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$\mathcal{A}$-quasiconvexity and weak lower semicontinuity of integral functionals

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Abstract. We state necessary and sufficient conditions for weak lower semicontinuity of $u \mapsto \int_\Omega h(x, u(x))\,\mathrm{d}x$ where $|h(x,s)| \le C(1+|s|^p)$ is continuous and possesses a recession function, and $u \in L^p(\Omega;\mathbb{R}^m)$, $p > 1$, lives in the kernel of a constant-rank first-order differential operator $\mathcal{A}$ which admits an extension property. Our newly defined notion coincides for $\mathcal{A} = \text{curl}$ with quasiconvexity at the boundary due to J.M. Ball and J. Marsden. Moreover, we give an equivalent condition for weak lower semicontinuity of the above functional along sequences weakly converging in $L^p(\Omega;\mathbb{R}^m)$ and approaching the kernel of $\mathcal{A}$ even if $\mathcal{A}$ does not have the extension property.

Key words. $\mathcal{A}$-quasiconvexity, concentrations, oscillations.

AMS (MOS) subject classification. 49J45, 35B05

1 Introduction

In this paper, we investigate the influence of concentration effects generated by sequences $\{u_k\}_{k\in\mathbb{N}} \subset L^p(\Omega;\mathbb{R}^m)$, which satisfy a linear differential constraint $\mathcal{A}u_k = 0$, or $\mathcal{A}u_k \to 0$ in $W^{-1,p}(\Omega;\mathbb{R}^d)$, $1 < p < +\infty$, where $\mathcal{A}$ is a first-order linear differential operator, on weak lower semicontinuity of integral functionals. To the best of our knowledge, the first such results were proved in [22] for nonnegative integrands. In this case, the crucial necessary and sufficient condition ensuring this property is the so-called $\mathcal{A}$-quasiconvexity; cf. (2.5) below. However, if we refrain from considering only nonnegative integrands, this condition is not necessarily sufficient. A prominent example is $\mathcal{A}$=curl, i.e., u has a potential. It is well known that the weak lower semicontinuity of $I(u) := \int_\Omega h(x, u(x))\,\mathrm{d}x$ for $|h(x,u)| \le C(1+|u|^p)$ (i.e. possibly negative and noncoercive) strongly depends, besides (Morrey's) quasiconvexity, also on the behavior of $h(\cdot, s)$ on the boundary of Ω. This was first observed by N. Meyers [34] and then elaborated more explicitly in [31]. Moreover, it turns out that for a special case where $h(x,\cdot)$ possesses a recession function, the precise condition is its so-called quasiconvexity at the boundary [6, 33]. Namely, if $\{u_k\}_{k\in\mathbb{N}} \subset L^p(\Omega;\mathbb{R}^m)$ is a weakly converging sequence, concentrations of $\{|u_k|^p\}_{k\in\mathbb{N}} \subset L^1(\Omega;\mathbb{R}^m)$ at the boundary of Ω can

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destroy weak lower semicontinuity. We refer to [25, 26] for a thorough analysis of oscillation and concentration effects in the gradient (curl-free) case. Hence, it is obvious that one should expect, besides $\mathcal{A}$-quasiconvexity, another condition to guarantee weak lower semicontinuity. Here we isolate an integral condition which additionally to $\mathcal{A}$-quasiconvexity is necessary and sufficient for I to be weakly lower semicontinuous along "asymptotically" $\mathcal{A}$-free sequences. It has already been observed in [19] that concentrations of the sequence at the boundary of the domain are exactly the reason for possible failure of this property. In comparison with the gradient case, the $\mathcal{A}$-free setting brings a few subtle features. First of all, we cannot always expect to have a continuous linear extension operator preserving the $\mathcal{A}$-free property at our disposal even for very smooth domains. Secondly, having Fourier analysis in its background, the treatment of problems with differential constraints typically relies on periodic test functions. On the other hand, (point) concentrations are closely related to sequences with vanishing support and values tending to infinity. This dilemma is resolved below by allowing for test functions which are in the kernel of the operator only approximately. As a result we get the condition stated in Definition 3.3 which precisely describes the behavior of the integrand at the boundary to ensure weak lower semicontinuity. The price we pay is that our condition is natural (at least as far as necessity is concerned) for sequences that are $\mathcal{A}$-free only in an asymptotical sense. For a full characterization of weak lower semicontinuity along genuinely $\mathcal{A}$-free sequences, we were forced to *assume* the existence of an $\mathcal{A}$-free extension operator in L^p, and in this case, we end up with a slightly modified condition given in Definition 3.1. Some links between those two settings are discussed in the final section.

Let us emphasize that variational problems with differential constraints naturally appear in hyperelasticity, electromagnetism, or in micromagnetics [13, 38, 39]. The concept of $\mathcal{A}$-quasiconvexity goes back to [11] and has been proved to be useful as a unified approach to variational problems with differential constraints. We refer to [9] for results concerning homogenization and to [21] for weak* lower semicontinuity results for functionals with nonstandard growth. The paper [40] treats the case of an operator $\mathcal{A}$ with nonconstant coefficients and the recent work [2] analyzes lower semicontinuity of functionals with linearly growing integrands. See also a very recent paper [3] where generalized Young measures were characterized in the $\mathcal{A}$-free setting. Finally, first results on $\mathcal{A}$-quasiaffine functions and weak continuity appeared recently in [24].

The plan of the paper is as follows. We first recall some needed definitions and results in Section 2. Our newly derived conditions which, together with $\mathcal{A}$-quasiconvexity precisely characterize weak lower semicontinuity are studied in Section 3. The main results are summarized in Theorem 3.8 and Theorem 3.14. After the concluding remarks in the final section, some auxiliary material is provided in the appendix.

2 Preliminaries

We recall some measure theory results and set the notation. Let X be a topological space. We denote by $C(X)$ the space of real-valued continuous functions in X. If X is a locally compact space then $C_0(X)$ denotes the closure of the subspace of $C(X)$ of functions with compact support. By the Riesz Representation Theorem, the dual space to $C_0(X)$, $C_0(X)'$, is isometrically isomorphic with $\mathcal{M}(X)$, the linear space of finite Radon measures supported on X, normed by the total variation. Moreover, if X is compact then the dual space to $C(X)$, $C(X)'$, is isometrically isomorphic with $\mathcal{M}(X)$. A positive Radon measure $\mu \in \mathcal{M}(X)$ with $\mu(X) = 1$ is called a probability measure. The n-dimensional Lebesgue measure is denoted $\mathcal{L}^n$.

Unless explicitly stated otherwise, we always work with a bounded domain $\Omega \subset \mathbb{R}^n$ such that $\mathcal{L}^n(\partial\Omega) = 0$, equipped with the Euclidean topology and the n-dimensional Lebesgue measure $\mathcal{L}^n$. $L^p(\Omega, \mathbb{R}^m)$, $1 \le p \le +\infty$, is a standard Lebesgue space. Furthermore, $W^{1,p}(\Omega; \mathbb{R}^m)$, $1 \le p < +\infty$, stands for the usual space of measurable mappings, which together with their first (distributional) derivatives, are integrable with the p-th power. The closure of $C_0^\infty(\Omega; \mathbb{R}^m)$ in $W^{1,p}(\Omega; \mathbb{R}^m)$ is denoted $W_0^{1,p}(\Omega; \mathbb{R}^m)$. If $1 < p < +\infty$ then $W^{-1,p}(\Omega; \mathbb{R}^m)$ denotes the dual space to $W_0^{1,p'}(\Omega; \mathbb{R}^m)$, where $p'^{-1} + p^{-1} = 1$. A sequence $\{u_k\}_{k\in\mathbb{N}}$ converges to zero in measure if $\mathcal{L}^n(\{x \in \Omega;\ u_k(x) \neq 0\}) \to 0$ as $k \to \infty$.

We say that $v \in \Upsilon^p(\mathbb{R}^m)$ if there exists a continuous and positively p-homogeneous function $v_\infty : \mathbb{R}^m \to \mathbb{R}$, i.e., $v_\infty(ts) = t^p v_\infty(s)$ for all $t \ge 0$ and $s \in \mathbb{R}^m$, such that

$$\lim_{|s|\to\infty} \frac{v(s) - v_\infty(s)}{|s|^p} = 0 \ . \tag{2.1}$$

Such a function is called the *recession function* of v.

2.1 The operator $\mathcal{A}$ and $\mathcal{A}$-quasiconvexity

Following [22], we consider linear operators $A^{(i)} : \mathbb{R}^m \to \mathbb{R}^d$, $i = 1, \dots, n$, and define $\mathcal{A} : L^p(\Omega; \mathbb{R}^m) \to W^{-1,p}(\Omega; \mathbb{R}^d)$ by

$$\mathcal{A}u := \sum_{i=1}^n A^{(i)} \frac{\partial u}{\partial x_i} \ , \text{where } \ u : \Omega \to \mathbb{R}^m \ ,$$

i.e., for all $w \in W_0^{1,p'}(\Omega; \mathbb{R}^d)$

$$\langle \mathcal{A}u, w\rangle = -\sum_{i=1}^n \int_\Omega A^{(i)} u(x) \cdot \frac{\partial w(x)}{\partial x_i} \,\mathrm{d}x \ .$$

For $w \in \mathbb{R}^n$ we define the linear map

$$\mathbb{A}(w) := \sum_{i=1}^n w_i A^{(i)} \ : \ \mathbb{R}^m \to \mathbb{R}^d \ .$$

Throughout this article, we assume that there is $r \in \mathbb{N} \cup \{0\}$ such that

$$\operatorname{rank} \mathbb{A}(w) = r \text{ for all } w \in \mathbb{R}^n \ , |w| = 1 \ ,$$

i.e., $\mathcal{A}$ has the so-called *constant-rank property.*

Below, we use $\ker \mathcal{A}$ to denote the set of all locally integrable functions u such that $\mathcal{A}u = 0$ in the sense of distributions, i.e., $\int u \cdot \mathcal{A}^* w \, dx = 0$ for all $w \in C^\infty$ compactly supported in the domain, where $\mathcal{A}^*$ is the formal adjoint of $\mathcal{A}$. Of course, this depends on the domain considered, which should be clear from the context. In particular, a periodic function u in the space

$$L^p_{\#}(\mathbb{R}^n;\mathbb{R}^m) := \{u \in L^p_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^m) : \ u \text{ is } Q\text{-periodic}\}$$

is in $\ker \mathcal{A}$ if and only if $\mathcal{A}u = 0$ on $\mathbb{R}^n$. Here and in the following, Q denotes the unit cube $(-1/2, 1/2)^n$ in $\mathbb{R}^n$, and we say that $u : \mathbb{R}^n \to \mathbb{R}^m$ is *Q-periodic* if for all $x \in \mathbb{R}^n$ and all $z \in \mathbb{Z}$

$$u(x+z) = u(x) \ .$$

We will use the following lemmas proved in [22, Lemma 2.14] and [22, Lemma 2.15], respectively.

