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Monotonous Segments of Algebraic  
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# Identifying and Approximating Monotonous Segments of Algebraic Curves Using Support Function Representation

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## Abstract

Algorithms describing the topology of real algebraic curves search primarily the singular points and they are usually based on algebraic techniques applied directly to the curve equation. We adopt a different approach, which is primarily based on the identification and approximation of smooth monotonous curve segments, which can in certain cases cross the singularities of the curve. We use not only the primary algebraic equation of the planar curve but also (and more importantly) its implicit support function representation. This representation is also used for an approximation of the segments. This way we obtain an approximate graph of the entire curve which has several nice properties. It approximates the curve within a given Hausdorff distance. The actual error can be measured efficiently and behaves as  $\mathcal{O}(N^{-3})$  where  $N$  is the number of segments. The approximate graph is rational and has rational offsets. In the simplest case it consists of arc segments which are efficiently represented via the support function. The question of topological equivalence of the approximate and precise graphs of the curve is also addressed and solved using bounding triangles and axis projections. The theoretical description of the whole procedure is accompanied by several examples which show the efficiency of our method.

*Keywords:* algebraic curve, support function, critical points, inflections approximation, arc-splines

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## 1. Introduction

Solution of many problems in Computer Aided Geometric Design depends on an approximation of a curve given by an implicitly defined bivariate polynomial with rational coefficients. It is very desirable to visualize the curve in any required precision, to find the number of components or to test to which component a given point belongs. All this information is fully contained in the planar graph topologically equivalent to the curve whose vertices are points of the algebraic curve and edges correspond to regular arcs of the curve.

Known algorithms studying the topology of an algebraic curve have always two parts. First we find out the critical points and then we connect them appropriately. There are two main types of algorithms. The first type uses the same principle as the Cylindrical Algebraic Decomposition (CAD) algorithm, cf. [5, page 159]. The other approach is based on a subdivision of the given region.

Cylindrical Algebraic Decomposition based algorithms are usually divided into three phases: First find the  $x$ -coordinates of critical points of  $\mathcal{C}$ , then for each  $x_i$  compute the intersection points  $P_{i,j}$  of  $\mathcal{C}$  and the vertical line  $x = x_i$  and finally for every  $P_{i,j}$  determine the number of branches of  $\mathcal{C}$  on the left and right and use this information to connect the points appropriately.

The main problem of these algorithms is the second phase, because the  $x$ -coordinates of the critical points are not necessarily rational numbers and therefore the polynomials  $f(x_i, y)$  have non-rational coefficients. There are several methods to deal with this problem for example in [6, 7, 8, 14, 19].

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The second type of algorithm is based on subdivision. The only certified algorithm (i.e. one which gives the correct output for every input) based on subdivision is [3]. This algorithm subdivides the region  $\mathcal{D}$  into *regular regions* (the curve is smooth inside) and *regions with singular points*, which can be made sufficiently small. The topology inside the regions containing a singular point is recovered from the information on the boundary using the topological degree.

Our approach is rather novel. It consists in a decomposition of the curve into smooth segments, which can be intersecting each other but which are *monotonous* in both  $x$  and  $y$  coordinates as well as with respect to the moving tangent lines. These segments are identified by their end-points and end-tangents. In order to determine such data (the points with their tangents) and their connectivity, we use both the primary curve equation and its support function representation, which is a kind of a dual equation. This representation is one of the classical tools in the field of convex geometry [12]. In this representation offsetting and convolution of curves correspond to simple algebraic operations of the corresponding support functions. In addition, it provides a computationally simple way to extract curvature information [9]. Applications of this representation to problems from Computer Aided Design were foreseen in the classical paper [18] and developed in several recent publications, see e.g., [1, 2, 4, 10, 11, 16, 20, 21].

The identified segments can not be parameterized rationally except for the zero-genus case. For this reason we interpolate the boundary  $G^1$  data with a suitable (piecewise) rational segments. For this interpolation we exploit again the support function representation and obtain segments with certified Hausdorff distance from the original curve. We are also able to evaluate efficiently the actual error. The collection of approximated segments provides a good approximation of the given algebraic curve, but does not necessarily possess the same topology. We provide an iterative test which in many cases proves that the right topology is obtained. In remaining cases it identifies (very small) regions, where the topology differs. These regions are then tested (or subdivided) using projections to contain only one critical point. Then the certified topology inside these boxes follows from the theory of the topological degree.

The remainder of the paper is organized as follows. In Section 2 we recall some basic definitions and facts about real planar algebraic curves. We also present our basic tools which are the support function representation of a curve and the decomposition of a boxed curve into smooth segments. In Section 3 we show how the collection of suitable  $G^1$  data can be identified and how to decide about their connectivity. In Section 4 we interpolate the  $G^1$  data with segments represented via their support functions. In the simplest case we use for each boundary data a bi-arc curve, which is very efficiently represented via its support function. We show that our procedure exhibits the approximation order 3. In Section 4 we also study the topological equivalence of the original curve with the union of the approximated segments. The most important tool in this section is a bounding triangle of a curve segment and the topological degree. Section 5 is devoted to a number of examples which carefully demonstrate the whole procedure. Eventually we conclude the paper.

## 2. Properties of algebraic curves

In this section we recall several basic definitions and results related to planar algebraic curves. We also prove several minor results useful in the remaining sections.

### 2.1. Curve, boxed curve and its segments

In the remainder of the paper we will suppose that  $f(x, y) \in \mathbb{Q}[x, y]$  is an irreducible polynomial of two variables with rational coefficients. We will denote by  $\mathcal{C}$  the set of corresponding affine points

$$\mathcal{C} = \{(x, y) \in \mathbb{R}^2 \mid f(x, y) = 0\}.$$

It will be sometimes useful to consider the projective closure of  $\mathcal{C}$  i.e. to add the points at infinity and consider the homogeneous version of the polynomial  $f$ . One can also consider the complex version of  $\mathcal{C}$  and take all complex points which nullify  $f$ .

We recall standard definitions of inflection points, regular and singular points, tangents and branches of an algebraic curve. More details can be found e.g. in [23].

We say that the point  $[x_0, y_0] \in \mathcal{C}$  is regular, if  $\nabla f(x_0, y_0) \neq 0$ , i.e. when at least one of the partial derivatives is non zero. Other points of  $\mathcal{C}$  are called singular. Multiplicity  $k$  of a point is defined as the order of the lowest nonzero partial derivative ( $k = 1$  for regular points).

Let us consider the graded bivariate Taylor expansion of  $f$  at  $[x_0, y_0]$ . All terms of the first non-vanishing degree  $k$  form together a homogeneous polynomial  $f_k$  in the variables  $(x - x_0), (y - y_0)$  which can be factored into linear factors over  $\mathbb{C}$ . Each of these factors defines a straight line (possibly imaginary) which is called tangent line to  $\mathcal{C}$  at  $[x_0, y_0]$ . Multiplicity of a tangent is defined as the multiplicity of the corresponding linear factor in the polynomial  $f_k$ . We say, that a singularity is *ordinary* if it has only tangents of multiplicity one.

Clearly the number of tangents (counted with the multiplicity) at a point is equal to the multiplicity of the point. At a regular point of  $\mathcal{C}$  we obtain precisely one tangent line, which coincides with usual tangent line defined in the differential geometry.

We say that a regular point  $[x_0, y_0] \in \mathcal{C}$  is an *inflection* if the tangent has at least a triple intersection with  $\mathcal{C}$  at  $[x_0, y_0]$ . Equivalently the linear term of the Taylor expansion divides the quadratic term.

