Polynomial-degree-robust a posteriori estimates in a unified setting for conforming, nonconforming, discontinuous Galerkin, and mixed discretizations*

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December 20, 2013

Abstract

We present equilibrated flux a posteriori error estimates in a unified setting for conforming, nonconforming, discontinuous Galerkin, and mixed finite element discretizations of the two-dimensional Poisson problem. Relying on the equilibration by the mixed finite element solution of patchwise Neumann problems, the estimates are guaranteed, locally computable, locally efficient, and robust with respect to polynomial degree. Maximal local overestimation is guaranteed as well.

Key words: a posteriori estimate, equilibrated flux, unified framework, robustness, polynomial degree, conforming finite element method, nonconforming finite element method, discontinuous Galerkin method, mixed finite element method

1 Introduction

A posteriori error estimates in a conforming finite element setting have already received a large attention. In particular, following the concept of Prager and Synge [60], cf. also Synge [67] and Aubin and Burchard [14], and invoking *fluxes* in the $\mathbf{H}(\mathrm{div}, \Omega)$ space, *guaranteed upper bounds* on the error can be obtained. A *local procedure* for the normal face fluxes reconstruction is a part of the equilibrated residual method of Ladevèze [50], Ladevèze and Leguillon [51], and Ainsworth and Oden [8, 9]. The method itself may, however, require the solution of infinite-dimensional element problems. Cheap *local equilibrations* leading to a fully computable guaranteed upper bound have been obtained in Destuynder and Métivet [36]. Later, *mixed finite elements* for the solution of local *Neumann problems* posed over *patches* of (sub)elements, where one minimizes locally the estimator contributions, were proposed, see Luce and Wohlmuth [52], Braess and Schöberl [20] and [72, 31, 74]. As a matter of fact, lifting the equilibrated normal face fluxes of the equilibrated residual method to $\mathbf{H}(\mathrm{div}, \Omega)$ immediately yields equilibrated fluxes, cf. Nicaise *et al.* [54]. Many closely related approaches exist, cf. Parés *et al.* [56, 57]; one may also cite the general numerical-method-independent framework of Repin [62, 63], where, however, local equilibration is not used, and the procedure can become costly if some global minimization is to be performed. Computational comparisons of some of these approaches in the lowest-order case can be found in Carstensen and Merdon [29].

The theory in the nonconforming finite element setting appears to be less developed. Here a key notion is that of *potentials* in the $H_0^1(\Omega)$ space. First contributions are those of Agouzal [2] and Dari *et al.* [34], whereas a guaranteed error upper bound in the lowest-order Crouzeix–Raviart case can be obtained along the lines of Destuynder and Métivet [35], see Ainsworth [3], Kim [47, 48], or [71]. Different equilibrations exist and tight links hold between them, see [43]. Higher-order methods have been treated in Ainsworth and Rankin [10] and a survey and a computational comparison in the lowest-order case can be found in Carstensen and Merdon [30].

*This work was supported by the ERC-CZ project MORE "MOdelling REvisited + MOdel REduction" LL1202.
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For the discontinuous Galerkin method, first guaranteed upper bounds by locally equilibrated fluxes have been obtained in Ainsworth [4], Kim [47, 48], Cochez-Dhondt and Nicaise [33], Ainsworth and Rankin [11], and [38, 40, 39], see also the references therein. Similar results for mixed finite elements can be found in Kim [47, 49], Ainsworth [5], Ainsworth and Ma [7], and [71, 73].

Recently, *unified frameworks* for different discretization methods have been conceived, see Carstensen *et al.* [23, 28, 27, 24], Ainsworth [6], and, using the equilibrated fluxes, in [41, 46, 42].

One particular and clearly distinctive feature of equilibrated fluxes with respect to other existing approaches, in the conforming finite element setting, is the *robustness* with respect to the approximation *polynomial degree* proven in Braess *et al.* [19]. This stands in particular in contrast to usual residual-based estimators, where such a robustness does not hold, see Melenk and Wohlmuth [53]. The first key ingredient for the proof in [19] are continuous-level problems on patches of elements around vertices featuring the hat functions, similar to those considered already in Carstensen and Funken [25]. The second key ingredient is then the polynomial-degree-robust stability of mixed finite elements of [19, Theorem 7].

In the present paper, we *unify* the potentials and equilibrated fluxes approach for most standard discretization schemes, including conforming, nonconforming, symmetric and incomplete discontinuous Galerkin, and mixed finite elements. The construction of the estimators becomes *method independent*, being close to that of Destuynder and Métivet [36] and coinciding with that of Braess and Schöberl [20] for fluxes in the conforming case, while being closely related to that of Carstensen and Merdon [30] for potentials in the nonconforming case. In the discontinuous Galerkin and mixed finite element cases, such an approach appears to be new. The potentials and fluxes are actually constructed by the same patchwise problems with different right-hand sides in the present two-dimensional setting. Most importantly, we prove the *polynomial-degree robustness* in this unified setting comprising all the discussed discretization schemes. Moreover, we can also guarantee a *maximal overestimation factor*, a feature which might be important in optimal convergence proofs.

The paper is organized as follows: The setting is described in Section 2. The main results together with their proofs are collected in Section 3. Applications to most standard numerical methods are showcased in Section 4. Concluding remarks are presented in Section 5.

2 Setting

We start by introducing the necessary continuous and discrete functional spaces and the model problem.

2.1 Sobolev spaces

Let $\Omega \subset \mathbb{R}^2$ be a polygonal domain (open, bounded, and connected set). We denote by $H^1(\Omega)$ the Sobolev space of $L^2(\Omega)$ functions with weak gradients in $[L^2(\Omega)]^2$ and by $H^1_0(\Omega)$ its subspace of functions with zero trace on the boundary $\partial\Omega$ of Ω, cf. Adams [1]. $\mathbf{H}(\mathrm{div},\Omega)$ stands for the space of $[L^2(\Omega)]^2$ functions with weak divergences in $L^2(\Omega)$, cf. Brezzi and Fortin [22] or Roberts and Thomas [64]. The notations ∇ and $\nabla\cdot$ will be used respectively for the weak gradient and divergence. Introduce the notation $\mathrm{R}_{\frac{\pi}{2}} := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ for the matrix representing the rotation by $\frac{\pi}{2}$; then $\mathrm{R}_{\frac{\pi}{2}}\nabla$ stands for the weak curl, i.e., the rotated gradient: for $v \in H^1(\Omega), \mathrm{R}_{\frac{\pi}{2}}\nabla v = (-\partial_y v, \partial_x v)$. For a subdomain ω of Ω, we denote by $(\cdot,\cdot)_\omega$ the $L^2(\omega)$-inner product, by $\|\cdot\|_\omega$ the associated norm (we omit the index ω when $\omega = \Omega$), and by $|\omega|$ the Lebesgue measure of ω. For $\omega \subset \mathbb{R}^1$, we let $\langle\cdot,\cdot\rangle_\omega$ stand for the 1-dimensional $L^2(\omega)$-inner product or for the appropriate duality pairing on ω.

2.2 Meshes

We shall work in this paper with partitions $\mathcal{T}_h$ of Ω which consist either of closed triangles or of closed rectangles K such that $\overline{\Omega} = \bigcup_{K\in\mathcal{T}_h} K$. We suppose that $\mathcal{T}_h$ are matching, i.e., such that for two distinct elements, their intersection is either an empty set or a common edge or a common vertex. For any $K \in \mathcal{T}_h$, $\mathbf{n}_K$ stands for the outward unit normal vector to K and h_K denotes the diameter of K. The edges of the mesh form the set $\mathcal{E}_h$ divided into interior edges $\mathcal{E}_h^{\mathrm{int}}$ and boundary edges $\mathcal{E}_h^{\mathrm{ext}}$. A generic edge is denoted by e and its diameter by h_e. For any $e \in \mathcal{E}_h$, $\mathbf{n}_e$ stands for the unit normal vector to e; the orientation is arbitrary but fixed for $e \in \mathcal{E}_h^{\mathrm{int}}$ and points outwards of Ω for $e \in \mathcal{E}_h^{\mathrm{ext}}$. The set of vertices will be denoted by $\mathcal{V}_h$; it is decomposed into interior vertices $\mathcal{V}_h^{\mathrm{int}}$ and vertices located on the boundary $\mathcal{V}_h^{\mathrm{ext}}$. For a vertex $\mathbf{a} \in \mathcal{V}_h$, $\mathcal{T}_{\mathbf{a}}$ denotes the patch of the elements of $\mathcal{T}_h$ which share $\mathbf{a}$ and $\omega_{\mathbf{a}}$ the corresponding open subdomain. For $K \in \mathcal{T}_h$, $\mathcal{V}_K$ denotes the set of vertices of K. Later on, we will need the shape-regularity assumption requesting the existence of a constant $\kappa_{\mathcal{T}} > 0$ such that $\max_{K\in\mathcal{T}_h} h_K/\varrho_K \le \kappa_{\mathcal{T}}$ for all triangulations $\mathcal{T}_h$, with ϱ_K being the diameter of the largest ball inscribed in K. We will also invoke the average operator $\{\!\{\cdot\}\!\}$ yielding the mean value of the traces from adjacent mesh elements on inner edges and the actual trace on boundary edges; similarly, the jump operator $[\![\cdot]\!]$ yields the difference evaluated along $\mathbf{n}_e$ of the traces of the argument from the two mesh elements that share $e \in \mathcal{E}_h^{\mathrm{int}}$ and the actual trace for $e \in \mathcal{E}_h^{\mathrm{ext}}$.

2.3 Finite element spaces

Let $p \ge 1$. We use $\mathbb{R}_p(\mathcal{T}_h) := \mathbb{P}_p(\mathcal{T}_h)$ (respectively, $\mathbb{Q}_p(\mathcal{T}_h)$) to denote piecewise polynomials of total degree at most p on triangles (respectively, at most p in each variable on rectangles), cf. Ciarlet [32]. Let $\mathbf{V}_h \times Q_h$ be finite-dimensional mesh-related subspaces of $\mathbf{H}(\mathrm{div},\Omega) \times L^2(\Omega)$. We consider the Raviart–Thomas–Nédélec (RTN) and the Brezzi–Douglas–Marini/Brezzi–Douglas–Durán–Fortin (BDM/BDDF) families, see Brezzi and Fortin [22], Roberts and Thomas [64], or [73]. At some places, we will also use mesh-related broken Sobolev spaces $H^1(\mathcal{T}_h) := \{v \in L^2(\Omega);\, v|_K \in H^1(K)$ for all $K \in \mathcal{T}_h\}$ and $\mathbf{H}(\mathrm{div},\mathcal{T}_h) := \{\mathbf{v} \in [L^2(\Omega)]^2;\, \mathbf{v}|_K \in \mathbf{H}(\mathrm{div},K)$ for all $K \in \mathcal{T}_h\}$. Then ∇ stands for the broken (elementwise) weak gradient, $\nabla\cdot$ for the broken (elementwise) weak divergence, and $\mathrm{R}_{\frac{\pi}{2}}\nabla$ for the broken (elementwise) weak curl. For a vertex $\mathbf{a} \in \mathcal{V}_h$, $\psi_{\mathbf{a}}$ stands for the "hat" function from $\mathbb{R}_1(\mathcal{T}_h) \cap H^1(\Omega)$ which takes value 1 at the vertex $\mathbf{a}$ and zero at the other vertices.

