ORTHOGONALITY PRINCIPLE FOR BILINEAR LITTLEWOOD-PALEY DECOMPOSITIONS

PETR HONZÍK

Abstract. We explore Littlewood-Paley like decompositions of bilinear Fourier multipliers. Grafakos and Li [5] showed that a bilinear symbol supported in an angle in the positive quadrant is bounded from $L^p \times L^q$ into L^r if its restrictions to dyadic annuli are bounded bilinear multipliers in the local L^2 case ($p \geq 2$, $q \geq 2$, $r = 1/(p^{-1}+q^{-1}) \leq 2$). We show that this range of indices is sharp and also discuss similar results for multipliers supported near axis and negative diagonal.

1. Introduction

Littlewood-Paley decomposition is an essential tool in the theory of Fourier multipliers. The function f is decomposed as

$$f = \sum_{i \in \mathbb{Z}} \Delta_i f,$$

where the support of $\widehat{\Delta_i f}$ is in an annulus A_i of radius 2^i. Similar tools are often used in the theory of the bilinear Fourier multipliers. Bilinear multiplier theorems have been studied by many authors, following the pioneereing work of Coifman and Meyer [2]. In contrast with the linear case, the behaviour of the bilinear multiplier is not invariant with respect to rotations. Therefore the space is decomposed into dyadic cubes with diameter equivalent to the distance to the origin, and these cubes are split into diagonal families ($|\xi_1| \sim |\xi_2|$) and families where one variable is dominant ($|\xi_1| << |\xi_2|$ or $|\xi_2| << |\xi_1|$). Decompositions of this type are used for example in recent articels by Tomita [8], Mayachi and Tomita [6] etc.

While the Littlewood-Paley decomposition is a key part in proofs of many multiplier theorems, L^p boundedness of a multiplier operator T_m on functions supported in the annuli A_i does not automatically imply the boundedness of T_m on L^p ($p \neq 2$). It is surprising that in the case of bilinear multiplier operator $T_m : L^{p_1} \times L^{p_2} \to L^p$ this is sometimes the case, provided the symbol is supported in the proximity of the positive diagonal, $2 \leq p_1, p_2 \leq \infty$ and $1 \leq p \leq 2$. This was noted by Grafakos and Li [5]. Similar theorem was also proved for symbol supported in proximity of the coordinate axis, but under the additional conditions on the symbol, by Diestel and

Date: December 9, 2013.
1991 *Mathematics Subject Classification.* Primary 42B20. Secondary 42E30.
Key words and phrases. Bilinear operators, Fourier multipliers.
The author was supported by the ERC CZ grant LL1203 of the Czech Ministry of Education. The author is a researcher in the University Centre for Mathematical Modelling, Applied Analysis and Computational Mathematics (Math MAC).

Grafakos [3]. In this article we study range of indices p_1 and p_2 for positive diagonal, negative diagonal and axis.

The author would like to thank Loukas Grafakos for his encouragement and helpful discussions.

2. Results

Let us have $m \in L^\infty(\mathbb{R}^2)$ and Schwartz functions f, g on $\mathbb{R}$. We define a bilinear multiplier operator

$$T_m(f,g)(x) = \int_{\mathbb{R}} \int_{\mathbb{R}} m(\xi,\nu) \hat{f}(\xi) \hat{g}(\nu) e^{2\pi i x(\xi+\nu)} d\xi d\nu.$$

On the positive diagonal, the following positive result is mostly proved in [5].

Theorem 1. *Let* $2 \le p_1, p_2 \le \infty$ *and* $p = (1/p_1 + 1/p_2)^{-1} \in [1,2]$. *Let* $m_j \in L^\infty(\mathbb{R}^2)$, $j \in \mathbb{Z}$ *be supported in* $[2^j, 2^{j+1}]^2$ *and let for any Schwartz functions* f, g

$$\|T_{m_j}(f,g)\|_p \le \|f\|_{p_1} \|g\|_{p_2},$$

then

$$\|\sum_{j\in\mathbb{Z}} T_{m_j}(f,g)\|_p \le C\|f\|_{p_1}\|g\|_{p_2}.$$

The results for axis and negative diagonal case are much weaker, in each case there is exactly one combination of p_1 and p_2 such that the theorem is valid.