The following definition is essential for our approach.

**Definition 1.** Let  $\mathbf{p}$  be a point of  $\mathcal{C}$  and  $\mathbf{t}$  a tangent at  $\mathbf{p}$ . We call the couple  $\mathbf{P} = (\mathbf{p}, \mathbf{t})$  a  $G^1$ -point of  $\mathcal{C}$ .

In the subsequent algorithm we will be looking for  $G^1$ -points rather than just for points of the curve. Clearly we have just one  $G^1$ -point at regular points of  $\mathcal{C}$ . In practice a  $G^1$ -point will be represented by the coordinates  $[x_p, y_p]$  of  $\mathbf{p}$  and by the coordinates of the unit normal of  $\mathbf{t}$ .

It follows from the theory of the local parameterization of algebraic curves [23, chapter IV], that the locus of the curve in a neighborhood of a singular point is a union of several branches, which can be parameterized (locally) using a Puiseux series [23, paragraph IV.3]. We call a branch through a point *regular* if its Puiseux series have a nonzero derivative vector at the point, we will call it *singular* otherwise. Each  $G^1$ -point is tangent to one or more branches at a singular point.

We will restrict our investigations of the curve locus only to the graph of a real algebraic curve within a given bounded box.

**Definition 2.** Let  $f \in \mathbb{Q}[x, y]$  be a polynomial in two variables with rational coefficients and  $B = [\underline{x}, \bar{x}] \times [\underline{y}, \bar{y}] \subset \mathbb{R}^2$  a finite two dimensional box. Suppose that the four lines  $x = \underline{x}$ ,  $x = \bar{x}$ ,  $y = \underline{y}$  and  $y = \bar{y}$  have only trivial intersections with  $\mathcal{C}$ , i.e. these lines does not pass through singularities nor inflections of  $\mathcal{C}$  and are not tangent to the curve. We define the *boxed curve* as the set

$$\tilde{\mathcal{C}} = \{(x, y) \in B \mid f(x, y) = 0\} = \mathcal{C} \cap B.$$

The regularity of the curve intersection with the box boundary is included only for the seek of simplicity. It would not be very difficult to relax this requirement by scanning for all particular points on the boundary.

Our strategy is based on the decomposition of  $\mathcal{C}$  into smooth segments.

**Definition 3.** Let  $[a, b] \subset \mathbb{R}$  be a closed real interval. We say, that the mapping  $\mathbf{c}(t) : [a, b] \rightarrow \mathbb{R}^2$  is a *smooth curve segment* if it is an injective smooth mapping and it has a nonzero derivative within  $(a, b)$ . Moreover we require, that the tangent line can be defined continuously on the whole interval  $t \in [a, b]$ . More precisely the segment unit normal  $\mathbf{n}(t) = \mathbf{c}'(t)^\perp / |\mathbf{c}'(t)|$ , which is continuous in  $(a, b)$  must have proper limits at the end-points

$$\mathbf{n}(a) := \lim_{t \rightarrow a^+} \mathbf{n}(t), \quad \mathbf{n}(b) := \lim_{t \rightarrow b^-} \mathbf{n}(t).$$

We say that  $\mathbf{c}$  is a smooth segment of  $\mathcal{C}$  if moreover  $f(\mathbf{c}(t)) = 0$  for  $t \in [a, b]$ .

**Proposition 1.** Any boxed curve  $\tilde{\mathcal{C}}$  can be decomposed into smooth segments with possibly some additional isolated points. More precisely, there exists a finite set  $\{\mathbf{c}_i\}$  of smooth segments and additional points  $\{P_k\} \subset B$  such that  $\tilde{\mathcal{C}} = \bigcup_i \mathbf{c}_i \cup \bigcup_k P_k$ ,  $\mathbf{c}_i \cap \mathbf{c}_j$  is either empty or consists of a finite number of points and  $P_k \notin \mathbf{c}_i$ .

PROOF. We can smoothly parameterize a neighborhood of every regular point due to the implicit function theorem. A real singularity with a real tangents has branches which can be locally parameterized using the Puiseux series. If a branch is regular, the segment goes smoothly through the singularity. If a branch is singular, the singularity can be taken as the end-point of the smooth segment. Points with no real branches are isolated singularities  $\{P_k\}$  and form a finite set. The infinite set of smooth segments covering  $\tilde{\mathcal{C}} - \{P_k\}$  can be reduced to a finite one, because the set  $\tilde{\mathcal{C}} - \{P_k\}$  is compact. Moreover the overlapping segments can be merged and re-parametrized together. As a result, we obtain a finite number of segments covering  $\tilde{\mathcal{C}} - \{P_k\}$  and having end-points at the boundary points and at the non-ordinary singularities.  $\square$

From now on, we will suppose that there are no isolated real singularities of  $\mathcal{C}$  in the box  $B$ . Our strategy will consist in the identification and approximation of suitable smooth segments  $\mathbf{c}_i$ . While doing so, we can ask two principal questions, namely how big is the approximation error (measured as the Hausdorff distance) and whether the topology of the curve is correctly represented.

**Definition 4.** Let  $\tilde{\mathcal{C}}$  be a boxed curve and  $\mathcal{G}$  be a subset of  $\mathbb{R}^2$  (typically a collection of segments and points  $\mathcal{G} = \bigcup_i \mathbf{c}_i \cup \bigcup_k P_k$ ). We say, that the curve  $\tilde{\mathcal{C}}$  and the set  $\mathcal{G}$  are *topologically equivalent* if and only if they are isotopic as curves of Euclidean space, i.e., there exists a continuous map  $H : \mathbb{R}^2 \times [0, 1] \rightarrow \mathbb{R}^2$ , such that  $H(x, t)$  is a homeomorphism for all  $t \in [0, 1]$ ,  $H(\tilde{\mathcal{C}}, 0) = \tilde{\mathcal{C}}$  and  $H(\tilde{\mathcal{C}}, 1) = \mathcal{G}$ .

## 2.2. Implicit support function representation of algebraic curves

For an algebraic planar curve  $\mathcal{C}$  defined by a bivariate polynomial equation  $f(x, y) = 0$  we define the support function  $h$  as a (possibly multivalued) function defined on a subset of the unit circle

$$h : \mathbb{S}^1 \supset U \rightarrow \mathbb{R}^1$$

by which is any unit normal  $\mathbf{n} = (n_1, n_2)$  associated with the distance(s) from the origin to the corresponding tangent line(s) of the curve.

As proved in [21] we can recover the curve  $\mathcal{C}$  from  $h$  as the envelope of the system of tangent lines  $\{\mathbf{n} \cdot \mathbf{x} - h(\mathbf{n}) = 0 : \mathbf{n} \in U\}$ . This envelope is locally parameterized via the formula

$$\mathbf{c}_h(\mathbf{n}) = h(\mathbf{n})\mathbf{n} + \nabla_{\mathbb{S}^1} h(\phi) = h(\mathbf{n})\mathbf{n}(\phi) + \dot{h}(\phi)\dot{\mathbf{n}}(\phi), \quad (1)$$

where  $\nabla_{\mathbb{S}^1}$  denotes the intrinsic gradient with respect to the unit circle, which is alternatively expressed using the following arc length parameterization of  $\mathbb{S}_1$

$$\mathbf{n}(\phi) = (\cos(\phi), \sin(\phi)), \quad \dot{\mathbf{n}}(\phi) = (-\sin(\phi), \cos(\phi)). \quad (2)$$

The curvature of the curve at the corresponding point can be expressed easily via the support function as

$$\kappa = -\frac{1}{h + \ddot{h}} \quad (3)$$

and points where  $h + \ddot{h} = 0$  correspond to cusps.

