# ON THE VARIATION OF THE HARDY-LITTLEWOOD MAXIMAL FUNCTION

ONDŘEJ KURKA

Abstract. We show that a function $f : \mathbb{R} \to \mathbb{R}$ of bounded variation satisfies

$$\operatorname{Var} Mf \leq C \operatorname{Var} f$$

where $Mf$ is the centered Hardy-Littlewood maximal function of $f$. Consequently, the operator $f \mapsto (Mf)'$ is bounded from $W^{1,1}(\mathbb{R})$ to $L^1(\mathbb{R})$. This answers a question of Hajłasz and Onninen in the one-dimensional case.

## 1. Introduction and main results

The centered Hardy-Littlewood maximal function of $f : \mathbb{R}^n \to \mathbb{R}$ is defined by

$$Mf(x) = \sup_{r>0} \fint_{B(x,r)} |f(y)| \, dy.$$

J. Kinnunen proved in [7] that the maximal operator $f \mapsto Mf$ is bounded in the Sobolev space $W^{1,p}(\mathbb{R}^n)$ for $1 < p \leq \infty$ (see also [6, Theorem 1]). Since then, regularity properties of the maximal function have been studied by many authors [1], [2], [3], [4], [5], [6], [8], [9], [10], [11], [12].

Because $Mf \notin L^1$ whenever $f$ is non-trivial, Kinnunen's result fails for $p = 1$. Still, one can ask whether the maximal function of $f \in W^{1,1}$ belongs locally to $W^{1,1}$. In [6], the authors posed the following question.

**Question 1.1** (Hajłasz and Onninen)**.** Is the operator $f \mapsto |\nabla Mf|$ bounded from $W^{1,1}(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$?

In the present work, we show that the answer is positive for $n = 1$. The question had been already answered positively in the non-centered one-dimensional case by H. Tanaka [12]. This result was sharpened later by J. M. Aldaz and J. Pérez Lázaro [2] who proved that, for an arbitrary $f : \mathbb{R} \to \mathbb{R}$ of bounded variation, its non-centered maximal function $\widetilde{M}f$ is weakly differentiable and

$$\operatorname{Var} \widetilde{M}f \leq \operatorname{Var} f.$$

We prove that such an inequality holds for the centered maximal function as well.

**Theorem 1.2.** *Let $f : \mathbb{R} \to \mathbb{R}$ be a function of bounded variation. Then*

$$\operatorname{Var} Mf \leq C \operatorname{Var} f$$

*for a universal constant $C$.*

---

2010 *Mathematics Subject Classification.* 42B25, 46E35.

*Key words and phrases.* Hardy-Littlewood maximal function, function of bounded variation, weak differentiability.

The author is a junior researcher in the University Centre for Mathematical Modelling, Applied Analysis and Computational Mathematics (MathMAC).

In this paper, we do not care how small the constant $C$ may be. We note that it is a plausible hypothesis that the inequality holds for $C = 1$, in the same way as in the non-centered case (see also [3, Question B]).

Once Theorem 1.2 is proven, it is not difficult to derive the weak differentiability of $Mf$. Note that $Mf$ needs not to be continuous for an $f$ of bounded variation, and so $M$ does not possess such strong regularity properties as $\widetilde{M}$. Anyway, for a weakly differentiable $f$, everything is all right.

**Corollary 1.3.** *Let $f : \mathbb{R} \to \mathbb{R}$ be a measurable function with $Mf \not\equiv \infty$. If $f$ is locally $AC$ on an open set $U$, then $Mf$ is also locally $AC$ on $U$.*

**Corollary 1.4.** *Let $f \in W^{1,1}(\mathbb{R})$. Then $Mf$ is weakly differentiable and*

$$\|(Mf)'\|_1 \leq C\|f\|_{1,1}$$

*for a universal constant $C$.*

We do not know whether $Mf$ is weakly differentiable for $f \in W^{1,1}(\mathbb{R}^n)$ if $n \geq 2$. However, it is known that $Mf$ is approximately differentiable a.e. [5].

## 2. A property of the maximal function

Throughout the whole proof of Theorem 1.2, a function $f : \mathbb{R} \to \mathbb{R}$ of bounded variation will be fixed. Without loss of generality, we will suppose that $f \geq 0$.

During the proof, we will make efforts to show that $f$ varies comparably with $Mf$. The basic tool for meeting of this objective is represented by Lemma 2.4. However, despite of the length of its proof, the simple idea presented in the following remark is behind.

*Remark* 2.1. Let two points $p < r$ be such that $Mf(p) < Mf(r)$ and $p$ is regular in the sense that $Mf(p) \geq f(p)$. Moreover, let there be a radius $\omega$ such that $⨍_{r-\omega}^{r+\omega} f = Mf(r)$ and $p \in (r - \omega, r)$. Then one can use $⨍_{r-\omega}^{2p-(r-\omega)} f \leq Mf(p)$ to compute that there is $t \in (2p - (r - \omega), r + \omega)$ such that

$$\frac{f(t) - f(p)}{\omega} \geq \frac{Mf(r) - Mf(p)}{r - p}.$$

Note that $\omega$ is close to $t - p$ if $p$ is close to $r$. Thus, the average increase of $f$ in $(p, t)$ is comparable to the average increase of $Mf$ in $(p, r)$. (Notice also that $\frac{Mf(p+\omega)-Mf(p)}{\omega} \geq \frac{Mf(r)-Mf(p)}{r-p}$.)

We expected at first that this idea might lead to a simple proof of Theorem 1.2, possibly with $C = 1$. Nevertheless, no simple proof was found at last.

We need to introduce some notation first.

**Definition 2.2.**
- A *peak* is the system consisting of three points $p < r < q$ such that $Mf(p) < Mf(r)$ and $Mf(q) < Mf(r)$,
- the *variation* of a peak $\mathbb{p} = \{p < r < q\}$ is given by
$$\operatorname{var} \mathbb{p} = Mf(r) - Mf(p) + Mf(r) - Mf(q),$$
- the *variation* of a system $\mathbb{P}$ of peaks is
$$\operatorname{var} \mathbb{P} = \sum_{\mathbb{p} \in \mathbb{P}} \operatorname{var} \mathbb{p},$$
- a peak $\mathbb{p} = \{p < r < q\}$ is *essential* if $\sup_{p<x<q} f(x) \leq Mf(r) - \frac{1}{4} \operatorname{var} \mathbb{p}$,

- for the top $r$ of an essential peak $p < r < q$, we define (see Lemma 2.3)

$$\omega(r) = \max\Big\{\omega > 0 : \fint_{r-\omega}^{r+\omega} f = Mf(r)\Big\}.$$

**Lemma 2.3.** *Let $p < r < q$ be an essential peak. Then $\omega(r)$ is well defined. Moreover,*

$$r - \omega(r) < p \quad \textit{and} \quad q < r + \omega(r).$$

*Proof.* We have

$$Mf(r) > \lim_{\omega\to\infty} \fint_{r-\omega}^{r+\omega} f \quad \text{and} \quad Mf(r) > \lim_{\omega\searrow 0} \fint_{r-\omega}^{r+\omega} f,$$

as $Mf(r) > Mf(p) \geq \lim_{\omega\to\infty} \fint_{p-\omega}^{p+\omega} f = \lim_{\omega\to\infty} \fint_{r-\omega}^{r+\omega} f$ and

$$(r-\omega, r+\omega) \subset (p,q) \quad \Rightarrow \quad \fint_{r-\omega}^{r+\omega} f \leq Mf(r) - \frac{1}{4}\operatorname{var}\mathbb{p}.$$

It follows that $\omega(r)$ is well defined. Moreover, at least one of the points $p, q$ belongs to $(r-\omega(r), r+\omega(r))$. We may assume that $p \in (r-\omega(r), r+\omega(r))$. It remains to realize that also $q \in (r-\omega(r), r+\omega(r))$.

Suppose that $q \geq r + \omega(r)$. Since $r - \omega(r) < p$, we can consider the interval $(r-\omega(r), 2p-(r-\omega(r)))$ centered at $p$. We have

$$\fint_{r-\omega(r)}^{r+\omega(r)} f = Mf(r) \quad \text{and} \quad \fint_{r-\omega(r)}^{2p-(r-\omega(r))} f \leq Mf(p) < Mf(r),$$

and so

$$\fint_{2p-(r-\omega(r))}^{r+\omega(r)} f \geq Mf(r).$$

On the other hand,

$$\fint_{2p-(r-\omega(r))}^{r+\omega(r)} f \leq \sup_{2p-(r-\omega(r))<x<r+\omega(r)} f(x) \leq \sup_{p<x<q} f(x) \leq Mf(r) - \frac{1}{4}\operatorname{var}\mathbb{p},$$

which is a contradiction. □

**Lemma 2.4.** *Let $(x,y)$ be an interval of length $L$. Let a non-empty system $\mathbb{p}_i = \{p_i < r_i < q_i\}, 1 \leq i \leq m,$ of essential peaks satisfy*

$$x \leq r_1 < q_1 \leq p_2 < r_2 < q_2 \leq \cdots \leq p_{m-1} < r_{m-1} < q_{m-1} \leq p_m < r_m \leq y$$

*and*

$$25L < \omega(r_i) \leq 50L, \quad 1 \leq i \leq m.$$

*Then there are $s < u < v < t$ such that*

$$x - 50L \leq s, \quad t \leq y + 50L,$$

$$u - s \geq 4L, \quad v - u = L, \quad t - v \geq 4L$$

*and*

$$\min\{f(s), f(t)\} - \fint_u^v f \geq \frac{1}{12}\sum_{i=1}^m \operatorname{var}\mathbb{p}_i.$$

*Proof.* We divide the proof into three parts. In parts I. and II., we consider two special cases and find appropriate numbers satisfying the improved inequality

$$(*)\qquad \min\{f(s), f(t)\} - \fint_u^v f \geq \frac{1}{4}\sum_{i=1}^{m} \operatorname{var} \mathbb{p}_i.$$

The general case is considered in part III.

I. Let us assume that the system consists of one peak $\mathbb{p} = \{p < r < q\}$. First, we find $s$ and $t$ such that

$$f(s) \geq Mf(r), \quad x - 50L \leq s \leq 2q - (r + \omega(r)),$$
$$f(t) \geq Mf(r), \quad 2p - (r - \omega(r)) \leq t \leq y + 50L.$$

Due to the symmetry, it is sufficient to find an $s$ only. By Lemma 2.3, we can consider the interval $(2q - (r + \omega(r)), r + \omega(r))$ centered at $q$. We have

$$\fint_{r-\omega(r)}^{r+\omega(r)} f = Mf(r) \quad \text{and} \quad \fint_{2q-(r+\omega(r))}^{r+\omega(r)} f \leq Mf(q) < Mf(r),$$

and so

$$\fint_{r-\omega(r)}^{2q-(r+\omega(r))} f \geq Mf(r).$$

So, there is a point $s \in (r - \omega(r), 2q - (r + \omega(r)))$ such that $f(s) \geq Mf(r)$. We have $x - 50L \leq r - \omega(r) \leq s$.

Now, we consider two possibilities.