Theorem 2. *Let* $p_1 = 2$, $p_2 = 2$ *and* $p = (1/p_1 + 1/p_2)^{-1} = 1$ *and* $M_j = [2^j, 2^{j+1}] \times [-2^{j+1}, -2^{-j}]$ *or let* $p_1 = 2$, $p_2 = \infty$ *and* $p = 2$ *and* $M_j = [2^j, 2^{j+1}] \times [-2^{j-1}, 2^{j-1}]$. *Let* $m_j \in L^\infty(\mathbb{R}^2)$, $j \in \mathbb{Z}$ *be supported in* M_j *and let for any Schwartz functions* f, g

$$\|T_{m_j}(f,g)\|_p \le \|f\|_{p_1} \|g\|_{p_2},$$

then

$$\|\sum_{j\in\mathbb{Z}} T_{m_j}(f,g)\|_p \le C\|f\|_{p_1}\|g\|_{p_2}.$$

Our main objective is to show that these results are sharp.

Theorem 3. *Suppose* $1 \le p_1, p_2 \le \infty$ *and* $p = (1/p_1 + 1/p_2)^{-1}$ *and either* $p_1 < 2$ *or* $p_2 < 2$ *or* $p \in [1/2, \infty] \setminus [1,2]$. *Then for any* $K > 0$ *there exits* $m \in L^\infty(\mathbb{R}^2)$ *supported in* $[1,2]^2$ *such that for any Schwartz functions* f, g

$$\|T_m(f,g)\|_p \le \|f\|_{p_1} \|g\|_{p_2},$$

and if we denote $m_j(\xi_1, \xi_2) = m(2^{-j}\xi_1, 2^{-j}\xi_2)$, *then there exist Schwartz functions* f, g *such that*

$$\|T_{\sum_{j\in\mathbb{Z}} m_j}(f,g)\|_p \ge K\|f\|_{p_1}\|g\|_{p_2}.$$

Theorem 4. *Suppose* $1 \le p_1, p_2 \le \infty$ *and* $p = (1/p_1 + 1/p_2)^{-1}$ *and either* $p_1 \ne 2$ *or* $p_2 \ne 2$. *Then for any* $K > 0$ *there exits* $m \in L^\infty(\mathbb{R}^2)$ *supported in* $[1,2] \times [-2,-1]$ *such that for any Schwartz functions* f, g

$$\|T_m(f,g)\|_p \le \|f\|_{p_1} \|g\|_{p_2},$$

and if we denote $m_j(\xi_1,\xi_2) = m(2^{-j}\xi_1, 2^{-j}\xi_2)$*, then there exist Schwartz functions* f, g *such that*

$$\|T_{\sum_{j\in\mathbb{Z}} m_j}(f,g)\|_p \geq K\|f\|_{p_1}\|g\|_{p_2}.$$

Theorem 5. *Suppose* $1 \leq p_1, p_2 \leq \infty$ *and* $p = (1/p_1 + 1/p_2)^{-1}$ *and either* $p_1 \neq 2$ *or* $p_2 \neq \infty$*. Then for any* $K > 0$ *there exits* $m \in L^\infty(\mathbb{R}^2)$ *supported in* $[1,2]\times[-1/2,1/2]$ *such that for any Schwartz functions* f, g

$$\|T_m(f,g)\|_p \leq \|f\|_{p_1}\|g\|_{p_2},$$

and if we denote $m_j(\xi_1,\xi_2) = m(2^{-j}\xi_1, 2^{-j}\xi_2)$*, then there exist Schwartz functions* f, g *such that*

$$\|T_{\sum_{j\in\mathbb{Z}} m_j}(f,g)\|_p \geq K\|f\|_{p_1}\|g\|_{p_2}.$$

3. Examples

Here we provide the examples from the Theorems 3, 4 and 5. They all use a similar principle, and therefore we construct them simultaneously. Let us fix a $N \in \mathbb{N}$ and a nonzero Schwartz function ψ with $\widehat{\psi}$ supported in $[1/8, 3/8]$. We choose M such that $\int_{\mathbb{R}\setminus[-M,M]} |\psi|(x)dx \leq \frac{1}{100N^2}$ and $\sup_{x\in\mathbb{R}\setminus[-M,M]} |\psi|(x) \leq \frac{1}{100N^2}$. We introduce test functions