2.4 The model problem

We study in this paper the Poisson problem for the Laplace equation: for $f \in L^2(\Omega)$, find u such that

$$-\Delta u = f \qquad \text{in } \Omega, \tag{2.1a}$$
$$u = 0 \qquad \text{on } \partial\Omega. \tag{2.1b}$$

The weak formulation consists in finding $u \in H_0^1(\Omega)$ such that

$$(\nabla u, \nabla v) = (f, v) \qquad \forall v \in H_0^1(\Omega). \tag{2.2}$$

Existence and uniqueness of the solution u to (2.2) follow from the Riesz representation theorem. We term the scalar-valued function u the *potential* and the vector-valued function $\boldsymbol{\sigma} := -\nabla u$ the *flux*. Owing to (2.2), $\boldsymbol{\sigma} \in \mathbf{H}(\mathrm{div},\Omega)$ with $\nabla\cdot\boldsymbol{\sigma} = f$.

Remark 2.1 (Setting). *The present setting is chosen for clearness of exposition. Extensions to inhomogeneous Dirichlet and Neumann boundary conditions, more general meshes, meshes with hanging nodes, and approximations with varying polynomial degree are possible modulo some technicalities.*

3 Main results

We present in this section our main results. The guaranteed error upper bound is presented in Section 3.1 and a lower bound robust with respect to the polynomial degree is stated in Section 3.2. Maximal overestimation is investigated in Section 3.3.

3.1 Guaranteed reliability

Let u_h denote the given approximate solution to problem (2.2). Throughout a large part of this section, we only need $u_h \in H^1(\mathcal{T}_h)$.

3.1.1 Equilibrated flux and potential reconstructions

At the approximation level, $u_h \notin H_0^1(\Omega)$, $-\nabla u_h \notin \mathbf{H}(\mathrm{div},\Omega)$, or $\nabla\cdot(-\nabla u_h) \neq f$ typically occurs. We begin by restoring/mimicking these three characterizing properties of the weak solution:

Definition 3.1 (Equilibrated flux reconstruction)**.** *We will call an* equilibrated flux reconstruction *any function* $\boldsymbol{\sigma}_h$ *constructed from* u_h *which satisfies*

$$\boldsymbol{\sigma}_h \in \mathbf{H}(\mathrm{div},\Omega), \tag{3.1a}$$

$$(\nabla\cdot\boldsymbol{\sigma}_h, 1)_K = (f,1)_K \qquad \forall K \in \mathcal{T}_h. \tag{3.1b}$$

Definition 3.2 (Potential reconstruction)**.** *We will call a* potential reconstruction *any function* s_h *constructed from* u_h *which satisfies*

$$s_h \in H_0^1(\Omega). \tag{3.2}$$

3.1.2 Guaranteed reliability

The error upper bound, even in the general setting $u_h \in H^1(\mathcal{T}_h)$, is straightforward:

Theorem 3.3 (A guaranteed a posteriori error estimate)**.** *Let u be the weak solution given by (2.2) and let $u_h \in H^1(\mathcal{T}_h)$ be arbitrary. Let $\boldsymbol{\sigma}_h$ and s_h be respectively the equilibrated flux and potential reconstructions of Definitions 3.1 and 3.2. Then*

$$\|\nabla(u-u_h)\|^2 \le \eta^2 := \sum_{K\in\mathcal{T}_h}\left(\|\nabla u_h + \boldsymbol{\sigma}_h\|_K + \frac{h_K}{\pi}\|f-\nabla\cdot\boldsymbol{\sigma}_h\|_K\right)^2 + \sum_{K\in\mathcal{T}_h}\|\nabla(u_h-s_h)\|_K^2.$$

Proof. The proof is straightforward along the lines of [60, 50, 34, 55, 62, 52, 3, 47, 71, 38, 63, 20, 40, 29, 30]. We sketch it for self-completeness. As in [55, 47], define the function $s \in H_0^1(\Omega)$ by

$$(\nabla s, \nabla v) = (\nabla u_h, \nabla v) \qquad \forall v \in H_0^1(\Omega). \tag{3.3}$$

Its existence and uniqueness follow from the Riesz representation theorem. From this projection-type construction result the Pythagorean equality

$$\|\nabla(u-u_h)\|^2 = \|\nabla(u-s)\|^2 + \|\nabla(s-u_h)\|^2 \tag{3.4}$$

and the minimization property

$$\|\nabla(s-u_h)\|^2 = \min_{v\in H_0^1(\Omega)}\|\nabla(v-u_h)\|^2. \tag{3.5}$$

As for the first term in (3.4), using that $u-s \in H_0^1(\Omega)$, (3.3) yields

$$\|\nabla(u-s)\| = \sup_{v\in H_0^1(\Omega);\,\|\nabla v\|=1}(\nabla(u-s),\nabla v) = \sup_{v\in H_0^1(\Omega);\,\|\nabla v\|=1}(\nabla(u-u_h),\nabla v).$$

Let $v \in H_0^1(\Omega)$ be fixed. The characterization (2.2) of the weak solution and adding and subtracting $(\boldsymbol{\sigma}_h,\nabla v)$ lead to

$$(\nabla(u-u_h),\nabla v) = (f-\nabla\cdot\boldsymbol{\sigma}_h, v) - (\nabla u_h + \boldsymbol{\sigma}_h, \nabla v),$$

where we have also employed the Green theorem $(\boldsymbol{\sigma}_h,\nabla v) = -(\nabla\cdot\boldsymbol{\sigma}_h, v)$. The Cauchy–Schwarz inequality yields

$$-(\nabla u_h + \boldsymbol{\sigma}_h, \nabla v) \le \sum_{K\in\mathcal{T}_h}\|\nabla u_h + \boldsymbol{\sigma}_h\|_K\|\nabla v\|_K,$$

whereas the approximate equilibrium property (3.1b), the Poincaré inequality

$$\|v - v_K\|_K \le C_{\mathrm{P},K} h_K \|\nabla v\|_K \qquad \forall v \in H^1(K) \tag{3.6}$$

with v_K the mean value of v on K and $C_{\mathrm{P},K} = 1/\pi$ thanks to the convexity of the mesh elements K (see Payne and Weinberger [58] and Bebendorf [18]), and the Cauchy–Schwarz inequality yield

$$(f-\nabla\cdot\boldsymbol{\sigma}_h, v) = \sum_{K\in\mathcal{T}_h}(f-\nabla\cdot\boldsymbol{\sigma}_h, v-v_K)_K \le \sum_{K\in\mathcal{T}_h}\frac{h_K}{\pi}\|f-\nabla\cdot\boldsymbol{\sigma}_h\|_K\|\nabla v\|_K. \tag{3.7}$$

Combining these results with the Cauchy–Schwarz inequality and since any $s_h \in H_0^1(\Omega)$ bounds the left-hand side of (3.5), we infer the assertion. □

3.1.3 Mixed finite element solution of Neumann problems on patches using the partition of unity

This section describes a practical way to obtain the equilibrated flux and potential reconstructions in the sense of Definitions 3.1 and 3.2. For the flux reconstruction, we rewrite equivalently the technique of [20], see also [36], proceeding as in [42]. The potential reconstruction is close to that of [30, Section 6.3]. For this section, we assume that u_h is a piecewise polynomial, typically $u_h \in \mathbb{R}_p(\mathcal{T}_h)$ for some fixed $p \geq 1$. The equilibration goes over patches of elements $\omega_{\mathbf{a}}$ sharing the generic vertex $\mathbf{a} \in \mathcal{V}_h$. Let $\mathbf{V}_h(\omega_{\mathbf{a}}) \times Q_h(\omega_{\mathbf{a}})$ be any of the mixed finite element spaces discussed in Section 2.3. We suppose their degree is such that the piecewise polynomial vectors $\boldsymbol{\tau}_h^{\mathbf{a}}$ defined below by (3.12a), (3.19a), or (3.22a) are such that

$$\boldsymbol{\tau}_h^{\mathbf{a}}|_K \in \mathbf{V}_h(K) \qquad \forall K \in \mathcal{T}_{\mathbf{a}}. \tag{3.8}$$

For example, on triangular meshes with $u_h \in \mathbb{P}_p(\mathcal{T}_h)$, the usual choice for $\mathbf{V}_h(\omega_{\mathbf{a}}) \times Q_h(\omega_{\mathbf{a}})$ is $\mathbf{RTN}_p(\omega_{\mathbf{a}}) \times \mathbb{P}_p(\omega_{\mathbf{a}})$.

Definition 3.4 (Construction of $\boldsymbol{\sigma}_h$)**.** *Let u_h satisfy the Galerkin orthogonality*

$$(\nabla u_h, \nabla \psi_{\mathbf{a}})_{\omega_{\mathbf{a}}} = (f, \psi_{\mathbf{a}})_{\omega_{\mathbf{a}}} \qquad \forall \mathbf{a} \in \mathcal{V}_h^{\mathrm{int}}. \tag{3.9}$$

For each $\mathbf{a} \in \mathcal{V}_h$, prescribe $\boldsymbol{\varsigma}_h^{\mathbf{a}} \in \mathbf{V}_h^{\mathbf{a}}$ and $\bar{r}_h^{\mathbf{a}} \in Q_h^{\mathbf{a}}$ by solving

$$(\boldsymbol{\varsigma}_h^{\mathbf{a}}, \mathbf{v}_h)_{\omega_{\mathbf{a}}} - (\bar{r}_h^{\mathbf{a}}, \nabla{\cdot}\mathbf{v}_h)_{\omega_{\mathbf{a}}} = -(\boldsymbol{\tau}_h^{\mathbf{a}}, \mathbf{v}_h)_{\omega_{\mathbf{a}}} \qquad \forall \mathbf{v}_h \in \mathbf{V}_h^{\mathbf{a}}, \tag{3.10a}$$

$$(\nabla{\cdot}\boldsymbol{\varsigma}_h^{\mathbf{a}}, q_h)_{\omega_{\mathbf{a}}} = (g^{\mathbf{a}}, q_h)_{\omega_{\mathbf{a}}} \qquad \forall q_h \in Q_h^{\mathbf{a}}, \tag{3.10b}$$

with the spaces

$$\left.\begin{aligned}\mathbf{V}_h^{\mathbf{a}} &:= \{\mathbf{v}_h \in \mathbf{V}_h(\omega_{\mathbf{a}});\ \mathbf{v}_h{\cdot}\mathbf{n}_{\omega_{\mathbf{a}}} = 0 \ \text{on}\ \partial\omega_{\mathbf{a}}\},\\ Q_h^{\mathbf{a}} &:= \{q_h \in Q_h(\omega_{\mathbf{a}});\ (q_h, 1)_{\omega_{\mathbf{a}}} = 0\},\end{aligned}\right. \qquad \mathbf{a} \in \mathcal{V}_h^{\mathrm{int}}, \tag{3.11a}$$

$$\left.\begin{aligned}\mathbf{V}_h^{\mathbf{a}} &:= \{\mathbf{v}_h \in \mathbf{V}_h(\omega_{\mathbf{a}});\ \mathbf{v}_h{\cdot}\mathbf{n}_{\omega_{\mathbf{a}}} = 0 \ \text{on}\ \partial\omega_{\mathbf{a}} \setminus \partial\Omega\},\\ Q_h^{\mathbf{a}} &:= Q_h(\omega_{\mathbf{a}}),\end{aligned}\right. \qquad \mathbf{a} \in \mathcal{V}_h^{\mathrm{ext}}, \tag{3.11b}$$

and the right-hand sides

$$\boldsymbol{\tau}_h^{\mathbf{a}} := \psi_{\mathbf{a}} \nabla u_h, \tag{3.12a}$$

$$g^{\mathbf{a}} := \psi_{\mathbf{a}} f - \nabla\psi_{\mathbf{a}}{\cdot}\nabla u_h. \tag{3.12b}$$

Then, set

$$\boldsymbol{\sigma}_h := \sum_{\mathbf{a}\in\mathcal{V}_h} \boldsymbol{\varsigma}_h^{\mathbf{a}}. \tag{3.13}$$

