Rotating compressible fluids under strong stratification

Eduard Feireisl * Yong Lu† Antonín Novotný‡

Institute of Mathematics of the Academy of Sciences of the Czech Republic
Žitná 25, 115 67 Praha 1, Czech Republic
and
Charles University in Prague, Faculty of Mathematics and Physics, Mathematical Institute
Sokolovská 83, 186 75 Praha 8, Czech Republic
and
IMATH, EA 2134, Université du Sud Toulon-Var BP 20132, 83957 La Garde, France

Abstract

We consider the Navier-Stokes system written in the rotational frame describing the motion of a compressible viscous fluid under strong stratification. The asymptotic limit for low Mach and Rossby numbers and large Reynolds number is studied on condition that the Froude number characterizing the degree of stratification is proportional to the Mach number. We show that, at least for the well prepared data, the limit system is the same as for the problem without stratification - a variant of the incompressible planar Euler system.

Key words: Rotating compressible fluid, singular limit, multiscale analysis, strong stratification

Contents

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1 Introduction

We study the asymptotic behavior of a multiple-parameter system of partial differential equations arising in the modelling of atmospheric flows, see e.g. Klein [7]. In the rotational coordinate frame, such a system reads:

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0, \tag{1.1}$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \frac{1}{\varepsilon} \varrho \mathbf{u} \times \omega + \frac{1}{\varepsilon^2} \nabla_x p(\varrho) = \nu \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u}) + \frac{1}{\varepsilon^2} \varrho \nabla_x G, \tag{1.2}$$

$$\mathbb{S}(\nabla_x \mathbf{u}) = \mu \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3} \mathrm{div}_x \mathbf{u} \mathbb{I} \right) + \eta \mathrm{div}_x \mathbf{u} \mathbb{I}, \ \mu > 0, \ \eta \geq 0. \tag{1.3}$$

Here, we have used the following notation:

$\varrho = \varrho(t, x)$ . . . . the fluid mass density
$\mathbf{u} = \mathbf{u}(t, x)$ . . . . the velocity field
$\omega = [0, 0, 1]$ . . . . the rotation axis
$p = p(\varrho)$ . . . . the pressure
$\mathbb{S} = \mathbb{S}(\nabla_x \mathbf{u})$ . . . . the Newtonian viscous stress tensor
μ, λ . . . . the viscosity coefficients
$G = -x_3$ . . . . the gravitational potential

1.1 Boundary conditions

The *Navier-Stokes system* (1.1 - 1.3) is written in its dimensionless form, with the Mach number = Rossby number = Froude number = ε, and the Reynolds number = ν^{-1}. The Mach and Froude number being of the same order, the asymptotic motion of the fluid is strongly stratified. We consider a highly simplified geometry, namely the infinite slab $\Omega \subset R^3$,

$$\Omega = R^2 \times (0,1), \tag{1.4}$$

with the complete slip boundary conditions

$$\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0, \ [\mathbb{S} \cdot \mathbf{n}]_{\tan}|_{\partial\Omega} = 0, \ \mathbf{n} - \text{ the outer normal vector to } \partial\Omega, \tag{1.5}$$

and the far field condition for the velocity

$$\mathbf{u} \to 0 \text{ as } |x| \to \infty. \tag{1.6}$$

Given this geometrical setting, it is convenient to introduce the horizontal variable $x_h = [x_1, x_2]$ and the vertical variable x_3, along with the associated differential operators div_h, ∇_h, and the Laplacean Δ_h acting on the horizontal variables. Moreover, we denote

$$\langle h \rangle = \int_0^1 h \ \mathrm{d}x_3$$

the vertical average of a function h.

1.2 Static solutions

The asymptotic distribution of the density is governed by the static system

$$\nabla_x p(\tilde{\varrho}) = \tilde{\varrho} \nabla_x G. \tag{1.7}$$

As $\nabla_x G = [0,0,-1]$, it is easy to see that $\tilde{\varrho}$ depends only on the vertical variable x_3, more specifically,

$$H'(\tilde{\varrho}) = -x_3 + c, \ H(\varrho) = \varrho \int_1^{\varrho} \frac{p(z)}{z^2} \ \mathrm{d}z \tag{1.8}$$

for a suitable constant c. Fixing c so that $\tilde{\varrho} > 0$ in $\overline{\Omega}$, we can prescribe the far field behavior of the density, namely

$$\varrho \to \tilde{\varrho} \text{ as } |x| \to \infty. \tag{1.9}$$

1.3 Target system

Our goal is to perform the simultaneous singular limit and to identify the target system as $\varepsilon,\ \nu \to 0$. This can be viewed as a continuation of our previous work [2], [3] in the context of strongly stratified fluids. We refer to the survey by Klein [7] for specific features of such multiscale problems in the context of atmospheric flows. There are several competing processes in the course of the singular limit. The density profile approaches the static distribution $\tilde{\varrho}$, while, as a consequence of the asymptotically infinite sound speed (low Mach number), the fluid flow becomes "incompressible". On the other hand, the fast rotation drives the system to the purely planar motion, and, last but not least, the limit fluid flow is inviscid due to the high Reynolds number.

