n (resp. v_n) in U (resp. V). By the Boundary Bumping Theorem (Fact 4) any continuum K_n (resp. L_n) intersects boundary of U (resp. V) which is a subset of E. Since the set E is finite we may suppose without loss of generality that there is a point $a \in E$ (resp. $b \in E$) such that $a \in K_n$ (resp. $b \in L_n$) for every $n \in \mathbb{N}$. Let K (resp. L) be the closure of $\bigcup\{K_n \colon n \in \mathbb{N}\}$ (resp. $\bigcup\{L_n \colon n \in \mathbb{N}\}$). Clearly K and L are continua because they are closures of connected sets. Moreover $e \in K \cap L$, $K \subseteq X \setminus V$ and $L \subseteq X \setminus U$ and thus $K \cap L \subseteq E$. Since

e is an end point we get by b) in Fact 1 that $K \subseteq L$ or $L \subseteq K$. This is a contradiction with the fact that $K \cap L$ is a subset of E which is a finite set and K as well as L are nondegenerate continua. $\square$

Remark 16. Let us note that Lemma 15 need not to be true if the set E of some end points is infinite. If X is a pseudo-arc (each point of which is an end point) and E is a suitable set for which $X \setminus E$ is not connected we get a counterexample.

Fact 17 (Theorem 10 from [BJ59])**.** Suppose that A and A' are arcs of pseudo-arcs and denote by E (resp. E') the set of all end points of A (resp. A'). Then any homeomorphism of E onto E' can be extended to a homeomorphism of A onto A'.

Lemma 18. Let X be a continuum for which there exists a continuous mapping $f\colon X \to [0,1]$ such that $f^{-1}(0)$ and $f^{-1}(1)$ are one point sets and $f^{-1}(c)$ is a pseudo-arc for any $c \in (0,1)$. Then X is an arcless-arc.

Proof. Let us denote by a (resp. b) the only point for which $f(a) = 0$ (resp. $f(b) = 1$). Let $g\colon A \to [0,1]$ be an onto continuous mapping of an arc of pseudo-arcs such that preimages of points are pseudo-arcs. Let $\{I_n\}_{n=1}^{\infty}$ be a sequence of nondegenerate compact intervals in $(0,1)$ whose union is $(0,1)$ such $I_{n+1} \cap (I_1 \cup \cdots \cup I_n)$ is a one point set. We denote $A_n = g^{-1}(I_n)$ and $X_n = f^{-1}(I_n)$ for $n \in \mathbb{N}$. Every continuum A_n as well as X_n is an arc of pseudo-arcs. With the use of Fact 17 we can find by induction a sequence of homeomorphisms $\{h_n\colon A_n \to X_n\}_{n=1}^{\infty}$ such that whenever $c \in I_m \cap I_n$ then $h_m(x) = h_n(x)$ for any $x \in A_m \cap A_n$ and $m, n \in \mathbb{N}$.

We define a mapping $h\colon A \to X$. For any $x \in X \setminus \{a, b\}$ there is $n \in \mathbb{N}$ such that $x \in A_n$ and we define $h(x)$ as $h_n(x)$. Moreover we define $h(x) = a$ for every $x \in g^{-1}(0)$ and $h(x) = b$ for $x \in g^{-1}(1)$. It is easily verified that h is a well defined continuous mapping. Since h is one-to-one on the set $A \setminus (g^{-1}(0) \cup g^{-1}(1))$ and it sends all points of the set $g^{-1}(0)$ to the point a and all points of the set $g^{-1}(1)$ to the point b we get that X is an arcless-arc. $\square$

Fact 19 (Theorem 4.2 from [Do08])**.** The set of end points of a chainable continuum is a G_δ-set.

3 Main results

Theorem 20. Let X be a chainable $\frac{1}{2}$-homogeneous continuum with exactly two end points. Then X is either an arc or an arcless-arc.

Proof. Let us denote by a and b the two distinct end points of X.

For any $c \in X$ we denote by A_c the intersection of all subcontinua of X which contain points a and c. Similarly we denote by B_c the intersection of all subcontinua of X which contain b and c. It follows that A_c as well as B_c is a continuum because any chainable continuum is hereditarily unicoherent by Fact 6. We denote $L_c = A_c \cap B_c$. Every space L_c is a continuum by the same reason. We call the sets L_c levels.

Claim 1. The sets $\{a, b\}$ and $X \setminus \{a, b\}$ are orbits in X.

Since X is $\frac{1}{2}$-homogeneous there are exactly two orbits. The end points $\{a, b\}$ of X form one of them and hence its complement $X \setminus \{a, b\}$ is the second one.