For an algebraic curve  $\mathcal{C}$  defined as the zero set of a polynomial  $f(x, y) = 0$  we typically do not obtain an explicit expression of  $h$  but rather an implicit one, which is closely related to the notion of dual curve.

**Definition 5.** Let  $\mathcal{C}$  be a curve in projective plane. The *dual of  $\mathcal{C}$*  is the Zariski closure of the set in the dual projective plane consisting of tangent lines of  $\mathcal{C}$ .

The equation of the dual curve  $D(h, \mathbf{n}) = 0$  can be computed by eliminating  $x$  and  $y$  from the following system of equations:

$$\mathbf{n} \cdot \nabla f = 0, \quad \mathbf{n} \cdot [x, y] = h. \quad (4)$$

**Definition 6.** The dual equation  $D(h, \mathbf{n}) = 0$  together with the algebraic constraint  $n_1^2 + n_2^2 = 1$  is called the implicit definition of the support function  $h$  or simply *the implicit support function*.

If the partial derivative  $\partial D/\partial h$  does not vanish at  $(\mathbf{n}_0, h_0)$  then  $D(h, \mathbf{n})$  implicitly defines the support function in a certain neighborhood of  $(\mathbf{n}_0, h_0) \in \mathbb{R}^3$ .

The (implicit) support function is obviously a kind of dual representation which takes into account the Euclidean metric. It has many nice properties and in particular simplifies the offset and convolution computation cf. [18, 21]. Let us remark that a  $G^1$ -point contains also an information of the corresponding support function, because  $h = \mathbf{n} \cdot [x_0, y_0]$ .

### 3. Determination of monotonous curve segments

In this section we will identify the monotonous segments of  $\tilde{\mathcal{C}}$  by their boundary  $G^1$ -points. We first define and identify the critical  $G^1$ -points which are the only possible points which can disturb the monotonicity of segments. Then we define a set of rules to determine which couples of  $G^1$ -points are actually connected by curve segments.

**Definition 7.** A smooth curve segment is called *monotonous* if it is monotonous in the  $x$  and  $y$  coordinates and with respect to its tangent. More precisely  $\mathbf{c}(t) = (c_x(t), c_y(t))$ , with the moving unit normal  $\mathbf{n}(\phi(t))$  is called monotonous if the functions  $c_x(t)$ ,  $c_y(t)$  and  $\phi(t)$  are monotonous.

#### 3.1. Critical $G^1$ -points

We want to detect the  $G^1$ -points of  $\tilde{\mathcal{C}}$ , where the smooth curve segments can possibly lose their monotonicity. These will be the boundary points, points with horizontal or vertical tangents, inflections and points with singular curve branches. Note that we do not need to detect all singularities of  $\tilde{\mathcal{C}}$ , because often all branches are regular and the segments go through such points smoothly.

For the determination of critical points we will exploit the support function representation, in particular in the search for cusps and points with a given tangent. On the other hand the determination of boundary points and inflection points is easier using the primary curve equation  $f(x, y) = 0$ .

Between the critical  $G^1$ -points we have to include the boundary points, i.e. the intersections of the curve  $\mathcal{C}$  with the sides of the box  $B$ . These are simply solutions of the univariate polynomial equations

$$f(\underline{x}, y) = 0, \quad f(\bar{x}, y) = 0, \quad f(x, \underline{y}) = 0, \quad f(x, \bar{y}) = 0.$$

As we assumed, on the boundary there are only the regular points, the corresponding  $G^1$ -points are therefore uniquely defined.

**Definition 8.** We say, that a  $G^1$ -point of  $\tilde{\mathcal{C}}$  is *x-extremal* or *y-extremal*, if it has a vertical or horizontal tangent, respectively. More precisely if the  $G^1$ -point normal is equal to  $(\pm 1, 0)$  or to  $(0, \pm 1)$ , respectively.

Note that compared to the standard definition of extremal points, we admit that extremal point is singular point which is extremal with respect to at least one branch of the given curve. For example the origin in tacnode  $x^4 + x^2y^2 - y^2 = 0$  with the normal  $(0, 1)$  is *y-extremal*.

Due to the dual nature of the (implicit) support function representation it is particularly easy to find any point with given normal vector, as shown in the following

**Lemma 1.** *Let  $h$  be the support function of curve  $\mathcal{C}$  implicitly defined by  $D(h, \mathbf{n}) = 0$ . The points on  $\mathcal{C}$  with given normal  $\mathbf{n}_0$  are the solutions of the polynomial univariate equation in  $h$*

$$D(h, \mathbf{n}_0) = 0. \tag{5}$$

The Lemma 1 can be obviously used to determine the extremal points and also points with another prescribed auxiliary tangent, which we will use for determination of the connectivity of critical  $G^1$ -points.

The monotonicity with respect to the tangent can be broken only at inflections or cusps.

**Definition 9.** We say that a  $G^1$ -point is an *inflection*, if it is tangent to an inflection branch in a (possibly singular) point.

To find these points we can use the well known theorem, see [13].

**Proposition 2.** *The intersections of a curve with its Hessian lie one at each inflexion, six at each node, and eight at each cusp of the given curve.*

We can use this theorem to find all standard inflexions, but also all inflection  $G^1$ -points, i.e. nodes which have at least on one branch inflection. To identify that a given point really contains inflection point at at least one branch, we can use the following proposition, see [13] or [23].

**Proposition 3.** *Let  $f$  be the equation of curve  $\mathcal{C}$  and let  $f_r$  be the first nonzero derivative of  $f$ . At least one branch of  $\mathcal{C}$  has an inflection point at  $p = [x_0, y_0] \in \mathcal{C}$  if and only if  $f_r(p)|f_{r+1}(p)$ .*

Monotonicity in  $x$ ,  $y$  or  $\mathbf{n}$  can be disturbed not only at extremal points or inflexions, but also at cusps.

**Definition 10.** We say that a  $G^1$ -point is a *cusp* point if it is tangent to a singular branch at a singular point of  $\tilde{\mathcal{C}}$ .

**Proposition 4.** *All cusp  $G^1$ -points can be determined via the support function by solving the equation*

$$h + \ddot{h} = 0 \quad (6)$$

or they are inflection  $G^1$ -points.

PROOF. Let  $B$  be a singular branch through cusp  $G^1$ -point  $\mathbf{p} = [0, 0]$  of order  $r > 1$  with tangent  $(1, 0)$ . Therefore  $h = 0$ . Assume that  $\mathbf{p}$  satisfies (6). Assume that  $s$  is class of  $\mathbf{p}$  (multiplicity of the tangent at  $\mathbf{p}$  minus one). From [23], we know that class and order are mutually dual. Let the tilde sign the duality. The implicit support function has  $\tilde{r} = s$ . We distinguish two cases. First  $s = 1$ , then the dual point is regular and  $\tilde{s} = r > 1$ . The dual point is inflection and  $\ddot{h} = 0$ . The equation (1) follows. Second case is  $s > 1$ . In this case  $\mathbf{p}$  has tangent of multiplicity at least three and the point  $\mathbf{p}$  is found as an inflection  $G^1$ -point.  $\square$

**Proposition 5.** *If only the implicit support function  $D(h, \mathbf{n}) = 0$  is available, the condition  $h + \ddot{h} = 0$  becomes*

$$\begin{aligned} h - \frac{n_1^2}{D_h^3} (D_h^2 D_{n_2 n_2} + D_{hh} D_{n_2}^2 - 2D_h D_{hn_2} D_{n_2}) - \frac{n_1 D_{n_1}}{D_h} + \\ + \frac{n_2^2}{D_h^3} (D_h^2 D_{n_1 n_1} + D_{hh} D_{n_1}^2 - 2D_h D_{hn_1} D_{n_1}) - \frac{n_2 D_{n_2}}{D_h} + \\ + \frac{2n_1 n_2}{D_h^3} (D_h D_{hn_2} D_{n_1} + D_h D_{hn_1} D_{n_2} + D_{hh} D_{n_1} D_{n_2} - D_h^2 D_{n_1 n_2}) = 0, \end{aligned} \quad (7)$$

where the subscripts denote corresponding partial derivatives.