Unlike the asymptotic density approaching the static profile $\tilde{\varrho}$, the limit velocity is not essentially influenced by stratification. Denoting $\varrho = \varrho_{\varepsilon,\nu}$, $\mathbf{u} = \mathbf{u}_{\varepsilon,\nu}$ the solutions of the scaled system, we show that

$$\frac{\varrho - \tilde{\varrho}}{\varepsilon} \to s,\ \mathbf{u} \to \mathbf{v} \text{ as } \varepsilon \to 0,\ \nu \to 0 \text{ in some sense,} \tag{1.10}$$

where s and $\mathbf{v}$ are interrelated through

$$\begin{bmatrix} v_2 \\ -v_1 \\ 0 \end{bmatrix} + \nabla_x \left(\frac{p'(\tilde{\varrho})}{\tilde{\varrho}} s \right) = 0, \text{ in particular, } \mathbf{v} = [v_1(t, x_h), v_2(t, x_h), 0], \tag{1.11}$$

whereas the limit system takes the form

$$\nabla_h^\perp q = \mathbf{v}_h,\quad \langle \tilde{\varrho} \rangle \, \partial_t \Delta_h q - \left\langle \frac{\tilde{\varrho}}{p'(\tilde{\varrho})} \right\rangle \partial_t q + \langle \tilde{\varrho} \rangle \, \nabla_h^\perp q \cdot \nabla_h \left(\Delta_h q \right) = 0,\ q = \frac{p'(\tilde{\varrho})}{\tilde{\varrho}} s, \tag{1.12}$$

$$\nabla_h^\perp q \equiv [\partial_{x_2} q, -\partial_{x_1} q].$$

Thus the limit is the same as in [3] - a "damped" variant of the planar incompressible Euler system occurring in certain meteorological models, see Zeitlin [12]. In particular, the problem (1.11), (1.12) admits global in time classical solutions for any smooth initial data, see Section 2 below.

Similarly to [2], [3] (cf. also Masmoudi [9], Wang and Jiang [11], Jiang et al. [5], [6]), our approach is based on uniform bounds derived by means of the *relative entropy* (modulated energy) inequality. We restrict ourselves to the case of *well-prepared* initial data for the Navier-Stokes system (1.1 - 1.3) approaching in the asymptotic limit the initial state of the target problem (1.11), (1.12). As a benefit, we obatin a rather exact *convergence rate* in terms of the singular parameters ε and ν.

The paper is organized as follows. In Section 2, we collect the necessary material concerning solvability of both primitive and target system and state our main result. The relative entropy inequality is introduced in Section 3. The necessary estimates and the proof of the main result are performed in Section 4.

2 Main result

We start with the standard technical hypothesis imposed on the pressure in order to ensure the existence of global-in-time weak solutions to the primitive Navier-Stokes system:

$$p \in C[0,\infty) \cap C^3(0,\infty),\ p(0)=0,\ p'(\varrho)>0 \text{ for } \varrho>0,\ \lim_{\varrho\to\infty}\frac{p'(\varrho)}{\varrho^{\gamma-1}} = p_\infty > 0 \tag{2.1}$$

for some $\gamma > 3/2$.

2.1 Weak solutions to the primitive system

We say that $[\varrho, \mathbf{u}]$ is a *finite energy weak solution* of the Navier-Stokes-Poisson system (1.1 - 1.3), supplemented with the boundary conditions (1.5), (1.6), (1.9), and the initial conditions

$$\varrho(0,\cdot)=\varrho_0,\ \mathbf{u}(0,\cdot)=\mathbf{u}_0, \tag{2.2}$$

in the space-time cylinder $(0,T)\times\Omega$ if:

- $\varrho \geq 0$ a.a. in $(0,T)\times\Omega$,

$$(\varrho-\tilde{\varrho}) \in C_{\rm weak}(0,T;L^2+L^\gamma(\Omega)),\ \varrho\mathbf{u}\in C_{\rm weak}(0,T;L^2+L^{2\gamma/(\gamma+1)}(\Omega;R^3)),$$

$$\mathbf{u}\in L^2(0,T;W^{1,2}(\Omega;R^3)),\ \mathbf{u}\cdot\mathbf{n}|_{\partial\Omega}\equiv u_3|_{\partial\Omega}=0;$$

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$$\int_0^\tau\int_\Omega [\varrho\partial_t\varphi + \varrho\mathbf{u}\cdot\nabla_x\varphi]\ {\rm d}x\ {\rm d}t = \int_\Omega \varrho(\tau,\cdot)\varphi(\tau,\cdot)\ {\rm d}x - \int_\Omega \varrho_0\varphi(0,\cdot)\ {\rm d}x \tag{2.3}$$

  for $0\leq\tau\leq T$ and any test function $\varphi\in C^\infty_c([0,T]\times\overline{\Omega})$;