(I.a) If $q - p < 10L$, then we have

$$s \leq 2q - (r + \omega(r)) < 2p + 20L - r - 25L < p - 5L < p - L/2 - 4L,$$

and it can be shown similarly that $q + L/2 + 4L \leq t$. We take

$$(u, v) = \begin{cases} (p - L/2, p + L/2), & Mf(p) \leq Mf(q), \\ (q - L/2, q + L/2), & Mf(p) > Mf(q). \end{cases}$$

We obtain

$$\min\{f(s), f(t)\} - \fint_u^v f \geq Mf(r) - \min\{Mf(p), Mf(q)\} \geq \frac{1}{2} \operatorname{var} \mathbb{p},$$

and $(*)$ is proven.

(I.b) If $q - p \geq 10L$, then we use

$$s \leq 2q - (r + \omega(r)) < q, \quad p < 2p - (r - \omega(r)) \leq t$$

(here, Lemma 2.3 is needed again). At the same time,

$$\min\{f(s), f(t)\} \geq Mf(r) > Mf(r) - \frac{1}{4} \operatorname{var} \mathbb{p} \geq \sup_{p<x<q} f(x),$$

and so $s$ and $t$ can not belong to $(p, q)$. It follows that

$$s \leq p, \quad q \leq t.$$

Let us realize that the choice

$$(u, v) = \big((p + q - L)/2, (p + q + L)/2\big)$$

works. Since $u - p = (q - p - L)/2 = q - v$, we have

$$u - s \geq u - p \geq 9L/2, \quad t - v \geq q - v \geq 9L/2.$$

One can verify $(*)$ by the computation

$$\min\{f(s),f(t)\} - \fint_u^v f \geq Mf(r) - \sup_{p<x<q} f(x) \geq \frac{1}{4}\operatorname{var}\mathbb{p}.$$

II. Let us assume that the peaks are contained in the interval $[x,y]$. (I.e., $x \leq p_1$ and $q_m \leq y$.) For $1 \leq i \leq m+1$, we define

$$e_i = \begin{cases} p_i, & i=1 \text{ or } Mf(p_i) \leq Mf(q_{i-1}), \\ q_{i-1}, & i=m+1 \text{ or } Mf(p_i) > Mf(q_{i-1}). \end{cases}$$

We work mainly with the modified system of peaks

$$\widetilde{\mathbb{p}}_i = \{e_i < r_i < e_{i+1}\}, \quad 1 \leq i \leq m.$$

We are going to prove that, for $1 \leq i \leq m$, there are points $s_i$ and $t_i$ such that

$$f(s_i) \geq Mf(e_{i+1}) + \frac{Mf(r_i) - Mf(e_{i+1})}{e_{i+1} - r_i} \cdot \omega(r_i), \quad x - 50L \leq s_i \leq x - 23L,$$

$$f(t_i) \geq Mf(e_i) + \frac{Mf(r_i) - Mf(e_i)}{r_i - e_i} \cdot \omega(r_i), \quad y + 23L \leq t_i \leq y + 50L.$$

Due to the symmetry, it is sufficient to find an $s_i$ only. We consider the interval $(2e_{i+1} - (r_i + \omega(r_i)), r_i + \omega(r_i))$ centered at $e_{i+1}$. We have

$$\fint_{r_i-\omega(r_i)}^{r_i+\omega(r_i)} f = Mf(r_i) \quad \text{and} \quad \fint_{2e_{i+1}-(r_i+\omega(r_i))}^{r_i+\omega(r_i)} f \leq Mf(e_{i+1}),$$

i.e.,

$$\int_{r_i-\omega(r_i)}^{r_i+\omega(r_i)} f = 2\omega(r_i)\cdot Mf(r_i), \qquad \int_{2e_{i+1}-(r_i+\omega(r_i))}^{r_i+\omega(r_i)} f \leq 2\big(r_i+\omega(r_i)-e_{i+1}\big)\cdot Mf(e_{i+1}).$$

It follows that

$$\begin{aligned} \int_{r_i-\omega(r_i)}^{2e_{i+1}-(r_i+\omega(r_i))} f &\geq 2\omega(r_i)\cdot Mf(r_i) - 2\big(r_i+\omega(r_i)-e_{i+1}\big)\cdot Mf(e_{i+1}) \\ &= 2(e_{i+1}-r_i)\cdot Mf(e_{i+1}) + 2\omega(r_i)\cdot\big(Mf(r_i) - Mf(e_{i+1})\big), \end{aligned}$$

i.e.,

$$\fint_{r_i-\omega(r_i)}^{2e_{i+1}-(r_i+\omega(r_i))} f \geq Mf(e_{i+1}) + \frac{Mf(r_i)-Mf(e_{i+1})}{e_{i+1}-r_i}\cdot\omega(r_i).$$

It is clear now that an appropriate $s_i \in (r_i - \omega(r_i), 2e_{i+1} - (r_i + \omega(r_i)))$ exists. We just realize that $x - 50L \leq r_i - \omega(r_i) \leq s_i$ and $s_i \leq 2e_{i+1} - (r_i + \omega(r_i)) \leq 2y - x - 25L = x - 23L$.

Similarly as in part I., we consider two possibilities.

(II.a) Assume that

$$\big|Mf(e_{m+1}) - Mf(e_1)\big| > \frac{1}{2}\sum_{i=1}^{m} \operatorname{var}\widetilde{\mathbb{p}}_i.$$

We may assume moreover that $Mf(e_{m+1}) > Mf(e_1)$. We observe that $f(s_m) \geq Mf(e_{m+1})$ and that

$$f(t_m) \geq Mf(e_m) + \frac{Mf(r_m)-Mf(e_m)}{r_m-e_m}\cdot(r_m-e_m) = Mf(r_m) \geq Mf(e_{m+1}).$$

We obtain

$$\min\{f(s_m), f(t_m)\} - Mf(e_1) \geq Mf(e_{m+1}) - Mf(e_1) \geq \frac{1}{2}\sum_{i=1}^{m} \operatorname{var} \widetilde{\mathbb{p}}_i \geq \frac{1}{2}\sum_{i=1}^{m} \operatorname{var} \mathbb{p}_i.$$

The required properties including $(*)$ are satisfied for

$$s = s_m, \quad (u, v) = (e_1 - L/2, e_1 + L/2), \quad t = t_m.$$

(II.b) Assume that

$$\big|Mf(e_{m+1}) - Mf(e_1)\big| \leq \frac{1}{2}\sum_{i=1}^{m} \operatorname{var} \widetilde{\mathbb{p}}_i.$$

We have

$$Mf(e_{m+1}) - Mf(e_1) = \sum_{i=1}^{m} \Big[\big(Mf(r_i) - Mf(e_i)\big) - \big(Mf(r_i) - Mf(e_{i+1})\big)\Big],$$

$$\sum_{i=1}^{m} \operatorname{var} \widetilde{\mathbb{p}}_i = \sum_{i=1}^{m} \Big[\big(Mf(r_i) - Mf(e_i)\big) + \big(Mf(r_i) - Mf(e_{i+1})\big)\Big],$$

and so the assumption can be written in the form

$$\sum_{i=1}^{m} \big(Mf(r_i) - Mf(e_i)\big) \geq \frac{1}{4}\sum_{i=1}^{m} \operatorname{var} \widetilde{\mathbb{p}}_i$$

$$\text{and} \quad \sum_{i=1}^{m} \big(Mf(r_i) - Mf(e_{i+1})\big) \geq \frac{1}{4}\sum_{i=1}^{m} \operatorname{var} \widetilde{\mathbb{p}}_i.$$

Let $j$ and $k$ be such that

$$\frac{Mf(r_j) - Mf(e_{j+1})}{e_{j+1} - r_j} = \max_{1 \leq i \leq m} \frac{Mf(r_i) - Mf(e_{i+1})}{e_{i+1} - r_i},$$

$$\frac{Mf(r_k) - Mf(e_k)}{r_k - e_k} = \max_{1 \leq i \leq m} \frac{Mf(r_i) - Mf(e_i)}{r_i - e_i}.$$

We have

$$\begin{aligned}
f(s_j) - Mf(e_{j+1}) &\geq \frac{Mf(r_j) - Mf(e_{j+1})}{e_{j+1} - r_j} \cdot \omega(r_j) \\
&\geq \frac{Mf(r_j) - Mf(e_{j+1})}{e_{j+1} - r_j} \cdot 25L \\
&\geq \frac{Mf(r_j) - Mf(e_{j+1})}{e_{j+1} - r_j} \cdot 25\sum_{i=1}^{m}(e_{i+1} - r_i) \\
&= 25\sum_{i=1}^{m} \frac{Mf(r_j) - Mf(e_{j+1})}{e_{j+1} - r_j} \cdot (e_{i+1} - r_i) \\
&\geq 25\sum_{i=1}^{m} \big(Mf(r_i) - Mf(e_{i+1})\big) \\
&\geq \frac{25}{4}\sum_{i=1}^{m} \operatorname{var} \widetilde{\mathbb{p}}_i,
\end{aligned}$$

and the same bound can be shown for $f(t_k) - Mf(e_k)$. Hence,

$$\min\{f(s_j), f(t_k)\} - Mf(e) \geq \frac{25}{4}\sum_{i=1}^{m} \operatorname{var} \widetilde{\mathbb{p}}_i \geq \frac{25}{4}\sum_{i=1}^{m} \operatorname{var} \mathbb{p}_i$$

for some $e \in \{e_{j+1}, e_k\}$. The required properties including $(*)$ are satisfied for

$$s = s_j, \quad (u,v) = (e - L/2, e + L/2), \quad t = t_k.$$

III. In the general case, the system of peaks can be divided into three subsystems

$$\mathbb{P}_1 = \{\mathbb{p}_i : p_i < x\}, \quad \mathbb{P}_2 = \{\mathbb{p}_i : x \leq p_i, q_i \leq y\}, \quad \mathbb{P}_3 = \{\mathbb{p}_i : x \leq p_i, y < q_i\}.$$

Each of these systems consists of at most one peak or of peaks contained in $[x,y]$. Thus, by parts I. and II. of the proof, if the system is non-empty, then there are appropriate numbers satisfying the improved inequality $(*)$. The numbers $s < u < v < t$ assigned to a $\mathbb{P}_k$ with

$$\sum_{\mathbb{p} \in \mathbb{P}_k} \operatorname{var} \mathbb{p} \geq \frac{1}{3}\sum_{i=1}^{m} \operatorname{var} \mathbb{p}_i$$

work. □

## 3. Basic setting for the proof

We are going to introduce the remaining notation needed for proving Theorem 1.2. Note that some notation was already introduced in Definition 2.2.

We recall that a function $f$ of bounded variation with $f \geq 0$ is fixed. We fix further a system

$$a_1 < b_1 < a_2 < b_2 < \dots < a_\sigma < b_\sigma < a_{\sigma+1}$$

such that

$$Mf(a_i) < Mf(b_i) \quad \text{and} \quad Mf(a_{i+1}) < Mf(b_i)$$

for $1 \leq i \leq \sigma$.

