SOME RESULTS ON MONOTONE METRIC SPACES

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Abstract. We give several new results on the recent topic of monotone metric spaces. First, we prove that every 1-monotone metric space in $\mathbb{R}^d$ has finite 1-dimensional Hausdorff measure. As a consequence we obtain that each continuous bounded curve has a finite length if and only if it can be written as a finite sum of 1-monotone continuous bounded curves. Second, we construct a continuous function f such that M has a zero Lebesgue measure provided graph$(f|M)$ is a monotone set in the plane. In the third part a differentiable function is found with a monotone graph and unbounded variation.

1. Introduction

The concept of monotone metric spaces was introduced in [4] (for more information and motivation of this definition see also [6]).

There exists a series of results on the concept of monotone metric spaces. For instance in [3] a Cantor set in $\mathbb{R}^2$ is found such that is not σ-monotone. In [1] it is proved that for each $c > 1$ there is a continuous, almost nowhere differentiable function with a symmetrically c-monotone graph. Consequently, such function has an unbounded variation. From [8] an interesting result follows. Let X be a compact metric space of Hausdorff dimension $\dim_H(X)$. Then for any $\varepsilon > 0$ there exists a monotone compact subset $S \subset X$ with $\dim_H(S) \geq \dim_H(X) - \varepsilon$. Further information can be found in [2], [5] and [7].

In this paper we are investigating some properties of the concept of monotone spaces. The paper is organized as follows. Section 2 contains basic notations, definitions and assertions.

In Section 3 we prove that every 1-monotone bounded subspace of a Euclidean space has finite length (see Theorem 3.8). Note at this moment that in [1, Theorem 6.5] it is proved that every real continuous function with 1-monotone graph has a bounded variation, which is a special case of our result. Moreover, as a consequence we prove that a continuous bounded curve in $\mathbb{R}^d$ has a finite length if and only if it can be expressed as a finite sum of continuous bounded 1-monotone curves.

Section 4 contains a construction of a continuous function f with small monotone subgraphs. More precisely, if graph$(f|M)$ is monotone then M is nowhere dense and has a zero Lebesgue measure. This example improves a known example of a

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function from [1] where M is nowhere dense provided the restriction of the function to M is monotone.

In Section 5 we give another example of a function. For $c > 1$ we find a continuous function defined on $[0, 1)$ with symmetrically c-monotone graph and unbounded variation such that $f'(0) = 0$ and $f \in C^\infty(0, 1]$. This answers [1, Question 8.4].

2. Notation and definitions

Given $d \in \mathbb{N}$ denote as usually by $\mathbb{R}^d$ the corresponding d-dimensional Euclidean space. We will use the symbol $B(x, r)$ for open ball with center x and radius $r > 0$ and $|z|$ will mean the Euclidean norm of z. Let $\lambda(M)$ stand for the d-dimensional Lebesgue measure of $M \subset \mathbb{R}^d$. Let $I \subset \mathbb{R}$ be interval and $f : I \to \mathbb{R}$ be a function. We denote $V_I(f)$ as a variation of the function f on the interval I.

Recall a definition of a monotone and symmetrically monotone metric space.

Definition 2.1. *Let $c \geq 1$. A metric space (X, ρ) is called c-monotone if there is an linear ordering $\prec$ such that for every $x, y, z \in X$ with $x \prec y \prec z$ we have $\rho(x, y) \leq c\rho(x, z)$. The space X is then called monotone, if it is c-monotone for some c.*

Definition 2.2. *Let $c \geq 1$. The metric space (X, ρ) is called symmetrically c-monotone if there is an linear ordering $\prec$ such that for every $x, y, z \in X$ with $x \prec y \prec z$ we have $\rho(x, y) \leq c\rho(x, z)$ and $\rho(z, y) \leq c\rho(z, x)$.*

We say that $A = \{a_i\}_{i=1}^N$ is (symmetrically) c-monotone sequence if A is (symmetrically) c-monotone with respect to the sequence ordering. We say that $A = \{a_i\}_{i=1}^N$ is α-separated if $|a_i - a_j| \geq \alpha$, for every $i \neq j$. Note that if A is 1-monotone then it is α-separated if and only if $|a_i - a_{i+1}| \geq \alpha$ for every suitable i.

We start with a definition introduced in [1].

Definition 2.3. *Let $c \geq 0$ and $I \subset \mathbb{R}$. We say that a function $f : I \to \mathbb{R}$ satisfy condition P_c if for every $x, y \in I$ such that $f(x) = f(y)$, we have*

$$\sup\{|f(t) - f(x)|;\ t \in (x, y)\} \leq c|x - y|. \tag{1}$$

It can be found in [1] that every continuous function satisfying condition P_c has symmetrically $(c + 1)$-monotone graph and also that every c-monotone set is symmetrically $(c + 1)$-monotone.

Lemma 2.4. *Let $A = \{a_i\}_{i=1}^N$ be 1-monotone sequence, then for every $1 \leq i \leq j \leq k \leq m \leq N$ we have*

$$|a_j - a_k| \leq 2|a_i - a_m|.$$

Proof. Since $\{a_i\}_{i=1}^N$ is 1-monotone and symmetrically 2-monotone we can write

$$|a_j - a_k| \leq |a_j - a_m| \leq 2|a_i - a_m|.$$

□

3. Hausdorff measure of 1-monotone spaces

As a main result of this section we prove that each 1-monotone bounded subset of $\mathbb{R}^d$ has a finite 1-dimensional Hausdorff outer measure.

Observation 3.1. *Let $d \in \mathbb{N}$. There is a constant $\frac{1}{2} > \Omega(d) > 0$ such that for every $z_1, ..., z_d \in \mathbb{R}^d \setminus \{0\}$ with the property that*

$$\left| \frac{z_i}{|z_i|} \cdot \frac{z_j}{|z_j|} \right| \leq \Omega(d) \quad \textit{for every} \quad i, j \in \{1, ..., d\},\ i \neq j$$

we can find a Cartesian system of coordinates $\tilde{e}_1, ..., \tilde{e}_d$ such that

$$\tilde{e}_i \cdot \frac{z_i}{|z_i|} \geq 1 - \frac{1}{32d^2} \tag{2}$$

for every $i = 1, ..., d$.

We will need for $j \in \mathbb{N}_0$ some additional notation:

$$r_j := \left(1 - \frac{\Omega(d)}{10}\right)^j,$$

$$\rho_j := r_j - r_{j+1} = \frac{\Omega(d)}{10} \cdot \left(1 - \frac{\Omega(d)}{10}\right)^j,$$

$$B(x, r, j) := B(x, r_j r) \setminus B(x, r_{j+1} r),$$

κ_d maximal cardinality of $2\rho_0$-separated subset of $B(x, 1, 0)$.

Lemma 3.2. *Let $x \in \mathbb{R}^d$ and $r > 0$. Let $A \subset B(x, r)$ be a set with a cardinality n. Then there is $j \in \mathbb{N}$ such that* $\operatorname{card}(A \cap B(x, r, j)) \geq \rho_j(n - 1)$.