Lemma 2.1 *(projection onto $\mathcal{A}$-free fields in the periodic setting) There is a linear bounded operator $\mathcal{T} : L^p_{\#}(\mathbb{R}^n;\mathbb{R}^m) \to L^p_{\#}(\mathbb{R}^n;\mathbb{R}^m)$ that vanishes on constant functions, $\mathcal{T}(\mathcal{T}u) = \mathcal{T}u$ for all $u \in L^p_{\#}(\mathbb{R}^n;\mathbb{R}^m)$, and $\mathcal{T}u \in \ker \mathcal{A}$. Moreover, for all $u \in L^p_{\#}(\mathbb{R}^n;\mathbb{R}^m)$ with $\int_Q u(x)\,\mathrm{d}x = 0$ it holds that*

$$\|u - \mathcal{T}u\|_{L^p_{\#}(\mathbb{R}^n;\mathbb{R}^m)} \leq C\|\mathcal{A}u\|_{W^{-1,p}_{\#}(\mathbb{R}^n;\mathbb{R}^d)} \ ,$$

where $C > 0$ is a constant independent of u and $W^{-1,p}_{\#}$ denotes the dual of $W^{1,p'}_{\#}$ ($\frac{1}{p'} + \frac{1}{p} = 1$), the Q-periodic functions in $W^{1,p'}_{loc}(\mathbb{R}^n;\mathbb{R}^m)$ equipped with the norm of $W^{1,p'}(Q;\mathbb{R}^m)$.

Remark 2.2 *For every $w \in W^{-1,p}_{\#}(\mathbb{R}^n)$, we have $\|w\|_{W^{-1,p}(Q)} \leq \|w\|_{W^{-1,p}_{\#}(\mathbb{R}^n)}$. The converse inequality does not hold, not even up to a constant. However, Lemma 2.1 is often applied to (a sequence of) functions supported in a fixed set $G \subset\subset Q$ (up to periodicity, of course). One can always find a constant $C = C(\Omega, p, G)$ such that*

$$\|\mathcal{A}u\|_{W^{-1,p}_{\#}(\mathbb{R}^n;\mathbb{R}^d)} \leq C\|\mathcal{A}u\|_{W^{-1,p}(Q;\mathbb{R}^d)} \quad \textit{for every } u \in L^p(Q;\mathbb{R}^m) \textit{ with } u = 0 \textit{ a.e. on } Q \setminus G.$$

To achieve this, the Q-periodic test functions used in the definition of the norm in $W^{-1,p}_{\#}$ can be multiplied with a fixed cut-off function $\eta \in C^\infty_0(Q;[0,1])$ with $\eta = 1$ on G to make them admissible (i.e., elements of $W^{1,p'}_0(Q)$) for the supremum defining the norm in $W^{-1,p}$. This enlarges their norm in $W^{1,p'}$ at most by a constant factor which only depends on p and $\|\nabla\eta\|_{L^\infty(Q)}$ (and thus the distance of G to ∂Q).

Lemma 2.3 *(Decomposition Lemma) Let $\Omega \subset \mathbb{R}^n$ be bounded and open, $1 < p < +\infty$, and let $\{u_k\} \subset L^p(\Omega;\mathbb{R}^m)$ be bounded and such that $\mathcal{A}u_k \to 0$ in $W^{-1,p}(\Omega;\mathbb{R}^d)$ strongly, and $u_k \rightharpoonup u$ in $L^p(\Omega;\mathbb{R}^m)$ weakly. Then there is a sequence $\{z_k\}_{k\in\mathbb{N}} \subset L^p(\Omega;\mathbb{R}^m) \cap \ker \mathcal{A}$, $\{|z_k|^p\}$ is equiintegrable in $L^1(\Omega)$ and $u_k - z_k \to 0$ in measure in Ω.*

We also point out the following simple observation made in the proof of Lemma 2.15 in [22], which is useful to truncate $\mathcal{A}$-free or "asymptotically" $\mathcal{A}$-free sequences:

Lemma 2.4 *Let $\Omega \subset \mathbb{R}^n$ be open and bounded, and let $\{u_k\} \subset L^p(\Omega;\mathbb{R}^m)$ be a bounded sequence such that $\mathcal{A}u_k \to 0$ in $W^{-1,p}(\Omega;\mathbb{R}^d)$ strongly and $u_k \rightharpoonup u$ in $L^p(\Omega;\mathbb{R}^m)$ weakly. Then for every $\eta \in C^\infty(\mathbb{R}^n)$, $\mathcal{A}(\eta u_k) \to 0$ in $W^{-1,p}(\Omega;\mathbb{R}^d)$.*

Proof. $\mathcal{A}(\eta u_k) = \eta \mathcal{A}u_k + \sum_{i=1}^n u_k A^{(i)} \frac{\partial \eta}{\partial x_i} \to 0$ in $W^{-1,p}$, the second term due to the compact embedding of L^p into $W^{-1,p}$. □

Definition 2.5 *(see [22, Def. 3.1, 3.2]) We say that a continuous function $v : \mathbb{R}^m \to \mathbb{R}$, $|v| \leq C(1 + |\cdot|^p)$ for some $C > 0$, is $\mathcal{A}$-quasiconvex if for all $s_0 \in \mathbb{R}^m$ and all $\varphi \in L^p_\#(Q;\mathbb{R}^m) \cap \ker \mathcal{A}$ with $\int_Q \varphi(x)\,\mathrm{d}x = 0$ it holds*

$$v(s_0) \leq \int_Q v(s_0 + \varphi(x))\,\mathrm{d}x \ .$$

Besides curl-free fields, admissible examples of $\mathcal{A}$-free mappings include solenoidal fields where $\mathcal{A} = \mathrm{div}$ and higher-order gradients where $\mathcal{A}u = 0$ if and only if $u = \nabla^{(s)}\varphi$ for some $\varphi \in W^{s,p}(\Omega;\mathbb{R}^\ell)$, and some $s \in \mathbb{N}$ (for more details see Subsection 4.3, where $s = 2$).

2.2 Weak lower semicontinuity

Let $I : L^p(\Omega;\mathbb{R}^m) \to \mathbb{R}$ be defined as

$$I(u) := \int_\Omega h(x, u(x))\,\mathrm{d}x \ . \tag{2.2}$$

We often restrict I to $\ker \mathcal{A}$ below.

Definition 2.6

(i) We say that a sequence $\{u_k\} \in L^p(\Omega;\mathbb{R}^m)$ is asymptotically $\mathcal{A}$-free if $\|\mathcal{A}u_k\|_{W^{-1,p}(\Omega;\mathbb{R}^m)} \to 0$ as $k \to \infty$.

(ii) A functional I as in (2.2) is called weakly sequentially lower semicontinuous (wslsc) along asymptotically $\mathcal{A}$-free sequences in $L^p(\Omega;\mathbb{R}^m)$ if $\liminf_{k\to\infty} I(u_k) \geq I(u)$ for all such sequences that weakly converge to some limit u in L^p.

We have the following result which was proved in [19, Theorem 2.4] in a slightly less general version. However, its original proof directly extends to this setting.

Theorem 2.7 *Let $h : \bar{\Omega} \times \mathbb{R}^m \to \mathbb{R}$ be continuous such that $h(x, \cdot) \in \Upsilon^p(\mathbb{R}^m)$ for all $x \in \bar{\Omega}$ and $h(x, \cdot)$ is $\mathcal{A}$-quasiconvex for almost every $x \in \Omega$, $1 < p < +\infty$. Then I is sequentially weakly lower semicontinuous in $L^p(\Omega; \mathbb{R}^m) \cap \ker \mathcal{A}$ if and only if for any bounded sequence $\{u_k\} \subset L^p(\Omega; \mathbb{R}^m) \cap \ker \mathcal{A}$ such that $u_k \to 0$ in measure there is*

$$\liminf_{k \to \infty} I(u_k) \geq I(0) \ . \tag{2.3}$$

The statement of Theorem 2.7 remains valid if we replace the sequences in $\ker \mathcal{A}$ with asymptotically $\mathcal{A}$-free sequences.

Theorem 2.8 *With h and p as in Theorem 2.7, I is wslsc along asymptotically $\mathcal{A}$-free sequences in $L^p(\Omega; \mathbb{R}^m)$ if and only if (2.3) holds for any bounded, asymptotically $\mathcal{A}$-free sequence $\{u_k\} \subset L^p(\Omega; \mathbb{R}^m)$ such that $u_k \to 0$ in measure.*

Proof. We only point out the differences to the proof [19, Theorem 2.4]. First, the result there is stated only for functions h of product form $h(x, \xi) = g(x)v(\xi)$, but as in the case of Theorem 2.7, it works verbatim also for our slightly more general class. "Only if" is trivial as before. For "if", we also rely on splitting a given sequence into a purely oscillating (p-equiintegrable) part and a purely concentrating part, which is still a straightforward application of the decomposition lemma (Lemma 2.3). Notice that the purely oscillating part $\{z_k\}$ lives in $\ker \mathcal{A}$, even if the sequence we started with is only asymptotically $\mathcal{A}$-free. The rest of the proof is completely analogous to the corresponding one in [19]. $\square$

Remark 2.9
(i) It follows from [19, (5.1)] that (2.3) can be replaced by

$$\liminf_{k \to \infty} I_\infty(u_k) \geq I_\infty(0) = 0, \quad \text{where } I_\infty(u) := \int_\Omega h_\infty(x, (u(x))\,\mathrm{d}x,$$

with $h_\infty(x, \cdot)$ denoting the recession function of $h(x, \cdot)$.
(ii) In fact, having an integrand $(x, s) \mapsto h(x, s)$ which is $\mathcal{A}$-quasiconvex in the second variable, weak lower semicontinuity can only fail due to sequences concentrating large values on small sets, and it even suffices to test that with sequences $\{u_k\}$ which tend to zero in measure and concentrate at the boundary in the sense that $\{|u_k|^p\}$ converges weakly to a measure $\sigma \in \mathcal{M}(\bar{\Omega})$ with $\sigma(\partial\Omega) > 0$.*

3 $\mathcal{A}$-quasiconvexity at the boundary

The two conditions introduced below play a crucial role in our characterization of weak lower semicontinuity of integral functionals. They are typically applied to the recession function h_∞ of an integrand h with p-growth.

Before we state them, we fix some additional notation frequently used in what follows:

$$L_0^p(\Omega;\mathbb{R}^m) := \{u \in L^p(\Omega;\mathbb{R}^m);\ \operatorname{supp} u \subset \Omega\},$$
$$C_{\mathrm{hom}}^p(\mathbb{R}^m) := \{v \in C(\mathbb{R}^m);\ v \text{ is positively } p\text{-homogeneous}\}.$$

A norm in C_{hom}^p is given by the supremum norm taken on the unit sphere in $\mathbb{R}^m$. Moreover, whenever a larger domain comes into play, functions in $L_0^p(\Omega;\mathbb{R}^m)$ are understood to be extended by zero to $\mathbb{R}^n \setminus \Omega$ without changing notation.