**Definition 3.1.**
- The system $\mathbb{P}$ consists of all peaks $\mathbb{p}_i = \{a_i < b_i < a_{i+1}\}$ where $1 \leq i \leq \sigma$,
- the system $\mathbb{E}$ consists of all essential peaks from $\mathbb{P}$,
- $L_0$ is given by $50L_0 = \max(\{\omega(b_i) : \mathbb{p}_i \in \mathbb{E}\} \cup \{0\})$,
- $L_n$ is given by $L_n = 2^{-n}L_0$ for $n \in \mathbb{N}$,
- the systems $\mathbb{E}_k^n, n \geq 0, k \in \mathbb{Z}$, are defined by

$$\mathbb{E}_k^n = \big\{\mathbb{p}_i \in \mathbb{E} : 25L_n < \omega(b_i) \leq 50L_n, kL_n \leq b_i < (k+1)L_n\big\}.$$

**Lemma 3.2.** *We have*

$$\operatorname{var}(\mathbb{P} \setminus \mathbb{E}) \leq 2 \operatorname{Var} f.$$

*Proof.* For every $\mathbb{p}_i \in \mathbb{P} \setminus \mathbb{E}$, we choose $x_i$ with $a_i < x_i < a_{i+1}$ such that

$$f(x_i) \geq Mf(b_i) - \frac{1}{4} \operatorname{var} \mathbb{p}_i.$$

We take a small enough $\varepsilon > 0$ such that the intervals $(a_i - \varepsilon, a_i + \varepsilon), 1 \leq i \leq \sigma + 1$, are pairwise disjoint and do not contain any $x_j$. For $1 \leq i \leq \sigma + 1$, we choose $y_i \in (a_i - \varepsilon, a_i + \varepsilon)$ so that

$$f(y_i) \leq Mf(a_i).$$

For $\mathbb{p}_i \in \mathbb{P} \setminus \mathbb{E}$, we have

$$\begin{aligned}|f(x_i) - f(y_i)| + |f(y_{i+1}) - f(x_i)| &\geq f(x_i) - f(y_i) + f(x_i) - f(y_{i+1}) \\ &\geq 2\Big[Mf(b_i) - \frac{1}{4}\operatorname{var}\mathbb{p}_i\Big] - Mf(a_i) - Mf(a_{i+1}) \\ &= \frac{1}{2}\operatorname{var}\mathbb{p}_i,\end{aligned}$$

and the lemma follows. □

To prove such a bound as in Lemma 3.2 for the peaks from $\mathbb{E}$ will not be so easy, and this is why we call them essential. Our first step is the application of Lemma 2.4 on every non-empty $\mathbb{E}^n_k$. It turns out that the system obtained directly from the lemma is not convenient for our purposes and an additional property is needed. In the following lemma, we show that there is a system with one of two additional properties. Unfortunately, we will be able to handle only with one property at the same time, and this will mean twice as much work for us.

**Lemma 3.3.** *Let $n \geq 0$ and $k \in \mathbb{Z}$. If $\mathbb{E}^n_k$ is non-empty (i.e., $\operatorname{var}\mathbb{E}^n_k > 0$), then at least one of the following two conditions takes place:*

(A) *There are $s < \alpha < \beta < \gamma < \delta < t$ such that*

$$(k-50)L_n \leq s, \quad t \leq (k+51)L_n,$$

$$\alpha - s \geq L_n, \quad \beta - \alpha \geq L_n, \quad \gamma - \beta = 2L_n, \quad \delta - \gamma \geq L_n, \quad t - \delta \geq L_n$$

*and*

$$\min\{f(s), f(t)\} - \max\Big\{\fint_\alpha^\beta f, \fint_\gamma^\delta f\Big\} \geq \frac{1}{24}\operatorname{var}\mathbb{E}^n_k.$$

(B) *There are $\alpha < \beta < u < v < \gamma < \delta$ such that*

$$(k-50)L_n \leq \alpha, \quad \delta \leq (k+51)L_n,$$

$$\beta - \alpha \geq L_n, \quad u - \beta \geq L_n, \quad v - u \geq L_n, \quad \gamma - v \geq L_n, \quad \delta - \gamma \geq L_n$$

*and*

$$\min\Big\{\fint_\alpha^\beta f, \fint_\gamma^\delta f\Big\} - \fint_u^v f \geq \frac{1}{24}\operatorname{var}\mathbb{E}^n_k.$$

*Proof.* Let $s < u < v < t$ be points which Lemma 2.4 gives for $\mathbb{E}^n_k$ and $x = kL_n, y = (k+1)L_n$. We define

$$\alpha = u - 3L_n, \quad \beta = u - 2L_n, \quad \gamma = v + 2L_n, \quad \delta = v + 3L_n$$

and look whether the inequality

$$\min\Big\{\fint_\alpha^\beta f, \fint_\gamma^\delta f\Big\} \geq \frac{1}{2}\Big(\min\{f(s), f(t)\} + \fint_u^v f\Big)$$

holds. If it holds, then (B) is satisfied. If it does not hold, then

$$\fint_I f \leq \frac{1}{2}\Big(\min\{f(s), f(t)\} + \fint_u^v f\Big)$$

where $I$ is one of the intervals $(\alpha, \beta), (\gamma, \delta)$. This inequality is fulfilled also for $I = (u, v)$. Hence, (A) is satisfied for one of the choices

$$\alpha' = \alpha, \beta' = \beta, \gamma' = u, \delta' = v,$$
$$\alpha' = u, \beta' = v, \gamma' = \gamma, \delta' = \delta.$$

□

**Definition 3.4.** We define

$$\mathcal{A} = \big\{(n,k) : \mathbb{E}^n_k \text{ is non-empty and (A) from Lemma 3.3 is satisfied for } (n,k)\big\},$$
$$\mathcal{A}^n_K = \big\{k \in \mathbb{Z} : k = K \bmod 200, (n,k) \in \mathcal{A}\big\}, \quad n \geq 0,\ 0 \leq K \leq 199,$$
$$\mathcal{B} = \big\{(n,k) : \mathbb{E}^n_k \text{ is non-empty and (B) from Lemma 3.3 is satisfied for } (n,k)\big\},$$
$$\mathcal{B}^n_K = \big\{k \in \mathbb{Z} : k = K \bmod 200, (n,k) \in \mathcal{B}\big\}, \quad n \geq 0,\ 0 \leq K \leq 199.$$

## 4. Dealing with group $\mathcal{A}$

**Proposition 4.1.** *Let $0 \leq N \leq 9$ and $0 \leq K \leq 199$. Let $\eta \in \mathbb{N} \cup \{0\}$ and let $n = 10\eta + N$. Then there is a system*

$$x_1 < u_1 < v_1 < x_2 < u_2 < v_2 < \cdots < x_m < u_m < v_m < x_{m+1}$$

*such that*

$$u_1 - x_1 \geq L_n, \quad v_1 - u_1 \geq L_n, \quad x_2 - v_1 \geq L_n, \quad \ldots$$

*and*

$$\sum_{i=1}^{m} \Big[f(x_i) + f(x_{i+1}) - 2\fint_{u_i}^{v_i} f\Big] \geq \frac{1}{60} \sum \Big\{ \operatorname{var} \mathbb{E}^o_k : o = N \bmod 10, o \leq n, k \in \mathcal{A}^o_K \Big\}.$$

To prove the proposition, we provide a method how to construct such a system for $\eta$ when a system for $\eta - 1$ is already constructed. We suppose that there is a system

$$X_1 < U_1 < V_1 < X_2 < U_2 < V_2 < \cdots < X_M < U_M < V_M < X_{M+1}$$

such that

$$U_1 - X_1 \geq 1024 L_n, \quad V_1 - U_1 \geq 1024 L_n, \quad X_2 - V_1 \geq 1024 L_n, \quad \ldots$$

and

$$\sum_{I=1}^{M} \Big[f(X_I) + f(X_{I+1}) - 2\fint_{U_I}^{V_I} f\Big]$$
$$\geq \frac{1}{60} \sum \Big\{ \operatorname{var} \mathbb{E}^o_k : o = N \bmod 10, o \leq n - 10, k \in \mathcal{A}^o_K \Big\}$$

(for $\eta - 1 = -1$, we consider $M = 0$ and $X_1 =$ anything). We want to construct a system

$$x_1 < u_1 < v_1 < x_2 < u_2 < v_2 < \cdots < x_m < u_m < v_m < x_{m+1}$$

such that

$$u_1 - x_1 \geq L_n, \quad v_1 - u_1 \geq L_n, \quad x_2 - v_1 \geq L_n, \quad \ldots$$

and

$$\sum_{i=1}^{m} \Big[f(x_i) + f(x_{i+1}) - 2\fint_{u_i}^{v_i} f\Big] \geq \sum_{I=1}^{M} \Big[f(X_I) + f(X_{I+1}) - 2\fint_{U_I}^{V_I} f\Big] + \frac{1}{60} \sum_{k \in \mathcal{A}^n_K} \operatorname{var} \mathbb{E}^n_k.$$

For every $k \in \mathcal{A}^n_K$, let us consider such a system as in (A) from Lemma 3.3. If we put $s_k = s, t_k = t$ and choose a $(\alpha_k, \beta_k) \in \{(\alpha, \beta), (\gamma, \delta)\}$, we obtain a system $s_k < \alpha_k < \beta_k < t_k$ such that

$$(k - 50) L_n \leq s_k, \quad t_k \leq (k + 51) L_n,$$
$$\alpha_k - s_k \geq L_n, \quad \beta_k - \alpha_k \geq L_n, \quad t_k - \beta_k \geq L_n$$

and

$$\min\{f(s_k), f(t_k)\} - \fint_{\alpha_k}^{\beta_k} f \geq \frac{1}{24} \operatorname{var} \mathbb{E}_k^n.$$

We require from the choice of $(\alpha_k, \beta_k) \in \{(\alpha, \beta), (\gamma, \delta)\}$ that

$$\operatorname{dist}(X_I, (\alpha_k, \beta_k)) \geq L_n, \quad 1 \leq I \leq M+1.$$

For an interval $(c, d)$ and a $k \in \mathbb{Z}$, we will denote

$$(c, d) \perp k \quad \Leftrightarrow \quad \operatorname{dist}\big((c, d), ((k-50)L_n, (k+51)L_n)\big) \geq L_n.$$

**Lemma 4.2.** *Let $(U, V)$ be an interval of length greater than $210L_n$. Then there are a subinterval $(U', V')$ and a $k$ with $k = K \bmod 200$ such that*

- $\fint_{U'}^{V'} f \leq \fint_U^V f$,
- $V' - U' \geq 5L_n$,
- $U' = (k-100)L_n$ *or* $V' = (k+100)L_n$,
- $(k-105)L_n \leq U'$ *and* $V' \leq (k+105)L_n$,
- $(U', V') \perp l$ *for every* $l \neq k$ *with* $l = K \bmod 200$.