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$$\int_0^\tau\int_\Omega \left[\varrho\mathbf{u}\cdot\partial_t\varphi + \varrho\mathbf{u}\otimes\mathbf{u}:\nabla_x\varphi - \frac{1}{\varepsilon}(\varrho\mathbf{u}\times\omega)\cdot\varphi + \frac{1}{\varepsilon^2}p(\varrho){\rm div}_x\varphi\right]\ {\rm d}x\ {\rm d}t \tag{2.4}$$

$$= \int_0^\tau\int_\Omega\left[\nu\mathbb{S}(\nabla_x\mathbf{u}):\nabla_x\varphi - \frac{1}{\varepsilon^2}\varrho\nabla_x G\cdot\varphi\right]\ {\rm d}x\ {\rm d}t + \int_\Omega \varrho\mathbf{u}(\tau,\cdot)\cdot\varphi(\tau,\cdot)\ {\rm d}x - \int_\Omega \varrho_0\mathbf{u}_0\cdot\varphi(0,\cdot)\ {\rm d}x$$

  for any $0\leq\tau\leq T$ and any test function $\varphi\in C^\infty_c([0,T]\times\overline{\Omega};R^3)$, $\varphi\cdot\mathbf{n}|_{\partial\Omega}=0$;

- the *energy inequality*

$$\int_\Omega \left[\frac{1}{2}\varrho|\mathbf{u}|^2 + \frac{1}{\varepsilon^2}\left(H(\varrho) - H'(\tilde\varrho)(\varrho - \tilde\varrho) - H(\tilde\varrho)\right)\right](\tau,\cdot)\ \mathrm{d}x \tag{2.5}$$

$$+\nu \int_0^\tau \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u}\ \mathrm{d}x\ \mathrm{d}t$$

$$\leq \int_\Omega \left[\frac{1}{2}\varrho_0|\mathbf{u}_0|^2 + \frac{1}{\varepsilon^2}\left(H(\varrho_0) - H'(\tilde\varrho)(\varrho_0 - \tilde\varrho) - H(\tilde\varrho)\right)\right] \mathrm{d}x$$

holds for a.a. $\tau \in (0,T)$, where we have set

$$H(\varrho) = \varrho \int_1^\varrho \frac{p(z)}{z^2}\ \mathrm{d}z.$$

It can be shown by the methods developed by Lions [8] and in [4] that the Navier-Stokes system admits global-in-time weak solutions for any finite energy initial data as long as $\varepsilon, \nu > 0$ and the pressure satisfies (2.1).

2.2 Solutions to the target problem

As for solvability of the problem (1.11), (1.12), with the initial data

$$q(0) = q_0, \tag{2.6}$$

we report the following result that can be shown within the abstract theory developed by Oliver [10, Theorem 3], cf. also [3, Section 2.1]:

Proposition 2.1 *Suppose that*

$$q_0 \in W^{m,2}(R^2) \ \textit{for}\ m \geq 4.$$

Then the problem (1.11), (1.12), (2.6) admits a solution q, unique in the class

$$q \in C([0,T]; W^{m,2}(R^2) \cap C^1([0,T]; W^{m-1,2}(R^2)).$$

2.3 Asymptotic limit - main result

The main result of the present paper can be stated as follows:

Theorem 2.1 *Let the pressure p satisfy the hypothesis (2.1) for some $\gamma > 3/2$. Let $q_0 \in W^{m,2}(R^2)$, $m \geq 4$, together with*

$$s_0, \mathbf{v}_{0,h},\ q_0 = \frac{p'(\tilde{\varrho})}{\tilde{\varrho}} s_0,\ \left[\begin{array}{c} v_{0,h}^2 \\ -v_{0,h}^1 \end{array} \right] + \nabla_x \left(\frac{p'(\tilde{\varrho})}{\tilde{\varrho}} s_0 \right) = 0$$

be given, and let q, s, $\mathbf{v} = [\mathbf{v}_h, 0]$ be the unique solution of the target problem (1.11), (1.12), (2.6). Let $\varrho = \varrho_{\varepsilon,\nu}$, $\mathbf{u} = \mathbf{u}_{\varepsilon,\nu}$ be a finite energy weak solution of the primitive Navier-Stokes system (1.1 - 1.3), (1.5), (1.6), (1.9), emanating from the initial data

$$\varrho(0,\cdot) = \varrho_{0,\varepsilon} = \tilde{\varrho} + \varepsilon \varrho_{0,\varepsilon}^{(1)},\ \mathbf{u}(0,\varepsilon) = \mathbf{u}_{0,\varepsilon}, \tag{2.7}$$

$$\|\varrho_{0,\varepsilon}^{(1)}\|_{L^2 \cap L^\infty(\Omega)} + \|\mathbf{u}_{0,\varepsilon}\|_{L^2(\Omega;R^3)} \leq d \tag{2.8}$$

uniformly in ε, ν.

