Global weak solutions to a class of non-Newtonian compressible fluids

E. Feireisl and X. Liao and J. Málek

Abstract. We consider a class of compressible fluids with nonlinear constitutive equations that guarantee that the divergence of the velocity field remains *bounded*. We study mathematical properties of unsteady three dimensional flows of such fluids in bounded domains. In particular, we show the long-time and large-data existence result of weak solutions with strictly positive density.

Mathematics Subject Classification (2000). Primary 35Q30,35Q72; Secondary 76D03,76A05.

Keywords. compressible fluid, generalized Newtonian fluids, power law fluid, bounded divergence of the velocity, weak solutions, positive density.

1. Introduction

The aim of this study is to develop a large data mathematical theory for the following problem. For a given $T > 0$, a bounded domain $\Omega \subset \mathbb{R}^3$, the density of the external body forces $\mathbf{b} = (b_1, b_2, b_3) : (0,T) \times \Omega \to \mathbb{R}^3$, an initial (mass) density $\varrho_0 : \Omega \to (0,\infty)$, an initial momentum $\mathbf{m}_0 = (m_{01}, m_{02}, m_{03}) : \Omega \to \mathbb{R}^3$, the viscosity function $\mu : \mathbb{R} \to (0,\infty)$ and positive model parameters a, b, to find $\varrho : (0,T) \times \Omega \to (0,\infty)$ and $\boldsymbol{v} = (v_1, v_2, v_3) : (0,T) \times \Omega \to \mathbb{R}^3$ satisfying, in $(0,T) \times \Omega$, the system of four partial differential equations

$$\frac{\partial \varrho}{\partial t} + \operatorname{div}(\varrho \boldsymbol{v}) = 0\,, \tag{1.1}$$

$$\frac{\partial(\varrho \boldsymbol{v})}{\partial t} + \operatorname{div}(\varrho \boldsymbol{v} \otimes \boldsymbol{v}) = \operatorname{div} \mathbb{T} + \varrho \mathbf{b}\,, \tag{1.2}$$

$$\text{with} \quad \mathbb{T} = -p(\varrho)\mathbb{I} + 2\mu(|\mathbb{D}^d|^2)\mathbb{D}^d + \frac{b \operatorname{div} \boldsymbol{v}}{(1 - b^a |\operatorname{div} \boldsymbol{v}|^a)^{1/a}}\mathbb{I}\,, \tag{1.3}$$

completed with no-slip boundary condition

$$\boldsymbol{v}|_{(0,T)\times\partial\Omega} = 0, \tag{1.4}$$

and the initial conditions

$$\varrho(0,\cdot) = \varrho_0 \quad \text{and} \quad (\varrho \boldsymbol{v})(0,\cdot) = \mathbf{m}_0 \quad \text{in } \Omega. \tag{1.5}$$

This research is supported by the project ERC-CZ LL12012 financed by MŠMT.

In (1.3), the symbol $\mathbb{T}$ stands for the Cauchy stress tensor, and $\mathbb{D}$ denotes the symmetric part of the velocity gradient $\nabla \boldsymbol{v}$. We also use $\mathbb{D}^d$ for the deviatoric (traceless) part of $\mathbb{D}$, i.e., $\mathbb{D}^d = \mathbb{D} - \frac{1}{3}(\operatorname{div} \boldsymbol{v})\mathbb{I}$. Later on, in order to indicate what velocity field generates the tensor $\mathbb{D}$ or $\mathbb{D}^d$ we will write $\mathbb{D}(\boldsymbol{v})$ or $\mathbb{D}^d(\boldsymbol{v})$, respectively. In general, for any tensor quantity $\mathbb{A}$, we set $\mathbb{A}^d := \mathbb{A} - \frac{1}{3}(\operatorname{tr} \mathbb{A})\mathbb{I}$.

In the present work we suppose that the scalar pressure $p = p(\varrho)$ depends on the density ϱ, more specifically

$$p \in C[0,\infty) \cap C^1(0,\infty), \quad p(0) = 0, \ p'(\varrho) > 0 \text{ for } \varrho > 0. \tag{1.6}$$

Referring to (1.3), the relation between $\mathbb{T}$ and $\mathbb{D}$ is nonlinear even if μ does not depend on $\mathbb{D}^d$. In fact, in this study we assume that

$$\mu(|\mathbb{D}^d|^2) = \mu_0 \left(1 + |\mathbb{D}^d|^2\right)^{(r-2)/2} \quad \text{with } \mu_0 > 0 \text{ and } r \in [11/5, \infty)\,. \tag{1.7}$$

The lower bound $\frac{11}{5}$ for r is a technical assumption ensuring a direct application of the monotonicity method in the form developed in [10, 11] within the context of the analysis of problems concerning *incompressible* fluids.

Our main goal in the present paper is to show that for any data, fulfilling certain natural conditions concerning their integrability (see (3.5) below), there exists a weak solution to the problem (1.1)–(1.7) that admits the strictly *positive* density in $(0,T) \times \Omega$ whenever $\varrho_0 > 0$ in Ω. Note that positivity of the density in the class of weak solutions of the compressible *Navier-Stokes system* in the general three-dimensional setting, where the relevant existence theory was developed in [5, 12], represents an outstanding open problem (cf. Hoff and Smoller [9]).

The main point in our approach is based on the fact that the specific form of the constitutive relation (1.3) gives rise to the following feature: the divergence of the velocity field is under all circumstances bounded; see Subsection 2.2 for details. This in turn implies that the density of the fluid remains strictly positive in the domain occupied by the fluid. The form of (1.3) is motivated by similar constitutive relations arising in the context of nonlinear elasticity, fitting in the general framework of the *implicit constitutive theory* developed in [16, 17]. The resulting evolutionary problem can be written in an elegant way as a variational inequality that can be handled by the methods of convex analysis. We note that similar problems, although in a rather different context of inhomogeneous *incompressible* fluids, were considered by [8], [21].

Note that we deal with both non-linear constitutive equations and a non-linear pressure law. To the best of our knowledge, there are very few studies concerning *compressible* fluids with non-linear relation between the Cauchy stress and the velocity gradient. Mamontov [14, 15] (see also the references therein) considered a compressible model with linear pressure equation and an exponential dependence of the viscosity on the velocity gradient in twodimensional domains. Zhikov and Pastukhova [22] made the first attempt to address the present problem in its full complexity - general nonlinear constitutive equations and a nonlinear pressure equation $p(\varrho) = \varrho^\gamma$. Although this paper introduces a number of original ideas, the proof contains an essential gap related to the (hypothetical) presence of vacuum that would hamper the main compactness argument.