Claim 2. For any homeomorphism $h \colon X \to X$ and $c \in X$ we obtain $L_{h(c)} = h(L_c)$.

Suppose first that $h(a) = a$ and thus $h(b) = b$. Since A_c is the least continuum containing a and c and h is a homeomorphism we get that $h(A_c)$ is the least continuum containing $a = h(a)$ and $h(c)$ and thus $h(A_c) = A_{h(c)}$. By the same reason $h(B_c) = B_{h(c)}$. Thus we get that

$$h(L_c) = h(A_c \cap B_c) = h(A_c) \cap h(B_c) = A_{h(c)} \cap B_{h(c)} = L_{h(c)}.$$

If $h(a) \neq a$ we get that $h(a) = b$ and thus $h(b) = a$. By a similar argument as in the first case we obtain that $h(A_c) = B_{h(c)}$ and $h(B_c) = A_{h(c)}$. Hence

$$h(L_c) = h(A_c \cap B_c) = h(A_c) \cap h(B_c) = B_{h(c)} \cap A_{h(c)} = L_{h(c)}.$$

Claim 3. Any two levels L_c and L_d for $c, d \in X \setminus \{a, b\}$ are homeomorphic.

For $c, d \in X \setminus \{a, b\}$ there is a homeomorphism $h \colon X \to X$ such that $h(c) = d$ by Claim 1. Using Claim 2 we get that $h(L_c) = L_{h(c)} = L_d$. Thus restriction of h to the level L_c is a homeomorphism of L_c onto L_d.

Claim 4. The space $X \setminus \{a, b\}$ is connected.

Since a and b are end points of X we get by Lemma 15 that $X \setminus \{a, b\}$ is connected.

Claim 5. For any pair $c, d \in X \setminus \{a, b\}$ there is a homeomorphism $h \colon X \to X$ such that $h(a) = a$, $h(b) = b$ and $h(c) = d$.

Let us consider a compatible metric ρ on the space X. We let $\varepsilon = \rho(a, b)$ and we denote by $\mathcal{H}$ the set of all homeomorphisms $h\colon X \to X$ such that $d(e, h(e)) < \varepsilon$ for every $e \in X$. By the choice of ε we get that $h(a) = a$ and $h(b) = b$ for any $h \in \mathcal{H}$. By Fact 13 applied to $G = X \setminus \{a, b\}$ we obtain that for every $c \in X \setminus \{a, b\}$ there is a neighborhood N_c of the point c such that for any $d \in N_c$ there is a homeomorphism $h \in \mathcal{H}$ such that $h(c) = d$.

Since $X \setminus \{a, b\}$ is connected by Claim 4 we get that for a fixed pair of points c and d in $X \setminus \{a, b\}$ there is a finite sequence of points $c_1, \dots, c_n \in X \setminus \{a, b\}$ such that $c_1 = c$, $c_n = d$ and $c_{i+1} \in N_{c_i}$ for any $i < n$. Thus for every $i < n$ there is a homeomorphism $h_i \in \mathcal{H}$ such that $h_i(c_i) = c_{i+1}$. Now it is enough to define $h = h_{n-1} \circ \dots \circ h_2 \circ h_1$. Clearly $h(c) = d$, $h(a) = a$ and $h(b) = b$.

Claim 6. Level L_c doesn't contain neither a nor b for $c \in X \setminus \{a, b\}$.

By the Boundary Bumping Theorem (Fact 4) there is a nondegenerate continuum $K \subseteq X$ which contains a and omit b. We denote by d any point in K different from a. By Claim 5 there is a homeomorphism $h\colon X \to X$ for which the points a and b are fixed and for which $h(d) = c$. Continuum $h(K)$ contains a and c and doesn't contain b. Thus we get that $L_c \subseteq A_c \subseteq h(K) \subseteq X \setminus \{b\}$. By the same reason we obtain that $B_c \subseteq X \setminus \{a\}$ and finally $L_c = A_c \cap B_c \subseteq X \setminus \{a, b\}$.

Claim 7. Let $\mathcal{L} = \{L_c\colon c \in X\}$. Then $\mathcal{L}$ forms a partition of X.