In (3.11), a homogeneous Neumann (or no-flux) boundary condition on the whole boundary of the patch $\omega_{\mathbf{a}}$ together with mean value zero is imposed for interior vertices, whereas the no-flux condition is only imposed on that part of the boundary of $\omega_{\mathbf{a}}$ which lies in the interior of Ω for boundary vertices. Also note that by (3.9), $(g^{\mathbf{a}}, 1)_{\omega_{\mathbf{a}}} = 0$ for interior vertices $\mathbf{a}$, which is the necessary Neumann compatibility condition. Existence and uniqueness of the solution to (3.10) is standard, see [22, 64, 73]. It is also clear that $\boldsymbol{\sigma}_h \in \mathbf{H}(\mathrm{div}, \Omega)$ as all $\boldsymbol{\varsigma}_h^{\mathbf{a}}$ belong to $\mathbf{H}(\mathrm{div}, \Omega)$. We actually have more than merely the second requirement (3.1b) of Definition 3.1:

Lemma 3.5 (Divergence of $\boldsymbol{\sigma}_h$)**.** *Let $\boldsymbol{\sigma}_h$ be given by Definition 3.4. Then*

$$(f - \nabla{\cdot}\boldsymbol{\sigma}_h, v_h)_K = 0 \qquad \forall v_h \in Q_h(K) \qquad \forall K \in \mathcal{T}_h. \tag{3.14}$$

Proof. The facts that $\boldsymbol{\varsigma}_h^{\mathbf{a}}{\cdot}\mathbf{n}_{\omega_{\mathbf{a}}} = 0$ on $\partial\omega_{\mathbf{a}}$ and $(g^{\mathbf{a}}, 1)_{\omega_{\mathbf{a}}} = 0$ for $\mathbf{a} \in \mathcal{V}_h^{\mathrm{int}}$ enable to take the constants as test functions in (3.10b) and thus (3.10b) actually holds for all functions from $Q_h(\omega_{\mathbf{a}})$. As the polynomials in $Q_h(\omega_{\mathbf{a}})$ are discontinuous, we infer that any $q_h \in Q_h(K)$ for a given $K \in \mathcal{T}_{\mathbf{a}}$ can be taken as a test function in (3.10b). Let $K \in \mathcal{T}_h$ and $v_h \in Q_h(K)$ be fixed. Employing that

$$\boldsymbol{\sigma}_h|_K = \sum_{\mathbf{a}\in\mathcal{V}_K} \boldsymbol{\varsigma}_h^{\mathbf{a}}|_K \tag{3.15}$$

and (3.10b) with (3.12b),

$$(f - \nabla\cdot\boldsymbol{\sigma}_h, v_h)_K = \sum_{\mathbf{a}\in\mathcal{V}_K} (\psi_{\mathbf{a}} f - \nabla\cdot\boldsymbol{\varsigma}_h^{\mathbf{a}}, v_h)_K = \sum_{\mathbf{a}\in\mathcal{V}_K} (\nabla\psi_{\mathbf{a}}\cdot\nabla u_h, v_h)_K = 0,$$

where we have also used the partition of unity

$$\sum_{\mathbf{a}\in\mathcal{V}_K} \psi_{\mathbf{a}}|_K = 1|_K. \tag{3.16}$$

□

Remark 3.6 (Data oscillation)**.** *The orthogonality* (3.14) *together with the mixed finite element spaces property* $\nabla\cdot\mathbf{V}_h = Q_h$ *imply that, for any* $K \in \mathcal{T}_h$*,*

$$\frac{h_K}{\pi}\|f - \nabla\cdot\boldsymbol{\sigma}_h\|_K = \frac{h_K}{\pi}\|f - \Pi_{Q_h} f\|_K$$

is actually the data oscillation term, where Π_{Q_h} *is the* $L^2(\Omega)$*-orthogonal projection onto* Q_h*. This typically yields convergence by two orders of magnitude faster than the energy error* $\|\nabla(u-u_h)\|$ *for flux reconstruction in the RTN spaces and by one order faster for the BDM/BDDF spaces.*

We now turn to the potential reconstruction s_h, necessary when $u_h \notin H_0^1(\Omega)$:

Definition 3.7 (Construction of s_h)**.** *For each* $\mathbf{a} \in \mathcal{V}_h$*, prescribe* $\boldsymbol{\varsigma}_h^{\mathbf{a}} \in \mathbf{V}_h^{\mathbf{a}}$ *and* $\bar{r}_h^{\mathbf{a}} \in Q_h^{\mathbf{a}}$ *by solving*

$$(\boldsymbol{\varsigma}_h^{\mathbf{a}}, \mathbf{v}_h)_{\omega_{\mathbf{a}}} - (\bar{r}_h^{\mathbf{a}}, \nabla\cdot\mathbf{v}_h)_{\omega_{\mathbf{a}}} = -(\boldsymbol{\tau}_h^{\mathbf{a}}, \mathbf{v}_h)_{\omega_{\mathbf{a}}} \qquad \forall \mathbf{v}_h \in \mathbf{V}_h^{\mathbf{a}}, \tag{3.17a}$$

$$(\nabla\cdot\boldsymbol{\varsigma}_h^{\mathbf{a}}, q_h)_{\omega_{\mathbf{a}}} = (g^{\mathbf{a}}, q_h)_{\omega_{\mathbf{a}}} \qquad \forall q_h \in Q_h^{\mathbf{a}}, \tag{3.17b}$$

with the spaces

$$\begin{aligned} \mathbf{V}_h^{\mathbf{a}} &:= \{\mathbf{v}_h \in \mathbf{V}_h(\omega_{\mathbf{a}});\ \mathbf{v}_h\cdot\mathbf{n}_{\omega_{\mathbf{a}}} = 0 \ \text{on}\ \partial\omega_{\mathbf{a}}\}, \\ Q_h^{\mathbf{a}} &:= \{q_h \in Q_h(\omega_{\mathbf{a}});\ (q_h, 1)_{\omega_{\mathbf{a}}} = 0\}, \end{aligned} \tag{3.18}$$

and the right-hand sides

$$\boldsymbol{\tau}_h^{\mathbf{a}} := \mathrm{R}_{\frac{\pi}{2}} \nabla(\psi_{\mathbf{a}} u_h), \tag{3.19a}$$

$$g^{\mathbf{a}} := 0. \tag{3.19b}$$

Then, set

$$-\mathrm{R}_{\frac{\pi}{2}} \nabla s_h^{\mathbf{a}} = \boldsymbol{\varsigma}_h^{\mathbf{a}}, \tag{3.20a}$$

$$s_h^{\mathbf{a}} = 0 \ \text{on}\ \partial\omega_{\mathbf{a}}, \tag{3.20b}$$

$$s_h := \sum_{\mathbf{a}\in\mathcal{V}_h} s_h^{\mathbf{a}}. \tag{3.20c}$$

The local mixed finite element problem (3.17) is the same as that of Definition 3.4; only the spaces $\mathbf{V}_h^{\mathbf{a}}$ and $Q_h^{\mathbf{a}}$ differ for boundary vertices, and the right-hand sides $\boldsymbol{\tau}_h^{\mathbf{a}}$ and $g^{\mathbf{a}}$ differ for all vertices. Existence and uniqueness of the solution to (3.17) is thus again granted. Moreover, the potential reconstruction from (3.20) is meaningful. Indeed. The fact that $\boldsymbol{\varsigma}_h^{\mathbf{a}}\cdot\mathbf{n}_{\omega_{\mathbf{a}}} = 0$ on $\partial\omega_{\mathbf{a}}$ and (3.19b) enable to take all test functions from $Q_h(\omega_{\mathbf{a}})$ in (3.17b). Thus $\nabla\cdot\boldsymbol{\varsigma}_h^{\mathbf{a}} = 0$ on each $K \in \mathcal{T}_{\mathbf{a}}$ and, consequently, there exists a piecewise polynomial $s_h^{\mathbf{a}}$ satisfying (3.20a) on each $K \in \mathcal{T}_{\mathbf{a}}$. The continuity of the normal trace of $\boldsymbol{\varsigma}_h^{\mathbf{a}}$ over the interior edges of $\mathcal{T}_{\mathbf{a}}$ then implies the continuity of the tangential trace of $s_h^{\mathbf{a}}$ over the interior edges of $\mathcal{T}_{\mathbf{a}}$. Finally, the normal trace of $\boldsymbol{\varsigma}_h^{\mathbf{a}}$ being zero on $\partial\omega_{\mathbf{a}}$, $s_h^{\mathbf{a}}$ is constant on $\partial\omega_{\mathbf{a}}$ and we can fix it to zero on $\partial\omega_{\mathbf{a}}$ by (3.20b); altogether, $s_h^{\mathbf{a}}$ is a piecewise polynomial in $H_0^1(\omega_{\mathbf{a}})$ for all $\mathbf{a} \in \mathcal{V}_h$ and thus $s_h \in H_0^1(\Omega)$ by (3.20c). Altogether, we have:

Lemma 3.8 (Properties of s_h)**.** *The potential reconstruction of Definition 3.7 satisfies* (3.2).

Remark 3.9 (Alternative potential reconstruction)**.** *An alternative potential reconstruction, close to that of [30, Section 6.3] is possible under the assumption*

$$(\mathrm{R}_{\frac{\pi}{2}}\nabla u_h, \nabla\psi_{\mathbf{a}})_{\omega_{\mathbf{a}}} = 0 \qquad \forall \mathbf{a} \in \mathcal{V}_h. \tag{3.21}$$

Set

$$\boldsymbol{\tau}_h^{\mathbf{a}} := \psi_{\mathbf{a}} \mathrm{R}_{\frac{\pi}{2}} \nabla u_h, \tag{3.22a}$$

$$g^{\mathbf{a}} := (\mathrm{R}_{\frac{\pi}{2}} \nabla\psi_{\mathbf{a}}) \cdot \nabla u_h, \tag{3.22b}$$

and use (3.17)–(3.18) *together with* $\varsigma_h := \sum_{\mathbf{a}\in\mathcal{V}_h} \varsigma_h^{\mathbf{a}}$. *This yields* $\varsigma_h \in \mathbf{V}_h$ *such that* $\varsigma_h\cdot\mathbf{n}_\Omega = 0$ *on* $\partial\Omega$. *Moreover, proceeding as in Lemma 3.5, one readily checks that* $\nabla\cdot\varsigma_h = 0$. *Thus, there exists a piecewise polynomial* s_h *in* $H_0^1(\Omega)$ *such that* $-\mathrm{R}_{\frac{\pi}{2}}\nabla s_h = \varsigma_h$*, to be taken as the potential reconstruction in Theorem 3.3. In contrast to [30, Section 6.3], but similarly to [30, Section 6.5],* $g^{\mathbf{a}}$ *is not zero but given by* (3.22b)*, which turns out to be essential to prove the local efficiency in Section 3.2 below. The advantage of the construction of Definition 3.7 is that the condition* (3.21) *is not requested at this stage.*

Remark 3.10 (Piecewise polynomial approximation)**.** *Assumption* (3.8) *indicates the practical choice of the degree of the spaces* $\mathbf{V}_h^{\mathbf{a}}$ *and* $Q_h^{\mathbf{a}}$ *for equilibration. It is not really necessary at the present stage where we could still go with* $u_h \in H^1(\mathcal{T}_h)$*; it is only necessary for the mixed finite element stability, see Corollary 3.15 below. Similarly, we will need in Corollary 3.15* $g^{\mathbf{a}} \in Q_h(\omega_{\mathbf{a}})$*, which requires* f *to be a piecewise polynomial for* (3.12b)*. The general case* $f \in L^2(\Omega)$ *makes additionally appear the usual data oscillation terms on the right-hand sides of* (3.38) *and* (3.39)*, cf. Remark 3.6.*

3.2 Polynomial-degree-robust efficiency

We show here that the a posteriori error estimate of Theorem 3.3, with the reconstructions of Definitions 3.4 and 3.7, is also a lower bound for the error $\|\nabla(u-u_h)\|$, up to a generic constant only depending on the shape-regularity parameter $\kappa_{\mathcal{T}}$. Some of our results, but not all, hinge on the additional assumption that

$$\langle [\![u_h]\!], 1\rangle_e = 0 \qquad \forall e \in \mathcal{E}_h, \tag{3.23}$$

i.e., the continuity in mean of the jumps of the approximate solution u_h. Note that (3.23) in particular implies (3.21).