Then there exists a constant $c(D,T)$ depending only on the data q_0, d, and T such that

$$\|\sqrt{\varrho}(\mathbf{u} - \mathbf{v})(\tau,\cdot)\|_{L^2(\Omega;R^3)}^2 + \left\| \left(\frac{\varrho - \tilde{\varrho}}{\varepsilon} - s \right) (\tau,\cdot) \right\|_{L^2 + L^\gamma(\Omega)}^2 \tag{2.9}$$

$$\leq c(D,T) \left(\|\mathbf{u}_{0,\varepsilon} - [\mathbf{v}_{0,h}, 0]\|_{L^2(\Omega;R^3)}^2 + \left\| \varrho_{0,\varepsilon}^{(1)} - s_0 \right\|_{L^2(\Omega)}^2 + \nu + \varepsilon \right)$$

for any $0 \leq \tau \leq T$.

Remark 2.1 *It follows from Theorem 2.1 that $\varrho = \varrho_{\varepsilon,\nu}$, $\mathbf{u} = \mathbf{u}_{\varepsilon,\nu}$ approach the solution of the targer system provided $\varepsilon, \nu \to 0$ and the initial data are* well prepared, *meaning, the right-hand side of the inequality (2.9) tends to zero.*

The rest of the paper will be devoted to the proof of Theorem 2.1.

3 Relative entropy

Similarly to [2], [3], we introduce the *relative entropy functional*

$$\mathcal{E}\left(\varrho,\mathbf{u}\Big|r,\mathbf{U}\right)=\int_\Omega\left[\frac{1}{2}\varrho|\mathbf{u}-\mathbf{U}|^2+\frac{1}{\varepsilon^2}\Big(H(\varrho)-H'(r)(\varrho-r)-H(r)\Big)\right]\ \mathrm{d}x, \tag{3.1}$$

along with the relative entropy inequality associated to the primitive Navier-Stokes system (1.1 - 1.3):

$$\begin{gathered}\mathcal{E}\left(\varrho,\mathbf{u}\ \Big|\ r,\mathbf{U}\right)(\tau)+\nu\int_0^\tau\int_\Omega\Big(\mathbb{S}(\nabla_x\mathbf{u})-\mathbb{S}(\nabla_x\mathbf{U})\Big):\Big(\nabla_x\mathbf{u}-\nabla_x\mathbf{U}\Big)\ \mathrm{d}x\ \mathrm{d}t\leq \\ \mathcal{E}\left(\varrho_{0,\varepsilon},\mathbf{u}_{0,\varepsilon}\ \Big|\ r(0,\cdot),\mathbf{U}(0,\cdot)\right) \\ +\int_0^\tau\int_\Omega\varrho\left(\partial_t\mathbf{U}+\mathbf{u}\cdot\nabla_x\mathbf{U}\right)\cdot(\mathbf{U}-\mathbf{u})\ \mathrm{d}x\ \mathrm{d}t \\ +\nu\int_0^\tau\int_\Omega\mathbb{S}(\nabla_x\mathbf{U}):\nabla_x(\mathbf{U}-\mathbf{u})\ \mathrm{d}x\ \mathrm{d}t-\frac{1}{\varepsilon}\int_0^\tau\int_\Omega\varrho(\omega\times\mathbf{u})\cdot(\mathbf{U}-\mathbf{u})\ \mathrm{d}x\ \mathrm{d}t \\ +\frac{1}{\varepsilon^2}\int_0^\tau\int_\Omega\Big[(r-\varrho)\partial_tH'(r)+\nabla_xH'(r)\cdot(r\mathbf{U}-\varrho\mathbf{u})\Big]\ \mathrm{d}x\ \mathrm{d}t-\frac{1}{\varepsilon^2}\int_0^\tau\int_\Omega\mathrm{div}_x\mathbf{U}\Big(p(\varrho)-p(r)\Big)\ \mathrm{d}x\ \mathrm{d}t \\ -\frac{1}{\varepsilon^2}\int_0^\tau\int_\Omega\varrho\nabla_xG\cdot(\mathbf{U}-\mathbf{u})\ \mathrm{d}x\ \mathrm{d}t\end{gathered} \tag{3.2}$$

for all (smooth) functions r, $\mathbf{U}$ such that

$$r>0,\ (r-\tilde{\varrho})\in C_c^\infty([0,T]\times\overline{\Omega}),\ \mathbf{U}\in C_c^\infty([0,T]\times\Omega). \tag{3.3}$$

As shown in [1], *any* finite energy weak solution $[\varrho,\mathbf{u}]$ satisfies the relative entropy inequality for any pair of "test" functions r, $\mathbf{U}$ as in (3.3).

3.1 Uniform bounds

The ansatz

$$r=\tilde{\varrho},\ \mathbf{U}=0,$$

together with the hypotheses (2.7), (2.8) imposed on the initial data, gives rise to the following bounds (cf. [3, Section 3.2]):

$$\operatorname*{ess\ sup}_{\tau\in(0,T)}\|\sqrt{\varrho}\mathbf{u}(\tau,\cdot)\|_{L^2(\Omega;R^3)}\leq c(D), \tag{3.4}$$

$$\operatorname*{ess\ sup}_{\tau\in(0,T)}\left\|\frac{\varrho-\tilde{\varrho}}{\varepsilon}\right\|_{L^2+L^\gamma(\Omega)}\leq c(D), \tag{3.5}$$

$$\nu\eta\int_0^T\int_\Omega|\mathrm{div}_x\mathbf{u}|^2\ \mathrm{d}x\,\mathrm{d}t+\nu\int_0^T\int_\Omega\left|\nabla_x\mathbf{u}+\nabla_x^t\mathbf{u}-\frac{2}{3}\mathrm{div}_x\mathbf{u}\mathbb{I}\right|^2\ \mathrm{d}x\ \mathrm{d}t\leq c(D), \tag{3.6}$$

where the constant $c(D)$ depends only on the norms of the initial data specified in Theorem 2.1.