Proof. We set $c_k = \operatorname{card}(A \cap B(x, r, k))$ for every $k \geq 0$. Clearly, $\bigcup_{k=0}^{\infty} B(x, r, k) = B(x, r) \setminus \{x\}$. Thus, we have $\sum_{k=0}^{\infty} c_k = \operatorname{card}(A \cap B(x, r) \setminus \{x\}) \geq n - 1$. So, we have

$$\sum_{k=0}^{\infty} \rho_k \frac{c_k}{\rho_k} \geq n - 1. \tag{3}$$

Clearly, $\sum_{k=0}^{\infty} \rho_k = 1$. Using this and formula (3) we have that there exists $j \in \mathbb{N}_0$ such that $\frac{c_j}{\rho_j} \geq n - 1$. So, we are done. □

Lemma 3.3. *Let $x \in \mathbb{R}^d$, $j \in \mathbb{N}_0$ and $r > 0$. Let $A \subset B(x, r, j)$ be a set with cardinality n. Then there is an $y \in A$ such that*

$$\operatorname{card}(A \cap B(y, 2r\rho_j)) \geq \frac{n}{\kappa_d}.$$

Proof. We can assume $x = 0$. Let C be some maximal $2r\rho_j$-separated subset of A. Then $\{\frac{y}{r_j r}; y \in C\}$ is $2\rho_0$-separated subset of $B(x, 1, 0)$. Thus, $\operatorname{card}(C) \leq \kappa_d$. By the maximality of C we have $\bigcup_{y \in C} A \cap B(y, 2r\rho_j) = A$. Thus, there exists $y \in C$ such that

$$\operatorname{card}(A \cap B(y, 2r\rho_j)) \geq \frac{\operatorname{card}(A)}{\operatorname{card}(C)} \geq \frac{n}{\kappa_d}$$

and we are done. □

Definition 3.4. *Let $x, y \in \mathbb{R}^d$. Define $C(x, y), D(x, y) \subset \mathbb{R}^d$ by formulas*

$$C(x, y) := \overline{\left\{ z \in \mathbb{R}^d : \frac{z - y}{|z - y|} \cdot \frac{x - y}{|x - y|} \leq -\frac{\Omega(d)}{2} \right\}}.$$

$$D(x, y) := \overline{\left\{ z \in \mathbb{R}^d : \frac{z - y}{|z - y|} \cdot \frac{x - y}{|x - y|} > -\frac{\Omega(d)}{2} \textit{ and } |x - y| \leq |x - z| \right\}}.$$

Lemma 3.5. *Suppose that* $w_1, ..., w_n \in \mathbb{R}^d$ *and* $A := \{a_i\}_{i=n+1}^l \subset \mathbb{R}^d$. *Put* $a_j = w_j$, $j = 1, .., n$ *and suppose that the sequence* $\{a_i\}_{i=1}^l$ *is 1-monotone. Then there are* $\gamma \in \mathbb{N}_0$ *and indices*

$$n+1 = i(0,+) \le i(1,-) \le \cdots \le i(\gamma,-) \le i(\gamma,+) \le i(\gamma+1,-) = l \tag{4}$$

such that for every $m = 1, ..., \gamma$

$$\text{if} \quad i(m,-) \le k < i(m,+) \quad \text{then} \quad a_{k+1} \in \bigcup_j C(w_j, a_k), \tag{5}$$

and for every $m = 0, ..., \gamma$

$$\text{if} \quad i(m,+) \le k < i(m+1,-) \quad \text{then} \quad a_k \in \bigcap_j D(w_j, a_{i(m,+)}), \tag{6}$$

and

$$\text{if} \quad m < \gamma \quad \text{then} \quad a_{i(m+1,-)} \in \bigcup_j C(w_j, a_{i(m,+)}). \tag{7}$$

Proof. Since a_k is 1-monotone we can easily see that either $a_{k+1} \in \bigcup_j C(w_j, a_k)$ or $a_{k+1} \in \bigcap_j D(w_j, a_k)$. Now, the proof can be done by straightforward induction. □

Lemma 3.6. *Suppose that* $w_1, ..., w_n \in \mathbb{R}^d$ *and* $A := \{a_i\}_{i=n+1}^l \subset \mathbb{R}^d$. *Put* $a_j = w_j$, $j = 1, .., n$ *and suppose that the sequence* $\{a_i\}_{i=1}^l$ *is* α*-separated 1-monotone. Pick* $\{b_i\}_{i=0}^L$ *be a subsequence of* $\{a_i\}_{i=n+1}^l$. *Suppose that* $b_{k+1} \in \bigcup_i C(w_i, b_k)$ *for every* $0 \le k < L$. *Then for every* k *there is some* i_k *such that*

$$|b_{k+1} - w_{i_k}| - |b_k - w_{i_k}| > \frac{\alpha\Omega(d)}{2}.$$

In particular, there is some i *such that*

$$|b_L - w_i| - |b_0 - w_i| > \frac{\alpha\Omega(d)}{2n} L.$$

Proof. The first inequality is a simple geometric fact. To see the second one set

$$W_i = \{k \in \{0, \dots, L-1\}; i_k = i\}$$

for every $i = 1, \dots, n$. Clearly there is a j such that $\mathrm{card}(W_j) \ge \frac{L}{n}$. Now, by 1-monotonicity we have

$$|b_L - w_j| - |b_0 - w_j| = \sum_{k=0}^{L-1} |b_k - w_j| - |b_k - w_j|$$

$$\ge \sum_{k \in W_j} |b_k - w_j| - |b_k - w_j| > \mathrm{card}(W_j) \frac{\alpha\Omega(d)}{2} \ge \frac{\alpha\Omega(d)L}{2n}.$$

□

Lemma 3.7. *Suppose that* $\{a_i\}_{i=0}^l$ *be an* α*-separated 1-monotone sequence. Choose* $N, M, p_1, ..., p_d \in \{1, ..., l\}$ *such that* $p_1 < p_2 < ... < p_d < N < M$. *Suppose that* $\frac{2}{\Omega(d)}|a_k - a_N| \le |a_{p_i} - a_N|$ *for every* $N < k \le M$ *and every* $i = 1, ..., d$.

Assume that for every $N \le k \le M$ *and every* $i, j \in \{1, ..., d\}$, $i \ne j$,

$$\left| \frac{a_{p_i} - a_k}{|a_{p_i} - a_k|} \cdot \frac{a_{p_j} - a_k}{|a_{p_j} - a_k|} \right| \le \Omega(d). \tag{8}$$

Then for every $N \le k < M$ *there is some* i *such that* $|a_{k+1} - a_{p_i}| - |a_k - a_{p_i}| > \frac{\alpha}{6d}$.

In particular, there is some i such that $|a_M - a_{p_i}| - |a_N - a_{p_i}| > \frac{\alpha(M-N)}{6d^2}$.

Proof. Using Observation 3.1 we can can find unit vectors $\widetilde{e}_i$ with

$$\cos(\gamma_i) = \tilde{e}_i \cdot \frac{a_{p_i}}{|a_{p_i}|} \geq 1 - \frac{1}{32d^2}, \tag{9}$$

where γ_i is the angle between a_{p_i} and $\tilde{e}_i$.

Take an arbitrary $N \leq k < M$ and consider $x = \sum_{j=1}^{d} x_j \widetilde{e}_j = a_k$ and $y = \sum_{j=1}^{d} y_j \widetilde{e}_j = a_{k+1}$. Without any loss of generality we can suppose that $a_k = 0$. First observe that there is some i with $|y_i| \geq \frac{|y|}{d}$. Without any loss of generality we can suppose that $i = 1$.

The fact above with the help of the monotonicity of $\{a_i\}$ means that

$$\cos(\beta) = \frac{y}{|y|} \cdot \tilde{e}_1 \leq -\frac{1}{d},$$

where β is the angle between y and $\tilde{e}_1$.

Let Δ be an angle between y and a_{p_1}, then

$$\begin{aligned}
\frac{y}{|y|} \cdot \frac{a_{p_1}}{|a_{p_1}|} &= \cos(\Delta) \leq \cos(\beta)\cos(\gamma_1) + |\sin(\beta)\sin(\gamma_1)| \\
&\leq -\frac{1}{d} + \frac{1}{32d^3} + |\sin(\gamma_1)| \leq -\frac{1}{2d} + \sqrt{1 - \cos^2(\gamma_1)} \\
&\leq -\frac{1}{2d} + \sqrt{1 - (1 - \frac{1}{32d^2})^2} = -\frac{1}{2d} + \sqrt{\frac{1}{16d^2} - \frac{1}{1024d^4}} \\
&\leq -\frac{1}{2d} + \frac{1}{4d} = -\frac{1}{4d}.
\end{aligned}$$

Now, with use of the cosine formula for triangle with vertices $a_{p_1}, 0$ and y we obtain

$$\begin{aligned}
|y - a_{p_1}| - |a_{p_1}| &= \frac{|y|^2 - 2|y| \cdot |a_{p_1}| \cdot \cos(\Delta)}{|a_{p_1}| + |y - a_{p_1}|} \\
&\geq |y| \left(\frac{|y|}{|a_{p_1}| + |y - a_{p_1}|} + \frac{2|a_{p_1}|}{4d(|a_{p_1}| + |y - a_{p_1}|)} \right) \\
&\geq \frac{|y|}{2d} \cdot \frac{|a_{p_1}|}{|a_{p_1}| + |y - a_{p_1}|} \geq \frac{|y|}{6d} \geq \frac{\alpha}{6d}.
\end{aligned}$$