PROOF. Let  $\mathbf{n}(s) = (n_1(s), n_2(s))$  be a parametrization of the unit circle by arc-length  $s$  and suppose that we locally have  $h(\mathbf{n}(s))$ . Using the chain rule we get following derivatives:

$$\dot{h} = h_{n_1} \dot{n}_1 + h_{n_2} \dot{n}_2 = -h_{n_1} n_2 + h_{n_2} n_1 \quad (8)$$

$$\begin{aligned} \ddot{h} &= h_{n_1 n_1} \dot{n}_1^2 + h_{n_1 n_2} \dot{n}_1 \dot{n}_2 + h_{n_1} \ddot{n}_1 + h_{n_2 n_2} \dot{n}_2^2 + h_{n_2 n_1} \dot{n}_1 \dot{n}_2 + h_{n_2} \ddot{n}_2 = \\ &= h_{n_1 n_1} n_2^2 - h_{n_1 n_2} n_2 n_1 - h_{n_1} n_1 + h_{n_2 n_2} n_1^2 - h_{n_2 n_1} n_1 n_2 - h_{n_2} n_2, \end{aligned} \quad (9)$$

where the dot denotes the derivative with respect to arc length  $s$  and the subscript denotes the partial derivative. The second equality in (8) and in (9) is deduced using the equality  $(\dot{n}_1, \dot{n}_2) = (-n_2, n_1)$ .

The partial derivatives of  $h$  can be deduced from its implicit definition. For example:

$$\frac{\partial}{\partial n_1} D(h(\mathbf{n}), n_1, n_2) = D_{n_1}(h, n_1, n_2) + h_{n_1} D_h(h, n_1, n_2) = 0.$$

And therefore

$$h_{n_1} = -\frac{D_{n_1}(h, n_1, n_2)}{D_h(h, n_1, n_2)}.$$

Similarly we can deduce all partial derivatives of  $h$  and substitute them into (9). That equation we substitute into (6) to get a necessary condition (7) for cusps in variable  $\mathbf{n}$ .  $\square$

In concrete computations the cusps will be found by simultaneous solving equation (6) and the fundamental equations  $D(h, \mathbf{n}) = 0$  and  $n_1^2 + n_2^2 - 1 = 0$ . Having the normal vector, the  $G^1$ -points are fully defined by (1). When the cusp corresponds in the implicit support function to the singular point, we have to define the tangent vector as described in Section 2. Using the envelope formula (1) we can recover  $G^1$ -points on  $\mathcal{C}$ .

**Proposition 6.** *If a smooth segment  $\mathbf{c}$  of  $\tilde{\mathcal{C}}$  does not contain in its interior any boundary points, extremal points, inflections nor cusps of  $\tilde{\mathcal{C}}$ , then it is a monotonous segment.*

PROOF. If the segment is not monotonous in its  $x$ -coordinate ( $y$ -coordinate) it must contain an  $x$ -extremal ( $y$ -extremal) point. If it is not monotonous with respect to its tangent it must contain an inflection. If there is no cusp in the interior of a segment it can be parameterized smoothly.  $\square$

The following proposition summarize the results of this subsection.

**Proposition 7.** *Any boxed curve  $\tilde{\mathcal{C}}$  can be decomposed into smooth monotonous segments connecting boundary points, extremal points, inflections and cusps.*

PROOF. There exists a decomposition into smooth segments due to Proposition 1. We can add as splitting points all the critical  $G^1$ -points.  $\square$

### 3.2. Connectivity of $G^1$ -points

In the previous section we have detected all critical  $G^1$ -points  $\mathbf{P}_i$ . Let us recall, that  $\mathbf{P}_i$  contains the point position  $[x_i, y_i]$  and the non-oriented unit normal  $\mathbf{n}_i$ . Along with the point and normal coordinates we also obtain information about the kind of curve branches passing through each  $G^1$ -point, in particular whether there is an inflection or a cusp.

A set of rules will be now formulated in order to decide which of  $\mathbf{P}_i$  are actually connected by smooth monotonous curve segments. From the equations of critical  $G^1$ -points we obtain the information about the number of smooth segments going to  $\mathbf{P}_i$ . Boundary points are connected only by one segment and the other  $G^1$ -points are always connected by a pair number of segments  $2m$ . The precise number is determined by the total multiplicity  $m$  of the corresponding roots of all equations of critical points described in the previous section.

The monotonicity condition together with the  $G^1$  nature of the information which we have for each  $\mathbf{P}_i$  provide following strong rules which allow us to determine the connectivity of the  $G^1$ -points.

1. *Two connected points has two consecutive normal vectors.* To make the whole procedure more systematic we add for any existing  $\mathbf{P}_i$  all  $G^1$ -points with the same normal. This can be done efficiently using the Lemma 1. After this addition only  $G^1$ -points with neighboring normals can be connected. Note that the normals are not oriented, we thus essentially choose an orientation of the projective line of directions in the plane and connect two consecutive normal directions. Clearly the “distance” of two possible normals is at most  $\pi/2$ .
2. *The normal determines the quadrants where the connected points can lie.* The normal of a  $G^1$ -point  $\mathbf{P}$  points towards one quadrant (with respect of the  $x, y$  coordinates). The opposite quadrant must be also considered as the normals are non-oriented. The connected points must lie in the two remaining quadrants. For the boundary points only one quadrant is possible because the other one falls outside the box  $B$ .
3. *A  $G^1$ -point which is not a cusp nor extremal point has branches in both opposite quadrants determined by the previous rule.* The cusps and extremal points are the only points where a branch does not continue smoothly - from one quadrant to the opposite one.



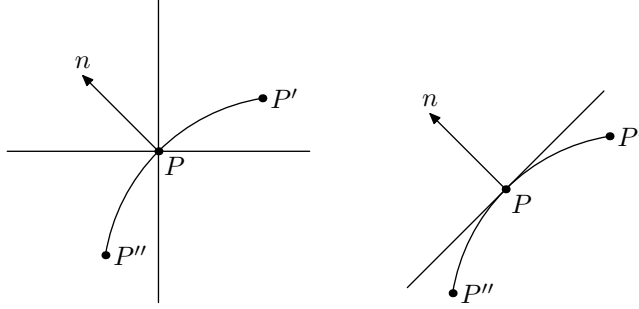


Figure 1: Illustration of rules 2 (left) and 4 (right).

4. *Every non-inflection  $G^1$ -point is connected to the points which are on the same side of the tangent line.* If the point is not an inflection, the monotonicity of the tangent direction will continue through the point and the both branches will wile on the same side of the tangent. On the other hand, in the first order inflection the side of branches will change.
5. *The interval between boundary normals must contain a vector perpendicular to the vector given by the points.* The distance of normals must be at most  $\pi/2$ . This can be obtained by possibly changing the orientation of the normal at one point. Now the two normals delimit an interval which contains a perpendicular vector to the difference vector. Indeed, the tangent at this point is parallel to the difference vector and such a point must occure in any smooth segment (not necessarily monotonous).