4 Convergence

Our goal is to derive (2.9) from the relative entropy inequality (3.2) using a new ansatz

$$r = \tilde{\varrho} + \varepsilon s, \ \mathbf{U} = \mathbf{v} \equiv [\mathbf{v}_h, 0], \tag{4.1}$$

where $\mathbf{v}_h$, s satisfy the limit system (1.11), (1.12). In the remaining part of this section, we examine term by term the integrals on the right-hand side of (3.2).

4.1 Initial data

Given (2.7), we easily compute

$$\mathcal{E}\left(\varrho_{0,\varepsilon}, \mathbf{u}_{0,\varepsilon} \Big| r(0), \mathbf{U}(0)\right) \leq c(D) \int_\Omega \left[|\mathbf{u}_{0,\varepsilon} - [\mathbf{v}_{0,h}, 0]|^2 + |\varrho^{(1)}_{0,\varepsilon} - s_0|^2 \right] \ \mathrm{d}x \tag{4.2}$$

4.2 "Viscous" terms

$$\nu \int_\Omega \mathbb{S}(\nabla_x \mathbf{v}) : \nabla_x(\mathbf{v} - \mathbf{u}) \ \mathrm{d}x \to 0 = 2\nu \int_\Omega \mathbb{S}(\nabla_x \mathbf{v}) : \mathbb{S}(\nabla_x(\mathbf{v} - \mathbf{u})) \ \mathrm{d}x \tag{4.3}$$
$$\leq \nu \|\mathbb{S}(\nabla_x \mathbf{v})\|^2_{L^2(\Omega;R^{3\times3})} + \nu \|\mathbb{S}(\nabla_x(\mathbf{v} - \mathbf{u}))\|^2_{L^2(\Omega;R^{3\times3})}$$
$$= \nu \|\mathbb{S}(\nabla_x \mathbf{v})\|^2_{L^2(\Omega;R^{3\times3})} + \frac{\nu}{2} \int_\Omega \left(\mathbb{S}(\nabla_x \mathbf{u}) - \mathbb{S}(\nabla_x \mathbf{v})\right) : \left(\nabla_x \mathbf{u} - \nabla_x \mathbf{v}\right) \ \mathrm{d}x$$

Combining (4.2), (4.3) we can rewrite (3.2) as

$$\mathcal{E}\left(\varrho, \mathbf{u} \ \Big| \ r, \mathbf{U}\right)(\tau) \leq c(D) \left(\int_\Omega \left[|\mathbf{u}_{0,\varepsilon} - [\mathbf{v}_{0,h}, 0]|^2 + |\varrho^{(1)}_{0,\varepsilon} - s_0|^2 \right] \ \mathrm{d}x + \nu \right) \tag{4.4}$$
$$+ \int_0^\tau \int_\Omega \varrho \left(\partial_t \mathbf{v} + \mathbf{u} \cdot \nabla_x \mathbf{v} \right) \cdot (\mathbf{v} - \mathbf{u}) \ \mathrm{d}x \ \mathrm{d}t$$
$$+ \frac{1}{\varepsilon} \int_0^\tau \int_\Omega \varrho(\omega \times \mathbf{u}) \cdot \mathbf{v} \ \mathrm{d}x \ \mathrm{d}t + \frac{1}{\varepsilon^2} \int_0^\tau \int_\Omega \left[(r - \varrho) \partial_t H'(r) + \nabla_x H'(r) \cdot (r\mathbf{v} - \varrho \mathbf{u}) \right] \mathrm{d}x \ \mathrm{d}t$$
$$- \frac{1}{\varepsilon^2} \int_0^\tau \int_\Omega \varrho \nabla_x G \cdot (\mathbf{v} - \mathbf{u}) \ \mathrm{d}x \ \mathrm{d}t.$$

4.3 Singular terms

Using the relation (1.11) we deduce

$$I_1 \equiv -\frac{1}{\varepsilon}\int_\Omega \varrho(\omega\times\mathbf{u})\cdot\mathbf{v}\ \mathrm{d}x = \frac{1}{\varepsilon}\int_\Omega \varrho(\omega\times\mathbf{v})\cdot\mathbf{u}\ \mathrm{d}x = \frac{1}{\varepsilon}\int_\Omega \varrho\mathbf{u}\cdot\nabla_x\left(\frac{p'(\tilde\varrho)}{\tilde\varrho}s\right)\ \mathrm{d}x. \tag{4.5}$$