Definition 3.1 *We say that $h_\infty \in C(\bar{\Omega}; C_{hom}^p(\mathbb{R}^m))$ is $\mathcal{A}$-quasiconvex at the boundary ($\mathcal{A}$-qcb) at $x_0 \in \partial\Omega$ if for every $\varepsilon > 0$ there are $\delta > 0$ and $\alpha > 0$ such that*

$$\int_{B(x_0,\delta)\cap\Omega} h_\infty(x,u(x)) + \varepsilon|u(x)|^p \,\mathrm{d}x \geq 0 \tag{3.1}$$

for every $u \in L_0^p(B(x_0,\delta);\mathbb{R}^m)$ with $\|\mathcal{A}u\|_{W^{-1,p}(\mathbb{R}^n;\mathbb{R}^d)} < \alpha\|u\|_{L^p(B(x_0,\delta)\cap\Omega;\mathbb{R}^m)}$.

Remark 3.2 *Above, $\mathcal{A}u$ is measured in the norm of $W^{-1,p}(\mathbb{R}^n;\mathbb{R}^d)$, but $\mathbb{R}^n$ can be replaced by any domain S_δ compactly containing $B(x_0,\delta)$, because for distributions supported on $B(x_0,\delta)$, the norms of $W^{-1,p}(\mathbb{R}^n;\mathbb{R}^d)$ and $W^{-1,p}(S_\delta;\mathbb{R}^d)$ are equivalent, with constants depending on δ. The latter is not a problem since α depends on ε and thus may also depend on $\delta = \delta(\varepsilon)$. In particular, $\mathcal{A}$-qcb can also be defined using the class of all $u \in L_0^p(B(x_0,\frac{\delta}{2});\mathbb{R}^m)$ with $\|\mathcal{A}u\|_{W^{-1,p}(B(x_0,\delta);\mathbb{R}^d)} < \alpha\|u\|_{L^p(B(x_0,\delta)\cap\Omega;\mathbb{R}^m)}$.*

Due to the fact that the test functions u and $\mathcal{A}u$ in Definition 3.1 are required to be defined on $B(x_0,\delta)$, a set which is not fully contained in Ω, $\mathcal{A}$-qcb as defined above is only natural if there is an $\mathcal{A}$-free extension operator on $L^p(\Omega;\mathbb{R}^m)$, cf. Definition 3.10 below. However, the existence of such an extension operator may require sufficient smoothness of $\partial\Omega$, and, worse, it strongly depends on $\mathcal{A}$. For instance, on the one hand, if $\partial\Omega$ is of class C^1, the extension operators are available for $\mathcal{A} = \operatorname{curl}$ and $\mathcal{A} = \operatorname{div}$ (essentially using a partition of unity and extension by a suitable reflection), but on the other hand, if we choose $\mathcal{A}$ to be the differential operator of the Cauchy–Riemann system ($n = m = 2$, identifying $\mathbb{C}$ with $\mathbb{R}^2$), no such extension operator exists, since holomorphic functions with singularities at the boundary of Ω can never be extended to holomorphic functions on a larger set including the singular point[5].

To circumvent this unpleasant dependence on the existence of $\mathcal{A}$-free extensions, we also introduce the following variant of $\mathcal{A}$-qcb, which is not equivalent in general. It turns out that it is related to weak lower semicontinuity along asymptotically $\mathcal{A}$-free sequences, instead of weak lower semicontinuity in $L^p \cap \ker \mathcal{A}$:

Definition 3.3 *We say that $h_\infty \in C(\bar{\Omega}; C^p_{hom}(\mathbb{R}^m))$ is $W^{-1,p}$-asymptotically $\mathcal{A}$-quasiconvex at the boundary (a$\mathcal{A}$-qcb) at $x_0 \in \partial\Omega$ if for every $\varepsilon > 0$ there are $\delta > 0$ and $\alpha > 0$ such that*

$$\int_{B(x_0,\delta)\cap\Omega} h_\infty(x, u(x)) + \varepsilon |u(x)|^p \,\mathrm{d}x \geq 0 \tag{3.2}$$

for every $u \in L^p_0(B(x_0,\delta); \mathbb{R}^m)$ with $\|\mathcal{A}u\|_{W^{-1,p}(\Omega;\mathbb{R}^d)} < \alpha \|u\|_{L^p(\Omega;\mathbb{R}^m)}$.

Remark 3.4 *For the reasons already outlined in Remark 3.2, the class of test functions above can be replaced by the set of all $u \in L^p_0(B(x_0, \frac{\delta}{2}); \mathbb{R}^m)$ such that $\|\mathcal{A}u\|_{W^{-1,p}(\Omega\cap B(x_0,\delta);\mathbb{R}^d)} < \alpha \|u\|_{L^p(\Omega\cap B(x_0,\delta);\mathbb{R}^m)}$.*

Notice that the smallness of $\mathcal{A}u$ is now measured in the $W^{-1,p}$-norm on Ω instead of a larger set as in Definition 3.1. To calculate this norm, we seek the largest possible value of $\int_{\mathbb{R}^n} u \cdot \mathcal{A}^*\varphi \mathrm{d}x$ among all functions $\varphi \in W^{1,p'}_0(\Omega; \mathbb{R}^d)$ with norm not larger than 1 in that space. In particular, each admissible φ is now required to vanish on $\partial\Omega$. This does make a difference, which can be easily be checked in a simplified setting: if we let B denote a ball in $\mathbb{R}^n$ and $D = \{x \in B | x \cdot \nu < 0\}$ the half ball in B determined by some (arbitrary but fixed) vector ν, then the norms of the dual spaces of $X := W^{1,p'}_0(D)$ and $Y := \{v \in W^{1,p'}_0(D) \mid v = u|_D$ for some $u \in W^{1,p'}_0(B)\}$ are *not equivalent* on their mutual subset $L^p_0(B)$.

Remark 3.5 *In Definition 3.1 as well as in Definition 3.3, if for a given $\varepsilon > 0$ the estimate holds for some $\delta > 0$, then it also holds for any $\tilde{\delta} < \delta$ in place of δ. Hence, both $\mathcal{A}$-qcb and a$\mathcal{A}$-qcb are local properties of h_∞ in the x variable, since it suffices to study arbitrarily small neighborhoods of x_0.*

We now focus on the link between $\mathcal{A}$-quasiconvexity at the boundary and weak lower semicontinuity. An equivalent variant of a$\mathcal{A}$-qcb and $\mathcal{A}$-qcb is discussed at the end of each subsection, respectively.

[5] In terms of integrability, the weakest possible point singularity of an elsewhere holomorphic function locally behaves like $z \mapsto 1/z$ ($z \in \mathbb{C} \setminus \{0\}$), which is not even in $L^1(\Omega)$ if $0 \in \partial\Omega$ and $\partial\Omega$ is smooth in a neighborhood, but using an appropriately weighted series of singular terms, each with a singularity slightly outside Ω, accumulating at a boundary point, examples in L^p are possible for arbitrary $1 \leq p < \infty$.

3.1 Asymptotically $\mathcal{A}$-free sequences

Proposition 3.6 *Let $h_\infty \in C(\bar{\Omega}; C^p_{hom}(\mathbb{R}^m))$. Then $I_\infty(u) := \int_\Omega h_\infty(x, u(x))\,\mathrm{d}x$ is weakly sequentially lower semicontinuous along asymptotically $\mathcal{A}$-free sequences in $L^p(\Omega; \mathbb{R}^m)$ if and only if*
(i) h_∞ is a$\mathcal{A}$-qcb at every $x_0 \in \partial\Omega$ and
(ii) $h_\infty(x, \cdot)$ is $\mathcal{A}$-quasiconvex at almost every $x \in \Omega$.

Proof. "only if": We show that a$\mathcal{A}$-qcb at $x_0 \in \partial\Omega$ is a necessary condition; the necessity of (ii) is well known. Suppose that h_∞ is not a$\mathcal{A}$-qcb at $x_0 \in \partial\Omega$. This means that there is $\varepsilon > 0$ such that for every $k \in \mathbb{N}$ there exists $u_k \in L^p_0(B(x_0, \frac{1}{k}); \mathbb{R}^m)$ with $\|\mathcal{A}u\|_{W^{-1,p}(\Omega;\mathbb{R}^d)} \leq \frac{1}{k}\|u_k\|_{L^p(\Omega;\mathbb{R}^m)}$ and

$$\int_{B(x_0,\frac{1}{k})\cap\Omega} h_\infty(x, u_k(x)) + \varepsilon|u_k(x)|^p\,\mathrm{d}x < 0\ .$$

In particular, u_k cannot be the zero function. Denote

$$\hat{u}_k := u_k/\|u_k\|_{L^p(B(x_0,\frac{1}{k})\cap\Omega;\mathbb{R}^m)} = u_k/\|u_k\|_{L^p(\Omega;\mathbb{R}^m)}.$$

Then $\hat{u}_k \in L^p_0(\Omega; \mathbb{R}^m)$ with $\|\hat{u}_k\|_{L^p} = 1$ and $\|\mathcal{A}\hat{u}_k\|_{W^{-1,p}(\Omega;\mathbb{R}^d)} \leq 1/k$. In addition, $\hat{u}_k$ vanishes outside of $B(x_0, \frac{1}{k})$, so that $\hat{u}_k \to 0$ in measure and weakly in $L^p(B(x_0, 1); \mathbb{R}^m)$. However,

$$\liminf_{k\to\infty} \int_\Omega h_\infty(x, \hat{u}_k(x))\,\mathrm{d}x \leq -\varepsilon < 0 = \int_\Omega h_\infty(x, 0)\,dx\ .$$

This means that $u \mapsto \int_\Omega h_\infty(x, u(x))\,\mathrm{d}x$ is not lower semicontinuous along $\{\hat{u}_k\}$.

"if": Let us now prove the sufficiency. Let $\{u_k\}_{k\in\mathbb{N}} \subset L^p(\Omega; \mathbb{R}^m)$ be an asymptotically $\mathcal{A}$-free sequence weakly converging to some u in L^p. As a first step, we assume that in addition, $\{u_k\}$ is purely concentrating in the sense that $u_k \rightharpoonup 0$ in $L^p(\Omega; \mathbb{R}^m)$ and $\mathcal{L}^n(\{x \in \Omega;\ u_k(x) \neq 0\}) \to 0$ as $k \to \infty$. It suffices to show that every subsequence of $\{u_k\}$ admits another subsequence along which I is lower semicontinuous. Using DiPerna-Majda measures as in (A.4) in the Appendix, and we get that for every $\delta > 0$, up to a subsequence,

$$\begin{aligned}\lim_{k\to\infty} &\int_{B(x_0,\delta)\cap\Omega} h_\infty(x, u_k(x))\,\mathrm{d}x \\ &= \int_{\overline{B(x_0,\delta)\cap\Omega}} \int_{\beta_\mathcal{S}\mathbb{R}^m\setminus\mathbb{R}^m} \frac{h_\infty(x,s)}{1+|s|^p}\mathrm{d}\lambda_x(s)\mathrm{d}\pi(x)\end{aligned} \tag{3.3}$$

for some $(\pi, \lambda) \in \mathcal{DM}^p_\mathcal{S}(\Omega; \mathbb{R}^m)$.