*Moreover, we can wish that $\fint_{U'}^{V'} f \geq \fint_U^V f$ instead of the first property.*

*Proof.* Let $g$ and $h$ be the uniquely determined integers with $g = h = K \bmod 200$ such that

$$(g-105)L_n \leq U < (g+95)L_n \quad \text{and} \quad (h-95)L_n < V \leq (h+105)L_n.$$

We have $g < h$ due to the assumption $V - U > 210L_n$. The system

$$U < (g+100)L_n < (g+300)L_n < \cdots < (h-100)L_n < V$$

is a partition of $(U, V)$ into intervals of length greater than $5L_n$. We choose a part the average value of $f$ over which is less or equal to the average value of $f$ over $(U, V)$. (Respectively, greater or equal to the average value of $f$ over $(U, V)$ if we want to prove the moreover statement.) Such a subinterval $(U', V')$ and the appropriate $k$ with $g \leq k \leq h$ and $k = K \bmod 200$ have the required properties. $\square$

**Claim 4.3.** *Let $(U, V)$ be an interval of length greater than $210L_n$. Then at least one of the following conditions is fulfilled:*

(i) *There is an interval $(c, d) \subset (U, V)$ with $d - c \geq L_n$ such that $(c, d) \perp l$ for every $l \in \mathcal{A}_K^n$ and*

$$\fint_c^d f \geq \fint_U^V f.$$

(ii) *There are an interval $(c, d) \subset (U, V)$ with $d - c \geq L_n$ and a $k \in \mathcal{A}_K^n$ such that $(c, d) \perp l$ for every $l \in \mathcal{A}_K^n \setminus \{k\}$ and*

$$\fint_c^d f \geq \fint_U^V f + \frac{1}{120} \operatorname{var} \mathbb{E}_k^n.$$

(iii) *There are a system*

$$c < d < y < c' < d'$$

*with $(c, d') \subset (U - 1023L_n, V + 1023L_n)$ and*

$$d - c \geq L_n, \quad y - d \geq L_n, \quad c' - y \geq L_n, \quad d' - c' \geq L_n$$

*and a* $k \in \mathcal{A}_K^n$ *such that* $(c,d') \perp l$ *for every* $l \in \mathcal{A}_K^n \setminus \{k\}$ *and*

$$f(y) - \fint_c^d f - \fint_{c'}^{d'} f \geq -\fint_U^V f + \frac{1}{120} \operatorname{var} \mathbb{E}_k^n.$$

*Proof.* Let $(U',V')$ and $k$ be as in Lemma 4.2. If $k \notin \mathcal{A}_K^n$, then (i) is fulfilled for $(c,d) = (U',V')$. So, let us assume that $k \in \mathcal{A}_K^n$ (and thus that we have $s_k < \alpha_k < \beta_k < t_k$ for this $k$).

Let us assume moreover that $U' = (k-100)L_n$ (the procedure is similar when $V' = (k+100)L_n$, see below). We put

$$W = U' + \frac{1}{5}(V' - U').$$

We have $W = \frac{4}{5}U' + \frac{1}{5}V' \leq \frac{4}{5}(k-100)L_n + \frac{1}{5}(k+105)L_n = (k-59)L_n \leq s_k - 9L_n$ and $\beta_k \leq t_k \leq (k+51)L_n = U' + 151L_n$. In particular,

$$s_k - W \geq L_n \quad \text{and} \quad \beta_k \leq V' + 1023L_n.$$

Further, we have

$$\fint_{U'}^{W} f \leq \fint_{U'}^{V'} f + \frac{4}{5} \cdot \frac{1}{24} \operatorname{var} \mathbb{E}_k^n \quad \text{or} \quad \fint_{W}^{V'} f \leq \fint_{U'}^{V'} f - \frac{1}{5} \cdot \frac{1}{24} \operatorname{var} \mathbb{E}_k^n.$$

If the second inequality takes place, then (ii) is fulfilled for $(c,d) = (W,V')$. If the first inequality takes place, then (iii) is fulfilled for

$$(c,d) = (U',W), \quad y = s_k, \quad (c',d') = (\alpha_k, \beta_k).$$

So, the claim is proven under the assumption $U' = (k-100)L_n$. The proof under the assumption $V' = (k+100)L_n$ can be done in a similar way. If we denote

$$W' = V' - \frac{1}{5}(V' - U'),$$

then one can show that (ii) is fulfilled for $(c,d) = (U',W')$ or (iii) is fulfilled for

$$(c,d) = (\alpha_k, \beta_k), \quad y = t_k, \quad (c',d') = (W',V').$$

□

**Claim 4.4.** *There is a subset* $\mathcal{S} \subset \mathcal{A}_K^n$ *for which there exists a system*

$$y_1 < c_1 < d_1 < y_2 < c_2 < d_2 < \cdots < y_j < c_j < d_j < y_{j+1}$$

*such that*

$$c_1 - y_1 \geq L_n, \quad d_1 - c_1 \geq L_n, \quad y_2 - d_1 \geq L_n, \quad \ldots,$$

$$\begin{aligned} l \in \mathcal{A}_K^n \setminus \mathcal{S} \quad &\Rightarrow \quad (c_i, d_i) \perp l, \ 1 \leq i \leq j, \\ l \in \mathcal{A}_K^n \setminus \mathcal{S} \quad &\Rightarrow \quad \operatorname{dist}(y_i, (\alpha_l, \beta_l)) \geq L_n, \ 1 \leq i \leq j+1, \end{aligned}$$

*and*

$$\sum_{i=1}^{j} \Big[ f(y_i) + f(y_{i+1}) - 2\fint_{c_i}^{d_i} f \Big] \geq \sum_{I=1}^{M} \Big[ f(X_I) + f(X_{I+1}) - 2\fint_{U_I}^{V_I} f \Big] + \frac{1}{60} \sum_{k \in \mathcal{S}} \operatorname{var} \mathbb{E}_k^n.$$

*Proof.* We apply Claim 4.3 on the intervals $(U_I, V_I), 1 \leq I \leq M$. We write the inequalities from Claim 4.3 in a form more familiar for our purposes:

$$\text{(i)} \quad f(X_I) + f(X_{I+1}) - 2\fint_c^d f \geq f(X_I) + f(X_{I+1}) - 2\fint_{U_I}^{V_I} f,$$

$$\text{(ii)} \quad f(X_I) + f(X_{I+1}) - 2\fint_c^d f \geq f(X_I) + f(X_{I+1}) - 2\fint_{U_I}^{V_I} f + \frac{1}{60} \operatorname{var} \mathbb{E}_k^n,$$

$$\text{(iii)} \quad \Big[f(X_I) + f(y) - 2\fint_c^d f\Big] + \Big[f(y) + f(X_{I+1}) - 2\fint_{c'}^{d'} f\Big]$$
$$\geq f(X_I) + f(X_{I+1}) - 2\fint_{U_I}^{V_I} f + \frac{1}{60} \operatorname{var} \mathbb{E}_k^n.$$

We define $\mathcal{S}$ as the set of those $k$'s which appeared in (ii) or (iii) for some $I$. One can construct the desired system by inserting the systems which we obtained from Claim 4.3 between $X_I$'s. □

To finish the proof of Proposition 4.1, it remains to show that, if a proper subset $\mathcal{S} \subset \mathcal{A}_K^n$ has such a system as in Claim 4.4, then $\mathcal{S} \cup \{k\}$ where $k \in \mathcal{A}_K^n \setminus \mathcal{S}$ has also such a system.

So, let $\mathcal{S}$ and

$$y_1 < c_1 < d_1 < y_2 < c_2 < d_2 < \cdots < y_j < c_j < d_j < y_{j+1}$$

be as in Claim 4.4 and let $k \in \mathcal{A}_K^n \setminus \mathcal{S}$. Let $\iota$ be the index such that $y_\iota$ belongs to the connected component of $\mathbb{R} \setminus \bigcup_{i=1}^j [c_i, d_i]$ which covers $((k-50)L_n, (k+51)L_n)$. We intend to obtain the desired system for $\mathcal{S} \cup \{k\}$ by replacing $y_\iota$ with

$$y < \alpha_k < \beta_k < y'$$

where

$$y = \begin{cases} y_\iota, & y_\iota \leq \alpha_k - L_n \text{ and } f(y_\iota) \geq f(s_k), \\ s_k, & \text{otherwise,} \end{cases}$$

$$y' = \begin{cases} y_\iota, & y_\iota \geq \beta_k + L_n \text{ and } f(y_\iota) \geq f(t_k), \\ t_k, & \text{otherwise.} \end{cases}$$

For every $l \neq k$ with $l = K \bmod 200$, we have

$$\operatorname{dist}\big(((k-50)L_n, (k+51)L_n), ((l-50)L_n, (l+51)L_n)\big) \geq 99L_n \geq L_n,$$

and thus

$$l \in \mathcal{A}_K^n \setminus (\mathcal{S} \cup \{k\}) \quad \Rightarrow \quad (\alpha_k, \beta_k) \perp l,$$
$$l \in \mathcal{A}_K^n \setminus (\mathcal{S} \cup \{k\}) \quad \Rightarrow \quad \operatorname{dist}(y, (\alpha_l, \beta_l)) \geq L_n \text{ and } \operatorname{dist}(y', (\alpha_l, \beta_l)) \geq L_n.$$

Let us prove the inequality for the modified system. We note that, if $j \geq 1$, then the left side of the inequality for the original system can be written in the form

$$f(y_1) - 2\fint_{c_1}^{d_1} f + 2f(y_2) - 2\fint_{c_2}^{d_2} f + \cdots + 2f(y_j) - 2\fint_{c_j}^{d_j} f + f(y_{j+1}).$$

We need to show that the modification of the system increased this quantity at least by $\frac{1}{60}\operatorname{var}\mathbb{E}_k^n$. What we need to show is

$$\text{when } 1<\iota<j+1: \qquad 2f(y)-2\fint_{\alpha_k}^{\beta_k} f+2f(y')\ge 2f(y_\iota)+\frac{1}{60}\operatorname{var}\mathbb{E}_k^n,$$

$$\text{when } 1=\iota<j+1: \qquad f(y)-2\fint_{\alpha_k}^{\beta_k} f+2f(y')\ge f(y_\iota)+\frac{1}{60}\operatorname{var}\mathbb{E}_k^n,$$

$$\text{when } 1<\iota=j+1: \qquad 2f(y)-2\fint_{\alpha_k}^{\beta_k} f+f(y')\ge f(y_\iota)+\frac{1}{60}\operatorname{var}\mathbb{E}_k^n,$$

$$\text{when } 1=\iota=j+1: \qquad f(y)-2\fint_{\alpha_k}^{\beta_k} f+f(y')\ge \frac{1}{60}\operatorname{var}\mathbb{E}_k^n.$$

These inequalities, even with $\frac{1}{12}$ instead of $\frac{1}{60}$, follow from

$$f(y)-\fint_{\alpha_k}^{\beta_k} f\ge\frac{1}{24}\operatorname{var}\mathbb{E}_k^n,\quad f(y')-\fint_{\alpha_k}^{\beta_k} f\ge\frac{1}{24}\operatorname{var}\mathbb{E}_k^n,$$
$$f(y)\ge f(y_\iota)\quad\text{or}\quad f(y')\ge f(y_\iota)$$

($f(y)\ge f(y_\iota)$ is implied by $y_\iota\le\alpha_k-L_n$ and $f(y')\ge f(y_\iota)$ is implied by $y_\iota\ge\beta_k+L_n$).

The proof of Proposition 4.1 is completed.