$$f_1(x) = \sum_{n=1}^{N} e^{2\pi i 2^{n-1}x}\psi(x),$$

$$f_2(x) = \sum_{n=1}^{N} e^{2\pi i 2^{n-1}x}\psi(x - 2^{N-n+1}M),$$

$$f_3(x) = \sum_{n=1}^{N} e^{-2\pi i(2^{n-1}+\frac{3}{2})x}\psi(x),$$

$$f_4(x) = \sum_{n=1}^{N} e^{-2\pi i(2^{n-1}+\frac{3}{2})x}\psi(x - 2^{N-n+1}M)$$

and

$$f_5(x) = e^{-2\pi i\frac{x}{4}}\psi(x).$$

The L^p norms of a lacunary Fourier series on an interval are equivalent for $1 \leq p < \infty$, therefore there is a constant C such that

$$C^{-1}N^{1/2} \leq \|f_1\|_p \leq CN^{1/2}$$

and

$$C^{-1}N^{1/2} \leq \|f_3\|_p \leq CN^{1/2}.$$

By the choice of M, we also get

$$C^{-1}N^{1/p} \leq \|f_2\|_p \leq CN^{1/p}$$

and

$$C^{-1}N^{1/p} \leq \|f_4\|_p \leq CN^{1/p}$$

for $1 \leq p \leq \infty$. (We take $1/\infty = 0$.) Obviously $\|f_5\|_p = \|\psi\|_p$.

Let us take a Schwartz function ϕ supported in $[0,1/2]$ with $\phi(\xi)=1$ for $\xi\in[1/8,3/8]$. We define symbols

$$s_0(\xi_1,\xi_2)=\phi(\xi_1-1)\phi(\xi_2+1/4)e^{2\pi iM2^N\xi_1},$$
$$s_1(\xi_1,\xi_2)=\phi(\xi_1-1)\phi(\xi_2+1/4)e^{-2\pi iM2^N\xi_1}e^{-2\pi iM2^N\xi_2},$$
$$s_2(\xi_1,\xi_2)=\phi(\xi_1-1)\phi(\xi_2-1)e^{2\pi iM2^N\xi_1}e^{2\pi iM2^N\xi_2}$$
$$s_3(\xi_1,\xi_2)=\phi(\xi_1-1)\phi(\xi_2-1)e^{-2\pi iM2^N\xi_1}$$
$$s_4(\xi_1,\xi_2)=\phi(\xi_1-1)\phi(-\xi_2-1)e^{2\pi iM2^N\xi_1}e^{2\pi iM2^N\xi_2},$$

and

$$s_5(\xi_1,\xi_2)=\phi(\xi_1-1)\phi(-\xi_2-1)e^{2\pi iM2^N\xi_1}.$$

all of these symbols are tensor products of modulated smooth bumps, therefore their norms as bilinear multipliers are bounded for any pair $1\le p_1,p_2\le\infty$ and $p=(1/p_1+1/p_2)^{-1}$. For example

$$T_{s_2}(f,g)=\mathcal{F}^{-1}(\phi(\xi_1-1)e^{2\pi iM2^N\xi_1}\widehat{f}(\xi_1))\mathcal{F}^{-1}(\phi(\xi_2-1)e^{2\pi iM2^N\xi_2}\widehat{g}(\xi_2)$$

and for $1\le p_1\le\infty$

$$\|\mathcal{F}^{-1}(\phi(\xi_1-1)e^{2\pi iM2^N\xi_1}\widehat{f}(\xi_1))\|_{p_1}\le C\|f\|_{p_1}$$

with similar inequality for f_2.

Now we construct the example from the Theorem 3, suppose first that $2<p\le\infty$. We set $m=s_2$. Then both $p_1>2$ and $p_2>2$ and we consider $T_m(f_2,f_2)$. As

$$\widehat{f_2}(\xi)=\sum_{n=1}^{N}e^{-2\pi iM2^{N-n+1}\xi}\widehat{\psi}(\xi-2^{n-1}), \tag{1}$$

we see that

$$T_{\sum_{j\in\mathbb{Z}}m_j}(f_2,f_2)(x)=\sum_{j\in\mathbb{Z}}T_{m_j}(f_2,f_2)(x)=\sum_{n=1}^{N}e^{4\pi i2^{n-1}x}\psi^2(x)$$

and using square function argument we get

$$\|T_{\sum_{j\in\mathbb{Z}}m_j}(f_2,f_2)\|\ge CN^{1/2}.$$

(This also holds in case $p=\infty$) On the other hand

$$\|f_2\|_{p_1}\|f_2\|_{p_2}\le CN^{1/p_1+1/p_2}$$

and as $1/p<1/2$ we can choose $K\approx N^{1/2-1/p_1-1/p_2}$. Next, suppose $p_1<2$. We set $m=s_3$. We consider $T_m(f_1,f_2)$. We have