The structure of the paper is the following. In Chapter 2, we provide comments concerning the theoretical justification of the considered model that is inspired by limiting strain models appearing naturally when applying implicit constitutive theory to elastic bodies; see Rajagopal [18, 19]. We also link the considered model with the Navier-Stokes model for compressible fluids. In Chapter 3, we fix notation of functions spaces, give the definition of weak solutions and formulate precisely the main existence result. In Chapter 4, we deduce the necessary *a priori bounds* and prove the existence theorem in detail.

2. Compressible Navier-Stokes fluids and fluids with bounded $\operatorname{div} \boldsymbol{v}$

In this section, we start with the classical definition of a compressible Navier-Stokes fluid and present several different forms how these constitutive equations can be formulated. Then we generalize one of these forms to non-Newtonian framework focusing on the models that have a priori bounded divergence of the velocity field.

2.1. Classical compressible Navier-Stokes fluid

A compressible Navier-Stokes fluid in the *barotropic* regime is characterized by the constitutive equation

$$\mathbb{T} = -p(\varrho)\mathbb{I} + 2\mu\,\mathbb{D}^d(\boldsymbol{v}) + \eta\,\operatorname{div}\boldsymbol{v}\,\mathbb{I}\,. \tag{2.1}$$

Setting $m := \frac{1}{3}\operatorname{tr}\mathbb{T}$, this relationship can be rewritten into the form

$$\begin{aligned} m + p(\varrho) &= \eta\,\operatorname{div}\boldsymbol{v},\\ \mathbb{T}^d &= 2\mu\,\mathbb{D}^d(\boldsymbol{v})\,, \end{aligned} \tag{2.2}$$

or

$$\begin{aligned} \operatorname{div}\boldsymbol{v} &= \frac{1}{\eta}(m + p(\varrho)),\\ \mathbb{D}^d(\boldsymbol{v}) &= \frac{1}{2\mu}\mathbb{T}^d, \end{aligned} \tag{2.3}$$

provided both μ and η are positive.

The form (2.3) will, in particular, lead to generalizations given in the next subsection. Before doing so, let us make a few observations; see for example [13] for more details.

In the absence of thermal effects, the dynamics of the process is often carried on by the so-called reduced thermodynamic identity

$$\mathbb{T}\cdot\mathbb{D} - \varrho\dot{\psi} = \xi\,. \tag{2.4}$$

Here ψ is the Helmholtz potential, ξ stands for the rate of dissipation, and, for any scalar quantity z, the symbol $\dot{z}$ denotes the material derivative of z that can be expressed as

$$\dot{z} = \frac{\partial z}{\partial t} + \sum_{k=1}^{3}\frac{\partial z}{\partial x_k}v_k\,.$$

In the barotropic regime, we have

$$\psi = \psi(\varrho)\,, \tag{2.5}$$

and, using (1.1) written in the compact form $\dot{\varrho} = -\varrho\operatorname{div}\boldsymbol{v}$, and denoting $p := \varrho^2\,\psi'(\varrho) := \varrho^2\,\frac{d\psi(\varrho)}{d\varrho}$, we observe that (2.4) and (2.5) lead to

$$\xi = \mathbb{T}^d\cdot\mathbb{D}^d + (m + p(\varrho))\operatorname{div}\boldsymbol{v}\,. \tag{2.6}$$

Thus, the rate of entropy production (the rate of dissipation) is split into two parts: while the first term represents the dissipation of energy due to isochoric processes, the second term is connected with the dissipation of energy due to volume changes.

Note that upon inserting (2.2) into (2.6) we get

$$\xi = 2\mu\,|\mathbb{D}^d(\boldsymbol{v})|^2 + \eta|\operatorname{div}\boldsymbol{v}|^2\,. \tag{2.7}$$

Hence, if μ and ν are non-negative, then ξ is non-negative and the second law of thermodynamics is fulfilled. Note also that inserting (2.3) into (2.6) gives

$$\xi = \frac{1}{2\mu}|\mathbb{T}^d|^2 + \frac{1}{\eta}|m + p(\varrho)|^2 \,. \tag{2.8}$$

As already noted in the above, the requirement that ξ of the form (2.8) is non-negative implies that $\mu > 0$ and $\eta > 0$, while the same requirement associated with (2.7) lead merely to the non-negativity of the (shear and bulk) viscosity coefficients. The forms (2.7) and (2.8) differ also by the answers to the question under what condition the rate of dissipation vanishes. While (2.7) implies that $\xi = 0$ (independently of considered processes) if $\mu = \eta = 0$, it follows from (2.8) that $\xi = 0$ if $m = -p(\varrho)$ and $\mathbb{T}^d = \mathbb{O}$. In both cases, however, we end up with the same form for the Cauchy stress:

$$\mathbb{T} = -p(\varrho)\mathbb{I} \,. \tag{2.9}$$

Recalling that $p(\varrho) = \varrho^2\psi'(\varrho)$, we see that $-p(\varrho)\mathbb{I}$ contains the information about the elastic (non-dissipative) processes of the fluid and $2\mu\,\mathbb{D}^d(\boldsymbol{v}) + \eta \operatorname{div} \boldsymbol{v}\mathbb{I}$ is linked with viscous (dissipative) properties of the material.

2.2. Generalizations of Compressible Navier-Stokes fluids

We now come to generalizations of the constitutive equation (2.1). As indicated by the above relations (2.2) or (2.3) but also by the structure of the dissipative mechanisms as appearing in (2.6), the quantities $m + p(\varrho)$ and $\operatorname{div} \boldsymbol{v}$ on one hand and $\mathbb{D}^d$ and $\mathbb{T}^d$ on the other hand are related. In the above relations (2.2) and (2.3) they are proportional. If one is interested in the generalizations of the constitutive equations (2.1) it sounds reasonable to relate the quantities $m + p(\varrho)$ and $\operatorname{div} \boldsymbol{v}$ on one hand and $\mathbb{D}^d$ and $\mathbb{T}^d$ on the other hand in a *nonlinear* way. This is the approach used in this study.