The last part of the statement of this Lemma is now straight forward application of the pigeonhole principle. □

Theorem 3.8. *Let $1 > \alpha > 0$. For every $d \in \mathbb{N}$ there is a constant $\Lambda(d)$ such that every α-separated 1-monotone sequence $\{a_i\}_{i=0}^{K}$ in $B(0,1) \subset \mathbb{R}^d$ with $a_0 = 0$ we have $\alpha K \leq \Lambda(d)$. In particular, every bounded 1-monotone set in $\mathbb{R}^d$ has finite 1-dimensional (outer) Hausdorff measure.*

Proof. We first prove the last part of the theorem. Suppose that $\Gamma \subset B(0, \frac{1}{2}) \subset \mathbb{R}^d$ is 1-monotone. Choose $1 > \alpha > 0$ and suppose that $\{\Gamma_i^\alpha\}_{i=1}^N$ is a maximal α-separated subset of Γ and 1-monotone sequence. Then

$$\Gamma \subset \bigcup_i B(\Gamma_i^\alpha, \alpha)$$

and $\{\Gamma_i^\alpha - \Gamma_1^\alpha\}_{i=1}^N \subset B(0,1)$. By the first part of the theorem we have $\alpha(N-1) \le \Lambda(d)$. Thus

$$\sum_i \operatorname{diam} B(\Gamma_i^\alpha, \alpha) \le 2\alpha N \le 2\alpha \cdot \frac{\Lambda(d)+\alpha}{\alpha} = 2\Lambda(d) + 2\alpha \le 2\Lambda(d) + 2.$$

Therefore $\mathcal{H}^1(\Gamma) \le 2\Lambda(d) + 2$.

Suppose that there is an α-separated 1-monotone sequence $\{a_i\}_{i=0}^K$, with K greater than $\frac{6d^2}{\alpha} \cdot \left(\frac{d\kappa_d}{\Omega(d)}\right)^d \cdot \left(\frac{100}{\Omega(d)}\right)^d$. Using a mathematical induction we will construct indices $p_i, N_i, M_i,\ i = 1, ..., d$ such that the following conditions hold for every $1 \le k \le d$:

(a) $N_{k-1} \le p_k < N_k < M_k \le M_{k-1}$, (for sake of completeness we put $N_0 = 0$, $M_0 = K$)

(b) $\frac{2}{\Omega(d)}|a_l - a_{M_k}| \le |a_{p_i} - a_{N_k}|$ for every $N_k \le l \le M_k$ and every $i = 1, ..., k$,

(c)
$$\left| \frac{a_{p_i} - a_l}{|a_{p_i} - a_l|} \cdot \frac{a_{p_j} - a_l}{|a_{p_j} - a_l|} \right| \le \Omega(d)$$
for every $i, j \in \{1, ..., k\}$, $i \ne j$ and every $N_k \le l \le M_k$,

(d) $\left(\frac{d\kappa_d}{\Omega(d)}\right)^{d-k} \cdot \left(\frac{100}{\Omega(d)}\right)^{d-k} |a_{M_k} - a_{N_k}| \le \frac{\alpha}{6d^2}(M_k - N_k)$.

(e) $10 \le M_k - N_k$.

Case $k = 1$: Put $p_1 = 0$.

Using Lemma 3.2 for $A = \{a_i\}_{i=N_0}^{M_0}$, $x = a_{N_0}$ and $r = |a_{M_0} - a_{N_0}|$ we obtain that there is some $q \in \mathbb{N}_0$ such that

$$\operatorname{card}(A \cap B(a_{N_0}, |a_{M_0} - a_{N_0}|, q)) \ge \frac{6d^2}{\alpha} \left(\frac{d\kappa_d}{\Omega(d)}\right)^d \left(\frac{100}{\Omega(d)}\right)^d \rho_q.$$

Since $\{a_i\}$ is 1-monotone we we can find such indices $N_0' \le M_0'$ that $\{a_i\}_{i=N_1'}^{M_1'} = A \cap B(a_{N_0}, |a_{M_0} - a_{N_0}|, q)$.

Then using Lemma 3.3 we can find some s, $N_1' \le s \le M_1'$, such that

$$\begin{aligned} &\operatorname{card}(B(a_s, 2|a_{M_0} - a_{N_0}|\rho_q) \cap \{a_i\}_{i=N_1'}^{M_1'}) \\ &\ge \frac{6d^2}{\alpha} \left(\frac{d}{\Omega(d)}\right)^d \kappa_d^{d-1} \left(\frac{100}{\Omega(d)}\right)^d \rho_q. \end{aligned} \tag{10}$$

Now, let N_1 be the first index for which $a_{N_1} \in B(a_s, 2|a_{M_0} - a_{N_0}|\rho_q) \cap \{a_i\}_{i=N_1'}^{M_1'}$ and M_1 be the last index for which $a_{M_1} \in B(a_s, 2|a_{M_0} - a_{N_0}|\rho_q) \cap \{a_i\}_{i=N_1'}^{M_1'}$. Then

$$\{a_i\}_{i=N_1}^{M_1} \subset B(a_{M_1}, 4|a_{M_0} - a_{N_0}|\rho_q). \tag{11}$$

To prove (e) note that

$$\begin{aligned} M_1 - N_1 + 1 \ge &\operatorname{card}(B(a_s, 2|a_{M_0} - a_{N_0}|\rho_q) \cap \{a_i\}_{i=N_1'}^{M_1'}) \\ &\overset{(10)}{\ge} \frac{6d^2}{\alpha} \left(\frac{d\kappa_d}{\Omega(d)}\right)^{d-1} \left(\frac{100}{\Omega(d)}\right)^d \rho_q \ge \frac{60}{\alpha} r_q \ge 11 \end{aligned} \tag{12}$$

Where the last inequality follows from

$$\alpha \le |a_{N_0} - a_{N_0+1}| \le r_q |a_{M_0} - a_{N_0}| \le r_q. \tag{13}$$

Condition (a) is easy, we only need to verify $N_1 < M_1$, which follows from (e). To prove (b) observe that $a_{N_1} \in B(a_{N_0}, |a_{M_0} - a_{N_0}|, q)$ and by (11) we have for every $N_1 \le j \le M_1$

$$\frac{2}{\Omega(d)}|a_j - a_{M_1}| \le \frac{8}{\Omega(d)}\rho_q|a_{M_0} - a_{N_0}| \le r_{q+1}|a_{M_0} - a_{N_0}| \le |a_{N_1} - a_{p_1}|. \tag{14}$$

Condition (c) is empty in this case. Using (e) and (10) we obtain

$$\begin{aligned} M_1 - N_1 \ge& \frac{10}{11} \cdot (M_1 - N_1 + 1) \\ \ge& \frac{10}{11} \cdot \mathrm{card}(B(a_s, 2|a_{M_0} - a_{N_0}|\rho_q) \cap \{a_i\}_{i=N_1'}^{M_1'}) \\ \overset{(10)}{\ge}& \frac{10}{11} \cdot \frac{6d^2}{\alpha} \cdot \left(\frac{d}{\Omega(d)}\right)^d \kappa_d^{d-1} \left(\frac{100}{\Omega(d)}\right)^d \rho_q \\ \overset{(11)}{\ge}& \frac{10}{11} \cdot \frac{6d^2}{\alpha} \cdot \left(\frac{d\kappa_d}{\Omega(d)}\right)^{d-1} \left(\frac{100}{\Omega(d)}\right)^d \frac{|a_{M_1} - a_{N_1}|}{4|a_{M_0} - a_{N_0}|} \\ \ge& \frac{6d^2}{\alpha} \cdot \left(\frac{d\kappa_d}{\Omega(d)}\right)^{d-1} \left(\frac{100}{\Omega(d)}\right)^{d-1} |a_{M_1} - a_{N_1}| \end{aligned} \tag{15}$$

which proves (d).

Induction step. Suppose that $p_i, N_i, M_i,\ i = 1, ..., k$ are already constructed for some $k < d$ we will now show how to construct $p_{k+1}, N_{k+1}, M_{k+1}$.