In many cases these rules yield directly the connectivity of the  $G^1$ -points, this is the case of examples 2, 3, 4. If there is a connectivity ambiguity, we add points with additional normal directions using Lemma 1. In example 1 the addition of one normal direction was sufficient and in example 5 the connectivity was resolved adding two normal directions. Although we were not able to provide a formal proof that the unique possible connectivity will always be obtained, the rules seem to be very strong (due to the  $G^1$  nature of the data) and resolved the connectivity in all examples we have studied.

#### 4. Approximate graph of the curve

As we have seen in the previous section, the boxed curve  $\tilde{\mathcal{C}}$  is decomposed into monotonous smooth segments and each of these segments is identified by its boundary  $G^1$ -points. In this section we will interpolate these  $G^1$ -points with suitable smooth segments which will thus approximate the actual curve segments. We also study the problem of the preservation of the curve topology.

##### 4.1. Support function based approximation of the curve segments

Let  $P_0, P_1$  be two  $G^1$ -points of  $\tilde{\mathcal{C}}$  connected by a smooth segment of  $\tilde{\mathcal{C}}$ . The normals are non oriented, but by the segment monotonicity we know, that the smooth transition along  $\mathbf{c}$  will rotate the normal by at most  $\pi/2$ . We can change the orientation of the unit normals of  $P_0$  and  $P_1$  so that their angle is  $\leq \pi/2$ . By possibly switching the order of the  $G^1$ -points and using the parameterization (2) we can put the boundary data to the form

$$P_0 = \{[x_0, y_0], \mathbf{n}(\phi_0)\} \text{ and } P_1 = \{[x_1, y_1], \mathbf{n}(\phi_1)\}, \text{ where } 0 < \phi_1 - \phi_0 \leq \pi/2. \quad (10)$$

We will interpolate this  $G^1$  boundary data using the support function. More precisely we look for a function  $h$  defined over the interval  $[\phi_0, \phi_1]$ , so that the curve segment obtained via the formula (1) interpolates the boundary points.

From the form of (1) it is clear, that the position of a point provides values of the support function and its derivative. The  $G^1$  interpolation by a smooth segment is thus transformed into the standard problem of

the functional Hermite interpolation of the first order at the two values  $\phi_0$  and  $\phi_1$ . We can use any suitable linear space of functions to obtain this interpolant. Natural choices of the basis functions are trigonometric polynomials [21], trigonometric polynomial with rational non-integer arguments [4] and rational trigonometric functions [11]. All these spaces will provide geometric invariance and the rationality of the resulting smooth segments.

It is a great advantage of the support function, that its approximation error will translate to the identical behavior of the Hausdorff distance of corresponding segments.

**Proposition 8.** *Let  $h, g$  be two support functions defined on the interval  $U = [\phi_0, \phi_1]$ , such that*

$$g(\phi_i) = h(\phi_i), \quad \dot{g}(\phi_i) = \dot{h}(\phi_i), \quad i \in 0, 1 .$$

*Suppose, that the corresponding curve segments  $\mathbf{c}_h, \mathbf{c}_g$  are cusp-free on  $U$ . Then their Hausdorff distance is equal to the error in support functions*

$$\|\mathbf{c}_h - \mathbf{c}_g\|_H = \|h - g\|_\infty . \quad (11)$$

PROOF. Due to boundary conditions and absence of singular points (cusps), the Hausdorff distance is realized by a common normal line to both curve segments. The distance of the points on this line is equal to the absolute value of the difference of the support functions. For a more formal proof see [21, Proposition 14].  $\square$

If we used functions which are at least  $C^2$  on the interval  $[\phi_0, \phi_1]$  we would obtain the approximation order 4, as it is usual in the case of the  $C^1$  Hermite interpolation at two values. In our examples we will however use a slightly different approach. We will give up one degree of the approximation order to obtain a greater geometric simplicity of the resulting curve graph. We will interpolate the boundary data with two first order trigonometric polynomials

$$p_0(\phi) = A_0 \cos(\phi) + B_0 \sin(\phi) + C_0 \quad \text{and} \quad p_1(\phi) = A_1 \cos(\phi) + B_1 \sin(\phi) + C_1, \quad (12)$$

which are smoothly connected (with  $C^1$  continuity) at an additional point. This problem has always one solution as proved in the following

**Proposition 9.** *Let two  $G^1$ -points (10) and  $\tilde{\phi} \in (\phi_0, \phi_1)$  are given. Then there is precisely one pair of polynomials (12) smoothly connected at  $\tilde{\phi}$  so that the resulting curve segment satisfies*

$$\mathbf{c}_{p_0}(\phi_0) = [x_0, y_0], \quad \mathbf{c}_{p_1}(\phi_1) = [x_1, y_1], \quad \mathbf{c}_{p_0}(\phi_0) = \mathbf{c}_{p_1}(\phi_1). \quad (13)$$

*These segments have the form of two circular arcs connected in a  $G^1$  manner.*

PROOF. The conditions (13) lead to the following system of 6 linear equations

$$\begin{pmatrix} 1 & 0 & \cos(\phi_0) & 0 & 0 & 0 \\ 0 & 1 & \sin(\phi_0) & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & \cos(\phi_1) \\ 0 & 0 & 0 & 0 & 1 & \sin(\phi_1) \\ 1 & 0 & \cos(\tilde{\phi}) & -1 & 0 & -\cos(\tilde{\phi}) \\ 0 & 1 & \sin(\tilde{\phi}) & 0 & -1 & -\sin(\tilde{\phi}) \end{pmatrix} \begin{pmatrix} A_0 \\ B_0 \\ C_0 \\ A_1 \\ B_1 \\ C_1 \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \\ x_1 \\ y_1 \\ 0 \\ 0 \end{pmatrix}. \quad (14)$$

The determinant of the matrix is equal to  $\sin(\tilde{\phi} - \phi_0) + \sin(\phi_1 - \tilde{\phi}) - \sin(\phi_1 - \phi_0)$ , which is non-zero unless at least two of the angles  $\phi_0, \tilde{\phi}, \phi_1$  coincide. The system of equations thus has solution for any right-hand side. A straightforward evaluation of the formula (1) provides

$$\mathbf{c}_{p_i}(\phi) = [A_i, B_i] + C_i[\cos(\phi), \sin(\phi)], \quad (15)$$

i.e. equation of two circles with one common point  $\mathbf{c}_{p_1}(\tilde{\phi}) = \mathbf{c}_{p_0}(\tilde{\phi})$  and the common normal  $\mathbf{n}(\tilde{\phi})$  at this point.  $\square$

The two circular arcs are connected in a  $G^1$  way, they can not be connected in an inflection (the normal is monotonous), but can be connected at a cusp. A cusp occur if and only if the two circles are differently oriented, i.e. if the signs of  $C_0$  and  $C_1$  are different. Indeed certain boundary data can not be connected in an inflection and cusp free way. This is however not the case of boundary data of a monotonous segment of  $\tilde{C}$ . We still have to be careful about the right choice of the breaking value  $\tilde{\phi}$ .