Inspired by implicit constitutive theory (see Rajagopal [16, 17]) one can consider, as a generalization of the constitutive equation (2.3) for the compressible Navier-Stokes fluid, the class of fluids characterized by the implicit relations

$$g(m + p(\varrho), \operatorname{div} \boldsymbol{v}) = 0 \,, \tag{2.10}$$

$$\mathbb{G}(\mathbb{T}^d, \mathbb{D}^d(\boldsymbol{v})) = \mathbb{O} \,, \tag{2.11}$$

where $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $\mathbb{G} : \mathbb{R}^{3\times3}_{\text{sym}} \times \mathbb{R}^{3\times3}_{\text{sym}} \to \mathbb{R}^{3\times3}_{\text{sym}}$ are given. We will not consider the full potential of this setting in this study[1]. We prefer to restrict ourselves to a subclass of the models characterized by

$$\operatorname{div} \boldsymbol{v} = \frac{1}{b}\frac{m + p(\varrho)}{(1 + |m + p(\varrho)|^a)^{1/a}} \,, \tag{2.12}$$

$$\mathbb{D}^d(\boldsymbol{v}) = \frac{1}{2\mu(|\mathbb{D}^d(\boldsymbol{v})|^2)}\mathbb{T}^d \qquad \text{with } \mu(|\mathbb{D}^d(\boldsymbol{v})|^2) = \mu_0(1 + |\mathbb{D}^d(\boldsymbol{v})|^2)^{\frac{r-2}{2}} \,, \tag{2.13}$$

where $a > 0$, $b > 0$, $\mu_0 > 0$ and $r \in [11/5, \infty)$. While the relation (2.13) represents the simplest generalization of the linear response between $\mathbb{T}^d$ and $\mathbb{D}^d$ to polynomial response (and simultaneously avoiding the possible degenerate or

[1]We can however refer the interested reader to [20] and [13] for issues concerning constitutive theory, and to [2], [1] and [3] for results concerning large data existence of weak solution to flows of *incompressible* fluids described as a subclass of fluids characterized through (2.11). To be more specific, this subclass for example includes responses that can be identified with maximal monotone graphs in $\mathbb{R}^{3\times3}_{\text{sym}} \times \mathbb{R}^{3\times3}_{\text{sym}}$ with $\mathbb{T}^d$, $\mathbb{D}^d$ fulfilling, for $r > 6/5$, the condition

$$\mathbb{T}^d \cdot \mathbb{D}^d \geq \alpha(|\mathbb{D}^d|^r + |\mathbb{T}^d|^{r'}) \,, \qquad \text{where } r' := r/(r-1) \text{ and } \alpha > 0 \,.$$

singular behavior of classical power-law fluid response given by $\mathbb{T}^d = 2\mu_0 |\mathbb{D}^d(\boldsymbol{v})|^{r-2}\mathbb{D}^d(\boldsymbol{v})$), the relation (2.12) a priori guarantees that $\operatorname{div} \boldsymbol{v}$ is bounded:

$$|\operatorname{div} \boldsymbol{v}| < \frac{1}{b}.$$

The form of this constitutive equation is inspired by the limiting strain models proposed by Rajagopal in several studies (see e.g. [18, 19]) in order to describe response of solid bodies where the stress can be very high yet the linearized strain remains small.

Note that (2.12) implies that

$$|\operatorname{div} \boldsymbol{v}|^a = \frac{1}{b^a} \frac{|m + p(\varrho)|^a}{1 + |m + p(\varrho)|^a} \,,$$

which further leads to

$$|m + p(\varrho)|^a = \frac{b^a |\operatorname{div} \boldsymbol{v}|^a}{1 - b^a |\operatorname{div} \boldsymbol{v}|^a} \,.$$

Inserting this relation back to (2.12) we obtain the inverse relation to (2.12) in the form

$$m + p(\varrho) = \frac{b \operatorname{div} \boldsymbol{v}}{(1 - b^a |\operatorname{div} \boldsymbol{v}|^a)^{1/a}} \,. \tag{2.14}$$

Then it follows from (2.13) and (2.14) that

$$\begin{aligned} \mathbb{T} &= \mathbb{T}^d + \frac{1}{3}(\operatorname{tr} \mathbb{T})\mathbb{I} = \mathbb{T}^d + m\mathbb{I} \\ &= -p(\varrho)\mathbb{I} + 2\mu_0(1 + |\mathbb{D}^d(\boldsymbol{v})|^2)^{(r-2)/2}\mathbb{D}^d(\boldsymbol{v}) + \frac{b \operatorname{div} \boldsymbol{v}}{(1 - b^a |\operatorname{div} \boldsymbol{v}|^a)^{1/a}}\mathbb{I} \,, \end{aligned} \tag{2.15}$$

which is the constitutive equation for the Cauchy stress as stated in (1.1).

Upon inserting (2.12) and (2.13) into (2.6) we observe that

$$\xi = \frac{1}{2\mu(|\mathbb{D}^d(\boldsymbol{v})|^2)} |\mathbb{T}^d|^2 + \frac{|m + p(\varrho)|^2}{b(1 + |m + p(\varrho)|^a)^{1/a}}, \tag{2.16}$$

or, alternatively, inserting (2.13) (now expressing $\mathbb{T}^d$ as a function of $\mathbb{D}^d$) and (2.14) we have

$$\xi = 2\mu(|\mathbb{D}^d(\boldsymbol{v})|^2)|\mathbb{D}^d|^2 + \frac{b\,|\operatorname{div} \boldsymbol{v}|^2}{(1 - b^a |\operatorname{div} \boldsymbol{v}|^a)^{1/a}}. \tag{2.17}$$

Thus the fluid model (2.12)-(2.13) is thermodynamically consistent as the rate of the entropy production is non-negative.

To summarize, we have derived the model described through (1.1)-(1.3).

3. Main results

In this section, we define the weak solutions to the problem (1.1)-(1.5) and state our main result on global existence. In what follows, we consider a slightly more general form of the stress tensor $\mathbb{T}$, specifically

$$\mathbb{T} = -p(\varrho)\mathbb{I} + \mathbb{S}(\boldsymbol{v}) + \eta(\operatorname{div} \boldsymbol{v}) \operatorname{div} \boldsymbol{v}\mathbb{I}, \tag{3.1}$$

where

- the deviatoric part of the Cauchy stress tensor $\mathbb{S}$ is specified through

$$\mathbb{S}(\boldsymbol{v}) := 2\mu_0(1+|\mathbb{D}^d(\boldsymbol{v})|^2)^{\frac{r-2}{2}}\mathbb{D}^d(\boldsymbol{v}),\ \ \mu_0>0 \text{ constant}, r\in[11/5,\infty); \tag{3.2}$$

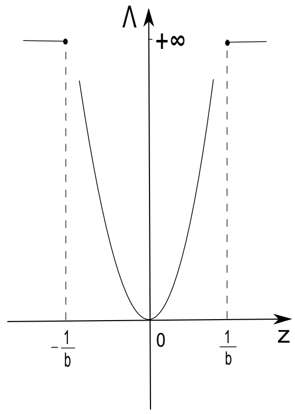
- the bulk viscosity coefficient η is a continuous function of $\operatorname{div}\boldsymbol{v}$, $\eta(\operatorname{div}\boldsymbol{v}) : \left(-\frac{1}{b},\frac{1}{b}\right)\to[0,\infty)$, such that there is a convex potential $\Lambda:\mathbb{R}\to[0,\infty]$,

$$\left\{\begin{array}{ll} \Lambda(0)=0, & \\ \Lambda'(z)=z\eta(z), & \\ \Lambda(z)\to\infty & \text{if } z\to\pm\frac{1}{b}, \\ \Lambda(z)=\infty & \text{if } |z|\geq\frac{1}{b}; \end{array}\right\} \tag{3.3}$$