Using Lemma 3.5 for $w_j = a_{p_j}$ and the sequence $A = \{a_i\}_{i=N_k}^{M_k}$ we can find indices

$$N_k = i(0,+) \le i(1,-) \le \cdots \le i(\gamma,-) \le i(\gamma,+) \le i(\gamma+1,-) = M_k \tag{16}$$

such that (5), (6) and (7) hold.

Consider

$$V := \{N_k \le i < M_k : i(\beta,-) \le i \le i(\beta,+),\ \ \beta = 1, ..., \gamma\}.$$

Define $W = \{N_k+1, ..., M_k-1\} \setminus V$. Using Lemma 3.6 for b_i being the subsequence obtained by restricting A to V and $w_j = a_{p_j}$ we obtain that either $\mathrm{card}(V) \le 1$ or there is some i such that

$$\begin{aligned} \mathrm{card}(V) \le& 2(\mathrm{card}(V) - 1) \\ \le& \frac{4k}{\alpha\Omega(d)} \cdot (|a_{\max V} - a_{p_i}| - |a_{\min V} - a_{p_i}|) \\ \le& \frac{4k}{\alpha\Omega(d)} \cdot (|a_{M_k} - a_{p_i}| - |a_{N_k} - a_{p_i}|) \\ \le& \frac{4k}{\alpha\Omega(d)} \cdot |a_{M_k} - a_{N_k}| \\ \overset{(d)}{\le}& \frac{4k\alpha(\Omega(d))^{d-k}}{6d^2\alpha\Omega(d)(d\kappa_d)^{d-k}\left(\frac{100}{\Omega(d)}\right)^{d-k}} \cdot (M_k - N_k) \\ \le& \frac{(M_k - N_k)}{10}\ . \end{aligned}$$

This and (e) from the induction step imply that $\mathrm{card}(W) \geq \frac{8(M_k-N_k)}{10}$. Clearly, we can find

$$N_k \leq \iota(0,-) < \iota(0,+) \leq \iota(1,-) < ... < \iota(\Upsilon-1,+) \leq \iota(\Upsilon,-) < \iota(\Upsilon,+) \leq M_k.$$

such that

$$W = \bigcup_{s=0}^{\Upsilon} \{i; \iota(s,-) < i < \iota(s,+)\}$$

and $\iota(s,-) < \iota(s,+) - 1$ for every $s = 0, \dots, \Upsilon$.

Now, we will prove that there is an index $0 \leq \widetilde{s} \leq \Upsilon$ such that

$$\begin{aligned} &\frac{1}{5}\left(\frac{d}{\Omega(d)}\right)^{d-k-1} (\kappa_d)^{d-k} \left(\frac{100}{\Omega(d)}\right)^{d-k} |a_{\iota(\tilde{s},+)-1} - a_{\iota(\tilde{s},-)}| \\ \leq &\frac{\alpha}{6d^2}(\iota(\tilde{s},+) - \iota(\tilde{s},-) - 1). \end{aligned} \tag{17}$$

First assume that $2(\iota(\Upsilon,+) - \iota(\Upsilon,-) - 1) \geq \mathrm{card}(W)$. Then we have

$$\begin{aligned} \frac{1}{5}\left(\frac{d}{\Omega(d)}\right)^{d-k-1} &(\kappa_d)^{d-k} \left(\frac{100}{\Omega(d)}\right)^{d-k} |a_{\iota(\Upsilon,+)-1} - a_{\iota(\Upsilon,-)}| \\ &\stackrel{Lemma\ 2.4}{\leq} \frac{1}{5}\left(\frac{d}{\Omega(d)}\right)^{d-k-1} (\kappa_d)^{d-k} \left(\frac{100}{\Omega(d)}\right)^{d-k} 2|a_{M_k} - a_{N_k}| \\ &\stackrel{(d)}{\leq} \frac{\alpha}{6d^2}\frac{2(M_k-N_k)}{5} \\ &\leq \frac{\alpha}{6d^2}(\iota(\Upsilon,+) - \iota(\Upsilon,-) - 1) \end{aligned}$$

and therefore we can put $\tilde{s} = \Upsilon$.

Now assume that $2(\iota(\Upsilon,+) - \iota(\Upsilon,-) - 1) \leq \mathrm{card}(W)$. We will prove that there is $0 \leq \widetilde{s} < \Upsilon$ such that for every $i = 1, ..., k$

$$\begin{aligned} &\frac{2}{5d}\left(\frac{d\kappa_d}{\Omega(d)}\right)^{d-k} \left(\frac{100}{\Omega(d)}\right)^{d-k} (|a_{\iota(\tilde{s},+)} - a_{p_i}| - |a_{\iota(\tilde{s},-)} - a_{p_i}|) \\ \leq &\frac{\alpha}{6d^2}(\iota(\tilde{s},+) - \iota(\tilde{s},-) - 1). \end{aligned} \tag{18}$$

For a contradiction suppose that for each $0 \leq s < \Upsilon$ there is some i_s such that

$$\begin{aligned} \frac{2}{5d}\left(\frac{d\kappa_d}{\Omega(d)}\right)^{d-k} \left(\frac{100}{\Omega(d)}\right)^{d-k} &(|a_{\iota(s,+)} - a_{p_{i_s}}| - |a_{\iota(s,-)} - a_{p_{i_s}}|) \\ > &\frac{\alpha}{6d^2}(\iota(s,+) - \iota(s,-) - 1). \end{aligned}$$

Define

$$W_i = \bigcup_{i_s=i} \{j; \iota(s,-) < j < \iota(s,+)\}.$$

Find i such that $\mathrm{card}(W_i)$ is maximal. Then $\mathrm{card}(W_i) \geq \frac{\mathrm{card}(W)}{2d} \geq \frac{2(M_k-N_k)}{5d}$

Now,

$$\begin{aligned}
\frac{\alpha}{6d^2}(M_k - N_k) \leq & \frac{5d}{2} \cdot \frac{\alpha}{6d^2} \operatorname{card}(W_i) \\
= & \frac{5d}{2} \cdot \frac{\alpha}{6d^2} \left(\sum_{s:i_s=i} (\iota(s,+) - \iota(s,-) - 1) \right) \\
< & \left(\frac{d\kappa_d}{\Omega(d)} \right)^{d-k} \left(\frac{100}{\Omega(d)} \right)^{d-k} \left(\sum_{s:i_s=i} (|a_{\iota(s,+)} - a_{p_i}| - |a_{\iota(s,-)} - a_{p_i}|) \right) \\
\leq & \left(\frac{d\kappa_d}{\Omega(d)} \right)^{d-k} \left(\frac{100}{\Omega(d)} \right)^{d-k} (|a_{M_k} - a_{p_i}| - |a_{N_k} - a_{p_i}|) \\
\leq & \left(\frac{d\kappa_d}{\Omega(d)} \right)^{d-k} \left(\frac{100}{\Omega(d)} \right)^{d-k} |a_{M_k} - a_{N_k}| \\
\overset{(d)}{\leq} & \frac{\alpha}{6d^2}(M_k - N_k),
\end{aligned}$$

which is not possible. Since $\tilde{s} < \Upsilon$ we have that $\iota(\tilde{s},-), \iota(\tilde{s},+)$ are consecutive elements of V. Thus by (7) we have $a_{\iota(\tilde{s},+)} \in \bigcup_{i=1}^k C(a_{p_i}, a_\iota(\tilde{s},-))$. So there exists $i \in \{1,\dots,k\}$ such that

$$\begin{aligned}
\Omega(d)|a_{\iota(\tilde{s},+)-1} - a_{\iota(\tilde{s},-)}| \leq & \Omega(d)|a_{\iota(\tilde{s},+)} - a_{\iota(\tilde{s},-)}| \\
\leq & 2(|a_{\iota(\tilde{s},+)} - a_{p_i}| - |a_{\iota(\tilde{s},-)} - a_{p_i}|).
\end{aligned}$$

Using this and (18) we obtain (17).