3.2.1 Continuous-level problems with hat functions on patches

In this subsection, we come temporarily back to the most general setting $u_h \in H^1(\mathcal{T}_h)$, see Remark 3.10. The following result has been shown in [25, 19]:

Lemma 3.11 (Continuous efficiency, flux reconstruction)**.** *Let* u *be the weak solution given by* (2.2) *and let* $u_h \in H^1(\mathcal{T}_h)$ *be arbitrary. Let* $\mathbf{a} \in \mathcal{V}_h$ *and let* $r_{\mathbf{a}} \in H_*^1(\omega_{\mathbf{a}})$ *solve*

$$(\nabla r_{\mathbf{a}}, \nabla v)_{\omega_{\mathbf{a}}} = -(\boldsymbol{\tau}_h^{\mathbf{a}}, \nabla v)_{\omega_{\mathbf{a}}} + (g^{\mathbf{a}}, v)_{\omega_{\mathbf{a}}} \qquad \forall v \in H_*^1(\omega_{\mathbf{a}}) \tag{3.24}$$

with the space

$$H_*^1(\omega_{\mathbf{a}}) := \{v \in H^1(\omega_{\mathbf{a}});\, (v,1)_{\omega_{\mathbf{a}}} = 0\}, \qquad \mathbf{a} \in \mathcal{V}_h^{\mathrm{int}}, \tag{3.25a}$$

$$H_*^1(\omega_{\mathbf{a}}) := \{v \in H^1(\omega_{\mathbf{a}});\, v = 0 \text{ on } \partial\omega_{\mathbf{a}} \cap \partial\Omega\}, \quad \mathbf{a} \in \mathcal{V}_h^{\mathrm{ext}}, \tag{3.25b}$$

and the right-hand side $\boldsymbol{\tau}_h^{\mathbf{a}}$ *and* $g^{\mathbf{a}}$ *given by* (3.12)*. Then there exists a constant* $C_{\mathrm{cont,PF}} > 0$ *only depending on the shape-regularity parameter* $\kappa_{\mathcal{T}}$ *such that*

$$\|\nabla r_{\mathbf{a}}\|_{\omega_{\mathbf{a}}} \le C_{\mathrm{cont,PF}} \|\nabla(u-u_h)\|_{\omega_{\mathbf{a}}}. \tag{3.26}$$

Proof. We include the proof for insight and later use. There holds

$$\|\nabla r_{\mathbf{a}}\|_{\omega_{\mathbf{a}}} = \sup_{v\in H_*^1(\omega_{\mathbf{a}});\, \|\nabla v\|_{\omega_{\mathbf{a}}}=1} (\nabla r_{\mathbf{a}}, \nabla v)_{\omega_{\mathbf{a}}}. \tag{3.27}$$

Fix $v \in H^1_*(\omega_{\mathbf{a}})$ with $\|\nabla v\|_{\omega_{\mathbf{a}}} = 1$. Definitions (3.24) and (3.12), the fact that $\psi_{\mathbf{a}} v \in H^1_0(\omega_{\mathbf{a}})$, the characterization (2.2) of the weak solution, and the Cauchy–Schwarz inequality imply

$$\begin{aligned}(\nabla r_{\mathbf{a}}, \nabla v)_{\omega_{\mathbf{a}}} &= -(\psi_{\mathbf{a}} \nabla u_h, \nabla v)_{\omega_{\mathbf{a}}} + (\psi_{\mathbf{a}} f - \nabla \psi_{\mathbf{a}} \cdot \nabla u_h, v)_{\omega_{\mathbf{a}}} \\ &= (f, \psi_{\mathbf{a}} v)_{\omega_{\mathbf{a}}} - (\nabla u_h, \nabla(\psi_{\mathbf{a}} v))_{\omega_{\mathbf{a}}} \\ &= (\nabla(u - u_h), \nabla(\psi_{\mathbf{a}} v))_{\omega_{\mathbf{a}}} \\ &\le \|\nabla(u - u_h)\|_{\omega_{\mathbf{a}}} \|\nabla(\psi_{\mathbf{a}} v)\|_{\omega_{\mathbf{a}}}.\end{aligned}$$

Next,

$$\begin{aligned}\|\nabla(\psi_{\mathbf{a}} v)\|_{\omega_{\mathbf{a}}} &= \|\nabla \psi_{\mathbf{a}} v + \psi_{\mathbf{a}} \nabla v\|_{\omega_{\mathbf{a}}} \\ &\le \|\nabla \psi_{\mathbf{a}}\|_{\infty,\omega_{\mathbf{a}}} \|v\|_{\omega_{\mathbf{a}}} + \|\psi_{\mathbf{a}}\|_{\infty,\omega_{\mathbf{a}}} \|\nabla v\|_{\omega_{\mathbf{a}}} \\ &\le 1 + C_{\mathrm{PF},\omega_{\mathbf{a}}} h_{\omega_{\mathbf{a}}} \|\nabla \psi_{\mathbf{a}}\|_{\infty,\omega_{\mathbf{a}}},\end{aligned} \tag{3.28}$$

employing $\|\psi_{\mathbf{a}}\|_{\infty,\omega_{\mathbf{a}}} = 1$, the Poincaré inequality (3.6) on the patch $\omega_{\mathbf{a}}$ for $\mathbf{a} \in \mathcal{V}_h^{\mathrm{int}}$, and the Friedrichs inequality

$$\|v\|_{\omega_{\mathbf{a}}} \le C_{\mathrm{F},\omega_{\mathbf{a}}} h_{\omega_{\mathbf{a}}} \|\nabla v\|_{\omega_{\mathbf{a}}} \qquad \forall v \in H^1_*(\omega_{\mathbf{a}}) \tag{3.29}$$

for $\mathbf{a} \in \mathcal{V}_h^{\mathrm{ext}}$. For the values of these constants in particular on nonconvex patches $\omega_{\mathbf{a}}$, we refer to Eymard *et al.* [44, 45], Carstensen and Funken [26], Veeser and Verfürth [68], Šebestová and Vejchodský [65], and to the references therein. Thus (3.26) follows with $C_{\mathrm{cont,PF}} := \max_{\mathbf{a} \in \mathcal{V}_h} \{1 + C_{\mathrm{PF},\omega_{\mathbf{a}}} h_{\omega_{\mathbf{a}}} \|\nabla \psi_{\mathbf{a}}\|_{\infty,\omega_{\mathbf{a}}}\}$ where $C_{\mathrm{PF},\omega_{\mathbf{a}}} = C_{\mathrm{P},\omega_{\mathbf{a}}}$ if $\mathbf{a} \in \mathcal{V}_h^{\mathrm{int}}$ and $C_{\mathrm{PF},\omega_{\mathbf{a}}} = C_{\mathrm{F},\omega_{\mathbf{a}}}$ if $\mathbf{a} \in \mathcal{V}_h^{\mathrm{ext}}$. ☐

A related but different problem (namely on the right-hand side) to (3.24) appears in [30, Section 6.5]. We have the following new crucial estimate for its solution:

Lemma 3.12 (Continuous efficiency, potential reconstruction). *Let u be the weak solution given by (2.2) and let $u_h \in H^1(\mathcal{T}_h)$ satisfying (3.23) be arbitrary. Let $\mathbf{a} \in \mathcal{V}_h$ and let $r_{\mathbf{a}} \in H^1_*(\omega_{\mathbf{a}})$ solve*

$$(\nabla r_{\mathbf{a}}, \nabla v)_{\omega_{\mathbf{a}}} = -(\boldsymbol{\tau}_h^{\mathbf{a}}, \nabla v)_{\omega_{\mathbf{a}}} + (g^{\mathbf{a}}, v)_{\omega_{\mathbf{a}}} \qquad \forall v \in H^1_*(\omega_{\mathbf{a}}) \tag{3.30}$$

with the space

$$H^1_*(\omega_{\mathbf{a}}) := \{v \in H^1(\omega_{\mathbf{a}});\ (v, 1)_{\omega_{\mathbf{a}}} = 0\} \tag{3.31}$$

and the right-hand side $\boldsymbol{\tau}_h^{\mathbf{a}}$ and $g^{\mathbf{a}}$ given by (3.19). Then there exists a constant $C_{\mathrm{cont,bPF}} > 0$ only depending on the shape-regularity parameter $\kappa_{\mathcal{T}}$ such that

$$\|\nabla r_{\mathbf{a}}\|_{\omega_{\mathbf{a}}} \le C_{\mathrm{cont,bPF}} \|\nabla(u - u_h)\|_{\omega_{\mathbf{a}}}. \tag{3.32}$$

Proof. We start again from (3.27) and then fix $v \in H^1_*(\omega_{\mathbf{a}})$ with $\|\nabla v\|_{\omega_{\mathbf{a}}} = 1$. For an arbitrary $\tilde{u} \in H^1(\omega_{\mathbf{a}})$ such that $(\tilde{u}, 1)_{\omega_{\mathbf{a}}} = (u_h, 1)_{\omega_{\mathbf{a}}}$ if $\mathbf{a} \in \mathcal{V}_h^{\mathrm{int}}$ and $\tilde{u} = 0$ on $\partial\omega_{\mathbf{a}} \cap \partial\Omega$ if $\mathbf{a} \in \mathcal{V}_h^{\mathrm{ext}}$, we observe that

$$(\mathrm{R}_{\frac{\pi}{2}} \nabla(\psi_{\mathbf{a}} \tilde{u}), \nabla v)_{\omega_{\mathbf{a}}} = 0.$$

Thus, using (3.30) with (3.19) and the Cauchy–Schwarz inequality, we arrive at

$$\begin{aligned}(\nabla r_{\mathbf{a}}, \nabla v)_{\omega_{\mathbf{a}}} &= -(\mathrm{R}_{\frac{\pi}{2}} \nabla(\psi_{\mathbf{a}} u_h), \nabla v)_{\omega_{\mathbf{a}}} = (\mathrm{R}_{\frac{\pi}{2}} \nabla(\psi_{\mathbf{a}} \tilde{u} - \psi_{\mathbf{a}} u_h), \nabla v)_{\omega_{\mathbf{a}}} \\ &\le \|\mathrm{R}_{\frac{\pi}{2}} \nabla(\psi_{\mathbf{a}} \tilde{u} - \psi_{\mathbf{a}} u_h)\|_{\omega_{\mathbf{a}}} \|\nabla v\|_{\omega_{\mathbf{a}}} = \|\nabla(\psi_{\mathbf{a}} \tilde{u} - \psi_{\mathbf{a}} u_h)\|_{\omega_{\mathbf{a}}}.\end{aligned}$$