**Proposition 10.** *Let two  $G^1$ -points (10) be boundary points of a smooth monotonous segment of  $\tilde{C}$ . Then there is precisely one value  $\tilde{\phi} \in (\phi_0, \phi_1)$  so that  $\mathbf{n}(\tilde{\phi}) \perp ([x_1, y_1] - [x_0, y_0])$  and the two arcs constructed using Proposition 9 are connected in a smooth cusp-free way. The whole interpolating curve is called parallel tangent bi-arc and its approximation order is  $\epsilon^3$ , where  $\epsilon$  is the distance of the boundary points.*

PROOF. The existence of the desired value of  $\tilde{\phi}$  follows from the rule 5 of the previous section. Its uniqueness is due to the size of the interval  $\phi_1 - \phi_0 \leq \pi/2$ . For this particular value of  $\tilde{\phi}$  a laborious but straightforward computation gives

$$C_0 \cdot C_1 = \frac{(x_1 - x_0)^2 + (y_1 - y_0)^2}{2(1 - \cos(\phi_1 - \phi_0))} > 0$$

and  $C_0, C_1$  thus have the same sign and the two circles are oriented in the same way. The tangent at the breaking point is obviously parallel to the difference vector  $([x_1, y_1] - [x_0, y_0])$ . This is a standard case in the bi-arc interpolation and it has approximation order equal 3 as shown in [17].  $\square$

Let us stress the fact, that bi-arcs are piecewise rational curves. Their union, which approximate the whole algebraic curve can therefore be represented in the NURBS format.

Let us conclude this paragraph by summarizing results obtained so far in the following

**Proposition 11.** *The procedure consisting in determination of the critical boundary  $G^1$ -points and properly connecting them with bi-arcs as described in Proposition 10 provides collection of  $N$  circular arcs. The union of circular arcs and  $\tilde{C}$  has the Hausdorff distance which decreases as  $\mathcal{O}(N^{-3})$ .*

#### 4.2. Verifying and enforcing the correct curve topology

The previous approximation procedure yields an approximation of  $\tilde{C}$  within any desired accuracy, but does not provide informations about the correct topology of the result. Indeed, while the approximated monotonous smooth segments start and end at critical points, it is still possible that they intersect in a different way than the exact curve segments. We propose the concept of a bounding triangle as a strong tool, which can prove the correct topology or localize the topology problems to small areas, where they can be handled using algebraic methods.

**Definition 11.** Let  $P_1, P_2$  be two  $G^1$ -points connected by a smooth monotonous segment of  $\tilde{C}$ . The *tangent triangle*  $T(P_1, P_2)$  is the triangle bounded by the tangents of  $P_1$  and  $P_2$  and by the segment  $P_1P_2$ .

This triangle provides a natural bounding area of the connecting segment.

**Proposition 12.** *The monotonous segment  $\mathbf{c}$  connecting  $P_1, P_2$  lies in the interior of the tangent triangle  $T(P_1, P_2)$ .*

PROOF. Denote by  $t_1$  and  $t_2$  the tangent vectors at  $P_1$  and  $P_2$  respectively. Due to the smoothness and monotonicity of  $\mathbf{c}$  in  $x, y$ , and  $\mathbf{n}$  it can be after suitable rotation parameterized as a graph of a function without inflections. Without loss of generality we can suppose that this function is strictly convex. From the definition of convexity, the arc lies above both tangents and below the segment  $P_1P_2$ .  $\square$

Due to this proposition both exact and approximate curve segments will lie with the same tangent triangle. If the tangent triangles of all segments are non-intersecting, the correct topology is ensured, because the continuous transition required in Definition 4 will be realized for each segment independently within the tangent triangle. This is e.g. the case of examples displayed on Figures 3, left and 4, right. Indeed the triangles meet only by their corners at the same  $G^1$ -point.

It turns out, that a transversal intersection of two triangles can also ensure the correct topology.

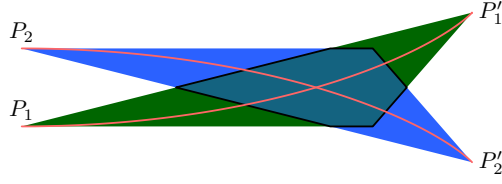


Figure 2: Two curve segments and their tangent triangles. The intersection of segments lies inside the intersection of tangent triangles.

**Proposition 13.** *Let  $\mathbf{c}_1$  and  $\mathbf{c}_2$  be two curve segments and their bounding triangles  $T_1 = T(P_1, P'_1)$  and  $T_2 = T(P_2, P'_2)$  intersect in the following way: The edge  $P_1P'_1$  intersects the edge  $P_2P'_2$ ,  $P_1, P'_1 \notin T_2$  and  $P_2, P'_2 \notin T_1$ . Then the segments have precisely one transversal intersection and it lies in  $T_1 \cap T_2$ .*

PROOF. Existence of the intersection follows from the transversal intersection of the triangles. The uniqueness is ensured by the convexity of both curve segments within the bounding tangent triangles.  $\square$

If we have only two triangle intersecting in the given area, we are sure, that the segments will intersect in one ordinary double point. The continuous transition required in Definition 4 can be realized for each of both segments independently within the tangent triangle, because it will not change the nature of the intersection due to Proposition 13. Often, this tool is sufficient to prove the correct topology. This is e.g. the case of the example displayed on Figure 5.

Other mutual positions of bounding tangent triangles must be handled more carefully. We must consider two cases

1. More then two triangles intersect at the same point.
2. Two triangles intersect in a different way then described in Proposition 13.

Both cases one must also consider if they would occur after slight perturbation of the triangle corners, because their position can be influenced by a computational error. Note, that this is not the case in all previously mentioned examples, because there the only proximity occurred between two triangles sharing the same  $G^1$ -point.

If one of previous cases occur, we can use two projections to detect boxes containing only one critical point. We realize the projection as the determinant (i.e. resultant of the function with its derivative) with respect to the direction of the projection. Let  $\mathcal{B} = \mathcal{I}_1 \times \mathcal{I}_2$  be a box, where we want to resolve the topology. We isolate roots of discriminant of  $f$  with respect to  $x$  (resp.  $y$ ) by intervals  $\mathcal{I}_1, \dots, \mathcal{I}_k$  resp.  $\mathcal{J}_1, \dots, \mathcal{J}_l$ , then in each smaller box  $\mathcal{I}_i \times \mathcal{J}_j$  there is at most one singular point. To handle the topology inside such a box we need the concept of topological degree used in [3].

**Definition 12.** Let  $\mathcal{B} \subset \mathbb{R}^2$ ,  $G : \mathcal{B} \rightarrow \mathbb{R}^2$  be  $C^2$  bivariate function and  $a \in \mathbb{R}^2$ . Suppose that no root of  $G(x, y) = a$  is on the boundary of  $\mathcal{B}$ . Let  $d$  be a regular point of  $G$  and let  $d$  be in the component  $\mathbb{R}^2 - G(\partial\mathcal{B})$  containing  $b$ . Then the topological degree of  $G$  at  $a$  inside  $\mathcal{B}$  is defined by

$$\text{Deg}(G, \mathcal{B}, a) = \sum_{b \in \mathcal{B}, G(b)=d} \text{sign det } J_G(b). \quad (16)$$

The topological degree can be computed using only the information on the boundary of the box. For more details and proof see [22].