Put $p_{k+1} = \tilde{N}_{k+1} = \iota(\tilde{s},-)$ and $\tilde{M}_{k+1} = \iota(\tilde{s},+) - 1$. This implies $p_{k+1} \geq N_k$. Observe that for every $i = 1, ..., k$

$$\{a_j\}_{\tilde{N}_{k+1}+1}^{\tilde{M}_{k+1}} \subset D(a_{p_i}, a_{p_{k+1}}). \tag{19}$$

Now, we will find N_{k+1} and M_{k+1}. Consider $B(a_{p_{k+1}}, |a_{\widetilde{M}_{k+1}} - a_{p_{k+1}}|)$. Then according to Lemma 3.2 there is some $q \in \mathbb{N}_0$ with

$$\operatorname{card}\left(\{a_j\}_{\widetilde{N}_{k+1}}^{\widetilde{M}_{k+1}} \cap B(a_{p_{k+1}}, |a_{\widetilde{M}_{k+1}} - a_{p_{k+1}}|, q)\right) \geq \rho_q(\widetilde{M}_{k+1} - \widetilde{N}_{k+1}).$$

Since $\{a_j\}_{\widetilde{N}_{k+1}}^{\widetilde{M}_{k+1}}$ is 1-monotone we have some indices N'_{k+1}, M'_{k+1} such that $\widetilde{N}_{k+1} < N'_{k+1} \leq M'_{k+1} \leq \widetilde{M}_{k+1}$ and

$$\{a_j\}_{N'_{k+1}}^{M'_{k+1}} = \{a_j\}_{\widetilde{N}_{k+1}}^{\widetilde{M}_{k+1}} \cap B(a_{p_{k+1}}, |a_{\widetilde{M}_{k+1}} - a_{p_{k+1}}|, q).$$

Further, due to Lemma 3.3 and (17) there is some index s with $N'_{k+1} \leq s \leq M'_{k+1}$ and

$$\begin{aligned}
& \operatorname{card}\left(\{a_i\}_{i=N'_{k+1}}^{M'_{k+1}} \cap B(a_s, 2|a_{\widetilde{M}_{k+1}} - a_{p_{k+1}}|\varrho_q)\right) \geq \frac{\rho_q}{\kappa_d}(\widetilde{M}_{k+1} - \widetilde{N}_{k+1}) \\
& \overset{(17)}{\geq} \frac{1}{5}\frac{6d^2}{\alpha}\left(\frac{d\kappa_d}{\Omega(d)}\right)^{d-k-1} \left(\frac{100}{\Omega(d)}\right)^{d-k} \rho_q |a_{\widetilde{M}_{k+1}} - a_{\widetilde{N}_{k+1}}|.
\end{aligned} \tag{20}$$

Let M_{k+1}, N_{k+1} be a smallest and greatest indeces from $\{j\}_{N'_{k+1}}^{M'_{k+1}}$ for which $a_{N_{k+1}}, a_{M_{k+1}} \in B(a_s, 2|a_{\widetilde{M}_{k+1}} - a_{p_{k+1}}|\varrho_q)$. Then

$$\{a_j\}_{N_{k+1}}^{M_{k+1}} \subset B(a_{M_{k+1}}, 4|a_{\widetilde{M}_{k+1}} - a_{p_{k+1}}|\varrho_q). \tag{21}$$

Evidently,

$$\tilde{N}_{k+1} < N_{k+1} \le M_{k+1} \le \tilde{M}_{k+1}. \tag{22}$$

Let us prove (c). Assume $N_{k+1} \le j \le M_{k+1}$ and $i = 1, ..., k$. By the 1-monotonicity we have

$$|a_j - a_{p_{k+1}}| \le |a_{p_{k+1}} - a_{M_k}| \overset{(b)}{\le} \frac{\Omega(d)}{2}|a_{p_i} - a_{N_k}| \le \frac{\Omega(d)}{2}|a_{p_i} - a_{p_{k+1}}|$$

which implies

$$\frac{\Omega(d)}{2}|a_{p_i} - a_{p_{k+1}}| + |a_j - a_{p_{k+1}}| \le \Omega(d)|a_{p_i} - a_{p_{k+1}}| \le \Omega(d)|a_{p_i} - a_j|. \tag{23}$$

Thus,

$$\begin{aligned} 0 \ge& (a_j - a_{p_{k+1}}) \cdot (a_{p_i} - a_j) \\ =& (a_j - a_{p_{k+1}}) \cdot ((a_{p_i} - a_{p_{k+1}}) + (a_{p_{k+1}} - a_j)) \\ =& (a_j - a_{p_{k+1}}) \cdot (a_{p_i} - a_{p_{k+1}}) + (a_j - a_{p_{k+1}}) \cdot (a_{p_{k+1}} - a_j) \\ \overset{(19)}{\ge}& - \frac{\Omega(d)|a_{p_{k+1}} - a_j| \cdot |a_{p_i} - a_{p_{k+1}}|}{2} - |a_j - a_{p_{k+1}}|^2 \\ \overset{(23)}{\ge}& - \Omega(d)|a_{p_{k+1}} - a_j| \cdot |a_{p_i} - a_j|, \end{aligned}$$

where the first inequality follows from 1-monotonicity of $\{a_j\}$. Now, the fact $N_k \le N_{k+1} \le M_{k+1} \le M_k$ completes (c).

Let us prove (b). Consider $N_{k+1} \le l \le M_{k+1}$. By (21) we have

$$\frac{2}{\Omega(d)}|a_l - a_{M_{k+1}}| \le \frac{8}{\Omega(d)}|a_{\widetilde{M}_{k+1}} - a_{p_{k+1}}|\varrho_q \le r_{q+1}|a_{\widetilde{M}_{k+1}} - a_{p_{k+1}}| \le |a_{N_{k+1}} - a_{p_{k+1}}|$$

which proves (b) for $i = k+1$. Assume now $1 \le i \le k$. Then by 1-monotonicity we obtain

$$\frac{2}{\Omega(d)}|a_l - a_{M_{k+1}}| \le \frac{2}{\Omega(d)}|a_l - a_{M_k}| \overset{(b)}{\le} |a_{p_i} - a_{N_k}| \le |a_{p_i} - a_{N_{k+1}}|$$

which finishes the proof of (b).

Using (20) and following the calculation showed in (12) and (13) we obtain

$$\begin{aligned} M_{k+1} - N_{k+1} + 1 \ge& \operatorname{card}(B(a_s, 2|a_{\tilde{M}_{k+1}} - a_{\tilde{N}_{k+1}}|\rho_q) \cap \{a_i\}_{i=N'_{k+1}}^{M'_{k+1}}) \\ \overset{(20)}{\ge}& \frac{1}{5} \cdot \frac{6d^2}{\alpha} \left(\frac{d\kappa_d}{\Omega(d)}\right)^{d-k-1} \left(\frac{100}{\Omega(d)}\right)^{d-k} \rho_q |a_{\tilde{M}_{k+1}} - a_{\tilde{N}_{k+1}}| \\ =& 2\frac{6d^2}{\alpha} \left(\frac{d\kappa_d}{\Omega(d)}\right)^{d-k-1} \left(\frac{100}{\Omega(d)}\right)^{d-k-1} r_q |a_{\tilde{M}_{k+1}} - a_{\tilde{N}_{k+1}}| \\ \ge& 11. \end{aligned} \tag{24}$$

Thus, $M_{k+1} - N_{k+1} \ge 10$ which proves (e).

Moreover, by (24) and (e) we obtain

$$\begin{aligned}
M_{k+1}-N_{k+1} \geq &\frac{10}{11}(M_{k+1}-N_{k+1}+1)\\
\overset{(24)}{\geq} &\frac{10}{11}\cdot\frac{1}{5}\cdot\frac{6d^2}{\alpha}\left(\frac{d\kappa_d}{\Omega(d)}\right)^{d-k-1}\left(\frac{100}{\Omega(d)}\right)^{d-k}\rho_q|a_{\tilde{M}_{k+1}}-a_{\tilde{N}_{k+1}}|\\
\overset{(21)}{\geq} &\frac{2}{11}\cdot\frac{6d^2}{\alpha}\cdot\left(\frac{d\kappa_d}{\Omega(d)}\right)^{d-k-1}\left(\frac{100}{\Omega(d)}\right)^{d-k}\frac{|a_{M_{k+1}}-a_{N_{k+1}}|}{4}\\
= &\frac{1}{22}\cdot\frac{6d^2}{\alpha}\cdot\left(\frac{d\kappa_d}{\Omega(d)}\right)^{d-k-1}\left(\frac{100}{\Omega(d)}\right)^{d-k}|a_{M_{k+1}}-a_{N_{k+1}}|\\
\geq &\frac{6d^2}{\alpha}\cdot\left(\frac{d\kappa_d}{\Omega(d)}\right)^{d-k-1}\left(\frac{100}{\Omega(d)}\right)^{d-k-1}|a_{M_{k+1}}-a_{N_{k+1}}|
\end{aligned}$$

which proves (d).