Next,

$$I_2 \equiv \frac{1}{\varepsilon^2}\int_\Omega (r-\varrho)\partial_t H'(r)\ \mathrm{d}x = \frac{1}{\varepsilon}\int_\Omega \frac{\tilde\varrho+\varepsilon s-\varrho}{r}p'(r)\partial_t s\ \mathrm{d}x \tag{4.6}$$

$$= \frac{1}{\varepsilon}\int_\Omega (\tilde\varrho+\varepsilon s-\varrho)\left(\frac{p'(r)}{r}-\frac{p'(\tilde\varrho)}{\tilde\varrho}\right)\partial_t s\ \mathrm{d}x + \frac{1}{\varepsilon}\int_\Omega (\tilde\varrho+\varepsilon s-\varrho)\frac{p'(\tilde\varrho)}{\tilde\varrho}\partial_t s\ \mathrm{d}x,$$

where, in view of the previously established uniform bounds (3.4), (3.5),

$$\left|\frac{1}{\varepsilon}\int_\Omega (\tilde\varrho+\varepsilon s-\varrho)\left(\frac{p'(r)}{r}-\frac{p'(\tilde\varrho)}{\tilde\varrho}\right)\partial_t s\ \mathrm{d}x\right| \le c(D)\varepsilon. \tag{4.7}$$

Finally,

$$I_3 \equiv \frac{1}{\varepsilon^2}\int_\Omega \nabla_x H'(r)\cdot(r\mathbf{v}-\varrho\mathbf{u})\ \mathrm{d}x - \frac{1}{\varepsilon^2}\int_\Omega \varrho\nabla_x G\cdot(\mathbf{v}-\mathbf{u})\ \mathrm{d}x \tag{4.8}$$

$$= \frac{1}{\varepsilon^2}\int_\Omega p'(r)\nabla_x r\cdot\left(\frac{r\mathbf{v}-\varrho\mathbf{u}}{r}\right)\ \mathrm{d}x + \frac{1}{\varepsilon^2}\int_\Omega \frac{p'(\tilde\varrho)}{\tilde\varrho}\varrho\mathbf{u}\cdot\nabla_x\tilde\varrho\ \mathrm{d}x$$

$$= \frac{1}{\varepsilon^2}\int_\Omega \left(\frac{p'(\tilde\varrho)}{\tilde\varrho}-\frac{p'(r)}{r}\right)\varrho\mathbf{u}\cdot\nabla_x\tilde\varrho\ \mathrm{d}x - \frac{1}{\varepsilon}\int_\Omega \frac{p'(r)}{r}\varrho\mathbf{u}\cdot\nabla_x s\ \mathrm{d}x$$

where we have used that

$$\nabla_x s\cdot\mathbf{v} = 0.$$

Thus we get

$$I_1+I_3 = \frac{1}{\varepsilon^2}\int_\Omega \left(\frac{p'(\tilde\varrho)}{\tilde\varrho}-\frac{p'(r)}{r}\right)\varrho\mathbf{u}\cdot\nabla_x\tilde\varrho\ \mathrm{d}x + \frac{1}{\varepsilon}\int_\Omega \left(\frac{p'(\tilde\varrho)}{\tilde\varrho}-\frac{p'(r)}{r}\right)\varrho\mathbf{u}\cdot[\nabla_h s, 0]\ \mathrm{d}x$$

$$-\frac{1}{\varepsilon}\int_\Omega \frac{p'(r)}{r}\varrho u^3\partial_{x_3}s\ \mathrm{d}x.$$

Denoting $h(z)=\frac{p'(z)}{z}$ we get

$$I_1+I_3 \le -\frac{1}{2}\int_\Omega h''(\tilde\varrho)s^2\varrho\mathbf{u}\cdot\nabla_x\tilde\varrho\ \mathrm{d}x - \frac{1}{\varepsilon}\int_\Omega h'(\tilde\varrho)s\varrho\mathbf{u}\cdot\nabla_x\tilde\varrho\ \mathrm{d}x$$

$$-\int_\Omega h'(\tilde\varrho) s\varrho\mathbf{u}\cdot[\nabla_h s,0]\ \mathrm{d}x - \frac{1}{\varepsilon}\int_\Omega \frac{p'(\tilde\varrho)}{\tilde\varrho}\varrho u^3\partial_{x_3} s\ \mathrm{d}x - \int_\Omega h'(\tilde\varrho)s\varrho u^3 \partial_{x_3} s\ \mathrm{d}x + c(D)\varepsilon$$
$$= -\frac{1}{2}\int_\Omega h''(\tilde\varrho)s^2\varrho\mathbf{u}\cdot\nabla_x\tilde\varrho\ \mathrm{d}x - \int_\Omega h'(\tilde\varrho)s\varrho\mathbf{u}\cdot[\nabla_h s,0]\ \mathrm{d}x - \int_\Omega h'(\tilde\varrho)s\varrho u^3\partial_{x_3}s\ \mathrm{d}x + c(D)\varepsilon.$$

where we have used (1.11), specifically,

$$h'(\tilde\varrho)\partial_{x_3}\tilde\varrho s + \frac{p'(\tilde\varrho)}{\tilde\varrho}\partial_{x_3}s = 0.$$