In the following, we only consider those $\delta > 0$ for which $\pi(\partial B(x_0, \delta) \cap \bar{\Omega}) = 0$, which is certainly true for a dense subset. Let $\{\eta_\ell\}_{\ell\in\mathbb{N}} \subset C^\infty_0(B(x_0, \delta))$ such that $0 \leq \eta_\ell \leq 1$ and $\eta_\ell \to \chi_{B(x_0,\delta)}$ as $\ell \to \infty$. Here, $\chi_{B(x_0,\delta)}$ is the characteristic function of $B(x_0, \delta)$ in $\mathbb{R}^n$ and $x_0 \in \partial\Omega$. By Lemma 2.4, $\mathcal{A}(\eta_\ell u_k) \to 0$ in $W^{-1,p}(\Omega; \mathbb{R}^d)$ as $k \to \infty$, for fixed ℓ. Take $\varepsilon > 0$, $x_0 \in \partial\Omega$, $\alpha, \delta > 0$ as in

Definition 3.3 and set $w_k := \eta_{\ell(k)} u_k$, where $\ell(k)$ tends to ∞ sufficiently slowly as $k \to \infty$ so that $\mathcal{A} w_k \to 0$ in $W^{-1,p}(\Omega;\mathbb{R}^d)$ and reasoning as in [19, Appendix], using that $\pi(\partial B(x_0,\delta)\cap\bar{\Omega}) = 0$, we see that $\{w_k\}$ also generates (π,λ), at least on $\overline{B(x_0,\delta)} \cap \Omega$. If w_k strongly converges to zero in L^p,

$$0 \le \lim_{k\to\infty} \int_{B(x_0,\delta)\cap\Omega} h_\infty(x, w_k(x)) + \varepsilon |w_k(x)|^p \, \mathrm{d}x, \tag{3.4}$$

by continuity (in that case, we even get equality). Otherwise, a subsequence of $\{w_k\}$ (not relabeled) is bounded away from zero in L^p, and since $\mathcal{A} w_k \to 0$ in $W^{-1,p}$, this implies that $\|\mathcal{A} w_k\|_{W^{-1,p}} \le \alpha \|w_k\|_{L^p}$, at least for k large enough. Hence, w_k is admissible as a test function in (3.2), and we end up again with (3.4). The right-hand side of (3.4) can be expressed using (A.4):

$$\begin{aligned} &\lim_{k\to\infty} \int_{B(x_0,\delta)\cap\Omega} h_\infty(x, w_k(x)) + \varepsilon |w_k(x)|^p \, \mathrm{d}x \\ &\quad = \int_{\overline{B(x_0,\delta)\cap\Omega}} \int_{\beta_{\mathcal{R}}\mathbb{R}^m \setminus \mathbb{R}^m} \frac{h_\infty(x,s) + \varepsilon |s|^p}{1+|s|^p} \mathrm{d}\lambda_x(s) \mathrm{d}\pi(x) \ . \end{aligned}$$

Hence,

$$0 \le \pi(\overline{B(x_0,\delta)\cap\Omega})^{-1} \int_{\overline{B(x_0,\delta)\cap\Omega}} \int_{\beta_{\mathcal{R}}\mathbb{R}^m \setminus \mathbb{R}^m} \frac{h_\infty(x,s) + \varepsilon |s|^p}{1+|s|^p} \mathrm{d}\lambda_x(s) \mathrm{d}\pi(x) \ .$$

Therefore, by the Lebesgue-Besicovitch differentiation theorem (see [17], e.g.) and by taking into account that $\varepsilon > 0$ is arbitrary we get that for π-almost every $x_0 \in \partial\Omega$

$$0 \le \int_{\beta_{\mathcal{R}}\mathbb{R}^m \setminus \mathbb{R}^m} \frac{h_\infty(x_0,s)}{1+|s|^p} \mathrm{d}\lambda_{x_0}(s) \ .$$

This together with Theorem A.2 and (A.4) implies that the inner integral on the right-hand side of (3.3) is nonnegative for π-almost every $x_0 \in \bar{\Omega}$. As a consequence, I_∞ is lower semicontinuous along $\{u_k\}$, i.e., all purely concentrating sequences. By Theorem 2.8 and Remark 2.9 (ii), we conclude that $u \mapsto \int_\Omega h(x,u(x))\,\mathrm{d}x$ is weakly lower semicontinuous along arbitrary asymptotically $\mathcal{A}$-free sequences. $\square$

It is possible to formulate several equivalent variants of the definition of $\mathcal{A}$-quasiconvexity at the boundary. In particular, the following proposition shows that the first variable of h can be "frozen" in Definition 3.3.

Proposition 3.7 *A function $h_\infty \in C(\bar{\Omega}; C^p_{hom}(\mathbb{R}^m))$ is $\mathcal{A}$-qcb at $x_0 \in \partial\Omega$ if and only if for all $\varepsilon > 0$ there are $\delta > 0$, $\alpha > 0$ such that for all $u \in L^p_0(B(x_0,\delta);\mathbb{R}^m)$ with $\|\mathcal{A}u\|_{W^{-1,p}} < \alpha \|u\|_{L^p}$ (all norms taken on $B(x_0,\delta)\cap\Omega$),*

$$\int_{B(x_0,\delta)\cap\Omega} h_\infty(x_0, u(x)) + \varepsilon |u(x)|^p \, \mathrm{d}x \ge 0 \ . \tag{3.5}$$

Proof. Let $\varepsilon > 0$ and recall that if (3.1) holds for some $\delta > 0$ then it holds also for any $0 < \tilde{\delta} < \delta$ in the place of δ. We have

$$\left| \int_{B(x_0,\delta)\cap\Omega} h_\infty\left(x, \frac{u(x)}{|u(x)|}\right) |u(x)|^p \,\mathrm{d}x - \int_{B(x_0,\delta)\cap\Omega} h_\infty\left(x_0, \frac{u(x)}{|u(x)|}\right) |u(x)|^p \,\mathrm{d}x \right|$$
$$\leq \int_{B(x_0,\delta)\cap\Omega} \mu(|x-x_0|, 0)|u(x)|^p \,\mathrm{d}x \leq M(\delta) \int_{B(x_0,\delta)\cap\Omega} |u(x)|^p \,\mathrm{d}x \ ,$$

where $\mu : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is a continuous modulus of continuity of the continuous function h_∞ restricted to the compact set $\bar{\Omega} \times S^{m-1}$ and $M(\delta) := \max_{x\in\overline{B(x_0,\delta)\cap\Omega}} \mu(|x-x_0|, 0)$. In particular, $M(\delta) \to 0$ as $\delta \to 0$. Hence, if (3.1) holds then we have that

$$\int_{B(x_0,\delta)\cap\Omega} h_\infty(x_0, u(x)) + (M(\delta)+\varepsilon)|u(x)|^p \,\mathrm{d}x \geq \int_{B(x_0,\delta)\cap\Omega} h_\infty(x, u(x)) + \varepsilon|u(x)|^p \,\mathrm{d}x \geq 0 \ .$$

This shows that (3.1) implies (3.5). Notice that $M(\delta)+\varepsilon$ can be made arbitrarily small if δ is small enough. The converse implication is proved analogously. $\square$

In view of Remark 2.9, our results obtained so far can be summarized as follows.

Theorem 3.8 *Let $1 < p < +\infty$, and let $h : \bar{\Omega} \times \mathbb{R}^m \to \mathbb{R}$ be continuous and such that $h(x,\cdot) \in \Upsilon^p(\mathbb{R}^m)$ for all $x \in \bar{\Omega}$, with recession function $h_\infty \in C(\overline{\Omega}; C^p_{hom})$. Then I is weakly lower semicontinuous along asymptotically $\mathcal{A}$-free sequences if and only if*
(i) $h(x,\cdot)$ is $\mathcal{A}$-quasiconvex for almost all $x \in \Omega$;
(ii) h_∞ is asymptotically $\mathcal{A}$-quasiconvex at the boundary for all $x_0 \in \partial\Omega$.

From its definition, it is not clear to what extent the notion of a$\mathcal{A}$-qcb depends on the local shape of $\partial\Omega$ near the boundary point under consideration. The proposition below shows that at least for domains with smooth boundary, the domain enters only via the outer normal to $\partial\Omega$ at this point.

Proposition 3.9 *Assume that $\Omega \subset \mathbb{R}^n$ has a C^1-boundary in a neighborhood of $x_0 \in \partial\Omega$. Let ν_{x_0} be the outer unit normal to $\partial\Omega$ at x_0 and*

$$D_{x_0} := \{x \in B(0,1) \mid \ x \cdot \nu_{x_0} < 0\}.$$

Then $v \in C^p_{hom}(\mathbb{R}^m)$ is a$\mathcal{A}$-qcb at x_0 if and only if

for every $\varepsilon > 0$ there exists $\beta > 0$ such that

$$\int_{D_{x_0}} v(\varphi(x)) + \varepsilon|\varphi(x)|^p \,\mathrm{d}x \geq 0 \tag{3.6}$$

for every $\varphi \in L^p_0(B(0,\frac{1}{2}); \mathbb{R}^m)$ with $\|\mathcal{A}\varphi\|_{W^{-1,p}(D_{x_0};\mathbb{R}^d)} \leq \beta\|\varphi\|_{L^p(D_{x_0};\mathbb{R}^m)}$.

Proof. Without loss of generality let us assume $x_0 = 0$. We adopt the proof which appeared already in [31] for the gradient case.