To finish the construction note that (a) follows from (e) and the induction procedure.

Now, using Lemma 3.7 for our choice of p_i, $N=N_d$, $M=M_d$ we obtain that for some i

$$|a_{M_d}-a_{N_d}| \geq |a_{M_d}-a_{p_i}|-|a_{N_d}-a_{p_i}| > \frac{\alpha}{6d^2}(M_d-N_d),$$

which is in contradiction with (d) for $k=d$. Note that we can use Lemma 3.7 due to $(a)-(c)$. □

Remark that an analogous theorem cannot hold in an infinite dimensional Hilbert space H because a 1-monotone space of Hausdorff dimension greater than 1 can be found in H.

Corollary 3.9. *Let $\Gamma : [0,1] \to \mathbb{R}^d$ be continuous curve. Then graph of Γ has finite 1-dimensional Hausdorff measure if and only if Γ is a linear combination of continuous curves with 1-monotone graphs.*

Proof. Let $\Gamma = (f_1,\dots,f_d) : [0,1] \to \mathbb{R}^d$ be continuous curve. Clearly, Γ has finite 1-dimensional Hausdorff measure if and only if $V_{[0,1]}(f_i)$ is finite for every $i=1,\dots,d$. This and Theorem 3.8 give that a linear combination of continuous curves with 1-monotone graphs has a finite 1-dimensional Hausdorff measure.

If Γ has a finite 1-dimensional Hausdorff measure then we can define functions $f_i^j : [0,1] \to \mathbb{R}$ for every $i=1,\dots,d$ and $j=0,1$ by

$$\begin{aligned}
f_i^0(t) &= V_{[0,t]}(f_i),\\
f_i^1(t) &= f_i(t) - V_{[0,t]}(f_i).
\end{aligned}$$

Now, we define $F^s : [0,1] \to \mathbb{R}^d$ for every $s \in \{0,1\}^d$ by

$$F^s(t) = \left(f_1^{s(1)}(t),\dots,f_d^{s(d)}(t)\right).$$

Since functions f_i^j are monotone we easily obtain that functions F^s are continuous and have 1-monotone graph. Clearly,

$$\Gamma = 2^{1-d}\sum_{s\in\{0,1\}^d} F^s.$$

So, we proved that Γ is a linear combination of continuous curves with 1-monotone graphs.

□

4. Function with small monotone subspaces

In this section we construct an example of a continuous function $f : [0,1] \to \mathbb{R}$ with the following property: if graph$(f|M)$ is a monotone set in the plane for some $M \subset [0,1]$ then $\lambda(M) = 0$ and M is nowhere dense.

Definition 4.1. *Let $F : [0,1] \to [0,1]$ be the standard (triadic) Cantor function and let $g : [0,1] \to [0,1]$ be a continuous function defined by*

$$g := \begin{cases} F(2x), & x \in [0, \frac{1}{2}] \\ F(2-2x), & x \in [\frac{1}{2}, 1]. \end{cases}$$

Let $I = [a, a+\varepsilon] \subset [0,1]$ be a nondegenerated interval $L > 0$ and $n \in \mathbb{N}$. Then we define a continuous function $f_I^{L,n} : [0,1] \to [0,L]$ by formula

$$f_I^{L,n} := \begin{cases} \varepsilon L g\left(\frac{n}{\varepsilon}\left(x - a - \frac{k\varepsilon}{n}\right)\right), & x \in \left[a + \frac{k\varepsilon}{n}, a + \frac{(k+1)\varepsilon}{n}\right], k = 0, ..., n-1, \\ 0, & \textit{otherwise} \end{cases}$$

Let Ω be a system of all continuous functions form $[0,1]$ to $\mathbb{R}$ that are locally constant on the set of full measure, i.e. there is a sequence of pairwise disjoint closed intervals I_k such that $\sum \lambda(I_k) = 1$ and f is constant on each I_k. Given $L > 0$ and $n \in \mathbb{N}$. define operator $\Upsilon_{L,n} : \Omega \to \Omega$ by the following procedure: For $h \in \Omega$ let $\mathcal{I}(h)$ be the system of all maximal nondegenerated intervals in which h is constant. Then we put

$$\Upsilon_{L,n}(h) = h + \sum_{I \in \mathcal{I}(h)} f_I^{L,n}.$$

Let $\{a_k\}_{k=1}^{\infty}$ be the sequence $3, 4, 3, 4, 5, 3, 4, 5, 6, 3, 4, 5, 6, 7, 3, 4, \ldots$ and put $L_n = \frac{1}{2a_n}$. We define $f_0 \equiv 0$ and put

$$f_{n+1} = \Upsilon_{L_{n+1}, 2a_{n+1}^2}(f_n).$$

Lemma 4.2. *Let $N \in \mathbb{N}$, $\Delta > 0$ and let*

$$X_k = \left[\Delta\left(2k - \frac{1}{9}\right), \Delta\left(2k + \frac{1}{9}\right)\right] \times \left[0, \frac{2\Delta}{3}\right],$$

$k = 0, ..., 2N^2$ *and*

$$Y_k = \left[\Delta\left(2k + 1 - \frac{1}{9}\right), \Delta\left(2k + 1 + \frac{1}{9}\right)\right] \times \left[\frac{\Delta(6N-2)}{3}, \frac{\Delta(6N+2)}{3}\right],$$

$k = 0, ..., 2N^2 - 1$.

Suppose that M is a symmetrically $\frac{N}{4}$-monotone set in $\mathbb{R}^2$, then there is some k such that $X_k \cap M = \emptyset$ or $Y_k \cap M = \emptyset$.

Proof. Without loss of generality we can assume $\Delta = 1$. Suppose for a contradiction that there are $x_k \in X_k \cap M$ and $y_k \in Y_k \cap M$ for every k and let $\prec$ is a witnessing ordering on M. Suppose that x_i and y_j are the minimal (with respect to $\prec$) among all x_k and y_k, respectively. We will additionally assume that $x_i \prec y_j$ the second case can be proved by the same way. There are two possibilities, either $x_k \prec y_j$ for every k or there are some $x_l \prec y_j \prec x_p$. In the second case it is not difficult to see that we can assume $|l-p|=1$ which implies $|x_l - x_p| < 4$, moreover $|x_l - y_j| > N$. This leads to

$$\frac{|x_l - y_j|}{|x_l - x_p|} > \frac{N}{4}$$

which is a contradiction. The first case implies that $x_0, x_{2N^2} \prec y_0, y_{2N^2}$. If $x_0 \prec x_{2N^2}$ we consider x_0, x_{2N^2}, y_0 with $x_0 \prec x_{2N^2} \prec y_0$. Then $|x_0 - x_{2N^2}| > 4N^2 - 1$ and $|x_0 - y_0| < 3N$. This leads to

$$\frac{|x_0 - x_{2N^2}|}{|x_0 - y_0|} > \frac{4N^2-1}{3N} > \frac{N}{4}$$

and we have again a contradiction. If $x_0 \succ x_{2N^2}$ we consider x_0, x_{2N^2}, y_{2N^2} with $x_{2N^2} \prec x_0 \prec y_{2N^2}$ and we continue analogously. □

Lemma 4.3. *Let $n \in \mathbb{N}$ and $I = [a, a + \Delta 4a_{n+1}^2] \in \mathcal{I}(f_n)$. Then*

(a) $0 \le f_{n+1} - f_n \le L_{n+1}|I| = 2\Delta a_{n+1}$ *on I,*

(b) *let $J \in \mathcal{I}(f_{n+1})$ such that $J \subset I$, then $|J| \le \frac{|I|}{12a_{n+1}^2} = \frac{\Delta}{3}$,*

(c) *for $i = 0, ..., 2a_{n+1}^2$*

$$(f_{n+1} - f_n)\left(\left[a + 2i\Delta - \frac{\Delta}{64a_{n+1}^2}, a + 2i\Delta + \frac{\Delta}{64a_{n+1}^2}\right] \cap I\right) \subset \left[0, \frac{\Delta}{4}\right],$$

(d) *for $i = 0, ..., 2a_{n+1}^2 - 1$*

$$(f_{n+1} - f_n)\left(\left[a + (2i+1)\Delta - \tfrac{\Delta}{64a_{n+1}^2}, a + (2i+1)\Delta + \tfrac{\Delta}{64a_{n+1}^2}\right]\right) \subset \left[\tfrac{(8a_{n+1}-1)\Delta}{4}, \tfrac{(8a_{n+1}+1)\Delta}{4}\right].$$

Proof. The first part is obvious and the fact that the biggest interval of constantness on the Cantor function has length $\frac{1}{3}$. The last two parts follow directly from the facts that for the standard Cantor function F we have $F(x) \le \sqrt{x}$ and therefore

$$F\left(\left[0, \frac{1}{64a_{n+1}^2}\right]\right) \subset \left[0, \frac{1}{8a_{n+1}}\right],$$

together with the symmetry of F. □

Due to Lemma 4.3 we know that the sequance $\{f_n\}$ is uniformly convergent (and monotone) and we can now define the continuous function $f = \sup_n f_n$.