Furthermore,

$$-\frac{1}{2}\int_0^\tau\int_\Omega h''(\tilde\varrho)s^2\varrho\mathbf{u}\cdot\nabla_x\tilde\varrho\ \mathrm{d}x\ \mathrm{d}t - \int_0^\tau\int_\Omega h'(\tilde\varrho)s\varrho\mathbf{u}\cdot[\nabla_h s,0]\ \mathrm{d}x\ \mathrm{d}t - \int_0^\tau\int_\Omega h'(\tilde\varrho)s\varrho u^3\partial_{x_3}s\ \mathrm{d}x\ \mathrm{d}t$$
$$= -\frac{1}{2}\int_0^\tau\int_\Omega \nabla_x\left(h'(\tilde\varrho)s^2\right)\cdot(\varrho\mathbf{u})\ \mathrm{d}x\ \mathrm{d}t = \frac{1}{2}\int_0^\tau\int_\Omega\partial_t\left(h'(\tilde\varrho)s^2\right)\varrho\ \mathrm{d}x\ \mathrm{d}t$$
$$+\frac{1}{2}\left[\int_\Omega h'(\tilde\varrho)s^2\varrho\ \mathrm{d}x\right]_{t=\tau}^{t=0} \le c(D)\varepsilon,$$

where the last inequality is guaranteed by the estimate (3.5) and the fact that $\tilde\varrho$ is independent of t.

Consequently, the relation (4.4) can be written as

$$\mathcal{E}\left(\varrho,\mathbf{u}\ \Big|\ r,\mathbf{U}\right)(\tau) \le c(D)\left(\int_\Omega\left[|\mathbf{u}_{0,\varepsilon} - [\mathbf{v}_{0,h},0]|^2 + |\varrho^{(1)}_{0,\varepsilon} - s_0|^2\right]\ \mathrm{d}x + \nu + \varepsilon\right) \tag{4.9}$$
$$+\int_0^\tau\int_\Omega\varrho\left(\partial_t\mathbf{v} + \mathbf{u}_\varepsilon\cdot\nabla_x\mathbf{v}\right)\cdot(\mathbf{v}-\mathbf{u})\ \mathrm{d}x\ \mathrm{d}t + \frac{1}{\varepsilon}\int_0^\tau\int_\Omega(\tilde\varrho + \varepsilon s - \varrho)\frac{p'(\tilde\varrho)}{\tilde\varrho}\partial_t s\ \mathrm{d}x\ \mathrm{d}t.$$

4.4 Remaining terms - conclusion

We write the last two integrals in (4.9) as

$$\int_0^\tau\int_\Omega\varrho\left(\partial_t\mathbf{v} + \mathbf{u}\cdot\nabla_x\mathbf{v}\right)\cdot(\mathbf{v}-\mathbf{u})\ \mathrm{d}x\ \mathrm{d}t + \frac{1}{\varepsilon}\int_0^\tau\int_\Omega(\tilde\varrho+\varepsilon s - \varrho)\frac{p'(\tilde\varrho)}{\tilde\varrho}\partial_t s\ \mathrm{d}x\ \mathrm{d}t \tag{4.10}$$
$$\le \int_0^\tau\int_\Omega\varrho|\mathbf{u}-\mathbf{v}|^2|\nabla_x\mathbf{v}|\ \mathrm{d}x\ \mathrm{d}t + I_1 + I_2,$$

where

$$I_1 \equiv \int_0^\tau\int_\Omega\varrho\left(\partial_t\mathbf{v} + \mathbf{v}\cdot\nabla_x\mathbf{v}\right)\cdot\mathbf{v}\ \mathrm{d}x\ \mathrm{d}t + \int_0^\tau\int_\Omega s\frac{p'(\tilde\varrho)}{\tilde\varrho}\partial_t s\ \mathrm{d}x\ \mathrm{d}t,$$

$$-I_2 \equiv \int_0^\tau \int_\Omega \varrho \left(\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla_x \mathbf{v} \right) \cdot \mathbf{u} \ \mathrm{d}x \ \mathrm{d}t + \int_0^\tau \int_\Omega \frac{\varrho - \tilde\varrho}{\varepsilon} \frac{p'(\tilde\varrho)}{\tilde\varrho} \partial_t s \ \mathrm{d}x.$$

Next, we observe that

$$I_1 \leq \int_0^\tau \int_\Omega \tilde\varrho \left(\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla_x \mathbf{v} \right) \cdot \mathbf{v} \ \mathrm{d}x \ \mathrm{d}t + \int_0^\tau \int_\Omega s \frac{p'(\tilde\varrho)}{\tilde\varrho} \partial_t s \ \mathrm{d}x \ \mathrm{d}t + c(D)\varepsilon \tag{4.11}$$

$$= \int_0^\tau \left\{ \frac{1}{2} \langle \tilde\varrho \rangle \frac{\mathrm{d}}{\mathrm{d}t} \int_{R^2} |\mathbf{v}_h|^2 \ \mathrm{d}x_h + \frac{1}{2} \left\langle \frac{\tilde\varrho}{p'(\tilde\varrho)} \right\rangle \frac{\mathrm{d}}{\mathrm{d}t} \int_{R^2} q^2 \ \mathrm{d}x_h \right\} \ \mathrm{d}t + c(D)\varepsilon = c(D)\varepsilon,$$

where we have use the energy balance for (1.12).