We next intend to proceed as in (3.28), with $\tilde{u} - u_h$ in place of v. Though $(\tilde{u} - u_h, 1)_{\omega_{\mathbf{a}}} = 0$ for $\mathbf{a} \in \mathcal{V}_h^{\mathrm{int}}$, the difference is that now $\tilde{u} - u_h$ does not belong to $H^1(\omega_{\mathbf{a}})$, with zero trace on $\partial\omega_{\mathbf{a}} \cap \partial\Omega$ for $\mathbf{a} \in \mathcal{V}_h^{\mathrm{ext}}$, but is a piecewise H^1 function from $H^1(\mathcal{T}_{\mathbf{a}})$. There is, fortunately, the continuity in mean of the jumps owing to the assumption (3.23), and in particular $\langle \tilde{u} - u_h, 1 \rangle_e = 0$ for all edges e located in $\partial\omega_{\mathbf{a}} \cap \partial\Omega$ for $\mathbf{a} \in \mathcal{V}_h^{\mathrm{ext}}$. Thus the Poincaré inequality (3.6) (on the patch $\omega_{\mathbf{a}}$) and the Friedrichs inequality (3.29) have to be replaced by their broken versions (see Brenner [21] or [70] and the references therein), leading to

$$\|\nabla(\psi_{\mathbf{a}} \tilde{u} - \psi_{\mathbf{a}} u_h)\|_{\omega_{\mathbf{a}}} \le (1 + C_{\mathrm{bPF},\omega_{\mathbf{a}}} h_{\omega_{\mathbf{a}}} \|\nabla \psi_{\mathbf{a}}\|_{\infty,\omega_{\mathbf{a}}}) \|\nabla(\tilde{u} - u_h)\|_{\omega_{\mathbf{a}}}. \tag{3.33}$$

Now it suffices to choose for $\tilde{u}$ the weak solution u shifted on interior patches by a constant such that $(\tilde{u} - u_h, 1)_{\omega_{\mathbf{a}}} = 0$ to infer (3.32) with $C_{\mathrm{cont,bPF}} := \max_{\mathbf{a} \in \mathcal{V}_h} \{1 + C_{\mathrm{bPF},\omega_{\mathbf{a}}} h_{\omega_{\mathbf{a}}} \|\nabla \psi_{\mathbf{a}}\|_{\infty,\omega_{\mathbf{a}}}\}$. ☐

Remark 3.13 (Efficiency for the potential reconstruction of Remark 3.9)**.** *Efficiency for the potential reconstruction of Remark 3.9 can be shown as above. In particular, problem* (3.30) *with the choice* (3.22) *reads (cf. [30, Section 6.5]): find $r_{\mathbf{a}} \in H^1_*(\omega_{\mathbf{a}})$ such that*

$$(\nabla r_{\mathbf{a}}, \nabla v)_{\omega_{\mathbf{a}}} = (\nabla u_h, \mathrm{R}_{\frac{\pi}{2}} \nabla(\psi_{\mathbf{a}} v))_{\omega_{\mathbf{a}}} \qquad \forall v \in H^1_*(\omega_{\mathbf{a}}).$$

An essential property is that $(\nabla u, \mathrm{R}_{\frac{\pi}{2}} \nabla(\psi_{\mathbf{a}} v))_{\omega_{\mathbf{a}}} = 0$. Thus,

$$(\nabla r_{\mathbf{a}}, \nabla v)_{\omega_{\mathbf{a}}} \le \|\nabla(u - u_h)\|_{\omega_{\mathbf{a}}} \|\mathrm{R}_{\frac{\pi}{2}} \nabla(\psi_{\mathbf{a}} v)\|_{\omega_{\mathbf{a}}} = \|\nabla(u - u_h)\|_{\omega_{\mathbf{a}}} \|\nabla(\psi_{\mathbf{a}} v)\|_{\omega_{\mathbf{a}}}$$

for any $v \in H^1_*(\omega_{\mathbf{a}})$ and we conclude by (3.28) *that* (3.26) *holds in this case, with $C_{\mathrm{cont,PF}}$ replaced by $C_{\mathrm{cont,P}} := \max_{\mathbf{a} \in \mathcal{V}_h} \{1 + C_{\mathrm{P},\omega_{\mathbf{a}}} h_{\omega_{\mathbf{a}}} \|\nabla \psi_{\mathbf{a}}\|_{\infty,\omega_{\mathbf{a}}}\}$, thereby requiring only the Poincaré inequality.*

Remark 3.14 (Dual and dual mixed formulations)**.** *For a vertex $\mathbf{a} \in \mathcal{V}_h$, consider the following dual formulation: Find $\boldsymbol{\varsigma}_{\mathbf{a}} \in \mathbf{H}_*(\mathrm{div}, \omega_{\mathbf{a}})$ with $\nabla \cdot \boldsymbol{\varsigma}_{\mathbf{a}} = g^{\mathbf{a}}$ such that*

$$(\boldsymbol{\varsigma}_{\mathbf{a}}, \mathbf{v})_{\omega_{\mathbf{a}}} = -(\boldsymbol{\tau}_h^{\mathbf{a}}, \mathbf{v})_{\omega_{\mathbf{a}}} \qquad \forall \mathbf{v} \in \mathbf{H}_*(\mathrm{div}, \omega_{\mathbf{a}}) \text{ with } \nabla \cdot \mathbf{v} = 0. \tag{3.34}$$

Here $\mathbf{H}_*(\mathrm{div}, \omega_{\mathbf{a}})$ stands for $\mathbf{H}(\mathrm{div}, \omega_{\mathbf{a}})$ functions with zero normal trace in the appropriate sense on $\partial\omega_{\mathbf{a}}$ for $\mathbf{a} \in \mathcal{V}_h^{\mathrm{int}}$ and for $\mathbf{H}(\mathrm{div}, \omega_{\mathbf{a}})$ functions with zero normal trace in the appropriate sense on $\partial\omega_{\mathbf{a}} \setminus \partial\Omega$ for $\mathbf{a} \in \mathcal{V}_h^{\mathrm{ext}}$. Similarly, consider the dual mixed formulation: Find $\boldsymbol{\varsigma}_{\mathbf{a}} \in \mathbf{H}_*(\mathrm{div}, \omega_{\mathbf{a}})$ and $\bar{r}^{\mathbf{a}} \in L^2_*(\omega_{\mathbf{a}})$ such that

$$(\boldsymbol{\varsigma}_{\mathbf{a}}, \mathbf{v})_{\omega_{\mathbf{a}}} - (\bar{r}^{\mathbf{a}}, \nabla \cdot \mathbf{v})_{\omega_{\mathbf{a}}} = -(\boldsymbol{\tau}_h^{\mathbf{a}}, \mathbf{v})_{\omega_{\mathbf{a}}} \qquad \forall \mathbf{v} \in \mathbf{H}_*(\mathrm{div}, \omega_{\mathbf{a}}), \tag{3.35a}$$

$$(\nabla \cdot \boldsymbol{\varsigma}_{\mathbf{a}}, q)_{\omega_{\mathbf{a}}} = (g^{\mathbf{a}}, q)_{\omega_{\mathbf{a}}} \qquad \forall q \in L^2_*(\omega_{\mathbf{a}}). \tag{3.35b}$$

Here $L^2_*(\omega_{\mathbf{a}})$ is the space of functions from $L^2(\omega_{\mathbf{a}})$ with mean value zero for $\mathbf{a} \in \mathcal{V}_h^{\mathrm{int}}$ and $L^2(\omega_{\mathbf{a}})$ for $\mathbf{a} \in \mathcal{V}_h^{\mathrm{ext}}$. It is classical that the problems (3.34) *and* (3.35) *are equivalent to the primal formulation* (3.24)*, with $\boldsymbol{\varsigma}_{\mathbf{a}} = -\nabla r_{\mathbf{a}} - \boldsymbol{\tau}_h^{\mathbf{a}}$. Then* (3.10) *is the natural finite element discretization of* (3.35)*. The same links hold true in the potential reconstruction cases.*

3.2.2 Uniform-in-polynomial-degree stability of mixed finite element methods

The following crucial result has been shown in [19, Theorem 7]:

Corollary 3.15 (Uniform stability of mixed finite element methods)**.** *Let $\mathbf{a} \in \mathcal{V}_h$ and let $r_{\mathbf{a}} \in H^1_*(\omega_{\mathbf{a}})$ either solve* (3.24) *with $H^1_*(\omega_{\mathbf{a}})$ given by* (3.25) *and $\boldsymbol{\tau}_h^{\mathbf{a}}$ and $g^{\mathbf{a}}$ by* (3.12)*, or* (3.30) *with $H^1_*(\omega_{\mathbf{a}})$ given by* (3.31) *and $\boldsymbol{\tau}_h^{\mathbf{a}}$ and $g^{\mathbf{a}}$ by* (3.19)*. Suppose in particular* (3.8) *and additionally that $g^{\mathbf{a}} \in Q_h(\omega_{\mathbf{a}})$. Let $\boldsymbol{\varsigma}_h^{\mathbf{a}}$ be the solution of duly* (3.10) *or* (3.17)*. Then there exists a constant $C_{\mathrm{st}} > 0$ only depending on the shape-regularity parameter $\kappa_{\mathcal{T}}$ such that*

$$\|\boldsymbol{\varsigma}_h^{\mathbf{a}} + \boldsymbol{\tau}_h^{\mathbf{a}}\|_{\omega_{\mathbf{a}}} \le C_{\mathrm{st}} \|\nabla r_{\mathbf{a}}\|_{\omega_{\mathbf{a}}}. \tag{3.36}$$

We have from (3.24) or (3.30), using (3.27),

$$\|\nabla r_{\mathbf{a}}\|_{\omega_{\mathbf{a}}} = \sup_{v \in H^1_*(\omega_{\mathbf{a}});\, \|\nabla v\|_{\omega_{\mathbf{a}}}=1} \{-(\boldsymbol{\tau}_h^{\mathbf{a}}, \nabla v)_{\omega_{\mathbf{a}}} + (g^{\mathbf{a}}, v)_{\omega_{\mathbf{a}}}\} \tag{3.37a}$$

$$= \sup_{v \in H^1_*(\omega_{\mathbf{a}});\, \|\nabla v\|_{\omega_{\mathbf{a}}}=1} \Bigg\{ \sum_{e \in \mathcal{E}_h,\, \mathbf{a} \in e} \langle \underbrace{[\![-\boldsymbol{\tau}_h^{\mathbf{a}} \cdot \mathbf{n}_e]\!]}_{r_e}, v\rangle_e + \sum_{K \in \mathcal{T}_{\mathbf{a}}} (\underbrace{\nabla \cdot \boldsymbol{\tau}_h^{\mathbf{a}} + g^{\mathbf{a}}}_{r_K}, v)_K \Bigg\}, \tag{3.37b}$$

so that $\|\nabla r_{\mathbf{a}}\|_{\omega_{\mathbf{a}}}$ in our notation is $\|r\|_{[H^1(\omega)/\mathbb{R}]^*}$ is the notation of [19]. Simultaneously,

$$\|\boldsymbol{\varsigma}_h^{\mathbf{a}} + \boldsymbol{\tau}_h^{\mathbf{a}}\|_{\omega_{\mathbf{a}}} = \inf_{\mathbf{v}_h \in \mathbf{V}_h^{\mathbf{a}},\, \nabla \cdot \mathbf{v}_h = g^{\mathbf{a}}} \|\mathbf{v}_h + \boldsymbol{\tau}_h^{\mathbf{a}}\|_{\omega_{\mathbf{a}}}$$

from (3.10) or (3.17), cf., e.g., [40]. Setting $\boldsymbol{\delta}_h^{\mathbf{a}} := \boldsymbol{\varsigma}_h^{\mathbf{a}} + \boldsymbol{\tau}_h^{\mathbf{a}}$, we see that

$$\|\boldsymbol{\varsigma}_h^{\mathbf{a}} + \boldsymbol{\tau}_h^{\mathbf{a}}\|_{\omega_{\mathbf{a}}} = \|\boldsymbol{\delta}_h\|_{\omega_{\mathbf{a}}} = \inf_{\mathbf{v}_h \in \mathbf{V}_h^{\mathbf{a}}(\mathcal{T}_{\mathbf{a}}),\, \nabla \cdot \mathbf{v}_h|_K = (\nabla \cdot \boldsymbol{\tau}_h^{\mathbf{a}} + g^{\mathbf{a}})|_K\ \forall K \in \mathcal{T}_{\mathbf{a}}} \|\mathbf{v}_h\|_{\omega_{\mathbf{a}}},$$

where $\mathbf{V}_h^{\mathbf{a}}(\mathcal{T}_{\mathbf{a}})$ is the broken version of $\mathbf{V}_h^{\mathbf{a}}$ with normal jumps imposed by $[\![\boldsymbol{\tau}_h^{\mathbf{a}} \cdot \mathbf{n}_e]\!]$, which is the form employed in [19, Theorem 7].