$$\widehat{f_1}(\xi)=\sum_{n=1}^{N}\widehat{\psi}(\xi-2^{n-1}), \tag{2}$$

and therefore

$$T_{\sum_{j\in\mathbb{Z}}m_j}(f_1,f_2)(x)=\sum_{j\in\mathbb{Z}}T_{m_j}(f_1,f_2)(x)=\sum_{n=1}^{N}e^{4\pi i2^{n-1}x}\psi^2(x-2^{N-n+1}M).$$

This is a sum of almost separated bumps and therefore

$$\|T_{\sum_{j\in\mathbb{Z}} m_j}(f_1, f_2)\|_p \geq CN^{1/p}.$$

On the other hand

$$\|f_1\|_{p_1}\|f_2\|_{p_2} \leq CN^{1/2+1/p_2}$$

and we choose $K \approx N^{1/p-1/2-1/p_2}$. (Again, if $p_2 = \infty$ we take $1/p_2 = 0$.)

Next we construct the example from the Theorem 4, we first suppose that $p \geq 1$, $p_1 > 2$ and $p_2 < \infty$. We set $m = s_5$ and observe that

$$T_{\sum_{j\in\mathbb{Z}} m_j}(f_2, f_3)(x) = e^{3\pi ix}\sum_{n=1}^{N}\psi^2(x)$$

and so

$$\|T_{\sum_{j\in\mathbb{Z}} m_j}(f_2, f_3)\|_p \geq CN$$

and we take $K \approx N^{1-1/p_1-1/2}$. The case $p_2 > 2$, $p_1 < \infty$ follows by symmetry. If $p < 1$, we take $m = s_4$ and we get

$$T_{\sum_{j\in\mathbb{Z}} m_j}(f_1, f_3)(x) = e^{3\pi ix}\sum_{n=1}^{N}\psi^2(x + 2^{N-n+1}M).$$

This gives

$$\|T_{\sum_{j\in\mathbb{Z}} m_j}(f_1, f_3)\|_p \geq CN^{1/p}$$

and we take $K \approx N^{1/p-1/2-1/2}$. The remaining case $p_1 = p_2 = p = \infty$ is very similar to 3, and may be done mirroring the example about the x axis.

Example from the theorem 5 in the case $p_1 \geq 2$ follows by taking $m = s_0$ and considering

$$T_{\sum_{j\in\mathbb{Z}} m_j}(f_2, f_5)(x) = \sum_{n=1}^{N} e^{2\pi i 2^{n-1}x}\psi^2(x).$$

We have

$$\|T_{\sum_{j\in\mathbb{Z}} m_j}(f_2, f_5)\|_p \geq CN^{1/2}$$

and we may take $K = N^{1/2-1/p_1}$. If $p_1 < 2$ we take $m = s_1$, consider

$$T_{\sum_{j\in\mathbb{Z}} m_j}(f_1, f_5)(x) = \sum_{n=1}^{N} e^{2\pi i 2^{n-1}x}\psi^2(x - 2^{N-n+1}M)$$

and

$$\|T_{\sum_{j\in\mathbb{Z}} m_j}(f_2, f_5)\|_p \geq CN^{1/p}.$$

So we may take $K = N^{1/p-1/2}$ and we are done.

4. The positive results

The positive cases are either done by Grafakos and Li in [5] or they are trivial. For the sake of completeness, we are going to sketch a proof here. The proof of the Theorem 1 relies upon the following lemma from [5].

Lemma 1. *Let $2 \leq p_1, p_2 < \infty$, $1 < p \leq 2$ and $1/p_1 + 1/p_2 = 1/p$ Suppose that $\{L_k\}_{k\in\mathbb{Z}}$ is a family of uniformly bounded bilinear operators from $L^{p_1} \times L^{p_2}$ into L^p. Furthermore, suppose that for all functions f, g, h on the line we have*

$$\langle L_k(f, g), h\rangle = \langle L_k(\Delta_k^1 f, \Delta_k^2 g), \Delta_k^3 h\rangle,$$