**Proposition 14.** *Suppose that  $G = (G_1, G_2)$  as above. Let the boundary of the box  $\mathcal{B}$  is counter-clockwise decomposed into segments  $[p_i, p_{i+1}]$ , where  $i \in \{1, 2, \dots, k\}$  and  $p_{k+1} = p_1$ . Assume that on each segment  $G_j$  has a constant nonzero sign. Then*

$$\text{Deg}(G, \mathcal{B}, (0, 0)) = \frac{1}{8} \sum_{i=1}^k (-1)^{\alpha_i - 1} \left| \begin{array}{cc} \text{sign}(G_{\alpha_i}(p_i)) & \text{sign}(G_{\alpha_i}(p_{i+1})) \\ \text{sign}(G_{\alpha_{i+1}}(p_i)) & \text{sign}(G_{\alpha_{i+1}}(p_{i+1})) \end{array} \right|, \quad (17)$$

where  $\alpha_i \in \{1, 2\}$  and  $G_3 = G_1$ .

Finally, using the combination of the previous and the following proposition we can deduce the topology inside the box.

**Proposition 15 (Khimšiašvili, [15]).** *Let  $p$  be the only singular point of curve  $\mathcal{C}$  in bounding box  $\mathcal{B}$ . Then the number of branches of  $\mathcal{C}$  at  $p$  is  $1 - \text{Deg}(\nabla f, \mathcal{B}, (0, 0))$ .*

At the examples displayed on Figures 3, right and 4, left several tangent triangles meet at the complicated singularity. Projection technique is used to determine that inside some small box is only one singularity with topological degree  $-1$ , i.e. the branches meets at one point.

## 5. Examples

In this section we demonstrate our algorithm on several examples. For each boxed curve we first determine its critical  $G^1$ -points as described in 3.1 and their connectivity using the rules of 3.2. Then we interpolate boundary  $G^1$ -points with bi-arcs following 4.1. and eventually check or modify the topology as explained in Section 4.2.

*Example 1 - see Figure 3, left.*

We consider the equation  $f(x, y) = 2x - 2x^3 + x^4 - 2y^2 + y^4$  inside the box  $[-2, 2] \times [-2, 2]$ . We compute the implicit support function and the critical  $G^1$ -points:

$$D(h, \mathbf{n}) = 16h^6 + 8n_1^4 n_2^2 - 3n_1^2 n_2^4 + 11n_2^6 - 48h^4(n_1^2 + n_2^2) + h^2(64n_1^4 + 24n_1^2 n_2^2 + 21n_2^4) + h(32n_1^5 + 54n_1 n_2^4),$$

point	point type	normal vector	coordinates
A	cuspid	(0, 1)	[1, 1]
B	cuspid	(0, 1)	[1, -1]
C	extremal	(0, 1)	[-0.5, -1.51626]
D	extremal	(0, 1)	[-0.5, 1.51626]
E	extremal	(1, 0)	[0, 0]
F	extremal	(1, 0)	[-0.83929, 0]
G	extremal	(1, 0)	[-1, 1]
H	extremal	(1, 0)	[-1, -1]
I	inflection	(0.43473, -0.90056)	[0.52589, -1.18459]
J	inflection	(0.43473, 0.90056)	[0.52589, 1.18459]
K	inflection	(-0.97192, 0.23529)	[-0.91579, -0.48011]
L	inflection	(-0.97192, -0.23529)	[-0.91579, 0.48011]
M	additional	(0.43473, -0.90056)	[-0.76059, 1.46391]
N	additional	(0.43473, -0.90056)	[0.78167, 0.92942]
O	additional	(0.43473, 0.90056)	[-0.76059, -1.46391]
P	additional	(0.43473, 0.90056)	[0.78167, -0.92942]
Q	additional	(-0.97192, 0.23529)	[0.01497, 0.12282]
R	additional	(-0.97192, 0.23529)	[-0.98067, 1.17794]
S	additional	(-0.97192, -0.23529)	[0.01497, -0.12282]
T	additional	(-0.97192, -0.23529)	[-0.98067, -1.17794]

The connectivity rules 1-4 provide the following chains of connected points

$$R - M - D - J - A - N - Q \quad \text{and} \quad S - P - B - I - C - O - T.$$

Let us have a look at the next connection of  $S$  in more detail. By rule 1 it must be connected to one of the points  $H, F, G$  or  $E$ . The point  $H$  is eliminated by rule 2 and the point  $G$  is eliminated by rule 5. To decide between  $F$  and  $E$  we will find all points with an intermediate normal e.g.  $\mathbf{n} = (0.11850, -0.99295)$ .

One of them is inside the tangent triangle  $T(S, E)$  but none is inside  $T(S, F)$ . We thus connect  $S - E$  and similarly  $E - Q$ . The connectivity of the remaining points follow in a simple way.

For each pair of connected  $G^1$ -points we now draw the tangent triangle (some of them are very small) and a the bi-arc interpolant. The arc-spline provide an approximation of the curve and the triangles its bounding region. Because we have no triangle intersection, the correct topology is certified.

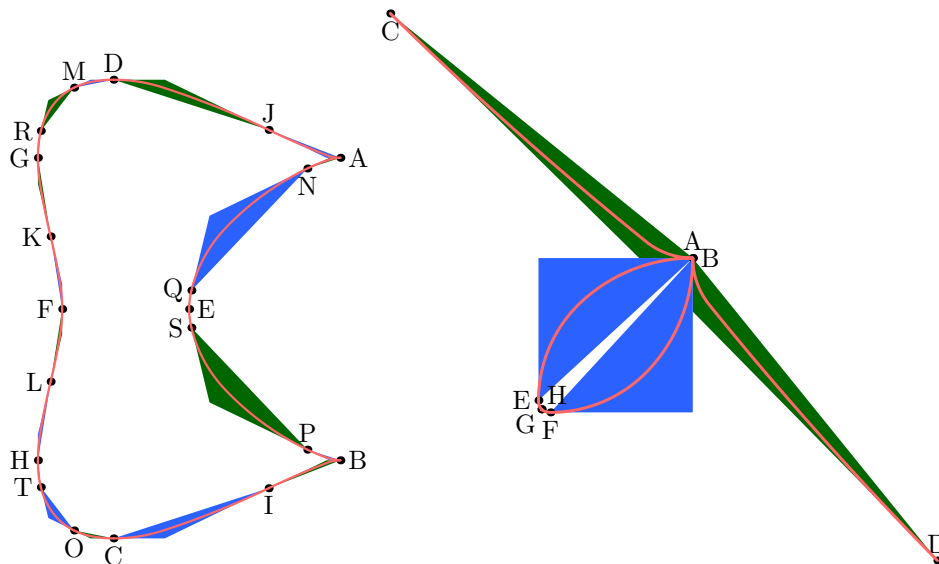


Figure 3: Results for Example 1 (left) and Example 2 (right) .

*Example 2 - see Figure 3, right.*

We consider the equation  $f(x, y) = x^2y^2 + x^5 + y^5$  inside the box  $[-1, 1] \times [-1, 1]$  and obtain

$$D(h, \mathbf{n}) = 3125h^6 - 3750h^4n_1n_2 + 825h^2n_1^2n_2^2 + 16n_1^3n_2^3 + 108h(n_1^5 + n_2^5),$$

point	point type	normal vector	coordinates
A	cuspid	(0, 1)	[0, 0]
B	cuspid	(1, 0)	[0, 0]
C	boundary	(0.70096, 0.71320)	[-1, 0.80873]
D	boundary	(0.71320, 0.70096)	[0.80873, -1]
E	extremal	(0, 1)	[-0.47043, -0.51017]
F	extremal	(1, 0)	[-0.51017, -0.47043]
G	additional	(0.70096, 0.71320)	[-0.49982, -0.50018]
H	additional	(0.71320, 0.70096)	[-0.50018, -0.49982]

The connectivity rules 1-5 gives all connected pairs immediately as shown at the figure. Tangent triangles indicates a topology issue at a neighborhood of the point  $[0, 0]$ . Because the  $G^1$ -points  $A, B$  were computed symbolically, we know that the point position coincide. If we considered their coordinates only as imprecise floats, we can use the  $x$  and  $y$  projections of  $f$  (taking the resultant of  $f$  and its partial derivatives)

$$\pi_x(x) = x^5(108x^{10} + 3125x^{15}) \quad \text{and} \quad \pi_y(y) = y^5(108y^{10} + 3125y^{15}).$$

Their roots guarantee that e.g. inside the box  $[-0.2, 0.2] \times [-0.2, 0.2]$  there is only one critical point. The curve has topological degree  $-1$  at this box and the curve has two branches at  $[0, 0]$ . Therefore points  $A$  and  $B$  coincides and we obtain the correct topology.