3.2.3 Polynomial-degree-robust efficiency

We are now ready to prove the main result of this paper:

Theorem 3.16 (Polynomial-degree-robust efficiency)**.** *Let u be the weak solution given by (2.2). Consider the construction of $\boldsymbol{\sigma}_h$ following Definition 3.4. Let f and u_h be piecewise polynomials so that $g^{\mathbf{a}}$ of (3.12b) satisfies $g^{\mathbf{a}} \in Q_h(\omega_{\mathbf{a}})$ and $\boldsymbol{\tau}_h^{\mathbf{a}}$ of (3.12a) satisfies (3.8). Then*

$$\|\nabla u_h + \boldsymbol{\sigma}_h\|_K \leq C_{\rm st} C_{\rm cont,PF} \sum_{\mathbf{a}\in\mathcal{V}_K} \|\nabla(u-u_h)\|_{\omega_{\mathbf{a}}} \tag{3.38}$$

for all $K \in \mathcal{T}_h$, with the constants $C_{\rm st}$ of (3.36) and $C_{\rm cont,PF}$ of (3.26), respectively. Consider the construction of s_h following Definition 3.7. Let u_h verify (3.23) and let it be a piecewise polynomial so that $\boldsymbol{\tau}_h^{\mathbf{a}}$ of (3.19a) satisfies (3.8). Then

$$\|\nabla(u_h - s_h)\|_K \leq C_{\rm st} C_{\rm cont,bPF} \sum_{\mathbf{a}\in\mathcal{V}_K} \|\nabla(u-u_h)\|_{\omega_{\mathbf{a}}} \tag{3.39}$$

for all $K \in \mathcal{T}_h$, with the constants $C_{\rm st}$ of (3.36) and $C_{\rm cont,bPF}$ of (3.32), respectively.

Proof. Let $K \in \mathcal{T}_h$ be given. Using Definition 3.4, (3.15), the partition of unity (3.16), and the triangle inequality, we infer that

$$\begin{aligned}\|\nabla u_h + \boldsymbol{\sigma}_h\|_K &= \left\| \sum_{\mathbf{a}\in\mathcal{V}_K} (\psi_{\mathbf{a}} \nabla u_h + \boldsymbol{\varsigma}_h^{\mathbf{a}})|_K \right\|_K \leq \sum_{\mathbf{a}\in\mathcal{V}_K} \|\psi_{\mathbf{a}} \nabla u_h + \boldsymbol{\varsigma}_h^{\mathbf{a}}\|_K \\ &\leq \sum_{\mathbf{a}\in\mathcal{V}_K} \|\psi_{\mathbf{a}} \nabla u_h + \boldsymbol{\varsigma}_h^{\mathbf{a}}\|_{\omega_{\mathbf{a}}}.\end{aligned}$$

Now Corollary 3.15 yields

$$\|\psi_{\mathbf{a}} \nabla u_h + \boldsymbol{\varsigma}_h^{\mathbf{a}}\|_{\omega_{\mathbf{a}}} \leq C_{\rm st} \|\nabla r_{\mathbf{a}}\|_{\omega_{\mathbf{a}}},$$

and Lemma 3.11 concludes the proof of (3.38).

Similarly, using Definition 3.7, $s_h|_K = \sum_{\mathbf{a}\in\mathcal{V}_K} s_h^{\mathbf{a}}|_K$, the partition of unity (3.16), and the triangle inequality, we infer that

$$\begin{aligned}\|\nabla(u_h - s_h)\|_K &= \left\| \sum_{\mathbf{a}\in\mathcal{V}_K} (\nabla(\psi_{\mathbf{a}} u_h - s_h^{\mathbf{a}}))|_K \right\|_K \leq \sum_{\mathbf{a}\in\mathcal{V}_K} \|\nabla(\psi_{\mathbf{a}} u_h - s_h^{\mathbf{a}})\|_K \\ &= \sum_{\mathbf{a}\in\mathcal{V}_K} \|{\rm R}_{\frac{\pi}{2}} \nabla(\psi_{\mathbf{a}} u_h - s_h^{\mathbf{a}})\|_K \leq \sum_{\mathbf{a}\in\mathcal{V}_K} \|{\rm R}_{\frac{\pi}{2}} \nabla(\psi_{\mathbf{a}} u_h) + \boldsymbol{\varsigma}_h^{\mathbf{a}}\|_{\omega_{\mathbf{a}}}.\end{aligned}$$

Thus Corollary 3.15 yields

$$\|{\rm R}_{\frac{\pi}{2}} \nabla(\psi_{\mathbf{a}} u_h) + \boldsymbol{\varsigma}_h^{\mathbf{a}}\|_{\omega_{\mathbf{a}}} \leq C_{\rm st} \|\nabla r_{\mathbf{a}}\|_{\omega_{\mathbf{a}}},$$

and Lemma 3.12 concludes the proof of (3.39). □

Remark 3.17 (Robustness for the potential reconstruction of Remark 3.9)**.** *It follows as above that for the potential reconstruction of Remark 3.9, there holds (3.39), with $C_{\rm cont,bPF}$ replaced by $C_{\rm cont,P}$ of Remark 3.13.*

3.3 Maximal overestimation

Guaranteed (local) maximal overestimation factors have been derived previously in, e.g., Babuška *et al.* [16], Carstensen and Funken [25], Babuška and Strouboulis [15, Section 5.1], Prudhomme *et al.* [61], or Repin [63, Section 4.1.1], see also the references therein, but not necessarily simultaneously with a guaranteed upper bound. In our setting, we obtain:

Lemma 3.18 (Maximal overestimation)**.** *Let the assumptions of Theorem 3.16 be verified. Then*

$$\begin{aligned}\|\nabla u_h + \boldsymbol{\sigma}_h\| &\leq 3 C_{\rm st} C_{\rm cont,PF} \|\nabla(u-u_h)\|,\\ \|\nabla(u_h - s_h)\| &\leq 3 C_{\rm st} C_{\rm cont,bPF} \|\nabla(u-u_h)\|.\end{aligned}$$

Proof. Employing (3.15), the partition of unity (3.16), the Cauchy–Schwarz inequality, and proceeding as in the proof of Theorem 3.16, we infer that

$$\begin{aligned} &\|\nabla u_h + \boldsymbol{\sigma}_h\|^2 \\ &= \sum_{K\in\mathcal{T}_h} \|\nabla u_h + \boldsymbol{\sigma}_h\|_K^2 = \sum_{K\in\mathcal{T}_h} \Big\| \sum_{\mathbf{a}\in\mathcal{V}_K} (\psi_{\mathbf{a}}\nabla u_h + \boldsymbol{\varsigma}_h^{\mathbf{a}})|_K \Big\|_K^2 \\ &\le 3 \sum_{K\in\mathcal{T}_h} \sum_{\mathbf{a}\in\mathcal{V}_K} \|\psi_{\mathbf{a}}\nabla u_h + \boldsymbol{\varsigma}_h^{\mathbf{a}}\|_K^2 = 3 \sum_{\mathbf{a}\in\mathcal{V}_h} \|\psi_{\mathbf{a}}\nabla u_h + \boldsymbol{\varsigma}_h^{\mathbf{a}}\|_{\omega_{\mathbf{a}}}^2 \\ &\le 3C_{\rm st}^2 C_{\rm cont,PF}^2 \sum_{\mathbf{a}\in\mathcal{V}_h} \|\nabla(u-u_h)\|_{\omega_{\mathbf{a}}}^2 = 9C_{\rm st}^2 C_{\rm cont,PF}^2 \|\nabla(u-u_h)\|^2. \end{aligned} \tag{3.40}$$

The bound for $\|\nabla(u_h - s_h)\|$ is similar. □

We finally present a local result indicating additionally how to asses the value of the unknown constant $C_{\rm st}$ of (3.36):

Lemma 3.19 (Guaranteed maximal local overestimation by auxiliary problems)**.** *Let the assumptions of Theorem 3.16 be verified. Fix $\mathbf{a}\in\mathcal{V}_h$ and consider an arbitrary conforming finite element approximation in $V_h^{\mathbf{a}} := \mathbb{R}_{\bar p}(\mathcal{T}_{\mathbf{a}}) \cap H_*^1(\omega_{\mathbf{a}})$ of (3.24) or (3.30) in the form: find $r_h^{\mathbf{a}} \in V_h^{\mathbf{a}}$ such that*

$$(\nabla r_h^{\mathbf{a}}, \nabla v_h)_{\omega_{\mathbf{a}}} = -(\boldsymbol{\tau}_h^{\mathbf{a}}, \nabla v_h)_{\omega_{\mathbf{a}}} + (g^{\mathbf{a}}, v_h)_{\omega_{\mathbf{a}}} \qquad \forall v_h \in V_h^{\mathbf{a}},$$

with the usual choices (3.12) or (3.19) for the right-hand side. Then,

$$\|\psi_{\mathbf{a}}\nabla u_h + \boldsymbol{\varsigma}_h^{\mathbf{a}}\|_{\omega_{\mathbf{a}}} \le C_{\rm cont,PF} \frac{\|\psi_{\mathbf{a}}\nabla u_h + \boldsymbol{\varsigma}_h^{\mathbf{a}}\|_{\omega_{\mathbf{a}}}}{\|\nabla r_h^{\mathbf{a}}\|_{\omega_{\mathbf{a}}}} \|\nabla(u-u_h)\|_{\omega_{\mathbf{a}}}, \tag{3.41a}$$

$$\|\nabla(\psi_{\mathbf{a}} u_h - s_h^{\mathbf{a}})\|_{\omega_{\mathbf{a}}} \le C_{\rm cont,bPF} \frac{\|\nabla(\psi_{\mathbf{a}} u_h - s_h^{\mathbf{a}})\|_{\omega_{\mathbf{a}}}}{\|\nabla r_h^{\mathbf{a}}\|_{\omega_{\mathbf{a}}}} \|\nabla(u-u_h)\|_{\omega_{\mathbf{a}}}. \tag{3.41b}$$

Proof. As $r_h^{\mathbf{a}}$ is the $(\nabla\cdot,\nabla\cdot)_{\omega_{\mathbf{a}}}$-orthogonal projection of $r_{\mathbf{a}}$ onto $V_h^{\mathbf{a}}$, $\|\nabla r_h^{\mathbf{a}}\|_{\omega_{\mathbf{a}}} \le \|\nabla r_{\mathbf{a}}\|_{\omega_{\mathbf{a}}}$. Thus the results follow respectively from Lemmas 3.11 and 3.12. □