Lemma 4.4. *Let $n \in \mathbb{N}$ and $I = [a, a + \Delta 4a_{n+1}^2] \in \mathcal{I}(f_n)$. Then*

(1) $0 \le f - f_n \le |I|$

(2) *for $i = 0, ..., 2a_{n+1}^2$*

$$(f - f_n)\left(\left[a + 2i\Delta - \frac{\Delta}{64a_{n+1}^2}, a + 2i\Delta + \frac{\Delta}{64a_{n+1}^2}\right] \cap I\right) \subset \left[0, \frac{2\Delta}{3}\right]$$

(3) *for $i = 0, ..., 2a_{n+1}^2 - 1$*

$$(f - f_n)\left(\left[a + (2i+1)\Delta - \tfrac{\Delta}{64a_{n+1}^2}, a + (2i+1)\Delta + \tfrac{\Delta}{64a_{n+1}^2}\right]\right) \subset \left[\tfrac{(6a_{n+1}-2)\Delta}{3}, \tfrac{(6a_{n+1}+2)\Delta}{3}\right].$$

Proof. Property (1) follows from properties (a) and (b) as follows

$$0 \le f - f_n = \sum_{i=1}^{\infty}(f_{n+i} - f_{n+i-1}) \le \frac{1}{2}\sum_{i=1}^{\infty}|I|2^{-i+1} = |I|.$$

To prove property (2) we write

$$0 \le f - f_n = f - f_{n+1} + f_{n+1} - f_n \overset{(1)\&(b)}{\le} \frac{\Delta}{3} + f_{n+1} - f_n \overset{(c)}{\le} \frac{\Delta}{3} + \frac{\Delta}{4} < \frac{2\Delta}{3}.$$

Property (3) can be proved following the same lines. □

Theorem 4.5. *Let $M \subset [0,1]$ and suppose that* graph$(f|_M)$ *is monotone. Then $\lambda(M) = 0$ and moreover, M is nowhere dense.*

Proof. Fix $c \ge 2$ and $M \subset [0,1]$ and suppose that graph$(f|_M)$ is c-monotone. Then graph$(f|_{\overline{M}})$ is symmetrically $(c+1)$-monotone.

Consider $A_n := [0,1] \setminus \bigcup_{I\in\mathcal{I}(f_n)} I$ and put $A = \bigcup A_n$. Then A has measure 0. Suppose for contradiction that $\overline{M}$ has positive measure. Then also $\overline{M} \setminus A$ has positive measure. This means that there is a Lebesgue point of $x \in \overline{M} \setminus A$. From the definition of the Lebesgue point we can find $\delta_0 > 0$ such that for every $\delta_0 > \delta > 0$ we have

$$\frac{\lambda(\overline{M} \cap [x-\delta, x+\delta])}{2\delta} \ge 1 - \frac{1}{2000000c^4}.$$

From the construction of the function f we can find n such that $4c+4 \le a_{n+1} \le 7c$ and such that there is some $I = [a,b] \in \mathcal{I}(f_n)$ with $x \in I \subset [x-\delta_0, x+\delta_0]$. Put $\delta = \max(|x-a|, |x-b|)$. Then $I \subset [x-\delta, x+\delta]$ and $|a-b| \ge \delta$. Now, by Lemma 4.2 and Lemma 4.4 we obtain that there is an interval J of length $\frac{|a-b|}{256a_{n+1}^4}$ such that $J \cap \overline{M} \setminus A = \emptyset$ and we can write

$$\begin{aligned} 1 - \frac{1}{2000000c^4} \le & \frac{\lambda(\overline{M} \cap [x-\delta, x+\delta])}{2\delta} \le \frac{2\delta - \frac{|a-b|}{256a_{n+1}^4}}{2\delta} \\ \le & \frac{2\delta - \frac{\delta}{256a_{n+1}^4}}{2\delta} \le \frac{2 - \frac{1}{256(7c)^4}}{2} = 1 - \frac{1}{512(7c)^4} < 1 - \frac{1}{2000000c^4}. \end{aligned}$$

Note that we proved $\lambda(\overline{M}) = 0$ in fact. Consequently, M is nowhere dense. □

Note that if we ask for a continuous function f such that no set $M \subset \operatorname{graph} f$ of positive 1-dimensional Hausdorff measure (equipped with the Euclidean metric) is monotone, the situation is completely different. In fact, for every such f there is always a monotone function $h : [\min f, \max f] \to \mathbb{R}$ such that $\operatorname{graph} h^{-1} \subset \operatorname{graph} f$ (see e.g. [5]). Note that for $M = \operatorname{graph} h$ we have $|M| \ge \max f - \min f$ and M is symmetrically 1-monotone.

5. Smooth function witt unbounded variation and monotone graph

In this section we will construct for every $c > 1$ a smooth function with symmetrically c-monotone graph and unbounded variation.

Definition 5.1. *Let $n \in \mathbb{N}$ and $I = [a, a+\Delta] \subset [0,1]$ be a closed nondegenerated interval. Put*

$$I_n^i := \left[a + i\Delta\frac{2n+3}{6n+6}, a + i\Delta\frac{2n+3}{6n+6} + \frac{\Delta n}{3n+3}\right]$$

for $i \in \{0,1,2\}$ *and define* $\mathcal{A}_n^I := \{I_n^0, I_n^1, I_n^2\}$.

Clearly, we can fix some $f_n^I \in C^\infty([0,1])$ *such that*

(a) $f_n^I(x) = 0$ *for* $x \in I_n^0 \cup I_n^2 \cup ([0,1] \setminus I)$,
(b) $f_n^I(x) = \frac{|I_n^1|}{2}$ *for* $x \in I_n^1$,
(c) $(f_n^I)'(x) \neq 0$ *for* $x \in I \setminus (I_n^0 \cup I_n^1 \cup I_n^2)$.

For every $n \geq 0$ *we inductively define functions* $f_n : [0,1] \to \mathbb{R}$ *and a collection of closed intervals* $\mathcal{A}_n$. *We put* $f_0 \equiv 0$ *and* $\mathcal{A}_0 = \{[0,1]\}$. *Assume that we already have* f_n *and* $\mathcal{A}_n$. *We define*