"only if": Suppose that v is a$\mathcal{A}$-qcb at 0. Take $\varepsilon > 0$ and get α, $\delta > 0$ such that

$$\int_{B(0,\delta)\cap\Omega} v(u(x)) + \varepsilon|u(x)|^p \,\mathrm{d}x \geq 0 \tag{3.7}$$

for every $u \in L_0^p(B(0, \frac{\delta}{2}); \mathbb{R}^m)$ satisfying $\|\mathcal{A}u\|_{W^{-1,p}(B(0,\delta)\cap\Omega;\mathbb{R}^d)} \leq \alpha\|u\|_{L^p(B(0,\delta)\cap\Omega;\mathbb{R}^m)}$. Introducing the scaling $\Phi_\delta \colon \overline{B(0,\delta)} \ni x \mapsto \delta^{-1}x \in \overline{B(0,1)}$, the inequality (3.7) can be rewritten as

$$\int_{\delta^{-1}(\Omega\cap B(0,\delta))} v(y(x')) + \varepsilon|y(x')|^p \,\mathrm{d}x' \geq 0 \;, \text{ where } y := \delta^{n/p} u \circ \Phi_\delta^{-1} \tag{3.8}$$

Due to the smoothness of the boundary near zero, there exists a transformation $\Psi_\delta \colon \overline{B(0,1)} \to \overline{B(0,1)}$ such that $\Psi_\delta(0) = 0$, $\Psi_\delta(B(0, \frac{1}{2})) = B(0, \frac{1}{2})$ and $\Psi_\delta(D_0) = \delta^{-1}(\Omega \cap B(0,\delta))$, while both Ψ_δ and its inverse Ψ_δ^{-1} converge to the identity in $C^1(\overline{B(0,1)}; \mathbb{R}^n)$ as $\delta \to 0$. Hence, (3.8) leads to

$$\int_{D_0} (v(\varphi(z)) + \varepsilon|\varphi(z)|^p)|\mathrm{det}\mathrm{D}_z\Psi_\delta(z)| \,\mathrm{d}z \geq 0 \;, \tag{3.9}$$

where $\varphi := y \circ \Psi_\delta$ and $[\mathrm{D}_z\Psi_\delta]_{ij} := \partial\Psi_{\delta i}/\partial z_j$ for i, $j = 1, \ldots, n$. Due to the boundedness of $v + \varepsilon|\cdot|^p$ and the (uniform) continuity of the transformation Ψ_δ on the unit sphere, we have the estimate

$$|(v(\varphi(z)) + \varepsilon|\varphi(z)|^p)(|\mathrm{det}\mathrm{D}_z\Psi_\delta(z)| - 1)| \leq \varepsilon|\varphi(z)|^p \;, \tag{3.10}$$

for $\delta > 0$ sufficiently small. Incorporating (3.10) into (3.9), we see that

$$\int_{D_0} (v(\varphi(z)) + 2\varepsilon|\varphi(z)|^p) \,\mathrm{d}z \geq 0 \;.$$

It remains to find some $\beta = \beta(\varepsilon, \delta, \alpha) > 0$, such that for any admissible φ in (3.6), the associated function $u = \delta^{-\frac{n}{p}} \varphi \circ \Psi_\delta^{-1} \circ \Phi_\delta$ is admissible as a test function in (3.7), i.e., we need that $\|\mathcal{A}\varphi\|_{W^{-1,p}(D_0;\mathbb{R}^d)} \leq \beta\|\varphi\|_{L^p(D_0;\mathbb{R}^m)}$ implies that $\|\mathcal{A}u\|_{W^{-1,p}(B(0,\delta)\cap\Omega;\mathbb{R}^d)} \leq \alpha\|u\|_{L^p(B(0,\delta)\cap\Omega;\mathbb{R}^m)}$.

We calculate

$$\begin{aligned}
&\| \mathcal{A}\varphi \|_{W_0^{-1,p}(D_0;\mathbb{R}^d)} \\
&= \sup_{\| w \|_{W_0^{1,p'}(D_0;\mathbb{R}^d)} \leq 1} \sum_{i=1}^n \int_{D_0} A^{(i)}\varphi(z) \cdot \frac{\partial w(z)}{\partial z_i} \mathrm{d}x \\
&= \sup_{\|w\| \leq 1} \sum_{i=1}^n \int_{\Psi_\delta(D_0)} A^{(i)}\varphi(\Psi_\delta^{-1}) \cdot \frac{\partial w}{\partial x_i'}(\Psi_\delta^{-1}(x')) \left|\mathrm{det}\mathrm{D}\Psi_\delta^{-1}(x')\right| \mathrm{d}x' \\
&= \sup_{\|w\| \leq 1} \sum_{i=1}^n \int_{\frac{1}{\delta}(B(0,\delta)\cap\Omega)} \sum_{j=1}^d \Big(A^{(i)}\varphi(\Psi_\delta^{-1}(x'))\Big)_j \Big(\mathrm{D}\big(w(\Psi_\delta^{-1}(x'))\big) \cdot \big(\mathrm{D}\Psi_\delta^{-1}(x')\big)^{-1}\Big)_{j,i} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \cdot \mathrm{det}|\mathrm{D}\Psi_\delta^{-1}(\mathrm{x}')|\mathrm{d}x'.
\end{aligned}$$

Denoting $w_\delta := w \circ \Psi_\delta^{-1}$, using the function y as in (3.8) and the convergence of Ψ_δ^{-1} to the identity in $C^1(\overline{B(0,1)};\mathbb{R}^n)$, we get

$$\begin{aligned}\|\mathcal{A}\varphi\|_{W_0^{-1,p}(D_0;\mathbb{R}^d)} &\geq \frac{1}{2} \sup_{\|w_\delta\|_{W_0^{1,p'}(\Psi_\delta(D_0);\mathbb{R}^d)} \leq 1} \sum_{i=1}^n \int_{\frac{1}{\delta}(B(0,\delta)\cap\Omega)} A^{(i)} y(x') \frac{\partial w_\delta(x')}{\partial x_i'} \mathrm{d}x' \\ &= \frac{1}{2} \sup_{\|w_\delta\|\leq 1} \sum_{i=1}^n \int_{B(0,\delta)\cap\Omega} A^{(i)} y(\delta^{-1}x) \frac{\partial w_\delta}{\partial x_i}(\delta^{-1}x)\mathrm{d}x \\ &= \frac{1}{2} \sup_{\|w_\delta\|\leq 1} \sum_{i=1}^n \int_{B(0,\delta)\cap\Omega} A^{(i)} \left(\delta^{n/p} u(x)\right) \delta \frac{\partial (w_\delta(\delta^{-1}x))}{\partial x_i}\mathrm{d}x\end{aligned}$$

for sufficiently small δ. With $\eta_\delta(x) := \delta^{1-\frac{n}{p'}} w_\delta(\delta^{-1}x)$ and due to

$$\|\mathrm{D}\eta_\delta\|_{L^{p'}(B(0,\delta)\cap\Omega;\mathbb{R}^d)} = \|\mathrm{D}w_\delta\|_{L^{p'}(\frac{1}{\delta}(B(0,\delta)\cap\Omega;\mathbb{R}^d)}$$

it follows that

$$\begin{aligned}\|\mathcal{A}\varphi\|_{W_0^{-1,p}(D_0;\mathbb{R}^d)} &\geq \frac{1}{2} \sup_{\|\eta_\delta\|_{W_0^{1,p'}(B(0,\delta)\cap\Omega;\mathbb{R}^d)} \leq 1} \sum_{i=1}^n \int_{B(0,\delta)\cap\Omega} A^{(i)} u(x) \cdot \frac{\partial \eta_\delta(x)}{\partial x_i} \delta^n \mathrm{d}x \\ &= \frac{1}{2}\delta^n \|\mathcal{A}u\|_{W^{-1,p}(B(0,\delta)\cap\Omega;\mathbb{R}^d)}.\end{aligned}$$

By a similar procedure as above, we compute

$$\begin{aligned}&\|u\|^p_{L^p(B(0,\delta)\cap\Omega;\mathbb{R}^m)} = \int_{B(0,\delta)\cap\Omega} |u(x)|^p \,\mathrm{d}x \\ &= \int_{\delta^{-1}(B(0,\delta)\cap\Omega)} |u(\Phi_\delta^{-1}(x')|^p |\mathrm{det}\mathrm{D}_{x'}\Phi_\delta^{-1}(x')| \,\mathrm{d}x' = \int_{\delta^{-1}(B(0,\delta)\cap\Omega)} |y(x')|^p \,\mathrm{d}x' \\ &= \int_{D_0} |y(\Psi_\delta(z))|^p |\mathrm{det}\mathrm{D}_z\Psi_\delta(z)| \,\mathrm{d}z \geq \frac{1}{2}\int_{D_0} |\varphi(z)|^p \,\mathrm{d}z = \frac{1}{2}\|\varphi\|^p_{L^p(D_0;\mathbb{R}^m)}\ .\end{aligned}$$

Hence, due to the assumption that u is $\mathcal{A}$-qcb at 0, we see that

$$\|\mathcal{A}\varphi\|_{W^{-1,p}(D_0;\mathbb{R}^d)} \lesssim \|\mathcal{A}u\|_{W^{-1,p}(B(0,\delta)\cap\Omega;\mathbb{R}^d)} \lesssim \|u\|_{L^p(B(0,\delta)\cap\Omega;\mathbb{R}^m)} \lesssim \|\varphi\|_{L^p(D_0;\mathbb{R}^m)}.$$

"if": The sufficiency of (3.6) for v to be $\mathcal{A}$-qcb at 0 can be shown by analogous computations, instead of the (uniform) convergence of Ψ_δ one uses the (uniform) convergence of Ψ_δ^{-1} as $\delta \to 0$. □

3.2 Genuinely $\mathcal{A}$-free sequences

We now focus on weak lower semicontinuity along sequences $\{u_k\}$ that satisfy $\mathcal{A}u_k = 0$ for each $k \in \mathbb{N}$. Since a substantial part of the arguments in this context is analogous to the ones in the preceding subsection, we do not always give full proofs. The main difference is that for the link to $\mathcal{A}$-quasiconvexity at the boundary ($\mathcal{A}$-qcb) as introduced in Definition 3.1, more precisely, for its sufficiency, we rely on an extension property:

Definition 3.10 ($\mathcal{A}$-free extension domain) *We say that* Ω *is an* $\mathcal{A}$-free extension domain *if there exists a larger domain* Ω' *with* $\Omega \subset\subset \Omega'$ *and an associated* $\mathcal{A}$-free extension operator, *i.e., a bounded linear operator* $E : L^p(\Omega;\mathbb{R}^m) \cap \ker \mathcal{A} \to L^p(\Omega';\mathbb{R}^m) \cap \ker \mathcal{A}$ *such that* $Eu = u$ *on* Ω.

As mentioned before, the existence of an $\mathcal{A}$-free extension operator not only depends on the smoothness of $\partial\Omega$, but also on $\mathcal{A}$ itself. If we are able to extend, especially to a periodic setting, the projection $\mathcal{T}$ of Lemma 2.1 can be used without changing the values of the functional in the limit due to its uniform continuity on bounded subsets of L^p:

Lemma 3.11 *Let* $h_\infty \in C(\bar{\Omega}; C^p_{hom}(\mathbb{R}^m))$. *Then for any pair* $\{u_k\}$, $\{v_k\}$ *of bounded sequences in* $L^p(\Omega;\mathbb{R}^m)$ *such that* $u_k - v_k \to 0$ *strongly in* L^p, $h_\infty(\cdot, u_k(\cdot)) - h_\infty(\cdot, v_k(\cdot)) \to 0$ *strongly in* L^1.