Example 3 - see Figure 4, left.

We have the equation  $f(x, y) = x^2y + x^4 + y^4$  inside the box  $[-1, 1] \times [-1, 1]$  and get

$$D(h, \mathbf{n}) = 256h^5 - 27hn_1^4 + 144h^2n_1^2n_2 - 128h^3n_2^2 - 4n_1^2n_2^3 + 16hn_2^4.$$

point	point type	normal vector	coordinates
A	cusp	(1, 0)	[0, 0]
B	extremal	(0, 1)	[0, 0]
C	extremal	(1, 0)	[0.56988, -0.43301]
D	extremal	(1, 0)	[-0.56988, -0.43301]
E	extremal	(0, 1)	[0.5, -0.5]
F	extremal	(0, 1)	[-0.5, -0.5]

Connectivity is resolved and the topological issue around the point  $B$  is resolved using the projections

$$\pi_x(x) = -27x^8 + 256x^{12}, \quad \pi_y(y) = 16y^4(4y^4 - y^2)^2$$

which indicates one singular point in the box  $[-0.2, 0.2]^2$ . This point has inside this box topological degree  $-1$  and the points  $A$  and  $B$  coincide.

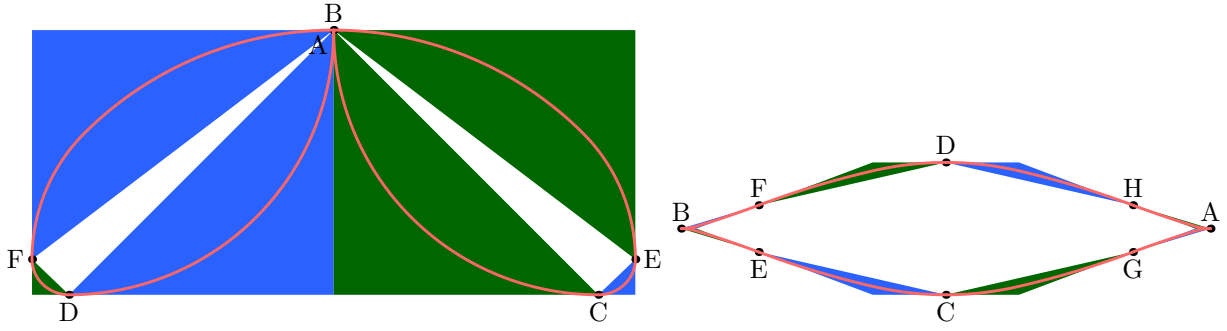


Figure 4: Results for Example 3 (left) and Example 4 (right).

Example 4 - see Figure 4, right.

Equation  $f(x, y) = y^2 + x^3(x + 1)^3$  inside box  $[-2, 2] \times [-1, 1]$  gives us

$$D(h, \mathbf{n}) = 1024n_1^6 + 54432h^2n_1^2n_2^2 + 7776hn_1^3n_2^2 - 108n_1^4n_2^2 + 729n_1(128h^3n_2^2 - hn_2^4) + 729(64h^4n_2^2 - h^2n_2^4).$$

point	point type	normal vector	coordinates
A	cusp	(1, 0)	[0, 0]
B	cusp	(1, 0)	[-1, 0]
C	extremal	(1, 0)	[-0.5, -0.125]
D	extremal	(1, 0)	[-0.5, 0.125]
E	inflection	(-0.35112, -0.93633)	[-0.85355, -0.04419]
F	inflection	(-0.35112, 0.93633)	[-0.85355, 0.04419]
G	inflection	(0.35112, -0.93633)	[-0.14645, -0.04419]
H	inflection	(0.35112, 0.93633)	[-0.14645, 0.04419]

In this case, the connectivity is clear, as well as the certified topology.

5.1. *Example 5 - see Figure 5.*

We want to draw the offset at distance  $-\frac{9}{10}$  to the ellipse  $f(x, y) = x^2 + 4y^2 - 4$  inside box  $[-2, 2] \times [-1, 1]$ . The implicit support function of the ellipse is

$$D(h, \mathbf{n}) = h^2 - 4n_1^2 - n_2^2$$

and therefore the implicit support function of the offset at distance  $-\frac{9}{10}$  is

$$D(h, \mathbf{n}) = \left(h - \frac{9}{10}\right)^2 - 4n_1^2 - n_2^2.$$

Using methods from Section 3 we get the following  $G^1$ -points.

point	point type	normal vector	coordinates
A	cusp	$(-0.75348, -0.65747)$	$[-1.15500, 0.19184]$
B	cusp	$(-0.75348, 0.65747)$	$[-1.15500, -0.19184]$
C	cusp	$(0.75348, -0.65747)$	$[1.15500, 0.19184]$
D	cusp	$(0.75348, 0.65747)$	$[1.15500, -0.19184]$
E	extremal	$(0, 1)$	$[0, -0.1]$
F	extremal	$(0, 1)$	$[0, 0.1]$
G	extremal	$(1, 0)$	$[-1.1, 0]$
H	extremal	$(1, 0)$	$[1.1, 0]$

Connectivity rule 1 indicates that  $A$  can be connected to  $E, F, G$  or  $H$ . Rule 5 discards the point  $H$ . We add points with normal  $\mathbf{n} = (-0.70710, -0.70710)$  and obtain a point inside the tangent triangles  $T(A, F)$  but not inside the tangent triangle  $T(A, E)$ . As  $A$  has to be connected to two points, the only possibility remains the chain  $G - A - E$ . Connectivity for other points is analogous. The bounding triangles have only transversal intersection, the correct topology is therefore certified due to Proposition 13.

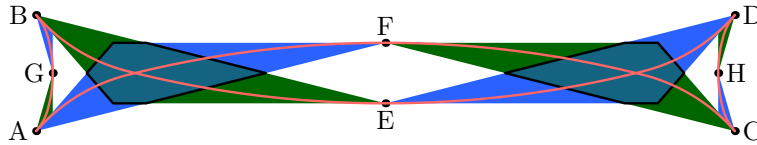


Figure 5: Result for Example 5.

## 6. Conclusion

To conclude, let us stress several interesting aspects of our new method for approximation of boxed algebraic curves. It is based on an identification of geometric features, including inflections, cusps and other critical points, which delimit the monotonous smooth curve segments. It exploits simultaneously the curve equation and its implicit support function representation. Unlike the other methods, our method focuses primarily on the high quality approximation of the segments. Due to the support function representation it controls the Hausdorff distance and not only its estimates such as the algebraic distance.

As a future research we plan to investigate the possibility to obtain symbolically higher order information about branches of algebraic curves. We also intend to exploit the support based construction and representation of circular arcs for other applications.

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