- the pressure $p=p(\varrho)$ and the Helmholtz free energy $\psi=\psi(\varrho)$ satisfy

$$p=\varrho^2\psi'(\varrho), \qquad p\in C[0,\infty)\cap C^1(0,\infty), \quad p(0)=0,\ p'(\varrho)>0 \text{ for } \varrho>0. \tag{3.4}$$

Remark 3.1. *(i) Notice that in the particular case considered in Subsection* 2.2*, the function η takes, for $z>0$, the form $\eta(z)=b(1-b^az^a)^{-\frac{1}{a}}$. The graph of the corresponding potential Λ is then sketched in the following figure:*



(ii) In Subsection 4.3 *below, we will introduce a function P by $P(\varrho):=\varrho\psi(\varrho)$. Since $p'(\varrho)=\varrho^2\psi''(\varrho)+2\varrho\psi'(\varrho)$, we observe that $P''(\varrho)=\frac{p'(\varrho)}{\varrho}$. Consequently, by* (3.4)*, $P''(\varrho)>0$ for all $\varrho>0$ and P is strictly convex on $(0,\infty)$.*

3.1. Weak solutions

Let the initial data $(\varrho_0,\boldsymbol{m}_0)$ satisfy

$$0<\underline{\varrho}\leq\varrho_0(x)\leq\overline{\varrho}, \text{ for a.a. } x\in\Omega, \quad \boldsymbol{m}_0\in(L^2(\Omega))^3, \tag{3.5}$$

and let $\boldsymbol{b}\in(L^{r'}((0,T)\times\Omega))^3$. The weak solutions of the problem (1.1)-(1.7) are defined as follows:

Definition 3.1. *A pair of functions $(\varrho,\boldsymbol{v})$ is called a weak solution of the problem* (1.1)-(1.7) *if:*

-

$$\begin{aligned} &\varrho\in C([0,T];L^1(\Omega))\cap L^\infty((0,T)\times\Omega), \quad \varrho(0)=\varrho_0, \\ &\sqrt{\varrho}\boldsymbol{v}\in L^\infty([0,T];(L^2(\Omega))^3), \quad \boldsymbol{v}\in L^r([0,T];(W_0^{1,r}(\Omega))^3),\ \eta(\operatorname{div}_x\boldsymbol{v})|\operatorname{div}_x\boldsymbol{v}|^2\in L^1((0,T)\times\Omega); \end{aligned} \tag{3.6}$$

- *the equation of continuity* (1.1) *holds in the distributional sense on* $(0,T)\times\mathbb{R}^3$*:*

$$\int_0^\tau\int_{\mathbb{R}^3}(\varrho\partial_t\varphi+\varrho\boldsymbol{v}\cdot\nabla\varphi)\;dxdt=\left[\int_{\mathbb{R}^3}\varrho\varphi\;dx\right]\Big|_0^\tau$$
$$\text{for any }\ \tau\in[0,T],\quad\text{for all }\ \varphi\in C_c^\infty([0,T]\times\mathbb{R}^3)\ \text{provided } \boldsymbol{v} \text{ was extended to be zero outside } \Omega; \tag{3.7}$$

- *the following weak formulation of the momentum equation holds:*

$$\begin{aligned}\int_0^\tau\int_\Omega\Lambda(\operatorname{div}_x\boldsymbol{\varphi})-\Lambda(\operatorname{div}_x\boldsymbol{v})\,dxdt\geq\left[\frac12\int_\Omega\varrho|\boldsymbol{v}|^2\right]\Big|_0^\tau-\left[\int_\Omega\varrho\boldsymbol{v}\cdot\boldsymbol{\varphi}\;dx\right]\Big|_0^\tau\\+\int_0^\tau\int_\Omega\varrho\boldsymbol{v}\cdot\partial_t\boldsymbol{\varphi}+\varrho\boldsymbol{v}\otimes\boldsymbol{v}:\nabla\boldsymbol{\varphi}\,dxdt+\int_0^\tau\int_\Omega\mathbb{S}(\boldsymbol{v}):\mathbb{D}^d(\boldsymbol{v}-\boldsymbol{\varphi})\,dxdt\\+\int_0^\tau\int_\Omega p(\varrho)\operatorname{div}(\boldsymbol{\varphi}-\boldsymbol{v})\,dxdt+\int_0^\tau\int_\Omega\varrho\boldsymbol{b}\cdot(\boldsymbol{\varphi}-\boldsymbol{v})\,dxdt,\\\text{for a.e. }\tau\in[0,T],\ \text{for all }\boldsymbol{\varphi}\in C_c^\infty([0,T]\times\Omega,\mathbb{R}^3).\end{aligned}\tag{3.8}$$

Remark 3.2. *It follows from* (3.6) *and the structural properties* (3.3) *of the function* Λ *that*

$$\Lambda(\operatorname{div}\boldsymbol{v})\in L^1((0,T)\times\Omega);\ \text{whence}$$
$$|\operatorname{div}\boldsymbol{v}|<\frac1b\ \text{ a.e. on }(0,T)\times\Omega.$$

Thus, applying the nowadays standard DiPerna-Lions theory [4] *to the equation of continuity* (1.1)*, we may infer the the density* ϱ *satisfies*

$$\underline{\varrho}\exp\left(-\frac tb\right)\leq\varrho(t,x)\leq\overline{\varrho}\exp\left(\frac tb\right),\tag{3.9}$$

together with the following renormalized equation

$$\partial_t[b(\varrho)]+\operatorname{div}_x[b(\varrho)\boldsymbol{v}]+(b'(\varrho)\varrho-b(\varrho))\operatorname{div}_x\boldsymbol{v}=0,\tag{3.10}$$

which is satisfied in the sense of distributions. Taking $\boldsymbol{\varphi}=0$ *in* (3.8)*, we get the energy inequality*

$$\left[\int_\Omega\frac12\varrho|\boldsymbol{v}|^2\;+\;\varrho\psi(\varrho)\right]\Big|_0^\tau+\int_0^\tau\int_\Omega\mathbb{S}(\boldsymbol{v}):\mathbb{D}^d\boldsymbol{v}+\Lambda(\operatorname{div}\boldsymbol{v})\leq\int_0^\tau\int_\Omega\varrho\boldsymbol{b}\cdot\boldsymbol{v}\,,\ \text{a.e. }\tau\in[0,T].\tag{3.11}$$