$$f_{n+1} = f_n + \sum_{I \in \mathcal{A}_n} f_{n+1}^I,$$

$$\mathcal{A}_{n+1} = \bigcup_{I \in \mathcal{A}_n} \mathcal{A}_{n+1}^I.$$

Lemma 5.2. *The following statements hold.*

(i) *Let* $n \in \mathbb{N}$, $i \in \{0,1,2\}$ *and* I *be a closed interval. Then* $I_n^i \subset I$.
(ii) *Let* $n \geq 0$. *Then the elements of* $\mathcal{A}_n$ *are mutually disjoint.*
(iii) $|\bigcup \mathcal{A}_n| = \frac{1}{n+1}$ *for every* $n \geq 0$.
(iv) *Let* $n \geq 0$ *and* $I \in \mathcal{A}_n$. *Then* $|I| = \frac{1}{(n+1)3^n}$.
(v) *Let* $n \geq 0$ *and* $I \in \mathcal{A}_n$. *Then* $0 \leq f_{n+1}^I(x) \leq \frac{1}{2 \cdot 3^{n+1}(n+2)}$ *for every* $x \in [0,1]$.
(vi) *Let* $n \geq 0$. *Then* $f_n(x) \leq \frac{1}{4}$ *for every* $x \in [0,1]$.
(vii) *Let* $n \geq 0$. *Then* $f_n \in C^\infty([0,1])$ *and* $(f_n)_+^{(i)}(0) = (f_n)_-^{(i)}(1) = 0$ *for every* $i \geq 0$.
(viii) *Let* $n \geq 0$ *and* $I \in \mathcal{A}_n$. *Then* f_n *is constant on* I.
(ix) $V_{[0,1]}(f_n) = \frac{1}{3}\sum_{i=1}^n \frac{1}{i+1}$ *for every* $n \in \mathbb{N}$.
(x) *Let* $0 \leq k < n$, $I \in \mathcal{A}_k$ *and* $x, y \in I$. *Then*

$$|f_n(x) - f_n(y)| \leq \sum_{i=k+1}^{n} \frac{1}{2 \cdot 3^i (i+1)}.$$

(xi) *The function* f_n *satisfy condition* P_1 *for every* $n \geq 0$.

Proof. Statements (i), (ii), (vii) and $(viii)$ are trivial.

We prove (iii) by induction. Clearly, $|\bigcup \mathcal{A}_0| = 1$. Assume, we had already shown $|\bigcup \mathcal{A}_n| = \frac{1}{n+1}$. Since $|\bigcup \mathcal{A}_{n+1}^I| = \frac{|I|(n+1)}{n+2}$ for every closed interval I we have

$$\left|\bigcup \mathcal{A}_{n+1}\right| = \frac{n+1}{n+2}\left|\bigcup \mathcal{A}_n\right| = \frac{1}{n+2}.$$

Clearly $\operatorname{card}(\mathcal{A}_n) = 3^n$ and all elements of $\mathcal{A}_n$ have same length. Thus, by (iii) and (ii) we obtain (iv).

Using (iv) we clearly obtain (v).

By (v) and (ii) we have $f_n \leq \sum_{i=1}^n \frac{1}{2 \cdot 3^i (i+1)} \leq \frac{1}{4}$. Thus we have (vi).

We prove (ix) by induction. Since $f_1 = f_1^{[0,1]}$ we have $V_{[0,1]}(f_1) = |[0,1]_1^1| = \frac{1}{6}$. Assume we had already shown $V_{[0,1]}(f_n) = \frac{1}{3}\sum_{i=1}^{n}\frac{1}{i+1}$. Clearly,

$$\begin{aligned} V_{[0,1]}(f_{n+1}) &\overset{(ii),(viii)}{=} V_{[0,1]}(f_n) + \sum_{I\in\mathcal{A}_n} V_I(f_{n+1}^I) = \frac{1}{3}\sum_{i=1}^{n}\frac{1}{i+1} + \sum_{I\in\mathcal{A}_n}|I_{n+1}^1| \\ &\overset{(iv)}{=} \frac{1}{3}\sum_{i=1}^{n}\frac{1}{i+1} + 3^n\frac{1}{(n+2)3^{n+1}} = \frac{1}{3}\sum_{i=1}^{n+1}\frac{1}{i+1}. \end{aligned}$$

Now we prove (x). Since $x, y \in I \in \mathcal{A}_k$ and $(viii)$ we have $f_k(x) = f_k(y)$. Since $f_n \geq f_k$ we have $|f_n(x) - f_n(y)| \leq \max\{f_n(t) - f_k(t);\ t \in I\}$. By (ii) and (v) we have

$$\max\{f_n(t) - f_k(t);\ t \in I\} \leq \sum_{i=k+1}^{n}\frac{1}{2\cdot 3^i(i+1)}.$$

Finally, we prove (xi). Let $x < y \in [0,1]$ be arbitrary such that $f_n(x) = f_n(y)$. We find $z \in (x,y)$ such that

$$|f_n(z) - f_n(x)| = \max\{|f_n(t) - f_n(x)|;\ t \in [x,y]\}. \tag{25}$$

By Definition 5.1(c) we have $z \in \bigcup\mathcal{A}_n$. We can assume $f_n(x) \neq f_n(z)$. Thus, $x, y \notin \bigcup\mathcal{A}_n$ and consequently, we can find maximal $0 \leq k < n$ such that there exists $I \in \mathcal{A}_k$ such that $x, z \in I$ or $z, y \in I$. By the maximality of k there exists $J \in \mathcal{A}_{k+1}$ such that $x, y \notin J$ and $z \in J$. Thus $J \subset (x,y)$ and

$$|x - y| > |J| = \frac{1}{(k+2)3^{k+1}}. \tag{26}$$

By (x) we have

$$|f(x) - f(z)| \leq \sum_{i=k+1}^{n}\frac{1}{2\cdot 3^i(i+1)} \leq \frac{1}{2\cdot 3^{k+1}(k+2)}\sum_{i=0}^{n-k-1}3^{-i} \leq \frac{1}{(k+2)3^{k+1}}.$$

Using this,(25) and (26) we are done.

□

Lemma 5.3. *Let $c > 0$ and $I \subset [0,1]$ be a closed non degenerated interval. Then there exists $g_c^I \in C^\infty([0,1])$ such that*

(a) *$g_c^I(x) = 0$ for every $x \in [0,1] \setminus I$,*
(b) *$0 \leq g_c^I \leq c$,*
(c) *g_c^I satisfy condition P_1,*
(d) *$V_{[0,1]}(g_c^I) \geq 1$.*

Proof. Let $I = [a,b]$. We can assume $c \leq 1$. By Lemma 5.2(ix) we can find $n \in \mathbb{N}$ such that $(b-a)cV_{[0,1]}(f_n) \geq 1$. We define

$$g_c^I(x) := \begin{cases} 0, & x \in [0,1] \setminus I, \\ c(b-a)f_n\left(\frac{x-a}{b-a}\right), & x \in I. \end{cases} \tag{27}$$

By Lemma 5.2 we have $g_c^I \in C^\infty([0,1])$ and conditions (a), (b), (c) and (d) are satisfied.

□

Theorem 5.4. *Let $c > 1$. Then there exists a continuous function $F : [0,1] \to \mathbb{R}$ such that*

(A) *F is infinitely differentiable at every $x \in (0,1]$,*
(B) *$F'_+(0) = 0$,*
(C) *$V_{[0,1]}(F) = \infty$,*
(D) *F has c-symmetrically monotone graph.*

Proof. For every $n \in \mathbb{N}$ we put $J_n := [2^{-2n+1}, 2^{-2n+2}]$. We define a function $G : [0,1] \to \mathbb{R}$ by

$$G(x) := \sum_{n=1}^{\infty} g^{J_n}_{4^{-2n+1}}(x),$$

where g^I_c are functions from Lemma 5.3.

By Lemma 5.3 we easily obtain (A).

If $x \in [0,1] \setminus \bigcup_{n=1}^{\infty} J_n$ then G(x)=0. If $x \in J_n$ then

$$0 \leq G(x) = g^{J_n}_{4^{-2n+1}}(x) \overset{L5.3(b)}{\leq} 4^{-2n+1} \leq x^2.$$

Thus, we have (B).

By Lemma 5.3(d) we have (C).

Now, we prove that G satisfy condition P_1. Let $x < y \in [0,1]$ such that $G(x) = G(y)$. We can assume that there is no $w \in (x,y)$ such that $G(w) = G(x)$. Thus, there exist $k \leq n$ such that $x \in J_n$ and $y \in J_k$. If $n = k$ then condition P_1 follows from Lemma 5.3(c). If $k < n$ then $y - x \geq 2^{-2n+1}$. Thus we have

$$\max\{|G(t) - G(x)|;\ t \in (x,y)\} \leq \max\{G(t);\ t \in J_n\} \overset{L5.3(b)}{\leq} 4^{-2n+1} \leq |x-y|$$

and condition P_1 is satisfied.

We put $F = (c-1)G$. Clearly F satisfy (A), (B), (C) and condition P_{c-1}. Thus F has c-symmetrically monotone graph.

□

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