Where $\widehat{\Delta_k^1 f} = \hat{f}\chi_{A_k}$, $\widehat{\Delta_k^2 g} = \hat{g}\chi_{B_k}$, $\widehat{\Delta_k^3 h} = \hat{h}\chi_{C_k}$, and $\{A_k\}$, $\{B_k\}$, $\{C_k\}$ are sets of intervals such that the A_k's being pairwise disjoint, the B_k's being pairwise disjoint and the C_k's being pairwise disjoint. Then there is a constant $C = C(p, p_1, p_2)$ such that for all functions f, g we have

$$\|\sum_k L_k(f, g)\|_p \leq C\|f\|_{p_1}\|g\|_{p_2}.$$

This Lemma gives proof of the Theorem 1 for $2 \leq p_1, p_2 < \infty$, $1 < p \leq 2$, because the operators T_{m_j} clearly satisfy the assumptions. The remaining cases are $p_1 = \infty$ or $p_2 = \infty$ or $p = 1$. First, if $p_1 = 2$, $p_2 = \infty$ and $p = 2$, we use the fact that the Fourier supports of the operators T_{m_j} are disjoint. We have

$$\|T_{\sum m_j}(f, g)\|_2^2 = \sum_j \|T_{m_j}(f, g)\|_2^2 \leq \sum_j \|\hat{f}\chi_{[2^j, 2^{j+1}]}\|_2^2 \|g\|_\infty^2 \leq \|\hat{f}\|_2^2 \|g\|_\infty^2.$$

The case $p_1 = \infty$ follows from symmetry. Finally if $p = 1$, $p_1 = p_2 = 2$ we write

$$\|T_{\sum m_j}(f, g)\|_1 = \sum_j \|T_{m_j}(f, g)\|_1 \leq \sum_j \|\hat{f}\chi_{[2^j, 2^{j+1}]}\|_2 \|g\chi_{[2^j, 2^{j+1}]}\|_2 \leq \|f\|_2 \|g\|_2.$$

To prove the Theorem 2, we note that for negative diagonal and $p_1 = p_2 = 2$ we may use exactly the same argument as for the positive diagonal. For the case of the axis, where we have $M_j = [2^j, 2^{j+1}] \times [-2^{j-1}, 2^{j-1}]$, we note that the Fourier support of T_{m_j} is $[2^{j-1}, 2^{j+1} + 2^{j-1}]$. Since these intervals may be organised into three disjoint systems, we may also repeat the previous argument for $p_1 = 2$, $p_2 = \infty$ and $p = 2$.

5. Notes

While there is no analogy to the Theorem 1 for the linear Fourier multipliers on L^p, analogous theorem holds for multipliers from L^p to L^q in the case $1 < p \leq 2 \leq q < \infty$, see for example [4], theorem 5.3.6.

Also, Seeger [7] and Carbery [1] proved that if m is a L^p multiplier on each dyadic annulus A_i and it satisfies some minimal smoothness estimate, then it is L^r multiplier for any r such that $(|1/2 - 1/r| < |1/2 - 1/p|.)$ It is a question if an analogous result may hold in the bilinear case.

References

[1] A. Carbery, Variants of the Calderón-Zygmund theory for L^p-spaces, *Rev. Mat. Iberoamericana* **2 no. 4** (1986), 381–396.

[2] R. Coifman and Y. Meyer, Commutateurs d'intgrales singulires et oprateurs multilinaires, *Ann. Inst. Fourier (Grenoble)* **28 no. 3** (1978), 177–202.

[3] G. Diestel and L. Grafakos, Maximal bilinear singular integral operators associated with dilations of planar sets, *J. Math. Anal. Appl.* **332 no. 2** (2007), 1482–1494.

[4] L. Grafakos, *Classical Fourier analysis*, Graduate Texts in Mathematics, 249, ed. 2, Springer, New York.

[5] L. Grafakos and X. Li, The disc as a bilinear multiplier, *Amer. J. Math.* **128 no. 1** (2006), 91–119.

[6] A. Miyachi and N. Tomita, Minimal smoothness conditions for bilinear Fourier multipliers, *Rev. Mat. Iberoam.* **29 no. 2** (2013), 495–530.

[7] A. Seeger, Some inequalities for singular convolution operators in L^p-spaces, *Trans. Amer. Math. Soc.* **308 no.1** (1988), 259–272.

[8] N. Tomita, Hörmander type multiplier theorem for multilinear operators, *J. Funct. Anal.* **259 no. 8** (2010), 2028–2044.

Petr Honzík,
E-mail address: honzik@gmail.com