Proof. For $\delta > 0$ let

$$A_k(\delta) := \{x \in \Omega : |u_k(x) - v_k(x)| \geq \delta(|u_k(x)| + |v_k(x)| + 1)\}.$$

Since $u_k - v_k \to 0$ in L^p, we see that

$$\int_{A_k(\delta)} (|u_k(x)| + |v_k(x)| + 1)^p \, dx \to 0 \text{ as } k \to \infty, \text{ for every } \delta. \tag{3.11}$$

In addition, h_∞ is uniformly continuous on the compact set $\overline{O} \times \overline{B(0,1)} \subset \mathbb{R}^n \times \mathbb{R}^m$, with a modulus of continuity μ, whence

$$\begin{aligned}
&\int_{\Omega \setminus A_k(\delta)} |h_\infty(x, u_k) - h_\infty(x, v_k)| \, dx \\
&= \int_{\Omega \setminus A_k(\delta)} \left| h_\infty\Big(x, \frac{u_k}{|u_k| + |v_k| + 1}\Big) - h_\infty\Big(x, \frac{v_k}{|u_k| + |v_k| + 1}\Big) \right| (|u_k(x)| + |v_k(x)| + 1)^p \, dx \\
&\leq \int_{\Omega \setminus A_k(\delta)} \mu(\delta)(|u_k(x)| + |v_k(x)| + 1)^p \, dx \\
&\leq \mu(\delta) C \underset{\delta \to 0}{\longrightarrow} 0 \quad \text{uniformly in } k,
\end{aligned} \tag{3.12}$$

where we also used that $\{u_k\}$ and $\{v_k\}$ are bounded in L^p. Combining (3.11) and (3.12), $\|h_\infty(\cdot, u_k(\cdot)) - h_\infty(\cdot, v_k(\cdot))\|_{L^1}$ can be made arbitrarily small, first choosing δ small enough and then k large, depending on δ. □

Proposition 3.6 can be adapted to the setting of genuinely $\mathcal{A}$-free sequences:

Proposition 3.12 *Suppose that* Ω *is an* $\mathcal{A}$*-free extension domain and let* $h_\infty \in C(\bar{\Omega}; C^p_{hom}(\mathbb{R}^m))$. *Then* $I_\infty(u) := \int_\Omega h_\infty(x, u(x))\, \mathrm{d}x$ *is weakly sequentially lower semicontinuous along* $\mathcal{A}$*-free sequences in* $L^p(\Omega;\mathbb{R}^m)$ *if and only if*

(i) h_∞ *is* $\mathcal{A}$*-qcb at every* $x_0 \in \partial\Omega$ *and*

(ii) $h_\infty(x, \cdot)$ *is* $\mathcal{A}$*-quasiconvex at almost every* $x \in \Omega$.

Proof. "only if": Again, necessity of (ii) is well known. If h_∞ is not $\mathcal{A}$-qcb at a point $x_0 \in \partial\Omega$, as in the proof of Proposition 3.6 we obtain an $\varepsilon > 0$ and a sequence $\{\hat{u}_k\} \subset L^p_0(B(x_0, \frac{1}{k}); \mathbb{R}^m)$ with $\|\hat{u}_k\|_{L^p(\Omega;\mathbb{R}^m)} = 1$ such that

$$\liminf_{k\to\infty} \int_\Omega h_\infty(x, \hat{u}_k(x))\,\mathrm{d}x \leq -\varepsilon < 0 = \int_\Omega h_\infty(x, 0)\,\mathrm{d}x,$$

and $\|\mathcal{A}\hat{u}_k\|_{W^{-1,p}(\mathbb{R}^n;\mathbb{R}^d)} \leq 1/k$. Each $\hat{u}_k$ can be interpreted as a Q-periodic function $\hat{u}_k^\#$ with respect to a cube Q compactly containing $\Omega \cup B(x_0, 1)$, by first extending $\hat{u}_k$ by zero to the rest of Q and then periodically to $\mathbb{R}^n$. We denote its cell average by

$$a_k := \frac{1}{|Q|} \int_Q \hat{u}_k\,\mathrm{d}x.$$

By Remark 2.2, we infer that $\|\mathcal{A}\hat{u}_k^\#\|_{W^{-1,p}_\#(\mathbb{R}^n;\mathbb{R}^d)} \leq C/k$ with a constant $C \geq 0$ independent of k. The projection of Lemma 2.1 now yields the sequence $\{\mathcal{T}\hat{u}_k^\#\} \subset L^p_\#(R^n; \mathbb{R}^m) \cap \ker \mathcal{A}$, which satisfies $\|a_k + \mathcal{T}\hat{u}_k^\# - \hat{u}_k\|_{L^p(Q;\mathbb{R}^m)} \to 0$ as $k \to \infty$. Consequently, $a_k + \mathcal{T}\hat{u}_k^\# \rightharpoonup 0$ weakly in L^p just like $\hat{u}_k$, and due to Lemma 3.11,

$$\liminf_{k\to\infty} \int_\Omega h_\infty(x, a_k + \mathcal{T}\hat{u}_k^\#(x))\,\mathrm{d}x \leq -\varepsilon < 0 = \int_\Omega h_\infty(x, 0)\,\mathrm{d}x.$$

Hence, I_∞ is not lower semicontinuous along the $\mathcal{A}$-free sequence $\{a_k + \mathcal{T}\hat{u}_k^\#\}$.

"if": The argument is completely analogous to that of Proposition 3.6, using Theorem 2.7 instead of Theorem 2.8. Observe that due to the extension operator, any given sequence $\{u_k\}$ along which we want to show lower semicontinuity is defined and $\mathcal{A}$-free on some set $\Omega' \supset\supset \Omega$. Hence, after the truncation argument of Proposition 3.6, we now end up with an admissible test function for Definition 3.1 (see also Remark 3.2). ◻

Exactly as in the case of Definition 3.3, the first variable of h_∞ can be "frozen" in Definition 3.1, and we arrive at the analogous main result:

Proposition 3.13 *A function $h_\infty \in C(\bar{\Omega}; C^p_{hom}(\mathbb{R}^m))$ is $\mathcal{A}$-qcb at $x_0 \in \partial\Omega$ if and only if for all $\varepsilon > 0$ there are $\delta > 0$, $\alpha > 0$ such that for all $u \in L^p_0(B(x_0, \delta); \mathbb{R}^m)$ with $\|\mathcal{A}u\|_{W^{-1,p}(\mathbb{R}^n;\mathbb{R}^d)} < \alpha\|u\|_{L^p(\Omega;\mathbb{R}^m)}$,*

$$\int_{B(x_0,\delta)\cap\Omega} h_\infty(x_0, u(x)) + \varepsilon|u(x)|^p\,\mathrm{d}x \geq 0\ . \tag{3.13}$$

Theorem 3.14 *Let $\Omega \subset \mathbb{R}^n$ be a bounded $\mathcal{A}$-free extension domain, let $1 < p < +\infty$, and let $h : \bar{\Omega} \times \mathbb{R}^m \to \mathbb{R}$ be continuous and such that $h(x, \cdot) \in \Upsilon^p(\mathbb{R}^m)$ for all $x \in \bar{\Omega}$, with recession function*

$h_\infty \in C(\overline{\Omega}; C^p_{hom})$. *Then I is sequentially weakly lower semicontinuous along $\mathcal{A}$-free sequences if and only if*

(i) $h(x,\cdot)$ is $\mathcal{A}$-quasiconvex for almost all $x \in \Omega$;

(ii) h_∞ is $\mathcal{A}$-quasiconvex at the boundary for all $x_0 \in \partial\Omega$.

Remark 3.15 *In general, the continuity of h_∞ in x cannot be dropped in Theorem 3.14. For a counterexample in the gradient case ($\mathcal{A}$ =curl) see [31, Section 4].*

Following the proof of Proposition 3.9, we are also able to give an equivalent variant of $\mathcal{A}$-qcb in the limit as $\delta \to 0$.

Proposition 3.16 *Assume that $\Omega \subset \mathbb{R}^n$ has a boundary of class C^1 in a neighborhood of $x_0 \in \partial\Omega$. Let ν_{x_0} be the outer unit normal to $\partial\Omega$ at x_0 and*

$$D_{x_0} := \{x \in B(0,1) \mid x \cdot \nu_{x_0} < 0\}.$$

Then $v \in C^p_{hom}(\mathbb{R}^m)$ is $\mathcal{A}$-qcb at x_0 if and only if

$$\begin{aligned}&\text{for every } \varepsilon > 0 \text{ there exists } \beta > 0 \text{ such that}\\ &\qquad \int_{D_{x_0}} v(\varphi(x)) + \varepsilon|\varphi(x)|^p \,\mathrm{d}x \geq 0 \qquad\qquad (3.14)\\ &\text{for every } \varphi \in L^p_0(B(0,\tfrac{1}{2});\mathbb{R}^m) \text{ with } \|\mathcal{A}\varphi\|_{W^{-1,p}(B(0,1);\mathbb{R}^d)} \leq \beta\|\varphi\|_{L^p(D_{x_0};\mathbb{R}^m)}.\end{aligned}$$

Unlike for a$\mathcal{A}$-qcb, it is possible to derive another version with periodic, precisely $\mathcal{A}$-free test functions and a much more obvious relationship to $\mathcal{A}$-quasiconvexity. It again illustrates the dependence on $\mathcal{A}$-free extension: for the Cauchy–Riemann system, the condition below would be trivial, because all periodic and thus bounded holomorphic functions on $\mathbb{C}$ are constant, and since γ can be chosen small enough so that $|Q \setminus \frac{1}{2}Q| > \gamma^{\frac{1}{p}}|Q|$, $\varphi = 0$ becomes the only admissible test function.