Furthermore, using the weak formulation of the momentum equation (2.4), we deduce that

$$\left| \int_0^\tau \int_\Omega \left[(\varrho \mathbf{u} \times \omega) \cdot \Psi - \frac{1}{\varepsilon} p(\varrho) \mathrm{div}_x \Psi - \frac{1}{\varepsilon} \varrho \nabla_x G \cdot \Psi \right] \ \mathrm{d}x \ \mathrm{d}t \right| \leq c(D)\varepsilon \|\Psi\|_{W^{1,\infty} \cap W^{1,2}((0,T)\times\Omega;R^3)}. \tag{4.12}$$

Next, we take

$$\Psi = \left(\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} \right) \times \omega$$

and verify by direct calculation using (1.12) that

$$\mathrm{div}_x \Psi = - \left\langle \frac{\tilde\varrho}{p'(\tilde\varrho)} \right\rangle \frac{1}{< \tilde\varrho >} \partial_t q, \quad \text{and} \quad (\varrho \mathbf{u} \times \omega) \cdot \Psi = \varrho \mathbf{u} \cdot (\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla_x \mathbf{v}). \tag{4.13}$$

Now, employing in (4.12) the definition of I_2 and (1.12), one gets

$$\left| - I_2 + \int_0^\tau \int_\Omega \Big[- \frac{\varrho - \tilde\varrho}{\varepsilon} \partial_t q - \frac{1}{\varepsilon} \Big(p(\varrho) - p'(\tilde\varrho)(\varrho - \tilde\varrho) + p(\tilde\varrho) \Big) \mathrm{div}_x \Psi - p'(\tilde\varrho) \frac{\varrho - \tilde\varrho}{\varepsilon} \mathrm{div}_x \Psi \ \mathrm{d}x \right| \leq c(D)\varepsilon,$$

where we have used $\Psi = [\Psi_h(t, x_h), 0]$, $G = G(x_3)$, $\tilde\varrho = \tilde\varrho(x_3)$. Consequently, expressing $\mathrm{div}_x \Psi$ in the last term at the left hand side by using (4.13), we infer that

$$|I_2| \leq \left| \int_0^\tau \int_\Omega \left[\frac{\tilde\varrho}{\langle \tilde\varrho \rangle} \left\langle \frac{\tilde\varrho}{p'(\tilde\varrho)} \right\rangle - \frac{\tilde\varrho}{p'(\tilde\varrho)} \right] \frac{p'(\tilde\varrho)}{\tilde\varrho} \frac{\varrho - \tilde\varrho}{\varepsilon} \partial_t q \ \mathrm{d}x \ \mathrm{d}t \right| + c(D)\varepsilon. \tag{4.14}$$

Finally, we use again (2.4) to obtain

$$\left| \int_0^\tau \int_\Omega \frac{p'(\tilde\varrho)}{\tilde\varrho} \left(\frac{\varrho_\varepsilon - \tilde\varrho}{\varepsilon} \right) \mathrm{div}_x(\tilde\varrho \Phi) \ \mathrm{d}x \ \mathrm{d}t \right| \leq \varepsilon c(D) \|\Phi\|_{W^{1,\infty} \cap W^{1,2}((0,T)\times\Omega)} \text{ whenever } \Phi = [0,0,\Phi]. \tag{4.15}$$

Thus, writing

$$\left[\frac{\tilde\varrho}{\langle \tilde\varrho \rangle} \left\langle \frac{\tilde\varrho}{p'(\tilde\varrho)} \right\rangle - \frac{\tilde\varrho}{p'(\tilde\varrho)} \right] \partial_t q = \frac{\partial}{\partial x_3} \left(\int_0^{x_3} \left[\frac{\tilde\varrho}{\langle \tilde\varrho \rangle} \left\langle \frac{\tilde\varrho}{p'(\tilde\varrho)} \right\rangle - \frac{\tilde\varrho}{p'(\tilde\varrho)} \right] \ \mathrm{d}z \ \partial_t q \right),$$

we may combine (4.14), (4.15) to obtain the desired estimate

$$|I_2| \leq c(D)\varepsilon. \tag{4.16}$$

Going back to (4.9) and exploiting (4.11), (4.16) we conclude that

$$\mathcal{E}\left(\varrho, \mathbf{u} \;\middle|\; r, \mathbf{U}\right)(\tau) \leq c(D,T)\left(\int_\Omega \left[|\mathbf{u}_{0,\varepsilon} - [\mathbf{v}_{0,h}, 0]|^2 + |\varrho^{(1)}_{0,\varepsilon} - s_0|^2\right] \, \mathrm{d}x + \nu + \varepsilon\right) \tag{4.17}$$

for any $\tau \in [0,T]$, which is nothing other than (2.9). Theorem 2.1 has been proved.

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