3.2. Global-in-time weak solutions

Now we state the main existence result:

Theorem 3.1. *Let* $\Omega\subset\mathbb{R}^3$ *be a bounded domain with a Lipschitz boundary. Suppose that the pressure* $p=p(\varrho)$ *and the Helmholtz free energy* $\psi=\psi(\varrho)$ *satisfy* (3.4) *and that the hypotheses* (3.1)-(3.3) *hold.*

Then, for any initial data (3.5) *and any* $T>0$*, there exists a weak solution to the problem* (1.1)-(1.7) *in* $(0,T)\times\Omega$*.*

The proof of Theorem 3.1 will be given in next section. Now, we fix the notation used in what follows:

- Notation $f_n\to f$ in X means that the sequence $\{f_n\}$ converges to the limit f strongly in the Banach space X, while $f_n\rightharpoonup f$ and $f_n\overset{*}{\rightharpoonup}f$ in X denote the weak convergence and weak-$*$ convergence in X, respectively;
- the symbol C denotes a generic positive constant, which may vary from time to time;

- the symbol $\overline{F(f)}$ denotes a weak limit of the sequence $\{F(f_n)\}$: for example, if $\{f_n\}$ is a bounded sequence in $L^r(\Omega)$ and $\{F(f_n)\}$ a bounded sequence in $L^q(\Omega)$ with $F \in C(\mathbb{R})$, $r, q > 1$, then there exists a subsequence $\{f_{n_k}\}$ such that
$$f_{n_k} \rightharpoonup f \text{ in } L^r(\Omega), \quad F(f_{n_k}) \rightharpoonup \overline{F(f)} \text{ in } L^q(\Omega);$$
- the scalar product $\langle \cdot\,;\,\cdot \rangle$ denotes the inner product in $L^2(\Omega)^3$.
- We set the source term $\boldsymbol{b} \equiv 0$ for the notational simplicity in the proof[2].

4. Proof of the main result

The basic ingredient of the proof of Theorem 3.1 represents the *a priori* bounds derived from the standard energy estimates. More specifically, taking the scalar product of (1.2) with $\boldsymbol{v}$ we obtain, with help of (1.1), the following identity (keeping in mind that the source term $\boldsymbol{b}$ is assumed to vanish in the present chapter)

$$\frac{1}{2}\Big[\frac{\partial(\varrho|\boldsymbol{v}|^2)}{\partial t} + \operatorname{div}(\varrho|\boldsymbol{v}|^2\boldsymbol{v})\Big] + \mathbb{T}\cdot\mathbb{D} = \operatorname{div}(\mathbb{T}\boldsymbol{v})\,. \tag{4.1}$$

Integrating this identity over Ω and using the boundary condition (1.4) we arrive at

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \varrho|\boldsymbol{v}|^2\ dx + \int_\Omega \mathbb{T}\cdot\mathbb{D}\ dx = 0. \tag{4.2}$$

Finally, using (2.4), (2.5) and (2.17) we obtain from (4.2) the identity

$$\frac{d}{dt}\int_\Omega \left(\frac{1}{2}\varrho|\boldsymbol{v}|^2 + \varrho\psi(\varrho)\right)\ dx + \int_\Omega 2\mu(|\mathbb{D}^d(\boldsymbol{v})|^2)|\mathbb{D}^d(\boldsymbol{v})|^2\ dx + \int_\Omega \eta(\operatorname{div}\boldsymbol{v})|\operatorname{div}\boldsymbol{v}|^2\ dx = 0\,.$$

If the initial data satisfy (3.5), then the following quantities are uniformly bounded:

$$\sup_{t\in(0,T)}\int_\Omega\Big[\frac{1}{2}\varrho|\boldsymbol{v}|^2 + \varrho\psi(\varrho)\Big](t,\cdot)\ dx, \quad \int_0^T\int_\Omega \mu(|\mathbb{D}^d(\boldsymbol{v})|^2)|\mathbb{D}^d(\boldsymbol{v})|^2\ dxdt, \quad \int_0^T\int_\Omega \eta(\operatorname{div}\boldsymbol{v})|\operatorname{div}\boldsymbol{v}|^2\ dxdt.$$

The proof of Theorem 3.1 is based on a two-level approximation scheme. We start by a *Galerkin type* approximation of the problem (1.1)-(1.7) in the spirit of [5, Chapter 6], with Λ replaced by a smooth regularization Λ_ε, see Section 4.1. Then, in Section 4.2, we perform the limit $\varepsilon \to 0$. Finally, the passage to the limit in the Galerkin approximation will be performed in Section 4.3.

4.1. Galerkin approximation

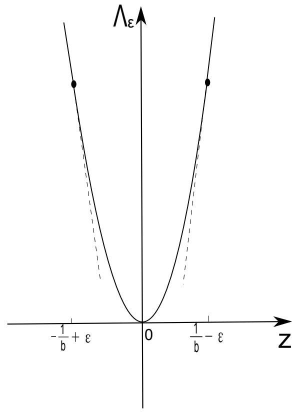
We define a regularization Λ_ε of the function Λ as follows:

$$\Lambda_\varepsilon(z) = \begin{cases} \Lambda(z) & \text{for } |z| \le \frac{1}{b} - \varepsilon, \\ \Lambda(\frac{1}{b}-\varepsilon) + \Lambda'(\frac{1}{b}-\varepsilon)\left(z - (\frac{1}{b}-\varepsilon)\right) & \text{if } z \ge \frac{1}{b} - \varepsilon, \end{cases} \tag{4.3}$$

$$\Lambda_\varepsilon(z) = \Lambda_\varepsilon(-z) \qquad \text{if } z \le -\frac{1}{b} + \varepsilon. \tag{4.4}$$

In the specific case with the Cauchy stress tensor (1.3), the function Λ_ε behaves as the following graph

[2]The source term $\boldsymbol{b} \in L^{r'}([0,T]\times\Omega)^3$ is of enough integrability and will not cost any real difficulty in the proof.



Next, since $C_0^\infty(\Omega)^3$ is compactly and densely embedded in the Hilbert space $L^2(\Omega)^3$, we can choose a countable set $\{\boldsymbol{w}_n\}_{n=1}^\infty \subset C_0^\infty(\Omega)^3$ as an orthonormal basis in the inner product $\langle\cdot;\cdot\rangle_{L^2(\Omega)^3}$. Let X_m be the linear span of $\{\boldsymbol{w}_1,\cdots,\boldsymbol{w}_m\}$.