Proposition 3.17 *Let $x_0 \in \partial\Omega$, assume that $\partial\Omega$ is of class C^1 in a neighborhood of x_0, and define $Q = Q(x_0) := \{y \in \mathbb{R}^n \mid |y \cdot e_j| < 1 \text{ for } j = 1,\dots,n\}$ and $Q^- := \{y \in Q \mid y \cdot e_1 < 0\}$, where $e_1,\dots,e_n$ of $\mathbb{R}^n$ is an orthonormal basis of $\mathbb{R}^n$ such that $e_1 = \nu_{x_0}$, the unit outer normal to $\partial\Omega$ at x_0. Then $v \in C^p_{hom}(\mathbb{R}^m)$ is $\mathcal{A}$-qcb at x_0 if and only if*

$$\begin{aligned}&\text{for every } \varepsilon > 0, \text{ there exists } \gamma > 0 \text{ such that}\\ &\qquad \int_{Q^-} v(\varphi(x)) + \varepsilon|\varphi(x)|^p \,\mathrm{d}x \geq 0 \qquad\qquad (3.15)\\ &\text{for every } \varphi \in L^p_\#(Q;\mathbb{R}^m) \text{ with } \mathcal{A}\varphi = 0 \text{ and } \|\varphi\|_{L^p(Q\setminus\frac{1}{2}Q;\mathbb{R}^m)} \leq \gamma\|\varphi\|_{L^p(Q;\mathbb{R}^m)}.\end{aligned}$$

Proof. "if": We claim that (3.15) implies (3.14). By p-homogeneity, it suffices to show the integral inequality in (3.14) for every $\varphi \in L_0^p(B(0,\frac{1}{2});\mathbb{R}^m)$ with $\|\varphi\|_{L^p} = 1$ and $\|\mathcal{A}\varphi\|_{W^{-1,p}} \leq \beta$, where $\beta = \beta(\varepsilon)$ is yet to be chosen. Below, the average of φ is denoted by

$$a_\varphi := \frac{1}{|Q|} \int_Q \varphi(x)\, dx.$$

By Lemma 2.1 and Remark 2.2, $\|\varphi - a_\varphi - \mathcal{T}\varphi\|_{L^p(Q;\mathbb{R}^m)}$ becomes arbitrarily small, provided that $\|\mathcal{A}\varphi\|_{W^{-1,p}} \leq \beta$ is small enough. In view of Lemma 3.11 (uniform continuity of $u \mapsto v(u)$ and $u \mapsto |u|^p$, $L^p \to L^1$, on bounded sets in L^p), this means that for every $\varepsilon > 0$, there exists a $\beta > 0$ such that

$$\int_{Q^-} v(\varphi(x)) + \varepsilon|\varphi(x)|^p \,\mathrm{d}x \geq \int_{Q^-} v(a_\varphi + \mathcal{T}\varphi(x)) + \frac{\varepsilon}{2}|a_\varphi + \mathcal{T}\varphi(x)|^p \,\mathrm{d}x,$$

and due to the inequality in (3.15) with $a_\varphi + \mathcal{T}\varphi$ instead of φ, the right-hand side above is non-negative. Hence,

$$\int_{D_{x_0}} v(\varphi(x)) + \varepsilon|\varphi(x)|^p \,\mathrm{d}x = \int_{Q^-} v(\varphi(x)) + \varepsilon|\varphi(x)|^p \,\mathrm{d}x \geq 0.$$

"only if": Suppose that (3.14) holds. Let $\varepsilon > 0$, and let φ denote an admissible test function for (3.15), i.e., $\varphi \in L^p_\#(Q_{x_0};\mathbb{R}^m)$ with $\mathcal{A}\varphi = 0$ and $\|\varphi\|_{L^p(Q\setminus\frac{1}{2}Q;\mathbb{R}^m)} \leq \gamma\|\varphi\|_{L^p(Q;\mathbb{R}^m)}$, with some γ still to be chosen. We may also assume that $\|\varphi\|_{L^p(Q)} = 1$. Let $\eta \in C_0^\infty(Q;[0,1])$ be a fixed function such that $\eta = 1$ on $\frac{1}{2}Q$ and $\eta = 0$ on $Q \setminus \frac{3}{4}Q$. Observe that $\|\varphi - \eta\varphi\|_{L^p(Q;\mathbb{R}^m)} \leq 2\|\varphi\|_{L^p(Q\setminus\frac{1}{2}Q;\mathbb{R}^m)} \leq 2\gamma\|\varphi\|_{L^p(Q;\mathbb{R}^m)}$, whence

$$\|\varphi - \eta\varphi\|_{L^p(Q;\mathbb{R}^m)} \leq 2\gamma\|\varphi\|_{L^p(Q;\mathbb{R}^m)} \leq \frac{2\gamma}{1-2\gamma}\|\eta\varphi\|_{L^p(Q;\mathbb{R}^m)}$$

In addition, there is a constant $C \geq 0$ depending on η and $\mathcal{A}$ such that

$$\|\mathcal{A}(\eta\varphi)\|_{W^{-1,p}(Q;\mathbb{R}^d)} \leq C\|\varphi\|_{L^p(\frac{3}{4}Q\setminus\frac{1}{2}Q;\mathbb{R}^m)} \leq C\gamma\|\varphi\|_{L^p(Q;\mathbb{R}^m)} \leq \frac{C\gamma}{1-2\gamma}\|\eta\varphi\|_{L^p(Q;\mathbb{R}^m)}.$$

Hence, for γ sufficiently small, $\eta\varphi$ is an admissible test function for (3.14) (which we apply with $\varepsilon/2$ instead of ε), up to the fact that the support of $\eta\varphi$, which is contained in $\frac{3}{4}Q$, might be larger than $B(0,\frac{1}{2})$. This, however, can be easily corrected by a change of variables, rescaling by a fixed factor. Consequently,

$$\int_{Q_{x_0}^-} v(\eta(x)\varphi(x)) + \frac{\varepsilon}{2}|\eta(x)\varphi(x)|^p \,\mathrm{d}x \geq 0,$$

and due to the uniform continuity shown in Lemma 3.11, we conclude that for γ small enough,

$$\int_{Q_{x_0}^-} v(\varphi(x)) + \varepsilon|\varphi(x)|^p \,\mathrm{d}x \geq 0.$$

□

4 Concluding remarks

4.1 $\mathcal{A}$-free versus asymptotically $\mathcal{A}$-free sequences

Clearly, weak lower semicontinuity along asymptotically $\mathcal{A}$-free sequences implies weak sequential lower semicontinuity for the functional restricted to ker $\mathcal{A}$. We do not know whether or not the converse is true in general. However, it holds at least in some special cases. More precisely, it suffices to have an extension property in the following sense. It trivially implies the $\mathcal{A}$-free extension property mentioned in Definition 3.10 (but the converse is not clear there, either):

Definition 4.1 (asymtotically $\mathcal{A}$-free extensions) *We say that Ω has the $\mathcal{A}$-(L^p,$W^{-1,p}$) extension property if there exists a domain Λ with $\bar{\Omega} \subset\subset \Lambda$ such that for every $u \in L^p(\Omega;\mathbb{R}^m)$, there is an extension $v \in L^p(\Lambda;\mathbb{R}^m)$ of u which satisfies*

$$\|v\|_{L^p(\Lambda;\mathbb{R}^m)} \leq C\|u\|_{L^p(\Omega;\mathbb{R}^m)} \quad \textit{and} \quad \|\mathcal{A}v\|_{W^{-1,p}(\Lambda;\mathbb{R}^d)} \leq C\|\mathcal{A}u\|_{W^{-1,p}(\Omega;\mathbb{R}^d)},$$

where $C \geq 0$ is a suitable constant only depending on Λ, Ω, p and $\mathcal{A}$.

If this holds, we can always reduce asymptotically $\mathcal{A}$-free sequences to genuinely $\mathcal{A}$-free sequences with arbitrarily small error in L^p. The argument can be sketched as follows: For a given approximately $\mathcal{A}$-free sequence $u_k \rightharpoonup u$ along which we want to show lower semicontinuity, it is possible to truncate the extension of $u_k - u$, multiplying with a cut-off function which is 1 on Ω and makes a transition down to zero in $\Lambda \setminus \Omega$ (this cannot be done inside, because u_k might concentrate a lot of mass near the boundary, and cutting off inside could then significantly alter the limit of the functional along the sequence). The modified sequence is still asymptotically $\mathcal{A}$-free due to Lemma 2.4, and since it is compactly supported in Λ by construction, we can further extend it periodically to $\mathbb{R}^n$, with a sufficiently large fundamental cell of periodicity containing the support of the cut-off function. We thus end up in the periodic setting where we can project onto $\mathcal{A}$-free fields with controllable error, essentially due to Lemma 2.1.

Even for smooth domains, the $\mathcal{A}$-(L^p,$W^{-1,p}$) extension property depends on $\mathcal{A}$ (and possibly on p), however; for instance, it holds for $\mathcal{A} = \operatorname{div}$ on domains of class C^1 using local maps and extension by an appropriate reflection for flat pieces of the boundary, but not for all $\mathcal{A}$. In particular, it fails to hold for the Cauchy-Riemann system.

Interestingly, the $\mathcal{A}$-(L^p,$W^{-1,p}$) extension property is unclear for $\mathcal{A}$ =curl, at least if $n \geq 3$. For a flat piece of the boundary, the natural extension for curl-free fields would of course also be by reflection, i.e., the one corresponding to an even extension of the scalar potential across the boundary (even in direction of the normal), but in this case, the required estimate in $W^{-1,p}$ for the curl seems to be nontrivial, if true at all. The problem appears for those of components of the curl that only contain partial derivatives in tangential directions, precisely the ones that "naturally" get extended to even functions, say, $\partial_2 u_3 - \partial_3 u_2$, if the normal to the boundary (locally) is the first unit vector.

4.2 The gradient case and classical quasiconvexity at the boundary

If $\varphi \in \ker \mathcal{A}$ then (3.6) as well as (3.14) implies that $\int_{D_{x_0}} v(\varphi(x))\,\mathrm{d}x \geq 0$. For $\mathcal{A} = \operatorname{curl}$, the differential constraint can also be encoded using potentials: If $\varphi \in L^p$ and $\operatorname{curl}\varphi = 0$ on the simply connected domain D_{x_0}, then there exists a potential vector field $\Phi \in W^{1,p}$ with $\varphi = \nabla\Phi$, and if $\varphi = 0$ on $D_{x_0} \setminus B(0,\frac{1}{2})$, then Φ inherits this property up to an appropriate choice of the constants of integration. Hence, we get that

$$\int_{D_{x_0}} v(\nabla\Phi(x))\,\mathrm{d}x \geq 0 \qquad \text{for every } \Phi \in W_0^{1,p}(B(0,\tfrac{1}{2});\mathbb{R}^m). \tag{4.1}$$

Taking into account that for p-homogeneous v, $v(0) = 0$ and $\mathrm{D}v(0) = 0$, the latter condition is the so-called quasiconvexity at the boundary [6] (at the zero matrix).

The converse, that is, going back from (4.1) to either (3.6) or (3.14), is not so obvious, however. In case of (3.14), this is true as a consequence of known characterizations of weak lower semicontinuity, on the one hand our Proposition 3.12 and the other hand Theorem 1.6 in [31]: Both results apply to functionals of the form $U \mapsto \int_{\Omega_{x_0}} \eta(x) v(\nabla U(x))\,dx$, where v is continuous and p-homogeneous, $\eta \in C^\infty(\mathbb{R}^n;[0,1])$ is supported in a small neighborhood of $x_0 \in \partial\Omega_{x_0}$, and Ω_{x_0} is a C^1-domain chosen in such a way that in a neighborhood of x_0 containing the support of η on $\partial\Omega_{x_0}$, ∂O_{x_0} is a hyperplane with constant normal ν_{x_0}. (A proof directly working with the two conditions is also possible, although slightly more technical.) As to the question whether (4.1) implies (3.6) for $\mathcal{A} = \operatorname{curl}$, we suspect that at least for $n \geq 3$, this is not true in general, but our attempts of constructing an example of a function v proving this so far did not succeed.