**Corollary 4.5.** *For $0\le N\le 9$ and $0\le K\le 199$, we have*

$$\sum\Big\{\operatorname{var}\mathbb{E}_k^n : n=N \bmod 10, k\in\mathcal{A}_K^n\Big\}\le 60\operatorname{Var} f.$$

*Proof.* Let $\eta$ be large enough such that

$$\mathcal{A}_K^o\ne\varnothing\quad\Rightarrow\quad o\le n$$

where $n=10\eta+N$. Let

$$x_1<u_1<v_1<x_2<u_2<v_2<\cdots<x_m<u_m<v_m<x_{m+1}$$

be the system which Proposition 4.1 gives for $N,K$ and $\eta$. For $1\le i\le m$, let $w_i\in(u_i,v_i)$ be chosen so that

$$f(w_i)\le\fint_{u_i}^{v_i} f.$$

We compute

$$\begin{aligned}
\operatorname{Var} f &\ge \sum_{i=1}^m\Big[|f(w_i)-f(x_i)|+|f(x_{i+1})-f(w_i)|\Big]\\
&\ge \sum_{i=1}^m\Big[f(x_i)-f(w_i)+f(x_{i+1})-f(w_i)\Big]\\
&\ge \sum_{i=1}^m\Big[f(x_i)+f(x_{i+1})-2\fint_{u_i}^{v_i} f\Big]\\
&\ge \frac{1}{60}\sum\Big\{\operatorname{var}\mathbb{E}_k^o : o=N \bmod 10, o\le n, k\in\mathcal{A}_K^o\Big\}\\
&= \frac{1}{60}\sum\Big\{\operatorname{var}\mathbb{E}_k^o : o=N \bmod 10, k\in\mathcal{A}_K^o\Big\}.
\end{aligned}$$

□

## 5. Dealing with group $\mathcal{B}$

**Proposition 5.1.** *Let $0 \leq N \leq 9$ and $0 \leq K \leq 199$. Let $\eta \in \mathbb{N} \cup \{0\}$ and let $n = 10\eta + N$. Then there is a system*

$$\varphi_1 < \psi_1 < s_1 < t_1 < \varphi_2 < \psi_2 < s_2 < t_2 < \cdots < s_m < t_m < \varphi_{m+1} < \psi_{m+1}$$

*such that*

$$\psi_1 - \varphi_1 \geq L_n, \quad s_1 - \psi_1 \geq L_n, \quad t_1 - s_1 \geq L_n, \quad \varphi_2 - t_1 \geq L_n, \quad \ldots$$

*and*

$$\sum_{i=1}^{m} \Big[ \fint_{\varphi_i}^{\psi_i} f + \fint_{\varphi_{i+1}}^{\psi_{i+1}} f - 2\fint_{s_i}^{t_i} f \Big] \geq \frac{1}{60} \sum \Big\{ \operatorname{var} \mathbb{E}_k^o : o = N \bmod 10, o \leq n, k \in \mathcal{B}_K^o \Big\}.$$

To prove the proposition, we provide a method how to construct such a system for $\eta$ when a system for $\eta - 1$ is already constructed. We suppose that there is a system

$$\Phi_1 < \Psi_1 < S_1 < T_1 < \Phi_2 < \Psi_2 < S_2 < T_2 < \cdots < S_M < T_M < \Phi_{M+1} < \Psi_{M+1}$$

such that

$$\Psi_1 - \Phi_1 \geq 1024L_n, \quad S_1 - \Psi_1 \geq 1024L_n, \quad T_1 - S_1 \geq 1024L_n, \quad \Phi_2 - T_1 \geq 1024L_n, \quad \ldots$$

and

$$\sum_{I=1}^{M} \Big[ \fint_{\Phi_I}^{\Psi_I} f + \fint_{\Phi_{I+1}}^{\Psi_{I+1}} f - 2\fint_{S_I}^{T_I} f \Big] \geq \frac{1}{60} \sum \Big\{ \operatorname{var} \mathbb{E}_k^o : o = N \bmod 10, o \leq n - 10, k \in \mathcal{B}_K^o \Big\}$$

(for $\eta - 1 = -1$, we consider $M = 0, \Phi_1 =$ anything and $\Psi_1 = \Phi_1 + 1024L_n$). We want to construct a system

$$\varphi_1 < \psi_1 < s_1 < t_1 < \varphi_2 < \psi_2 < s_2 < t_2 < \cdots < s_m < t_m < \varphi_{m+1} < \psi_{m+1}$$

such that

$$\psi_1 - \varphi_1 \geq L_n, \quad s_1 - \psi_1 \geq L_n, \quad t_1 - s_1 \geq L_n, \quad \varphi_2 - t_1 \geq L_n, \quad \ldots$$

and

$$\sum_{i=1}^{m} \Big[ \fint_{\varphi_i}^{\psi_i} f + \fint_{\varphi_{i+1}}^{\psi_{i+1}} f - 2\fint_{s_i}^{t_i} f \Big] \geq \sum_{I=1}^{M} \Big[ \fint_{\Phi_I}^{\Psi_I} f + \fint_{\Phi_{I+1}}^{\Psi_{I+1}} f - 2\fint_{S_I}^{T_I} f \Big] + \frac{1}{60} \sum_{k \in \mathcal{B}_K^n} \operatorname{var} \mathbb{E}_k^n.$$

For every $k \in \mathcal{B}_K^n$, let us consider such a system as in (B) from Lemma 3.3. We obtain a system $\alpha_k < \beta_k < u_k < v_k < \gamma_k < \delta_k$ such that

$$(k-50)L_n \leq \alpha_k, \quad \delta_k \leq (k+51)L_n,$$

$$\beta_k - \alpha_k \geq L_n, \quad u_k - \beta_k \geq L_n, \quad v_k - u_k \geq L_n, \quad \gamma_k - v_k \geq L_n, \quad \delta_k - \gamma_k \geq L_n$$

and

$$\min \Big\{ \fint_{\alpha_k}^{\beta_k} f, \fint_{\gamma_k}^{\delta_k} f \Big\} - \fint_{u_k}^{v_k} f \geq \frac{1}{24} \operatorname{var} \mathbb{E}_k^n.$$

Again, for an interval $(c, d)$ and a $k \in \mathbb{Z}$, we denote

$$(c, d) \perp k \quad \Leftrightarrow \quad \operatorname{dist}\big((c, d), ((k-50)L_n, (k+51)L_n)\big) \geq L_n.$$

**Claim 5.2.** *Let $(S,T)$ be an interval of length greater than $210L_n$. Then at least one of the following conditions is fulfilled:*

(i) *There is an interval $(c,d) \subset (S,T)$ with $d-c \geq L_n$ such that $(c,d) \perp l$ for every $l \in \mathcal{B}_K^n$ and*

$$-\!\!\!\!\!\!\int_c^d f \geq -\!\!\!\!\!\!\int_S^T f.$$

(ii) *There are an interval $(c,d) \subset (S,T)$ with $d-c \geq L_n$ and a $k \in \mathcal{B}_K^n$ such that $(c,d) \perp l$ for every $l \in \mathcal{B}_K^n \setminus \{k\}$ and*

$$-\!\!\!\!\!\!\int_c^d f \geq -\!\!\!\!\!\!\int_S^T f + \frac{1}{120} \operatorname{var} \mathbb{E}_k^n.$$

(iii) *There are a system*

$$c < d < \mu < \nu < c' < d'$$

*with $(c,d') \subset (S-500L_n, T+500L_n)$ and*

$$d-c \geq L_n, \quad \mu - d \geq L_n, \quad \nu - \mu \geq L_n, \quad c' - \nu \geq L_n, \quad d' - c' \geq L_n$$

*and a $k \in \mathcal{B}_K^n$ such that $(c,d') \perp l$ for every $l \in \mathcal{B}_K^n \setminus \{k\}$ and*

$$-\!\!\!\!\!\!\int_\mu^\nu f - -\!\!\!\!\!\!\int_c^d f - -\!\!\!\!\!\!\int_{c'}^{d'} f \geq -\!\!\!\!\!\!\int_S^T f + \frac{1}{120} \operatorname{var} \mathbb{E}_k^n.$$

*Proof.* This can be proven in the same way as Claim 4.3. □

The main difference between proofs of Propositions 4.1 and 5.1 is that we need one more analogy of Claim 4.3 because there are intervals $(\Phi_I, \Psi_I)$ instead of points $X_I$. Even, two versions of this analogy are provided. Both versions are written at once in the manner that the inequalities belonging to the second version are written in square brackets (this concerns also the proof of the claim).

**Claim 5.3.** *Let $(\Phi, \Psi)$ be an interval of length greater than $210L_n$. Then at least one of the following conditions is fulfilled:*

(i*) *There is an interval $(\mu,\nu) \subset (\Phi,\Psi)$ with $\nu - \mu \geq L_n$ such that $(\mu,\nu) \perp l$ for every $l \in \mathcal{B}_K^n$ and*

$$-\!\!\!\!\!\!\int_\mu^\nu f \geq -\!\!\!\!\!\!\int_\Phi^\Psi f.$$

(ii*) *There are an interval $(\mu,\nu) \subset (\Phi,\Psi)$ with $\nu - \mu \geq L_n$ and a $k \in \mathcal{B}_K^n$ such that $(\mu,\nu) \perp l$ for every $l \in \mathcal{B}_K^n \setminus \{k\}$ and*

$$-\!\!\!\!\!\!\int_\mu^\nu f \geq -\!\!\!\!\!\!\int_\Phi^\Psi f + \frac{1}{120} \operatorname{var} \mathbb{E}_k^n \quad \left[ \textit{resp.}\ -\!\!\!\!\!\!\int_\mu^\nu f \geq -\!\!\!\!\!\!\int_\Phi^\Psi f + \frac{1}{60} \operatorname{var} \mathbb{E}_k^n \right].$$

(iii*) *There are a system*

$$\mu < \nu < c < d < \mu' < \nu'$$

*with $(\mu,\nu') \subset (\Phi - 500L_n, \Psi + 500L_n)$ and*

$$\nu - \mu \geq L_n, \quad c - \nu \geq L_n, \quad d - c \geq L_n, \quad \mu' - d \geq L_n, \quad \nu' - \mu' \geq L_n$$

*and a $k \in \mathcal{B}_K^n$ such that $(\mu,\nu') \perp l$ for every $l \in \mathcal{B}_K^n \setminus \{k\}$ and*

$$-\!\!\!\!\!\!\int_\mu^\nu f - -\!\!\!\!\!\!\int_c^d f + -\!\!\!\!\!\!\int_{\mu'}^{\nu'} f \geq -\!\!\!\!\!\!\int_\Phi^\Psi f + \frac{1}{120} \operatorname{var} \mathbb{E}_k^n$$

$$\Big[\, \textit{resp.}\ \fint_\mu^\nu f - 2\fint_c^d f + 2\fint_{\mu'}^{\nu'} f \ge \fint_\Phi^\Psi f + \frac{1}{60}\operatorname{var}\mathbb{E}_k^n$$
$$\textit{and}\ 2\fint_\mu^\nu f - 2\fint_c^d f + \fint_{\mu'}^{\nu'} f \ge \fint_\Phi^\Psi f + \frac{1}{60}\operatorname{var}\mathbb{E}_k^n \,\Big].$$

*Proof.* By Lemma 4.2, there are a subinterval $(\Phi', \Psi')$ and a $k$ with $k = K \bmod 200$ such that

- $\fint_{\Phi'}^{\Psi'} f \ge \fint_\Phi^\Psi f$,
- $\Psi' - \Phi' \ge 5L_n$,
- $\Phi' = (k-100)L_n$ or $\Psi' = (k+100)L_n$,
- $(k-105)L_n \le \Phi'$ and $\Psi' \le (k+105)L_n$,
- $(\Phi', \Psi') \perp l$ for every $l \ne k$ with $l = K \bmod 200$.