Remark 3.20 (Size of overestimation, comparison with [25])**.** *The above lemma together with Remark 3.14 suggest that the constant $C_{\rm st}$ approaches 1 as the polynomial degrees p, $\bar p$ are increased. Next, for convex patches $\omega_{\mathbf{a}}$ around interior vertices $\mathbf{a}$, $C_{{\rm P},\omega_{\mathbf{a}}} = 1/\pi$, whereas $h_{\omega_{\mathbf{a}}}\|\nabla\psi_{\mathbf{a}}\|_{\infty,\omega_{\mathbf{a}}} \approx 2$ for "nice" meshes. Thus we may expect $C_{\rm cont,PF} \approx 1 + 2/\pi$ from the proof of Lemma 3.11 in such a case. Then Lemma 3.18 gives $3C_{\rm st}C_{\rm cont,PF} \approx 4.9$ for the maximal theoretical overestimation factor. In practice, however, the effectivity indices of the present estimates are quite close to the optimal value of one, see [19, 39, 59, 42] for some numerical experiments. For the conforming finite element method, Carstensen and Funken [25, Example 3.1] obtain a maximal theoretical overestimation factor 2.34 for "nice" meshes, which is roughly twice better than our result. This can be attributed to the localization of the estimators around mesh vertices with a specific use of the partition of unity of [25], see equation (3.7) in this reference and also the next remark, whereas we loose roughly a factor 3 in the estimate (3.40). Note, however, that the upper bound in [25] is, in contrast to the lower one, not guaranteed.*

Remark 3.21 (Localization on the patches $\omega_{\mathbf{a}}$)**.** *In [25], see in particular Theorem 3.2 therein, the following local problems similar to (3.24) are considered: find $r_{\mathbf{a}} \in \bar H_*^1(\omega_{\mathbf{a}})$ such that, with the choice (3.12) for the right-hand side,*

$$(\psi_{\mathbf{a}}\nabla r_{\mathbf{a}}, \nabla v)_{\omega_{\mathbf{a}}} = -(\boldsymbol{\tau}_h^{\mathbf{a}}, \nabla v)_{\omega_{\mathbf{a}}} + (g^{\mathbf{a}}, v)_{\omega_{\mathbf{a}}} \qquad \forall v \in \bar H_*^1(\omega_{\mathbf{a}}),$$

where $\bar H_*^1(\omega_{\mathbf{a}})$ are $\psi_{\mathbf{a}}^{1/2}$-weighted versions of the spaces (3.25), and the (unfortunately not computable) a posteriori error estimator is simply $\big\|\psi_{\mathbf{a}}^{1/2}\nabla r_{\mathbf{a}}\big\|_{\omega_{\mathbf{a}}}$. Adjusting the equilibration of Definition 3.4, its computable upper bound may be constructed via local problems consisting in finding $\boldsymbol{\varsigma}_h^{\mathbf{a}} \in \mathbf{V}_h^{\mathbf{a}}$ and $\bar r_h^{\mathbf{a}} \in Q_h^{\mathbf{a}}$ such that

$$\begin{aligned} (\psi_{\mathbf{a}}\boldsymbol{\varsigma}_h^{\mathbf{a}}, \mathbf{v}_h)_{\omega_{\mathbf{a}}} - (\bar r_h^{\mathbf{a}}, \nabla\cdot(\psi_{\mathbf{a}}\mathbf{v}_h))_{\omega_{\mathbf{a}}} &= -(\boldsymbol{\tau}_h^{\mathbf{a}}, \mathbf{v}_h)_{\omega_{\mathbf{a}}} \qquad && \forall \mathbf{v}_h \in \mathbf{V}_h^{\mathbf{a}}, \\ (\nabla\cdot(\psi_{\mathbf{a}}\boldsymbol{\varsigma}_h^{\mathbf{a}}), q_h)_{\omega_{\mathbf{a}}} &= (g^{\mathbf{a}}, q_h)_{\omega_{\mathbf{a}}} && \forall q_h \in Q_h^{\mathbf{a}}. \end{aligned}$$

4 Applications to discretization methods

We show here how to apply our results to common discretizations via the verification of the assumptions of Section 3.

4.1 Conforming finite elements

Let $V_h := \mathbb{P}_p(\mathcal{T}_h) \cap H_0^1(\Omega)$, $p \geq 1$. The conforming finite element method for (2.2), cf. [32], reads: find $u_h \in V_h$ such that

$$(\nabla u_h, \nabla v_h) = (f, v_h) \qquad \forall v_h \in V_h. \tag{4.1}$$

The application of our framework is straightforward: (3.9) is nothing but the Galerkin orthogonality with respect to the hat basis function $\psi_{\mathbf{a}}$ which follows immediately from (4.1). The approximate solution u_h is conforming, included in $H_0^1(\Omega)$, so that we set $s_h := u_h$, the nonconformity estimators $\|\nabla(u_h - s_h)\|_K$ disappear, and there is nothing to verify in this respect.

4.2 Nonconforming finite elements

Let V_h stand for functions from $\mathbb{P}_p(\mathcal{T}_h)$, $p \geq 1$, satisfying (3.23) for all polynomials up to degree $p-1$ on each edge instead of merely the constants function 1. The full family nonconforming finite element method for (2.2), cf. Stoyan and Baran [66] or [10], reads: find $u_h \in V_h$ such that

$$(\nabla u_h, \nabla v_h) = (f, v_h) \qquad \forall v_h \in V_h. \tag{4.2}$$

Nonconforming finite elements again fit perfectly into our framework: (3.9) follows immediately from (4.2) as $\psi_{\mathbf{a}} \in V_h$. The approximate solution u_h is not included in $H_0^1(\Omega)$ but satisfies (3.23) from the definition of the space V_h, so that both the constructions from Definition 3.7 and Remark 3.9 are possible.

Remark 4.1 (Different flux reconstructions, evaluation cost)**.** *It has been recently shown that several seemingly different flux reconstructions coincide for the lowest-order Crouzeix–Raviart case on simplices, including that of Definition 3.4 with the lowest-order RTN space, see [43]. So, at least in this particular case, this smears the conceptual difference between the present implicit estimators (where solutions of local problems are necessary) and the a priori cheaper explicit (directly computable) estimators like those of Verfürth [69].*

4.3 Interior penalty discontinuous Galerkin finite elements

Set $V_h := \mathbb{P}_p(\mathcal{T}_h)$, $p \geq 1$; thus V_h is the space of piecewise polynomials of order p without any continuity requirement. The discontinuous Galerkin method, cf. [37] and the references therein, reads: find $u_h \in V_h$ such that

$$\begin{aligned} &\sum_{K\in\mathcal{T}_h} (\nabla u_h, \nabla v_h)_K - \sum_{e\in\mathcal{E}_h} \{\langle \{\!\{\nabla u_h\}\!\}\cdot\mathbf{n}_e, [\![v_h]\!]\rangle_e + \theta\langle \{\!\{\nabla v_h\}\!\}\cdot\mathbf{n}_e, [\![u_h]\!]\rangle_e\} \\ &+ \sum_{e\in\mathcal{E}_h} \langle \alpha h_e^{-1} [\![u_h]\!], [\![v_h]\!]\rangle_e = (f, v_h) \qquad \forall v_h \in V_h, \end{aligned} \tag{4.3}$$

where α is a positive stabilization parameter and $\theta = 0, 1$ correspond respectively to the incomplete and symmetric versions. The nonsymmetric version with $\theta = -1$ does not seem to fall easily into our framework and is not considered herein.

4.3.1 The incomplete version

We start with the incomplete version with $\theta = 0$. Then taking $v_h = \psi_{\mathbf{a}}$ in (4.3) yields the Galerkin orthogonality (3.9), as $\psi_{\mathbf{a}}$ has no jumps. So the flux reconstruction of Definition 3.4 is possible. Condition (3.21) does not hold true, so that Definition 3.7 seems to be the right choice for the potential reconstruction and we do have the guaranteed estimate of Theorem 3.3. As, unfortunately, (3.23) happens not to be satisfied, we cannot directly use Lemma 3.12. The inspection of its proof, however, shows that we merely need to replace the estimate (3.33) by

$$\begin{aligned}\|\nabla(\psi_{\mathbf{a}}\tilde{u}-\psi_{\mathbf{a}}u_h)\|_{\omega_{\mathbf{a}}} &\leq \|\nabla\psi_{\mathbf{a}}\|_{\infty,\omega_{\mathbf{a}}}\|\tilde{u}-u_h\|_{\omega_{\mathbf{a}}}+\|\psi_{\mathbf{a}}\|_{\infty,\omega_{\mathbf{a}}}\|\nabla(\tilde{u}-u_h)\|_{\omega_{\mathbf{a}}}\\ &\leq (1+C_{\mathrm{bPF},\omega_{\mathbf{a}}}h_{\omega_{\mathbf{a}}}\|\nabla\psi_{\mathbf{a}}\|_{\infty,\omega_{\mathbf{a}}})\|\nabla(\tilde{u}-u_h)\|_{\omega_{\mathbf{a}}}\\ &\quad + C_{\mathrm{bPF},\omega_{\mathbf{a}}}h_{\omega_{\mathbf{a}}}\|\nabla\psi_{\mathbf{a}}\|_{\infty,\omega_{\mathbf{a}}}\Bigg\{\sum_{e\in\mathcal{E}_h,\,\mathbf{a}\in e}h_e^{-1}\|\Pi_0[\![u_h]\!]\|_e^2\Bigg\}^{1/2},\end{aligned}$$

with Π_0 the $L^2(e)$-orthogonal projection onto constants, using the discrete Poincaré–Friedrichs inequalities of [21, Remark 1.1] and the fact that $\tilde{u}$ has no jumps. Thus,

$$\|\nabla r_{\mathbf{a}}\|_{\omega_{\mathbf{a}}}\leq C_{\mathrm{cont,bPF}}\left(\|\nabla(u-u_h)\|_{\omega_{\mathbf{a}}}+\Bigg\{\sum_{e\in\mathcal{E}_h,\,\mathbf{a}\in e}h_e^{-1}\|\Pi_0[\![u-u_h]\!]\|_e^2\Bigg\}^{1/2}\right)$$

in place of (3.32). The local efficiency result (3.39) then gets

$$\|\nabla(u_h-s_h)\|_K\leq C_{\mathrm{st}}C_{\mathrm{cont,bPF}}\sum_{\mathbf{a}\in\mathcal{V}_K}\left(\|\nabla(u-u_h)\|_{\omega_{\mathbf{a}}}+\Bigg\{\sum_{e\in\mathcal{E}_h,\,\mathbf{a}\in e}h_e^{-1}\|\Pi_0[\![u-u_h]\!]\|_e^2\Bigg\}^{1/2}\right). \tag{4.4}$$

It is still polynomial degree robust, but features the additional jump term. The classical option to obtain both upper and lower bounds for the same error measure is to resort to the jumps-augmented energy seminorm to get

$$\begin{aligned}\|\nabla(u-u_h)\|^2+\sum_{e\in\mathcal{E}_h}h_e^{-1}\|\Pi_0[\![u-u_h]\!]\|_e^2 &\leq \sum_{K\in\mathcal{T}_h}\left(\|\nabla u_h+\boldsymbol{\sigma}_h\|_K+\frac{h_K}{\pi}\|f-\Pi_{Q_h}f\|_K\right)^2\\ &\quad+\sum_{K\in\mathcal{T}_h}\|\nabla(u_h-s_h)\|_K^2+\sum_{e\in\mathcal{E}_h}h_e^{-1}\|\Pi_0[\![u_h]\!]\|_e^2,\end{aligned}$$

and the local lower bounds (3.38), (4.4), and efficiency with constant 1 for the jump term as $[\![u-u_h]\!]=-[\![u_h]\!]$. Remark that the projection Π_0 above can be removed, obtaining the usual discontinuous Galerkin norm.