Finally, for given $\boldsymbol{v}\in C([0,T];X_m)$, let $\varrho=\varrho[\boldsymbol{v}]$ be the unique solution of the equation of continuity (1.1), with the regularized initial data $\varrho_{\varepsilon,0}\in C_0^\infty(\Omega)$,

$$\varrho_{\varepsilon,0}\to\varrho_0 \text{ in } L^1(\Omega) \text{ as } \varepsilon\to 0,\ \underline{\varrho}\le\varrho_{0,\varepsilon}\le\overline{\varrho}. \tag{4.5}$$

Thanks to the standard results for the transport equation (see e.g. [4]), the mapping assigning the velocity field $\boldsymbol{v}$ to the solution $\varrho=\varrho[\boldsymbol{v}]$ of (1.1) is continuous, from $L^1([0,T];X_n)$ to $C([0,T];W^{1,q}(\Omega))$, $q<\infty$. Moreover, if $\|\operatorname{div}\boldsymbol{v}\|_{L^1([0,T];L^\infty(\Omega))}\le L$ for some positive constant L, then

$$\varrho\in[\underline{\varrho}e^{-L},\overline{\varrho}e^{L}] \text{ on } [0,T]\times\Omega.$$

Consequently, using the same arguments as in [5, Chapter 6], we may find an approximate solution $\boldsymbol{v}_{n,\varepsilon}\in C([0,T];X_n)$ such that $\partial_t\boldsymbol{v}_{n,\varepsilon}\in L^1([0,T];X_n)$, and

$$\Big[\int_\Omega \varrho\boldsymbol{v}_{n,\varepsilon}\cdot\boldsymbol{\varphi}\Big]\Big|_0^\tau+\int_0^\tau\int_\Omega -\varrho\boldsymbol{v}_{n,\varepsilon}\otimes\boldsymbol{v}_{n,\varepsilon}:\nabla\boldsymbol{\varphi}+\mathbb{S}(\boldsymbol{v}_{n,\varepsilon}):\mathbb{D}^d\boldsymbol{\varphi}+\Lambda_\varepsilon'(\operatorname{div}\boldsymbol{v}_{n,\varepsilon})\operatorname{div}\boldsymbol{\varphi}-p(\varrho)\operatorname{div}\boldsymbol{\varphi}=0, \text{ for all } \boldsymbol{\varphi}\in X_n, \tag{4.6}$$

where $\varrho=\varrho_{n,\varepsilon}=\varrho[\boldsymbol{v}_{n,\varepsilon}]$.

Differentiating the above equation (4.6) with respect to t, integrating by parts and then choosing the test function $\boldsymbol{\varphi}=\boldsymbol{v}_n$, we obtain the following energy equality for $(\varrho_{n,\varepsilon},\boldsymbol{v}_{n,\varepsilon})$:

$$\Big[\int_\Omega\Big(\frac{1}{2}\varrho_{n,\varepsilon}|\boldsymbol{v}_{n,\varepsilon}|^2+\varrho_{n,\varepsilon}\psi(\varrho_{n,\varepsilon})\Big)\Big]\Big|_0^\tau+\int_0^\tau\int_\Omega\mathbb{S}(\boldsymbol{v}_{n,\varepsilon}):\mathbb{D}^d(\boldsymbol{v}_{n,\varepsilon})+\Lambda_\varepsilon'(\operatorname{div}\boldsymbol{v}_{n,\varepsilon})\operatorname{div}\boldsymbol{v}_{n,\varepsilon}=0,\ \tau\in[0,T]. \tag{4.7}$$

4.2. The limit $\varepsilon\to 0$

Fixing $n>0$ and denoting by $(\varrho_\varepsilon,\boldsymbol{v}_\varepsilon)$ the family of approximate solutions obtained Section 4.1, our goal in this section is to perform the limit $\varepsilon\to 0$. To begin with, we recall the following bounds independent of the parameter ε:

$$\varrho_\varepsilon\in[\underline{\varrho}e^{-L},\overline{\varrho}e^{L}],\ \text{with } L=L(n), \tag{4.8}$$
$$\|\sqrt{\varrho_\varepsilon}\boldsymbol{v}_\varepsilon\|_{L^\infty([0,T];(L^2(\Omega))^3)},\ \|\mathbb{D}^d\boldsymbol{v}_\varepsilon\|_{L^r([0,T];(L^r(\Omega))^9)},\ \|\Lambda_\varepsilon'(\operatorname{div}\boldsymbol{v}_\varepsilon)\operatorname{div}\boldsymbol{v}_\varepsilon\|_{L^1([0,T];L^1(\Omega))}\le C(\underline{\varrho},\overline{\varrho},\boldsymbol{m}_0,T).$$

Moreover, taking the above estimates into account, we deduce from the equation (4.6) that

$$\|\langle\partial_t\varrho_\varepsilon\boldsymbol{v}_\varepsilon;\boldsymbol{w}_m\rangle\|_{L^1(0,T)}\le C(\underline{\varrho},\overline{\varrho},\boldsymbol{m}_0,T),\ 1\le m\le n, \tag{4.9}$$

where $\langle\cdot;\cdot\rangle$ denotes the scalar product in $L^2(\Omega)^3$.

Thanks to the stability result for the equation of continuity (1.1) (see e.g. [4]), we can extract subsequences such that

$$\begin{aligned}&\boldsymbol{v}_\varepsilon \to \boldsymbol{v} \text{ in } C([0,T]\times\Omega)\cap L^r([0,T];W^{1,r}(\Omega)^3) \text{ as } \varepsilon\to 0,\\ &\varrho_\varepsilon \to \varrho \text{ in } C([0,T];L^q(\Omega)),\ q\in[1,\infty), \quad \text{as } \varepsilon\to 0,\end{aligned}$$

where the limit $(\varrho,\boldsymbol{v})$ satisfies the equation of continuity (1.1), supplemented with the initial condition (3.5). Furthermore, as (4.9) is independent of m, we deduce that

$$\|\partial_t\langle\varrho\boldsymbol{v};\boldsymbol{w}_m\rangle\|_{\mathcal{M}[0,T]} \le C(\underline{\varrho},\overline{\varrho},\boldsymbol{m}_0,T,m) \text{ for any fixed } m, \tag{4.10}$$

with $\mathcal{M}[0,T]$ denoting the space of measures in $[0,T]$. This boundedness result will be the key point to get the time compactness of $\varrho\boldsymbol{v}$ when $n\to\infty$ in the next section.