If $k \notin \mathcal{B}_K^n$, then (i*) is fulfilled for $(\mu, \nu) = (\Phi', \Psi')$. So, let us assume that $k \in \mathcal{B}_K^n$ (and thus that we have $\alpha_k < \beta_k < u_k < v_k < \gamma_k < \delta_k$ for this $k$).

We provide the proof under the assumption $\Phi' = (k-100)L_n$ only (the procedure is similar when $\Psi' = (k+100)L_n$). We put

$$\Theta = \Phi' + \frac{1}{5}(\Psi' - \Phi').$$

We have $\Theta = \frac{4}{5}\Phi' + \frac{1}{5}\Psi' \le \frac{4}{5}(k-100)L_n + \frac{1}{5}(k+105)L_n = (k-59)L_n \le \alpha_k - 9L_n \le u_k - 9L_n$ and $\delta_k \le (k+51)L_n = \Phi' + 151L_n$. In particular,

$$u_k - \Theta \ge L_n \quad \text{and} \quad \delta_k \le \Psi' + 500L_n.$$

Further, we have

$$\fint_{\Phi'}^{\Theta} f \ge \fint_{\Phi'}^{\Psi'} f - \frac{4}{5}\cdot\frac{1}{24}\operatorname{var}\mathbb{E}_k^n \quad \text{or} \quad \fint_{\Theta}^{\Psi'} f \ge \fint_{\Phi'}^{\Psi'} f + \frac{1}{5}\cdot\frac{1}{24}\operatorname{var}\mathbb{E}_k^n$$

$$\Big[\,\text{resp.}\ \fint_{\Phi'}^{\Theta} f \ge \fint_{\Phi'}^{\Psi'} f - \frac{8}{5}\cdot\frac{1}{24}\operatorname{var}\mathbb{E}_k^n \quad \text{or} \quad \fint_{\Theta}^{\Psi'} f \ge \fint_{\Phi'}^{\Psi'} f + \frac{2}{5}\cdot\frac{1}{24}\operatorname{var}\mathbb{E}_k^n \,\Big].$$

If the second inequality takes place, then (ii*) is fulfilled for $(\mu, \nu) = (\Theta, \Psi')$. If the first inequality takes place, then (iii*) is fulfilled for

$$(\mu,\nu) = \begin{cases} (\Phi', \Theta), & \fint_{\Phi'}^{\Theta} f \ge \fint_{\alpha_k}^{\beta_k} f, \\ (\alpha_k, \beta_k), & \fint_{\Phi'}^{\Theta} f < \fint_{\alpha_k}^{\beta_k} f, \end{cases} \qquad (c,d) = (u_k, v_k), \quad (\mu', \nu') = (\gamma_k, \delta_k).$$

The inequalities in (iii*) follow from

$$\fint_\mu^\nu f \ge \fint_{\Phi'}^{\Theta} f \ge \fint_\Phi^\Psi f - \frac{4}{5}\cdot\frac{1}{24}\operatorname{var}\mathbb{E}_k^n \quad \Big[\,\text{resp.}\ \cdots - \frac{8}{5}\cdot\frac{1}{24}\operatorname{var}\mathbb{E}_k^n\,\Big],$$

$$\fint_\mu^\nu f - \fint_c^d f \ge \frac{1}{24}\operatorname{var}\mathbb{E}_k^n, \quad \fint_{\mu'}^{\nu'} f - \fint_c^d f \ge \frac{1}{24}\operatorname{var}\mathbb{E}_k^n.$$

□

**Claim 5.4.** *There is a subset $\mathcal{T} \subset \mathcal{B}_K^n$ for which there exists a system*

$$\mu_1 < \nu_1 < c_1 < d_1 < \mu_2 < \nu_2 < c_2 < d_2 < \cdots < c_j < d_j < \mu_{j+1} < \nu_{j+1}$$

*such that*

$$\nu_1 - \mu_1 \ge L_n, \quad c_1 - \nu_1 \ge L_n, \quad d_1 - c_1 \ge L_n, \quad \mu_2 - d_1 \ge L_n, \quad \ldots,$$

$$
\begin{aligned}
l \in \mathcal{B}_K^n \setminus \mathcal{T} \quad &\Rightarrow \quad (c_i, d_i) \perp l, \ 1 \le i \le j, \\
l \in \mathcal{B}_K^n \setminus \mathcal{T} \quad &\Rightarrow \quad (\mu_i, \nu_i) \perp l, \ 1 \le i \le j+1,
\end{aligned}
$$

*and*

$$
\sum_{i=1}^{j} \Big[ \fint_{\mu_i}^{\nu_i} f + \fint_{\mu_{i+1}}^{\nu_{i+1}} f - 2\fint_{c_i}^{d_i} f \Big] \ge \sum_{I=1}^{M} \Big[ \fint_{\Phi_I}^{\Psi_I} f + \fint_{\Phi_{I+1}}^{\Psi_{I+1}} f - 2\fint_{S_I}^{T_I} f \Big] + \frac{1}{60} \sum_{k \in \mathcal{T}} \operatorname{var} \mathbb{E}_k^n.
$$

We note that, if $j \ge 1$ and $M \ge 1$, then the inequality can be written in the form

$$
\begin{aligned}
&\fint_{\mu_1}^{\nu_1} f - 2\fint_{c_1}^{d_1} f + 2\fint_{\mu_2}^{\nu_2} f - 2\fint_{c_2}^{d_2} f + \cdots + 2\fint_{\mu_j}^{\nu_j} f - 2\fint_{c_j}^{d_j} f + \fint_{\mu_{j+1}}^{\nu_{j+1}} f \\
&\ge \fint_{\Phi_1}^{\Psi_1} f - 2\fint_{S_1}^{T_1} f + 2\fint_{\Phi_2}^{\Psi_2} f - \cdots - 2\fint_{S_M}^{T_M} f + \fint_{\Phi_{M+1}}^{\Psi_{M+1}} f + \frac{1}{60} \sum_{k \in \mathcal{T}} \operatorname{var} \mathbb{E}_k^n.
\end{aligned}
$$

*Proof.* If $M = 0$, then we can put $\mathcal{T} = \varnothing, j = 0$ and find a suitable interval $(\mu_1, \nu_1)$ of length $L_n$. So, let us assume that $M \ge 1$.

We apply Claim 5.2 on the intervals $(S_I, T_I), 1 \le I \le M$, and Claim 5.3 on the intervals $(\Phi_I, \Psi_I), 1 \le I \le M+1$, (the first version for $1 < I < M+1$, the second version for $I = 1, I = M+1$). We write the inequalities from Claim 5.2 in a form more familiar for our purposes:

$$
\begin{aligned}
&\text{(i)} \quad -2\fint_c^d f \ge -2\fint_{S_I}^{T_I} f, \\
&\text{(ii)} \quad -2\fint_c^d f \ge -2\fint_{S_I}^{T_I} f + \frac{1}{60} \operatorname{var} \mathbb{E}_k^n, \\
&\text{(iii)} \quad -2\fint_c^d f + 2\fint_\mu^\nu f - 2\fint_{c'}^{d'} f \ge -2\fint_{S_I}^{T_I} f + \frac{1}{60} \operatorname{var} \mathbb{E}_k^n.
\end{aligned}
$$

Concerning the inequalities from Claim 5.3, we moreover specify which inequality will be applied for $I$:

$$
\begin{aligned}
&\text{(i*)} \quad 1 < I < M+1: && 2\fint_\mu^\nu f \ge 2\fint_{\Phi_I}^{\Psi_I} f, \\
&\quad I = 1 \text{ or } I = M+1: && \fint_\mu^\nu f \ge \fint_{\Phi_I}^{\Psi_I} f, \\
&\text{(ii*)} \quad 1 < I < M+1: && 2\fint_\mu^\nu f \ge 2\fint_{\Phi_I}^{\Psi_I} f + \frac{1}{60} \operatorname{var} \mathbb{E}_k^n, \\
&\quad I = 1 \text{ or } I = M+1: && \fint_\mu^\nu f \ge \fint_{\Phi_I}^{\Psi_I} f + \frac{1}{60} \operatorname{var} \mathbb{E}_k^n, \\
&\text{(iii*)} \quad 1 < I < M+1: && 2\fint_\mu^\nu f - 2\fint_c^d f + 2\fint_{\mu'}^{\nu'} f \ge 2\fint_{\Phi_I}^{\Psi_I} f + \frac{1}{60} \operatorname{var} \mathbb{E}_k^n, \\
&\quad I = 1: && \fint_\mu^\nu f - 2\fint_c^d f + 2\fint_{\mu'}^{\nu'} f \ge \fint_{\Phi_I}^{\Psi_I} f + \frac{1}{60} \operatorname{var} \mathbb{E}_k^n, \\
&\quad I = M+1: && 2\fint_\mu^\nu f - 2\fint_c^d f + \fint_{\mu'}^{\nu'} f \ge \fint_{\Phi_I}^{\Psi_I} f + \frac{1}{60} \operatorname{var} \mathbb{E}_k^n.
\end{aligned}
$$

We define $\mathcal{T}$ as the set of those $k$'s which appeared in (ii), (iii), (ii*) or (iii*) for some $I$. One can construct the desired system by collecting the systems which we obtained from Claims 5.2 and 5.3. □

To finish the proof of Proposition 5.1, it remains to show that, if a proper subset $\mathcal{T} \subset \mathcal{B}_K^n$ has such a system as in Claim 5.4, then $\mathcal{T} \cup \{k\}$ where $k \in \mathcal{B}_K^n \setminus \mathcal{T}$ has also such a system.