4.3 Examples for the case of higher order derivatives

The following example shows that $I(u) := \int_\Omega \det \nabla^2 u(x)\,\mathrm{d}x$ is not weakly lower semicontinuous on $W^{2,2}(\Omega)$. Consequently, the determinant is not $\mathcal{A}$-qcb for suitably defined $\mathcal{A}$. As to the definition of $\mathcal{A}$, we recall [22]: The functional I fits into our framework, if instead of $\nabla^2 u$, we define I on fields $v = (v)_{ij}$, $1 \leq i \leq j \leq n$, in L^2, satisfying $\mathcal{A}v := \operatorname{curl}\ v = 0$, with the understanding that for each x, $v(x)$ (the upper triangular part of a matrix) is identified with a symmetric matrix in $\mathbb{R}^{n\times n}$ still denoted v, both for the application of the (row-wise) curl and the evaluation of I, where $\nabla^2 u$ is replaced by v. One can check that $\mathcal{A}v = 0$ if and only if there exists a scalar-valued $u \in W^{2,2}$ with $v = \nabla^2 u$, at least as long as the domain is simply connected.

Example 4.2 *Consider $\Omega := (-1,1)^2$ and for $F \in \mathbb{R}^{2\times 2}$ the function $v_\infty(F) := \det F$ and the operator $\mathcal{A}$ such that $\mathcal{A}w = 0$ if and only if for some $u \in W^{2,2}(\Omega)$, w is the upper (or lower) triangular part of $\nabla^2 u$, which takes values in the symmetric matrices; cf. [22, Example 3.10(d)]. Here $\nabla^2 u$ denotes the Hessian matrix of u. Then v_∞ is not $\mathcal{A}$-qcb. Indeed, take $u \in W^{2,2}_0(\Omega)$ extended by zero to the whole $\mathbb{R}^2$. Define $u_k(x) := k^{-1}u(kx)$. Then $u_k \rightharpoonup 0$ in $W^{2,2}(\Omega)$. We have that*

$$\lim_{k\to\infty} \int_{(0,1)\times(-1,1)} \det \nabla^2 u_k(x)\,\mathrm{d}x = \int_{(0,1)\times(-1,1)} \det \nabla^2 u(y)\,\mathrm{d}y\ . \tag{4.2}$$

Hence, it remains to find u for which the integral on the right-hand side is negative. Let $u(x_1,x_2) := f(x_1)g(x_2)$ where $f,g : [-1,1] \to \mathbb{R}$ are smooth and such that $g(\pm 1) = g'(\pm 1) = f(1) = f'(1) = 0$, $f'(0)f(0) > 0$, and g' does vanish identically. Then

$$\begin{aligned}
\int_{(0,1)\times(-1,1)} \det \nabla^2 u(y)\,\mathrm{d}y &= \int_{(0,1)\times(-1,1)} f(x_1)g(x_2)f''(x_1)g''(x_2) - f'(x_1)^2 g'(x_2)^2\,\mathrm{d}x \\
&= \int_{(0,1)\times(-1,1)} f'(x_1)^2 g'(x_2)^2\,\mathrm{d}x + [f'(x)f(x)]_0^1 \int_{-1}^1 g''(x_2)g(x_2)\,\mathrm{d}x_2 \\
&\quad - [g'(x)g(x)]_{-1}^1 \int_0^1 f'(x)^2\,\mathrm{d}x_1 - \int_{(0,1)\times(-1,1)} f'(x_1)^2 g'(x_2)^2\,\mathrm{d}x \\
&= -f'(0)f(0)\int_{-1}^1 g'(x_2)^2\,\mathrm{d}x < 0\ .
\end{aligned}$$

Example 4.3 *Consider $\Omega := B(0,1) \subset \mathbb{R}^3$ and $\mathcal{A}$ such that $\mathcal{A}w = 0$ if and only if $w = \nabla^2 u$ for some $u \in W^{2,2}(\Omega)$, and the mapping $h(x,F) := a(x)\cdot(\mathrm{Cof}F)\nu(x)$, where $a \in C(\bar\Omega;\mathbb{R}^3)$ is arbitrary and $\nu(x) \in C(\bar\Omega)$ coincides with the outer unit normal to $\partial\Omega$ for $x \in \partial\Omega$. Notice that by definition of the Cofactor matrix ($(\mathrm{Cof})_{ij}$ is $(-1)^{i+j}$ times the determinant of the 2×2 submatrix of F obtained by erasing the i-th row and j-th column), $(\mathrm{Cof}\nabla u(x))\nu(x)$ effectively only depends on directional derivatives of u in directions perpendicular to $\nu(x)$.*

For this h,

$$\int_\Omega h(x,\nabla^2 u_k(x))\,\mathrm{d}x \to \int_\Omega h(x,\nabla^2 u_0(x))\,\mathrm{d}x$$

whenever $u_k \rightharpoonup u_0$ in $W^{2,2}(\Omega)$.

To see that consider $z_k := \nabla u_k$ for $k \in \mathbb{N}\cup\{0\}$. Then $\{z_k\} \in W^{1,2}(\Omega;\mathbb{R}^3)$ and the result follows from[25].

A Appendix

A.1 DiPerna-Majda measures

Consider a complete (i.e. containing constants, separating points from closed subsets and closed with respect to the supremum norm), separable (i.e. containing a dense countable subset) ring $\mathcal{S}$ of continuous bounded functions from $\mathbb{R}^m$ into $\mathbb{R}$ defined as

$$\mathcal{S} := \left\{ v_0 \in C(\mathbb{R}^m) \,\middle|\, \text{ there exist } c \in \mathbb{R}\ ,\ v_{0,0} \in C_0(\mathbb{R}^m),\ \text{and } v_{0,1} \in C(S^{m-1}) \text{ s.t.} \right.$$
$$\left. v_0(s) = c + v_{0,0}(s) + v_{0,1}\left(\frac{s}{|s|}\right)\frac{|s|^p}{1+|s|^p} \text{ if } s \neq 0 \text{ and } v_0(0) = c + v_{0,0}(0) \right\}, \tag{A.1}$$

where S^{m-1} denotes the $(m-1)$-dimensional unit sphere in $\mathbb{R}^m$. Then $\beta_{\mathcal{S}}\mathbb{R}^m$ is homeomorphic to the unit ball $\overline{B(0,1)} \subset \mathbb{R}^m$ via the mapping $f : \mathbb{R}^m \to B(0,1)$, $f(s) := s/(1+|s|)$ for all $s \in \mathbb{R}^m$. Note that $f(\mathbb{R}^m)$ is dense in $\overline{B(0,1)}$. It is known that there is a one-to-one correspondence $\mathcal{S} \mapsto \beta_{\mathcal{R}}\mathbb{R}^m$ between such ring and a (metrizable) compactification of $\mathbb{R}^m$ by the sphere [16]; by a compactification we mean here a compact set, denoted by $\beta_{\mathcal{S}}\mathbb{R}^m$, into which $\mathbb{R}^m$ is embedded homeomorphically and densely. For simplicity, we will not distinguish between $\mathbb{R}^m$ and its image in $\beta_{\mathcal{S}}\mathbb{R}^m$.

DiPerna and Majda [14] proved the following theorem:

Theorem A.1 *Let Ω be an open domain in $\mathbb{R}^n$ with $\mathcal{L}^n(\partial\Omega) = 0$, and let $\{y_k\}_{k\in\mathbb{N}} \subset L^p(\Omega;\mathbb{R}^m)$, with $1 \le p < +\infty$, be bounded. Then there exists a subsequence (not relabeled), a positive Radon measure $\pi \in \mathcal{M}(\bar\Omega)$ and a family of probability measures on $\beta_{\mathcal{S}}\mathbb{R}^m$ $\lambda := \{\lambda_x\}_{x\in\bar\Omega}$ such that for all $h_0 \in C(\bar\Omega \times \beta_{\mathcal{S}}\mathbb{R}^m)$ it holds that*

$$\lim_{k\to\infty} \int_\Omega h_0(x, y_k(x))(1 + |y_k(x)|^p)dx \ = \int_{\bar\Omega}\int_{\beta_{\mathcal{S}}\mathbb{R}^m} h_0(x,s)d\lambda_x(s)d\pi(x)\ . \tag{A.2}$$

If (A.2) holds we say that $\{y_k\}$ generates (π, λ) and we denote the set of all such pairs of measures generated by some sequence in $L^p(\Omega;\mathbb{R}^m)$ by $\mathcal{DM}^p_{\mathcal{S}}(\Omega;\mathbb{R}^m)$.

For any $h(x,s) := h_0(x,s)(1+|s|^p)$ with $h_0 \in C(\bar\Omega \times \beta_{\mathcal{S}}\mathbb{R}^m)$ then there exists a continuous and positively p-homogeneous function $h_\infty : \bar\Omega \times \mathbb{R}^m \to \mathbb{R}$, i.e., $h_\infty(x, ts) = t^p h_\infty(x,s)$ for all $t \ge 0$, all $x \in \bar\Omega$, and $s \in \mathbb{R}^m$, such that

$$\lim_{|s|\to\infty} \frac{h(x,s) - h_\infty(x,s)}{|s|^p} = 0\ . \tag{A.3}$$

It is already mentioned in [19, 33] that if $\{y_k\} \subset L^p(\Omega;\mathbb{R}^m)$ is bounded and $\mathcal{L}^n(\{x \in \Omega;\ y_k(x) \neq 0\}) \to 0$ as $k \to \infty$ then (A.2) can be replaced by

$$\lim_{k\to\infty} \int_\Omega h_\infty(x, y_k(x))dx \ = \int_{\bar\Omega}\int_{\beta_{\mathcal{S}}\mathbb{R}^m\setminus\mathbb{R}^m} \frac{h_\infty(x,s)}{1+|s|^p)} d\lambda_x(s)d\pi(x)\ , \tag{A.4}$$

where $(x,s) \mapsto h_0(x,s) := h_\infty(x,s)/(1+|s|^p)$ belongs to $C(\bar{\Omega} \times \beta_{\mathcal{S}}\mathbb{R}^m)$.

The following theorem is a direct consequence of [19, Thms. 2.1, 2.2].

Theorem A.2 *Let $\{y_k\} \subset L^p(\Omega;\mathbb{R}^m) \cap \ker \mathcal{A}$ generates $(\pi,\lambda) \in \mathcal{DM}^p_{\mathcal{S}}(\Omega;\mathbb{R}^m)$ and let $y_k \to 0$ in measure. Then for π-almost every $x \in \Omega$ and all $h \in C(\bar{\Omega}; C^p_{hom}(\mathbb{R}^m))$ such that $h(x,\cdot)$ is $\mathcal{A}$-quasiconvex for all $x \in \bar{\Omega}$ it holds that*

$$0 \leq \int_{\beta_{\mathcal{S}}\mathbb{R}^m \setminus \mathbb{R}^m} \frac{h(x,s)}{1+|s|^p} \mathrm{d}\lambda_x(s) \ . \tag{A.5}$$

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