Our ultimate goal in this section is to show that the integral identity (3.8) holds for the limit $(\varrho,\boldsymbol{v})$, with the test function $\boldsymbol{\varphi}\in C^1([0,T];X_n)$, in particular,

$$|\operatorname{div}\boldsymbol{v}| < \frac{1}{b} \text{ a.e. on } [0,T]\times\Omega. \tag{4.11}$$

Differentiating (4.6) with respect to t and then testing by $\boldsymbol{\varphi}-\boldsymbol{v}$ we get the following inequality for $(\varrho_\varepsilon,\boldsymbol{v}_\varepsilon)$ (keeping in mind the convexity of Λ_ε):

$$\begin{aligned}\int_0^\tau\int_\Omega \Lambda_\varepsilon(\operatorname{div}\boldsymbol{\varphi})-\Lambda_\varepsilon(\operatorname{div}\boldsymbol{v}_\varepsilon) \ge{}& \Big[\frac12\int_\Omega \varrho_\varepsilon|\boldsymbol{v}_\varepsilon|^2\Big]\Big|_0^\tau - \Big[\int_\Omega \varrho_\varepsilon\boldsymbol{v}_\varepsilon\cdot\boldsymbol{\varphi}\Big]\Big|_0^\tau\\ &+\int_0^\tau\int_\Omega \varrho_\varepsilon\boldsymbol{v}_\varepsilon\cdot\partial_t\boldsymbol{\varphi}+\varrho_\varepsilon\boldsymbol{v}_\varepsilon\otimes\boldsymbol{v}_\varepsilon:\nabla\boldsymbol{\varphi}\,dxdt+\int_0^\tau\int_\Omega \mathbb{S}(\boldsymbol{v}_\varepsilon):\mathbb{D}^d(\boldsymbol{v}_\varepsilon-\boldsymbol{\varphi})\,dxdt\\ &+\int_0^\tau\int_\Omega p(\varrho_\varepsilon)\operatorname{div}(\boldsymbol{\varphi}-\boldsymbol{v}_\varepsilon) \text{ for a.e. } \tau\in[0,T], \text{ for all } \boldsymbol{\varphi}\in C^1([0,T];X_n).\end{aligned} \tag{4.12}$$

Note that

$$\Lambda(\operatorname{div}\boldsymbol{\varphi}) \ge \Lambda_\varepsilon(\operatorname{div}\boldsymbol{\varphi}) \text{ for all } \varepsilon\in(0,\frac1b) \text{ and all } \boldsymbol{\varphi}\in C^1([0,T]\times\Omega,\mathbb{R}^3).$$

Moreover, for the regularized function sequence $\{\Lambda_\varepsilon\}_\varepsilon$, one has

$$\liminf_{\varepsilon\to 0}\Lambda_\varepsilon(\operatorname{div}\boldsymbol{v}_\varepsilon) \ge \Lambda(\operatorname{div}\boldsymbol{v}). \tag{4.13}$$

Indeed, the lower semi-continuous convex function Λ can be written as a supremum of its affine minorants:

$$\Lambda(z) = \sup\{a(z)\,|\,a \text{ an affine function on } \mathbb{R},\ a\le\Lambda \text{ on } \mathbb{R}\}.$$

For any affine function $a\le\Lambda$, there exists Λ_ε such that $a\le\Lambda_\varepsilon$. Consequently, $\Lambda_\varepsilon(\operatorname{div}\boldsymbol{v}_\varepsilon)\ge a(\operatorname{div}\boldsymbol{v}_\varepsilon)$. Since $a(\operatorname{div}\boldsymbol{v}_\varepsilon)\to a(\operatorname{div}\boldsymbol{v})$, (4.13) follows.

Taking the limit in (4.12) entails the weak formulation (3.8), with the test function $\boldsymbol{\varphi}\in C^1([0,T];X_n)$. In particular, if $\boldsymbol{\varphi}\equiv 0$, then one has the energy inequality (3.11) for $(\varrho_\varepsilon,\boldsymbol{v}_\varepsilon)$. According to the definition of Λ in (3.3), the bound on $\operatorname{div}\boldsymbol{v}$ in (4.11) follows.

4.3. Limit in the Galerkin approximation

Denoting the family of solutions obtained in Section 4.2 by $(\varrho_n, \boldsymbol{v}_n)$, we pass to the limit for $n \to \infty$. The energy inequality (3.11), together with the bound (4.11) on $\operatorname{div} \boldsymbol{v}_n$, give rise to the following estimates:

$$\begin{aligned}&\varrho_n \in [\underline{\varrho} e^{-T/b}, \overline{\varrho} e^{T/b}],\\ &\|\boldsymbol{v}_n\|_{L^\infty([0,T];L^2(\Omega)^3)},\ \|\mathbb{D}^d \boldsymbol{v}_n\|_{L^r([0,T];L^r(\Omega)^9)} \leq C(\underline{\varrho}, \overline{\varrho}, \boldsymbol{m}_0, T),\end{aligned}$$

which implies (noticing that the lower bound 11/5 for the index r plays role in getting the following bound on $\boldsymbol{v}_n$)

$$\|\boldsymbol{v}_n\|_{L^{2r'}([0,T]\times\Omega)^3}, \quad \|\partial_t \varrho_n\|_{L^r([0,T];W^{-1,r}(\Omega))} \leq C(\underline{\varrho}, \overline{\varrho}, \boldsymbol{m}_0, T).$$

Consequently, up to a subsequence, we get (see [5, Chapter 6])

$$\begin{aligned}\varrho_n &\to \varrho \text{ in } C([0,T]; L^q_{\text{weak}}(\Omega)) \text{ for all } q \in [1,\infty),\\ \boldsymbol{v}_n &\rightharpoonup \boldsymbol{v} \text{ in } L^r([0,T]; W^{1,r}(\Omega)^3),\\ \operatorname{div} \boldsymbol{v}_n &\overset{*}{\rightharpoonup} \operatorname{div} \boldsymbol{v} \text{ in } L^\infty([0,T]\times\Omega).\end{aligned} \tag{4.14}$$

Moreover, by virtue of (4.10), one has

$$\langle \varrho_n \boldsymbol{v}_n; \boldsymbol{w}_m \rangle \to \langle \varrho \boldsymbol{v}; \boldsymbol{w}_m \rangle \text{ in, say, } L^2(0,T) \text{ for any fixed } m = 1, 2, \dots;$$

whence

$$\varrho_n \boldsymbol{v}_n \to \varrho \boldsymbol{v} \text{ in } L^r([0,T]; W^{-1,2}(\Omega)^3),$$

which, combined with (4.14), gives rise to

$$\varrho_n \boldsymbol{v}_n \otimes \boldsymbol{v}_n \rightharpoonup \varrho \boldsymbol{v} \otimes \boldsymbol{v} \text{ in } L^{r'}([0,T]\times\Omega)^9.$$