So, let $\mathcal{T}$ and

$$\mu_1 < \nu_1 < c_1 < d_1 < \mu_2 < \nu_2 < c_2 < d_2 < \cdots < c_j < d_j < \mu_{j+1} < \nu_{j+1}$$

be as in Claim 5.4 and let $k \in \mathcal{B}_K^n \setminus \mathcal{T}$. Let $\iota$ be the index such that $(\mu_\iota, \nu_\iota)$ is covered by the same connected component of $\mathbb{R} \setminus \bigcup_{i=1}^j [c_i, d_i]$ as $((k-50)L_n, (k+51)L_n)$. We intend to obtain the desired system for $\mathcal{T} \cup \{k\}$ by replacing $\mu_\iota < \nu_\iota$ with

$$\mu < \nu < u_k < v_k < \mu' < \nu'$$

where

$$(\mu, \nu) = \begin{cases} (\mu_\iota, \nu_\iota), & \nu_\iota \leq u_k - L_n \text{ and } \fint_{\mu_\iota}^{\nu_\iota} f \geq \fint_{\alpha_k}^{\beta_k} f, \\ (\alpha_k, \beta_k), & \text{otherwise,} \end{cases}$$

$$(\mu', \nu') = \begin{cases} (\mu_\iota, \nu_\iota), & \mu_\iota \geq v_k + L_n \text{ and } \fint_{\mu_\iota}^{\nu_\iota} f \geq \fint_{\gamma_k}^{\delta_k} f, \\ (\gamma_k, \delta_k), & \text{otherwise.} \end{cases}$$

For every $l \neq k$ with $l = K \bmod 200$, we have

$$\operatorname{dist}\big(((k-50)L_n, (k+51)L_n), ((l-50)L_n, (l+51)L_n)\big) \geq 99L_n \geq L_n,$$

and thus

$$\begin{aligned} l \in \mathcal{B}_K^n \setminus (\mathcal{T} \cup \{k\}) \quad &\Rightarrow \quad (u_k, v_k) \perp l, \\ l \in \mathcal{B}_K^n \setminus (\mathcal{T} \cup \{k\}) \quad &\Rightarrow \quad (\mu, \nu) \perp l \text{ and } (\mu', \nu') \perp l. \end{aligned}$$

Let us prove the inequality for the modified system. We need to show that the modification of the system increased the left side at least by $\frac{1}{60} \operatorname{var} \mathbb{E}_k^n$. What we need to show is

when $1 < \iota < j+1$: $$2\fint_\mu^\nu f - 2\fint_{u_k}^{v_k} f + 2\fint_{\mu'}^{\nu'} f \geq 2\fint_{\mu_\iota}^{\nu_\iota} f + \frac{1}{60} \operatorname{var} \mathbb{E}_k^n,$$

when $1 = \iota < j+1$: $$\fint_\mu^\nu f - 2\fint_{u_k}^{v_k} f + 2\fint_{\mu'}^{\nu'} f \geq \fint_{\mu_\iota}^{\nu_\iota} f + \frac{1}{60} \operatorname{var} \mathbb{E}_k^n,$$

when $1 < \iota = j+1$: $$2\fint_\mu^\nu f - 2\fint_{u_k}^{v_k} f + \fint_{\mu'}^{\nu'} f \geq \fint_{\mu_\iota}^{\nu_\iota} f + \frac{1}{60} \operatorname{var} \mathbb{E}_k^n,$$

when $1 = \iota = j+1$: $$\fint_\mu^\nu f - 2\fint_{u_k}^{v_k} f + \fint_{\mu'}^{\nu'} f \geq \frac{1}{60} \operatorname{var} \mathbb{E}_k^n.$$

These inequalities, even with $\frac{1}{12}$ instead of $\frac{1}{60}$, follow from

$$\fint_\mu^\nu f - \fint_{u_k}^{v_k} f \geq \frac{1}{24} \operatorname{var} \mathbb{E}_k^n, \quad \fint_{\mu'}^{\nu'} f - \fint_{u_k}^{v_k} f \geq \frac{1}{24} \operatorname{var} \mathbb{E}_k^n,$$

$$\fint_\mu^\nu f \geq \fint_{\mu_\iota}^{\nu_\iota} f \quad \text{or} \quad \fint_{\mu'}^{\nu'} f \geq \fint_{\mu_\iota}^{\nu_\iota} f$$

($\fint_{\mu}^{\nu} f \geq \fint_{\mu_\iota}^{\nu_\iota} f$ is implied by $\nu_\iota \leq u_k - L_n$ and $\fint_{\mu'}^{\nu'} f \geq \fint_{\mu_\iota}^{\nu_\iota} f$ is implied by $\mu_\iota \geq v_k + L_n$).

The proof of Proposition 5.1 is completed.

**Corollary 5.5.** *For $0 \leq N \leq 9$ and $0 \leq K \leq 199$, we have*

$$\sum \Big\{ \operatorname{var} \mathbb{E}_k^n : n = N \bmod 10, k \in \mathcal{B}_K^n \Big\} \leq 60 \operatorname{Var} f.$$

*Proof.* Let $\eta$ be large enough such that

$$\mathcal{B}_K^o \neq \varnothing \quad \Rightarrow \quad o \leq n$$

where $n = 10\eta + N$. Let

$$\varphi_1 < \psi_1 < s_1 < t_1 < \varphi_2 < \psi_2 < s_2 < t_2 < \cdots < s_m < t_m < \varphi_{m+1} < \psi_{m+1}$$

be the system which Proposition 5.1 gives for $N, K$ and $\eta$. For $1 \leq i \leq m+1$, let $\lambda_i \in (\varphi_i, \psi_i)$ be chosen so that

$$f(\lambda_i) \geq \fint_{\varphi_i}^{\psi_i} f.$$

For $1 \leq i \leq m$, let $z_i \in (s_i, t_i)$ be chosen so that

$$f(z_i) \leq \fint_{s_i}^{t_i} f.$$

We compute

$$\begin{aligned}
\operatorname{Var} f &\geq \sum_{i=1}^{m} \Big[ |f(z_i) - f(\lambda_i)| + |f(\lambda_{i+1}) - f(z_i)| \Big] \\
&\geq \sum_{i=1}^{m} \Big[ f(\lambda_i) - f(z_i) + f(\lambda_{i+1}) - f(z_i) \Big] \\
&\geq \sum_{i=1}^{m} \Big[ \fint_{\varphi_i}^{\psi_i} f + \fint_{\varphi_{i+1}}^{\psi_{i+1}} f - 2\fint_{s_i}^{t_i} f \Big] \\
&\geq \frac{1}{60} \sum \Big\{ \operatorname{var} \mathbb{E}_k^o : o = N \bmod 10, o \leq n, k \in \mathcal{B}_K^o \Big\} \\
&= \frac{1}{60} \sum \Big\{ \operatorname{var} \mathbb{E}_k^o : o = N \bmod 10, k \in \mathcal{B}_K^o \Big\}.
\end{aligned}$$

□

## 6. Proof of Theorem 1.2

We are going to finish the proof of Theorem 1.2. We recall that Theorem 1.2 is being proven for a fixed function $f$ of bounded variation with $f \geq 0$. Let us summarize our conclusions first (the required notation was introduced in Definition 2.2 and during Section 3).

Let $0 \le N \le 9$ and $0 \le K \le 199$. Using Lemma 3.3 and Corollaries 4.5 and 5.5, we can write

$$\begin{aligned}\sum \Big\{ \operatorname{var} \mathbb{E}_k^n : n = N \bmod 10, k = K \bmod 200 \Big\} \\ &= \sum \Big\{ \operatorname{var} \mathbb{E}_k^n : n = N \bmod 10, k = K \bmod 200 \text{ and } \mathbb{E}_k^n \text{ is non-empty} \Big\} \\ &= \sum \Big\{ \operatorname{var} \mathbb{E}_k^n : n = N \bmod 10, k \in \mathcal{A}_K^n \cup \mathcal{B}_K^n \Big\} \\ &\le 60 \operatorname{Var} f + 60 \operatorname{Var} f.\end{aligned}$$

Further, using Lemma 3.2, we can compute

$$\begin{aligned}\sum_{i=1}^{\sigma} \Big[ Mf(b_i) - Mf(a_i) + Mf(b_i) - Mf(a_{i+1}) \Big] \\ &= \operatorname{var} \mathbb{P} = \operatorname{var}(\mathbb{P} \setminus \mathbb{E}) + \operatorname{var} \mathbb{E} \\ &= \operatorname{var}(\mathbb{P} \setminus \mathbb{E}) + \sum_{N=0}^{9} \sum_{K=0}^{199} \sum \Big\{ \operatorname{var} \mathbb{E}_k^n : n = N \bmod 10, k = K \bmod 200 \Big\} \\ &\le 2 \operatorname{Var} f + 10 \cdot 200 \cdot 120 \operatorname{Var} f.\end{aligned}$$

So, we have proven the following statement.

**Proposition 6.1.** *Let*

$$a_1 < b_1 < a_2 < b_2 < \cdots < a_\sigma < b_\sigma < a_{\sigma+1}$$

*be such that*

$$Mf(a_i) < Mf(b_i) \quad \textit{and} \quad Mf(a_{i+1}) < Mf(b_i)$$

*for $1 \le i \le \sigma$. Then*

$$\sum_{i=1}^{\sigma} \Big[ Mf(b_i) - Mf(a_i) + Mf(b_i) - Mf(a_{i+1}) \Big] \le (2 + 10 \cdot 200 \cdot 120) \operatorname{Var} f.$$

Once we have Proposition 6.1, the proof of Theorem 1.2 is easy. Nevertheless, we provide the final argument for completeness.

*Proof of Theorem 1.2.* Let $x_1 < x_2 < \cdots < x_l$ be given. We want to show that

$$\sum_{j=1}^{l-1} \big| Mf(x_{j+1}) - Mf(x_j) \big| \le C \operatorname{Var} f.$$

After eliminating unnecessary points and possible repeating of the first and the last point, we obtain a system

$$b_0 \le a_1 < b_1 < a_2 < b_2 < \cdots < a_\sigma < b_\sigma < a_{\sigma+1} \le b_{\sigma+1}$$

such that

$$Mf(a_i) < Mf(b_i) \quad \text{and} \quad Mf(a_{i+1}) < Mf(b_i)$$

for $1 \le i \le \sigma$ and

$$\sum_{i=0}^{\sigma} \big( Mf(b_i) - Mf(a_{i+1}) \big) + \sum_{i=1}^{\sigma+1} \big( Mf(b_i) - Mf(a_i) \big) = \sum_{j=1}^{l-1} \big| Mf(x_{j+1}) - Mf(x_j) \big|.$$

We have $Mf(b_0) - Mf(a_1) \leq \sup f - \inf f \leq \operatorname{Var} f$. Similarly, $Mf(b_{\sigma+1}) - Mf(a_{\sigma+1}) \leq \operatorname{Var} f$. It follows now from Proposition 6.1 that

$$\sum_{j=1}^{l-1} \big|Mf(x_{j+1}) - Mf(x_j)\big| \leq (2 + 2 + 10 \cdot 200 \cdot 120) \operatorname{Var} f,$$

and the proof of the theorem is completed! □

## 7. Proof of Corollaries 1.3 and 1.4

In this section, we follow methods from [2] and [12]. We recall that a function $f : A \subset \mathbb{R} \to \mathbb{R}$ is said to have *Lusin's property* $(N)$ (or is called an $N$*-function*) on $A$ if, for every set $N \subset A$ of measure zero, $f(N)$ is also of measure zero. The well-known Banach-Zarecki theorem states that $f : [a, b] \to \mathbb{R}$ is absolutely continuous if and only if it is a continuous $N$-function of bounded variation.

**Lemma 7.1.** *Let $f : \mathbb{R} \to \mathbb{R}$ be a measurable function with $Mf \not\equiv \infty$ and let $r > 0$. Then the function*

$$M_{\geq r} f(x) = \sup_{\omega \geq r} \fint_{x-\omega}^{x+\omega} |f|$$

*is locally Lipschitz. In particular, $M_{\geq r} f$ is a continuous $N$-function.*

We prove a claim first.