Suppose for contradiction that there are points $c, d \in X$ such that $L_c \cap L_d \neq \emptyset$ and $L_c \neq L_d$. By Claim 6 we obtain that $L_c, L_d \subseteq X \setminus \{a, b\}$. Using Zorn's lemma we will prove, that there is a minimal level which is a subset of $L_c \cap L_d$. We denote by $\mathcal{S}$ the system of all levels contained in $L_c \cap L_d$. Clearly $\mathcal{S}$ is nonempty because there is a point $x \in L_c \cap L_d$ and thus the level $L_x = A_x \cap B_x$ is a subset of $A_c \cap B_c \cap A_d \cap B_d = L_c \cap L_d$. Thus $L_x \in \mathcal{S}$. For any nonempty chain $\mathcal{E} \subseteq \mathcal{S}$ we denote by K the intersection of $\mathcal{E}$. The space K is a continuum because it is an intersection of a chain of continua. There is a set $E \subseteq X$ such that $\mathcal{E} = \{L_e\colon e \in E\}$. We fix some $x \in K$. Clearly

$$L_x = A_x \cap B_x \subseteq \bigcap_{e \in E} A_e \cap \bigcap_{e \in E} B_e = \bigcap_{e \in E} L_e = \bigcap \mathcal{E}.$$

Hence any chain is bounded from below. By Zorn's lemma there is a minimal level $L_m \in \mathcal{S}$. We get that $L_m \subseteq L_c \cap L_d$ and since $L_c \neq L_d$ we obtain that L_m is either a proper subset of L_c or a proper subset of L_d. Without loss of generality we may suppose the first case holds. Since $L_c \subseteq X \setminus \{a,b\}$, we obtain by Claim 1 that there is a homeomorphism $h\colon X \to X$ such that $h(c) = m$. By Claim 2 we get that $L_m = h(L_c)$. Since L_m is a proper subset of L_c we get also that $h(L_m)$ is a proper subset of $h(L_c)$. Thus the level $L_{h(m)} = h(L_m)$ is a proper subset of L_m. This contradicts minimality of the level L_m. Thus $\mathcal{L}$ is a partition of the continuum X.

Claim 8. Every level L_c is a homogeneous continuum.

Since L_a and L_b are one-point sets, they are clearly homogeneous. Next suppose that $c \in X \setminus \{a,b\}$ and let $d \in L_c$ be an arbitrary point. Since $X \setminus \{a,b\}$ is an orbit in X by Claim 1 there is a homeomorphism $h\colon X \to X$ such that $h(c) = d$. By Claim 2 we have that $L_d = h(L_c)$. Moreover $L_c \cap L_d \neq \emptyset$ and thus by Claim 7 we obtain that $L_c = L_d$. If we restrict homeomorphism h to the level L_c we obtain a homeomorphism onto L_c such that $h(c) = d$. Thus L_c is a homogeneous continuum.

Claim 9. Every level L_c is either a point or a pseudo-arc.

Suppose that L_c is nondegenerate. Then L_c is a chainable continuum by Fact 5 and it is homogeneous by Claim 8. It follows that it is a pseudo-arc by Fact 10.

Claim 10. We define a binary relation $\preceq$ on $\mathcal{L}$ by $L_c \preceq L_d$ if and only if $A_c \subseteq A_d$ for $c, d, \in X$. We claim that the relation $\preceq$ is an order.

The relation $\preceq$ is clearly reflexive and transitive. It remains to verify that it is antisymmetric. Thus suppose for contradiction that we have $c, d \in X$ such that $L_c \preceq L_d$ and $L_d \preceq L_c$, but $L_c \neq L_d$. By the definition of $\preceq$ we get that $A_c = A_d$. Since $\mathcal{L}$ is a partition by Claim 7, we get that $L_c \cap L_d = \emptyset$. Let us denote by B the union of B_c and B_d. Clearly B is a continuum because $B_c \cap B_d$ contains a common point b. But $B \cap A_c = B \cap A_d = L_c \cup L_d$ is not connected. This is a contradiction with Fact 6 which provides the hereditarily unicoherence of X .

Claim 11. The pair $(\mathcal{L}, \preceq)$ is a linearly ordered set.

We take any L_c and L_d in $\mathcal{L}$. Continuum A_c as well as A_d contain the end point a. Thus by b) in Fact 1 we get that $A_c \subseteq A_d$ or $A_d \subseteq A_c$. Thus $L_c \preceq L_d$ or $L_d \preceq L_c$.

Claim 12. Suppose that $L_c \preceq L_d$ and $L_c \neq L_d$ for some $c, d \in X$. Then $A_c \cap B_d = \emptyset$.

Suppose not. Then there is a point $e \in A_c \cap B_d$. Since $A_c \subseteq A_d$ we get that $e \in A_d \cap B_d = L_d$ and hence $L_e = L_d$ by Claim 7. We get that A_c is a proper subset containing the end point a and the point d which is a contradiction with the minimality of A_d.