4.3.2 The symmetric version

To cover the symmetric version, we proceed as already suggested in [42, Section 6.4]. Introduce the discrete gradient $\mathfrak{G}(u_h):=\nabla u_h-\theta\sum_{e\in\mathcal{E}_h}\mathfrak{l}_e([\![u_h]\!])$ where the lifting operator $\mathfrak{l}_e:L^2(e)\to[\mathbb{P}_{p-1}(\mathcal{T}_h)]^2$ is such that $(\mathfrak{l}_e([\![u_h]\!]),\mathbf{v}_h)=\langle\{\!\{\mathbf{v}_h\}\!\}{\cdot}\mathbf{n}_e,[\![u_h]\!]\rangle_e$ for all $\mathbf{v}_h\in[\mathbb{P}_{p-1}(\mathcal{T}_h)]^2$, see [37, Section 4.3]. Observe that $\mathfrak{G}(v)=\nabla v$ for any function v with zero jumps. Then, taking $v_h=\psi_{\mathbf{a}}$ in (4.3) and since $\psi_{\mathbf{a}}$ has no jumps yields $(\mathfrak{G}(u_h),\nabla\psi_{\mathbf{a}})_{\omega_{\mathbf{a}}}=(f,\psi_{\mathbf{a}})_{\omega_{\mathbf{a}}}$ for all $\mathbf{a}\in\mathcal{V}_h^{\mathrm{int}}$ instead of (3.9). Thus the flux reconstruction of Definition 3.4 is possible with right-hand sides $\boldsymbol{\tau}_h^{\mathbf{a}}:=\psi_{\mathbf{a}}\mathfrak{G}(u_h)$ and $g^{\mathbf{a}}:=\psi_{\mathbf{a}}f-\nabla\psi_{\mathbf{a}}{\cdot}\mathfrak{G}(u_h)$. Concerning the potential reconstruction, the discrete gradient $\mathfrak{G}$ leads to the satisfaction of the following modification of condition (3.21):

$$(\mathrm{R}_{\frac{\pi}{2}}\mathfrak{G}(u_h),\nabla\psi_{\mathbf{a}})_{\omega_{\mathbf{a}}}=0\qquad\forall\mathbf{a}\in\mathcal{V}_h. \tag{4.5}$$

Indeed, using the definition of the discrete gradient and the Green theorem, we have

$$\begin{aligned}(\mathrm{R}_{\frac{\pi}{2}}\mathfrak{G}(u_h),\nabla\psi_{\mathbf{a}})_{\omega_{\mathbf{a}}} &= -(\mathfrak{G}(u_h),\mathrm{R}_{\frac{\pi}{2}}\nabla\psi_{\mathbf{a}})_{\omega_{\mathbf{a}}}\\ &= -(\nabla u_h,\mathrm{R}_{\frac{\pi}{2}}\nabla\psi_{\mathbf{a}})_{\omega_{\mathbf{a}}}+\theta\sum_{e\in\mathcal{E}_h}\langle\{\!\{\mathrm{R}_{\frac{\pi}{2}}\nabla\psi_{\mathbf{a}}\}\!\}{\cdot}\mathbf{n}_e,[\![u_h]\!]\rangle_e\\ &= -\sum_{K\in\mathcal{T}_{\mathbf{a}}}\langle u_h,(\mathrm{R}_{\frac{\pi}{2}}\nabla\psi_{\mathbf{a}}){\cdot}\mathbf{n}_K\rangle_{\partial K}+\theta\sum_{e\in\mathcal{E}_h}\langle\{\!\{\mathrm{R}_{\frac{\pi}{2}}\nabla\psi_{\mathbf{a}}\}\!\}{\cdot}\mathbf{n}_e,[\![u_h]\!]\rangle_e.\end{aligned}$$

Now for $\theta=1$, the two above terms cancel. Thus we can use here the procedure of Remark 3.9, where we systematically replace $\mathrm{R}_{\frac{\pi}{2}}\nabla v$ by $\mathrm{R}_{\frac{\pi}{2}}\mathfrak{G}(v)$.

These flux and potential reconstructions lead to the guaranteed estimate of Theorem 3.3, which, using the discrete gradient, takes the form

$$\|\mathfrak{G}(u-u_h)\|^2 \le \sum_{K\in\mathcal{T}_h}\left(\|\mathfrak{G}(u_h)+\boldsymbol{\sigma}_h\|_K + \frac{h_K}{\pi}\|f-\nabla\cdot\boldsymbol{\sigma}_h\|_K\right)^2 + \sum_{K\in\mathcal{T}_h}\|\mathfrak{G}(u_h-s_h)\|_K^2.$$

Then the local efficiency result (3.38) for the flux reconstruction takes the form

$$\|\mathfrak{G}(u_h)+\boldsymbol{\sigma}_h\|_K \le C_{\rm st}C_{\rm cont,PF}\sum_{\mathbf{a}\in\mathcal{V}_K}\|\mathfrak{G}(u-u_h)\|_{\omega_{\mathbf{a}}}$$

and the local efficiency result (3.39) for the potential reconstruction takes the form

$$\|\mathfrak{G}(u_h-s_h)\|_K \le C_{\rm st}C_{\rm cont,P}\sum_{\mathbf{a}\in\mathcal{V}_K}\|\mathfrak{G}(u-u_h)\|_{\omega_{\mathbf{a}}}$$

and both are polynomial degree robust.

Remark 4.2 (Elementwise flux reconstructions)**.** *In [17, 4, 47, 38, 33, 40, 39, 11], see also the references therein, different elementwise flux reconstructions leading to* (3.14) *are designed. Although they are cheaper (typically no local linear system is to be solved), it is not clear whether they are robust with respect to polynomial degree.*

4.4 Mixed finite elements

The application of our framework to mixed finite elements is again rather straightforward. Let $\mathbf{V}_h \times Q_h$ be any of the usual mixed finite element spaces, see Section 2.3; we consider here the polynomial degree $p' \ge 0$. We look for the couple $\boldsymbol{\sigma}_h \in \mathbf{V}_h$ and $\bar{u}_h \in Q_h$ such that, cf. [22, 64, 73],

$$(\boldsymbol{\sigma}_h,\mathbf{v}_h)-(\bar{u}_h,\nabla\cdot\mathbf{v}_h)=0 \qquad \forall \mathbf{v}_h\in\mathbf{V}_h, \tag{4.6a}$$

$$(\nabla\cdot\boldsymbol{\sigma}_h,q_h)=(f,q_h) \qquad \forall q_h\in Q_h. \tag{4.6b}$$

We have written the formulation explicitly with $\boldsymbol{\sigma}_h$ as this computed flux can serve directly as the equilibrated flux reconstruction of Definition 3.1. Flux equilibration following Definition 3.4 is useless here (and unfeasible as (3.9) does not hold true in general); remark also that we directly have (3.14) by (4.6b).

The original potential approximation $\bar{u}_h$ has low regularity (it is only piecewise constant in lowest-order methods); local postprocessing is usually employed to improve it. In particular, following Arnold and Brezzi [13], Arbogast and Chen [12], and [71], there exists for each couple $\mathbf{V}_h \times Q_h$ a piecewise polynomial space M_h such that $u_h \in M_h$ can be prescribed by

$$\Pi_{Q_h(K)}(u_h|_K)=\bar{u}_h|_K \qquad \forall K\in\mathcal{T}_h, \tag{4.7a}$$

$$\Pi_{\mathbf{V}_h(K)}((-\nabla u_h)|_K)=\boldsymbol{\sigma}_h|_K \qquad \forall K\in\mathcal{T}_h, \tag{4.7b}$$

where $\Pi_{Q_h(K)}$ is the $L^2(K)$-orthogonal projection on $Q_h(K)$ and $\Pi_{\mathbf{V}_h(K)}$ is the $[L^2(K)]^2$-orthogonal projection on $\mathbf{V}_h(K)$. Plugging (4.7) into (4.6a), it follows that

$$-(\nabla u_h,\mathbf{v}_h)-(u_h,\nabla\cdot\mathbf{v}_h)=0 \qquad \forall \mathbf{v}_h\in\mathbf{V}_h.$$

An immediate consequence of the Green theorem and the structure of $\mathbf{V}_h$ is that

$$\langle [\![u_h]\!], v_h\rangle_e = 0 \qquad \forall e\in\mathcal{E}_h,\ \forall v_h\in\mathbf{V}_h\cdot\mathbf{n}_e(e), \tag{4.8}$$

i.e., the orthogonality of the jumps of u_h to all polynomials from $\mathbf{V}_h\cdot\mathbf{n}$. We let p denote the polynomial degree of functions in M_h, so that u_h, as throughout this paper, is a p-degree piecewise polynomial. With respect to the present a posteriori analysis, the crucial feature is that (4.8) implies (3.23).

For u_h of (4.7), the upper bound of Theorem 3.3 holds true, with $\boldsymbol{\sigma}_h$ obtained directly from (4.6) and s_h constructed by Definition 3.7 or Remark 3.9. The local lower bound (3.39) holds true but (3.38) cannot be verified, as $\boldsymbol{\sigma}_h$ was not constructed from u_h by Definition 3.4. This, fortunately, does not appear obstructive, as $\|\nabla u_h + \boldsymbol{\sigma}_h\|$ by (4.7b) takes small values and can be seen as a numerical quadrature. Alternatively, proceeding as in [73], we may estimate simultaneously the error in both the flux and potential approximations $\boldsymbol{\sigma}_h$ and u_h. This yields

$$\begin{aligned}\|\nabla(u-u_h)\|^2 + \|\nabla u + \boldsymbol{\sigma}_h\|^2 \le & \sum_{K\in\mathcal{T}_h}\left(\|\nabla u_h + \boldsymbol{\sigma}_h\|_K + \frac{h_K}{\pi}\|f - \Pi_{Q_h} f\|_K\right)^2 \\ & + \sum_{K\in\mathcal{T}_h}\|\nabla(u_h - s_h)\|_K^2 \\ & + \sum_{K\in\mathcal{T}_h}\|\nabla s_h + \boldsymbol{\sigma}_h\|_K^2 + \sum_{K\in\mathcal{T}_h}\left(\frac{h_K}{\pi}\|f - \Pi_{Q_h} f\|_K\right)^2.\end{aligned}$$

The efficiency result is then derived by using (3.39) for $\|\nabla(u_h - s_h)\|_K$, $\|\nabla s_h + \boldsymbol{\sigma}_h\|_K \le \|\nabla(u_h - s_h)\|_K + \|\nabla u_h + \boldsymbol{\sigma}_h\|_K$, and $\|\nabla u_h + \boldsymbol{\sigma}_h\|_K \le \|\nabla(u-u_h)\|_K + \|\nabla u + \boldsymbol{\sigma}_h\|_K$.

5 Concluding remarks

For any numerical method for the approximation of (2.1), its stiffness matrix needs to be assembled. From a practical viewpoint, the present a posteriori error estimates can be seen similarly: we need to assemble the block-diagonal matrix with (3.10)/(3.17) for each mesh vertex as entry. Then the computation of the degrees of freedom of the flux and potential reconstructions corresponds to solving a block-diagonal system (or to a matrix-vector multiplication if the inverse of the block-diagonal matrix is prepared). Similarly, the actual evaluation of the estimators of Theorem 3.3 can be implemented as a matrix-vector multiplication formula stemming from the appropriate quadrature rule and the above degrees of freedom. Thus, the slightly increased cost of this approach seems to be largely compensated by its advantages: it offers a unified setting for a large spectrum of numerical methods, a guaranteed upper bound, a robust lower bound with respect to polynomial degree, and no parameter to tune. Moreover different error components can be distinguished, see [42] and the references therein, leading to fully adaptive strategies with adaptive stopping criteria for linear and nonlinear solvers, adaptive time step choice, and adaptive mesh refinement. The present theory extends immediately to d space dimensions, $d \ge 3$, except for the potential reconstruction in Definition 3.7 or Remark 3.9, which are the subject of ongoing work. Numerical experiments as a part of *hp*-refinement strategies relying on the present novel estimators are in preparation.

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