**Claim 7.2.** *For $x, y \in \mathbb{R}$, we have*

$$M_{\geq r} f(y) \geq M_{\geq r} f(x) - \frac{M_{\geq r} f(x)}{r} |y - x|.$$

*Proof.* Due to the symmetry, we may assume that $y > x$. Let $\varepsilon > 0$. There is an $\omega \geq r$ for which

$$\fint_{x-\omega}^{x+\omega} |f| \geq M_{\geq r} f(x) - \varepsilon.$$

We can compute

$$\begin{aligned} M_{\geq r} f(y) &\geq \fint_{x-\omega}^{2y-(x-\omega)} |f| = \frac{1}{2(y - x + \omega)} \int_{x-\omega}^{2y-(x-\omega)} |f| \\ &\geq \frac{1}{2(y - x + \omega)} \int_{x-\omega}^{x+\omega} |f| \geq \frac{2\omega}{2(y - x + \omega)} \big(M_{\geq r} f(x) - \varepsilon\big) \\ &\geq M_{\geq r} f(x) - \varepsilon - \frac{y - x}{y - x + \omega} M_{\geq r} f(x) \geq M_{\geq r} f(x) - \varepsilon - \frac{y - x}{r} M_{\geq r} f(x). \end{aligned}$$

As $\varepsilon > 0$ could be chosen arbitrarily, the claim is proven. □

*Proof of Lemma 7.1.* We realize first that $M_{\geq r} f$ is locally bounded. If $y \in \mathbb{R}$, then $M_{\geq r} f$ is bounded on a neighbourhood of $y$ by Claim 7.2, as

$$|y - x| < r \quad \Rightarrow \quad M_{\geq r} f(x) \leq \frac{r}{r - |y - x|} M_{\geq r} f(y).$$

Now, let $I$ be a bounded interval. There is a $B > 0$ such that $M_{\geq r} f(x) \leq B$ for $x \in I$. Using Claim 7.2 again, we obtain, for $x, y \in I$,

$$M_{\geq r} f(y) \geq M_{\geq r} f(x) - \frac{B}{r} |y - x|.$$

Hence, $M_{\geq r} f$ is Lipschitz with the constant $B/r$ on $I$. □

**Lemma 7.3.** *Let $f : \mathbb{R} \to \mathbb{R}$ be a measurable function with $Mf \not\equiv \infty$ and let $x \in \mathbb{R}$. If $f$ is continuous at $x$, then $Mf$ is continuous at $x$, too.*

*Proof.* The assumption $Mf \not\equiv \infty$ is sufficient for $Mf$ to be lower semicontinuous. Assume that $Mf$ is not upper semicontinuous at $x$. There is a sequence $x_k$ converging to $x$ such that $\inf_{k\in\mathbb{N}} Mf(x_k) > Mf(x)$. We choose $c$ so that

$$\inf_{k\in\mathbb{N}} Mf(x_k) > c > Mf(x).$$

For each $k \in \mathbb{N}$, we choose $\omega_k > 0$ such that

$$\fint_{x_k-\omega_k}^{x_k+\omega_k} |f| \geq c, \quad k = 1, 2, \ldots .$$

Now,

- the possibility $\omega_k \to 0$ contradicts the continuity of $f$ at $x$, since then $\limsup_{y\to x} |f(y)| \geq c > Mf(x) \geq \liminf_{y\to x} |f(y)|$,
- the possibility $\limsup_{k\to\infty} \omega_k > r > 0$ contradicts the continuity of the function $M_{\geq r}f$ from Lemma 7.1, since then $\limsup_{k\to\infty} M_{\geq r}f(x_k) \geq c > Mf(x) \geq M_{\geq r}f(x)$.

□

**Lemma 7.4.** *Let $f : \mathbb{R} \to \mathbb{R}$ be a measurable function with $Mf \not\equiv \infty$ which is continuous on an open set $U$. If $f$ has $(N)$ on $U$, then $Mf$ has also $(N)$ on $U$.*

*Proof.* Note that the set $E = \{x \in U : Mf(x) > |f(x)|\}$ fulfills $E = \bigcup_{k=1}^{\infty} E_{1/k}$ where

$$E_r = \Big\{x \in U : Mf(x) > \sup_{|y-x|<r} |f(y)|\Big\}, \quad r > 0.$$

For $x \in E_r$, we have $Mf(x) = M_{\geq r}f(x)$ where $M_{\geq r}f$ is as in Lemma 7.1. At the same time, for $x \in U \setminus E$, we have $Mf(x) = |f(x)|$. Hence,

$$\begin{aligned} |Mf(N)| &\leq |Mf(N \setminus E)| + \sum_{k=1}^{\infty} |Mf(N \cap E_{1/k})| \\ &\leq |f(N \setminus E)| + \sum_{k=1}^{\infty} |M_{\geq 1/k}f(N \cap E_{1/k})| = 0 \end{aligned}$$

for every null set $N \subset U$. □

*Proof of Corollary 1.3.* By Lemma 7.3, $Mf$ is continuous on $U$. By Lemma 7.4, $Mf$ has $(N)$ on $U$. So, it is sufficient to show that $Mf$ has bounded variation on a given $[a, b] \subset U$ because then the Banach-Zarecki theorem can be applied to prove that $Mf$ is absolutely continuous on $[a, b]$.

Let $r > 0$ be chosen so that $[a - r, b + r] \subset U$ and let $g : \mathbb{R} \to \mathbb{R}$ be defined by

$$g(x) = \begin{cases} f(x), & x \in [a - r, b + r], \\ 0, & x \notin [a - r, b + r]. \end{cases}$$

Then $g$ has bounded variation, as $f$ is absolutely continuous on $[a - r, b + r]$. By Theorem 1.2, $Mg$ has bounded variation. It remains to realize that

$$Mf(x) = \max\{Mg(x), M_{\geq r}f(x)\}, \quad x \in [a, b],$$

for the function $M_{\geq r}f$ from Lemma 7.1. □

*Proof of Corollary 1.4.* Assume that $f \in W^{1,1}(\mathbb{R})$. Then $f$ is represented by an absolutely continuous function of bounded variation (which will be also denoted by $f$). By Corollary 1.3, $Mf$ is locally AC, and thus weakly differentiable. Using Theorem 1.2, we can write

$$\|(Mf)'\|_1 = \operatorname{Var} Mf \le C \operatorname{Var} f = C\|f'\|_1.$$

□

## 8. Remarks

*Remark* 8.1. It is possible to formulate an abstract statement which covers a significant part of the proof of Theorem 1.2.

*Let $f : \mathbb{R} \to \mathbb{R}$ be a function of bounded variation and let $Q_{n,k}, n \ge 0, k \in \mathbb{Z}$, be non-negative numbers. Let $L_0 > 0$ and $L_n = 2^{-n}L_0$ for $n \in \mathbb{N}$. Assume that, for every $(n, k)$ with $Q_{n,k} > 0$, there are $s < u < v < t$ such that*

$$(k - 50)L_n \le s, \quad t \le (k + 51)L_n,$$

$$u - s \ge 4L_n, \quad v - u \ge L_n, \quad t - v \ge 4L_n$$

*and*

$$\min\{f(s), f(t)\} - \fint_u^v f \ge \frac{1}{12} Q_{n,k}.$$

*Then*

$$\sum_{n,k} Q_{n,k} \le 10 \cdot 200 \cdot 120 \operatorname{Var} f.$$

It is sufficient to prove this under the assumption that $Q_{n,k} > 0$ for finitely many $Q_{n,k}$'s only. In such a case, just consider $Q_{n,k}$ instead of $\operatorname{var} \mathbb{E}^n_k$ in Lemma 3.3 and in Sections 4&5.

*Remark* 8.2. The proof of Theorem 1.2 works also for the local Hardy-Littlewood maximal function. More precisely, if $\Omega \subset \mathbb{R}$ is open and $d : \Omega \to (0,\infty)$ is Lipschitz with the constant 1 such that $d(x) \le \operatorname{dist}(x, \mathbb{R} \setminus \Omega)$, then the function

$$M_{\le d} f(x) = \sup_{0<\omega\le d(x)} \fint_{x-\omega}^{x+\omega} |f|$$

fulfills $\operatorname{Var}_\Omega M_{\le d} f \le C \operatorname{Var}_\Omega f$. Here, by $\operatorname{Var}_\Omega$ we mean $\sum_n \operatorname{Var}_{I_n}$ where $\Omega = \bigcup_n I_n$ is a decomposition of $\Omega$ into open intervals. The inequality $\operatorname{Var}_{I_n} M_{\le d} f \le C \operatorname{Var}_{I_n} f$ can be proven in the same way as Theorem 1.2. It is sufficient just to modify appropriately the formula for $\omega(r)$ in Definition 2.2.

The version of Corollary 1.4 for $M_{\le d} f$ can be proven as well. If $f \in W^{1,1}(\Omega)$, then $M_{\le d} f$ is weakly differentiable and

$$\|(M_{\le d} f)'\|_{1,\Omega} \le C\|f\|_{1,1,\Omega}.$$

## Acknowledgment

The author is grateful to Jan Malý for suggesting the problem.

## References


[1] J. M. Aldaz and J. Pérez Lázaro, *Boundedness and unboundedness results for some maximal operators on functions of bounded variation*, J. Math. Anal. Appl. **337**, no. 1 (2008), 130–143.

[2] J. M. Aldaz and J. Pérez Lázaro, *Functions of bounded variation, the derivative of the one dimensional maximal function, and applications to inequalities*, Trans. Amer. Math. Soc. **359**, no. 5 (2007), 2443–2461.

[3] J. Bober, E. Carneiro, K. Hughes and L. B. Pierce, *On a discrete version of Tanaka's theorem for maximal functions*, Proc. Amer. Math. Soc. **140**, no. 5 (2012), 1669–1680.

[4] S. M. Buckley, *Is the maximal function of a Lipschitz function continuous?*, Ann. Acad. Sci. Fenn. Math. **24**, no. 2 (1999), 519–528.

[5] P. Hajłasz and J. Malý, *On approximate differentiability of the maximal function*, Proc. Amer. Math. Soc. **138**, no. 1 (2010), 165–174.

[6] P. Hajłasz and J. Onninen, *On boundedness of maximal functions in Sobolev spaces*, Ann. Acad. Sci. Fenn. Math. **29**, no. 1 (2004), 167–176.

[7] J. Kinnunen, *The Hardy-Littlewood maximal function of a Sobolev function*, Israel J. Math. **100** (1997), 117–124.

[8] J. Kinnunen and E. Saksman, *Regularity of the fractional maximal function*, Bull. London Math. Soc. **35**, no. 4 (2003), 529–535.

[9] S. Korry, *Boundedness of Hardy-Littlewood maximal operator in the framework of Lizorkin-Triebel spaces*, Rev. Mat. Complut. **15**, no. 2 (2002), 401–416.

[10] H. Luiro, *Continuity of the maximal operator in Sobolev spaces*, Proc. Amer. Math. Soc. **135**, no. 1 (2007), 243–251.

[11] H. Luiro, *On the size of the set of non-differentiability points of maximal function*, arXiv:1208.3971v1.

[12] H. Tanaka, *A remark on the derivative of the one-dimensional Hardy-Littlewood maximal function*, Bull. Austral. Math. Soc. **65**, no. 2 (2002), 253–258.



Department of Mathematical Analysis, Faculty of Mathematics and Physics, Charles University, Sokolovská 83, 186 75 Prague 8, Czech Republic

*E-mail address*: `kurka.ondrej@seznam.cz`