Claim 13. The family $\mathcal{L}$ is an upper semi-continuous decomposition of X.

Suppose that U is an open set containing a set $L_c \in \mathcal{L}$. We would like to show that there is an open set $V \subseteq U$ which contains L_c such that any level intersecting V is a subset of U. Suppose that this is not true. Then for every $n \geq 1$ there is a level $L_{c(n)}$ which intersects $\frac{1}{n}$ neighborhood of L_c and it intersects also $X \setminus U$. Without loss of generality we may suppose that $L_{c(n)} \preceq L_c$ and by Claim 11 we may suppose that $L_{c(n)} \preceq L_{c(n+1)}$ for every n. By compactness of $X \setminus U$, there is a point $d \in X \setminus U$ whose every neighborhood intersects infinitely many levels $L_{c(n)}$. Clearly $L_c \neq L_d$ because $c \notin L_d$.

We distinguish two cases. First suppose that $L_c \preceq L_d$. We get that $L_{c(n)} \subseteq A_{c(n)} \subseteq A_c$. Since A_c is closed we get that $d \in A_c$ and thus $A_d \subseteq A_c$ which means $L_d \preceq L_c$. This is a contradiction.

Suppose that $L_d \preceq L_c$. Then A_d is disjoint with B_c by Claim 12 and thus there is some $N \geq 1$ for which $L_d \preceq L_{c(N)}$ and $L_d \neq L_{c(N)}$ (otherwise $L_{c(n)} \subseteq A_d$ for every n and thus $c(n)$ could not converge to the point $c \in L_c$). Then $B_{c(N)}$ contains any level $L_{c(n)}$ for $n \geq N$. But $B_{c(N)}$ is a closed set disjoint with A_d by Claim 12. Hence $L_{c(N)}$ is a subset of $B_{c(N)}$ for $n \geq N$ and thus we get a contradiction with the assumption that any neighborhood of d intersects infinitely many levels $L_{c(n)}$.

Thus the family $\mathcal{L}$ is an upper semi-continuous decomposition.

Claim 14. The levels L_a and L_b are not cut points of the decomposition space $\mathcal{L}$.

The point $\{a\}$ is not a cut point of $\mathcal{L}$ because its complement in $\mathcal{L}$ is a continuous image under the quotient mapping of the set $X \setminus \{a\}$ which is connected by Lemma 15.

Claim 15. $A_c \cup B_c = X$ for any $c \in X$.

Suppose not. Then there is a point $d \in X$ such that $d \in X \setminus (A_c \cup B_c)$. Since a is an end point we get that eiter $A_c \subseteq A_d$ or $A_d \subseteq A_c$. Without loss of generality we may suppose that $A_c \subseteq A_d$. Then $B_d \subseteq B_c$. Thus $d \in B_c$ which is a contradiction with the choice of the point d.

Claim 16. Any level L_c is a cut point of the decomposition space $\mathcal{L}$ for $c \in X \setminus \{a, b\}$.

We define open sets $U = X \setminus B_c$ and $V = X \setminus A_c$. Since $A_c \cup B_c = X$ by Claim 15 we get that $X \setminus L_c = U \cup V$. Since $a \in U$ and $b \in V$ we get that $X \setminus L_c$ is a disjoint union of two nonempty open sets and thus it is not connected. Thus L_c is a cutpoint of $\mathcal{L}$.

Claim 17. The decomposition space $\mathcal{L}$ is an arc.

By Claim 13 we know that $\mathcal{L}$ is an upper semi-continuous decomposition and thus $\mathcal{L}$ is a continuum. Using Claim 14 and Claim 16 we get that $\mathcal{L}$ contains exactly two points which are not cut points and thus by Fact 9 we obtain that $\mathcal{L}$ is an arc.

Claim 18. X is either an arc or an arcless-arc.

By Claim 9 and Claim 3 there are two possible cases. Suppose first that L_c is a one-point set for every $c \in X$. Then $\mathcal{L}$ is a decomposition into singletons and thus X is homeomorphic to the decomposition space $\mathcal{L}$ which is an arc by Claim 17.

Now suppose that L_c is a pseudo-arc for every $c \in X \setminus \{a, b\}$. The quotient mapping $f \colon X \to \mathcal{L}$ satisfies assumptions of Lemma 18 and thus X is an arcless-arc. □

Corollary 21. A continuum is an arcless-arc if and only if it is a chainable $\frac{1}{2}$-homogeneous continuum with exactly two end points, but which is not an arc.

Now we will study chainable continua with exactly one end point. It is obvious that if we try to find such a continuum which is homogeneous, the only one is a degenerate continuum. In the next theorem we prove that there is no possibility if we are looking for a $\frac{1}{2}$-homogeneous one.

Proposition 22. There is no chainable $\frac{1}{2}$-homogeneous continuum with one end point.

Proof. Suppose for contradiction that X is a chainable $\frac{1}{2}$-homogeneous continuum with one end point a. Thus the orbits of X are $\{a\}$ and $X \setminus \{a\}$. By Fact 14 we get that X has to be decomposable. Thus there are proper subcontinua A and B of X such that $X = A \cup B$. If $a \in A \cap B$ then by b) in Fact 1 we get $A \subseteq B$ or $B \subseteq A$ which is a contradiction. Thus the end point a is an element of exactly one of the sets A and B. Without loss of generality we may suppose that $a \in A$ and $a \notin B$. Then $a \in X \setminus B \subseteq A$ and hence a is in the interior of A. Let us fix any point c in the interior of A distinct from a. Now for any point $d \in X \setminus \{a\}$ there is a homeomorphism $h\colon X \to X$ for which $h(c) = d$ and of course $h(a) = a$. Thus the point d is contained in the interior of $h(A)$ which is a proper subcontinuum containing a. Hence

$$\bigcup\{\mathrm{int}B\colon a \in B, B \subsetneq X, B \text{ is a continuum}\} = X.$$

Since X is compact there is a finite family $B_1, \ldots, B_n$ of proper subcontinua of X such that $a \in B_i$ for every $i \leq n$ and

$$\bigcup\{\mathrm{int}B_i\colon i \leq n\} = X.$$

Since continua containing the end point a are comparable by b) in Fact 1, there exists $i \leq n$ such that the interior of B_i is equal to X. Hence $B_i = X$ which is a contradiction with the choice of B_i as a proper subcontinuum of X. Thus there is no continuum X with the given properties. □

Proposition 23. There is no chainable $\frac{1}{2}$-homogeneous continuum with exactly n end points for an integer $n \geq 3$.

Proof. Suppose for contradiction that there is such a continuum. Denote by E the set of all end points of X. Similarly as in Theorem 20 we can define for every $c \in X$ level L_c as an intersection of all continua containing a and some of the end points $e \in E$.

We can prove straightforward generalizations of Claims 1.–6. We fix three distinct points $a, b, c \in E$. Now we fix any point $x \in X \setminus E$. Let A, B and C be subcontinua of X containing the point x such that $a \in A$, $b \in B$ and $c \in C$. Since X is hereditarily unicoherent by Fact 6 we can assume that A, B and C are minimal continua with these properties. By the natural generalization of the proof of Claim 6 of Theorem 20 we get that $a \in A \setminus (B \cup C)$, $b \in B \setminus (A \cup C)$ and $c \in C \setminus (A \cup B)$. Thus $A \cup B \cup C$ is a weak triod which is a contradiction with Fact 8. □

Corollary 24. Let X be a $\frac{1}{2}$-homogeneous chainable continuum. Then the set of end points of X is either empty, or contains exactly two points, or it is infinite.

Proof. The result follows immediately by Proposition 22 and Proposition 23. □

4 Questions

We have just described all $\frac{1}{2}$-homogeneous chainable continua with a nonempty finite set of end points. It is natural to ask for the case when the set of end points is either empty or infinite. If there are no end points it is hard to say something constructive. On the other hand if we suppose that X is a $\frac{1}{2}$-homogeneous chainable continuum whose set of end points E is infinite we can distinguish three cases. If $E = X$ we get that X is a pseudo-arc by Fact 11 which is contradiction with $\frac{1}{2}$-homogeneity. If E is a proper dense subset of X we can observe that E is a homogeneous dense G_δ set by Fact 19, but we don't know how to proceed further. If the closure of E is a proper subset of X we can easily prove using $\frac{1}{2}$-homogeneity that E is a closed set with an empty interior. Moreover since E is a homogeneous compact subset of a chainable continuum we get that components of E are either points or pseudo-arcs and consequently E is homeomorphic either to a Cantor space or to the product of a finite set and a pseudo-arc or to the product of a Cantor space and a pseudo-arc (by Theorem 1 from [Le83]).

Question 1. Does there exist a $\frac{1}{2}$-homogeneous chainable continuum without end points?

Question 2. Does there exist a $\frac{1}{2}$-homogeneous chainable continuum with infinitely many end points which is not an arc of pseudo-arcs?

The most ambitious question of this paper follows.

Question 3. What are the $\frac{1}{2}$-homogeneous chainable continua?

We know three of them, namely an arc, an arc of pseudo-arcs and an arcless-arc.

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