It is easy to check that the limit $(\varrho, \boldsymbol{v})$ satisfies the equation of continuity (1.1), together with the renormalized equation (3.10). Moreover, taking to the limit in (3.8), we obtain the following inequality

$$\begin{aligned}\int_0^\tau \int_\Omega \Lambda(\operatorname{div} \boldsymbol{\varphi}) - \Lambda(\operatorname{div} \boldsymbol{v})\, dxdt \geq \Big[\frac{1}{2}\int_\Omega \varrho |\boldsymbol{v}|^2\Big]\Big|_0^\tau - \Big[\int_\Omega \varrho \boldsymbol{v} \cdot \boldsymbol{\varphi}\Big]\Big|_0^\tau\\ + \int_0^\tau \int_\Omega \varrho \boldsymbol{v} \cdot \partial_t \boldsymbol{\varphi} + \varrho \boldsymbol{v} \otimes \boldsymbol{v} : \nabla \boldsymbol{\varphi}\, dxdt + \int_0^\tau \int_\Omega \overline{\mathbb{S}(\boldsymbol{v}) : \mathbb{D}^d \boldsymbol{v}} - \overline{\mathbb{S}(\boldsymbol{v})} : \mathbb{D}^d \boldsymbol{\varphi}\, dxdt\\ + \int_0^\tau \int_\Omega \overline{p(\varrho)} \operatorname{div} \boldsymbol{\varphi} - \overline{p(\varrho) \operatorname{div} \boldsymbol{v}}\, dxdt \text{ for a.e. } \tau \in [0,T], \text{ for all } \boldsymbol{\varphi} \in C_c^\infty([0,T]\times\Omega, \mathbb{R}^3),\end{aligned} \tag{4.15}$$

with the weak limit $\overline{\mathbb{S}(\boldsymbol{v}) : \mathbb{D}^d \boldsymbol{v}}$ of the sequence $\{\mathbb{S}(\boldsymbol{v}_n) : \mathbb{D}^d \boldsymbol{v}_n\}$ being a measure on $[0,T]\times\Omega$.

Our next goal is to show

$$\begin{aligned}\varrho_n &\to \varrho \text{ in } L^q([0,T]\times\Omega), q \in [1,\infty),\\ \mathbb{D}^d \boldsymbol{v}_n &\to \mathbb{D}^d \boldsymbol{v} \text{ in } L^r([0,T]\times\Omega; \mathbb{R}^{3\times 3}).\end{aligned} \tag{4.16}$$

To this end, we first justify $\boldsymbol{\varphi} = \boldsymbol{v}$ as a test function in (4.15). Using the idea of Zhikov and Pastukhova [22][3], we consider a family of regularizing kernels

$$\eta_h(t) := \frac{1}{h} 1_{[-h,0]}(t), \quad \eta_{-h}(t) := \frac{1}{h} 1_{[0,h]}(t), \quad h > 0,$$

[3] A similar idea appears in the papers [6] and [7, Section 4] within the context of analysis of flows of incompressible fluids with variable density.

together with the cut-off functions

$$\zeta_\delta \in C_c^\infty(0,\tau),\ 0 \le \zeta \le 1,\ \zeta_\delta(t) = 1 \text{ whenever } t \in [\delta, \tau-\delta],\ \delta > 0.$$

Noticing that $\eta_h * \boldsymbol{v}(t) = \frac{1}{h}\int_t^{t+h} \boldsymbol{v} \in W^{1,r}([0,T]; W_0^{1,r}(\Omega)^3)$, we can take the quantities

$$\boldsymbol{\varphi}_{h,\delta} = \zeta_\delta\, \eta_{-h} * \eta_h * (\zeta_\delta \boldsymbol{v}), \quad \delta,\, h > 0.$$

as test functions in (4.15). Obviously one has

$$\begin{aligned}
&\lim_{\delta\to 0}\lim_{h\to 0}\int_0^\tau \int_\Omega \Lambda(\operatorname{div}\boldsymbol{\varphi}_{h,\delta}) - \Lambda(\operatorname{div}\boldsymbol{v})\, dxdt = 0,\\
&\Big[\int_\Omega \varrho \boldsymbol{v}\cdot\boldsymbol{\varphi}_{h,\delta}\Big]\Big|_0^\tau \equiv 0 \text{ for all } \delta,\, h > 0,\\
&\lim_{\delta\to 0}\lim_{h\to 0}\int_0^\tau \int_\Omega \overline{\mathbb{S}(\boldsymbol{v}):\mathbb{D}^d\boldsymbol{v}} - \overline{\mathbb{S}(\boldsymbol{v})}:\mathbb{D}^d\boldsymbol{\varphi}_{h,\delta}\, dxdt \ge 0.
\end{aligned}$$

We further analyze the second term (involving $\partial_t\boldsymbol{\varphi}_{h,\delta}$) on the left hand side of (4.15) and observe that

$$\begin{aligned}
\int_0^\tau\int_\Omega \varrho\boldsymbol{v}\cdot\partial_t\boldsymbol{\varphi}_{h,\delta}\, dxdt = &\int_0^\tau\int_\Omega \varrho\boldsymbol{v}\cdot\partial_t\zeta_\delta\, \eta_{-h}*\eta_h*(\zeta_\delta\boldsymbol{v})\, dxdt\\
&+ \int_{\mathbb{R}^1}\int_\Omega [\eta_h*(\varrho\zeta_\delta\boldsymbol{v})]\cdot\partial_t[\eta_h*(\zeta_\delta\boldsymbol{v})]\, dxdt.
\end{aligned} \tag{4.17}$$

Moreover, we easily get the limit of the first member on the right-hand side of (4.17):

$$\begin{aligned}
\lim_{\delta\to 0}\lim_{h\to 0}\int_0^\tau\int_\Omega \varrho\boldsymbol{v}\cdot\partial_t\zeta_\delta\, \eta_{-h}*\eta_h*(\zeta_\delta\boldsymbol{v}) &= \lim_{\delta\to 0}\int_0^\tau \Big(\frac{1}{2}\int_\Omega \varrho|\boldsymbol{v}|^2\, dx\Big)\partial_t|\zeta_\delta|^2\, dt\\
&= -\Big[\frac{1}{2}\int_\Omega \varrho|\boldsymbol{v}|^2\, dx\Big]\Big|_0^\tau.
\end{aligned}$$

The second member on the right-hand side of (4.17) rewrites as follows

$$\begin{aligned}
\int_{\mathbb{R}^1}\int_\Omega [\eta_h*(\varrho\zeta_\delta\boldsymbol{v})]\cdot\partial_t[\eta_h*(\zeta_\delta\boldsymbol{v})]\, dxdt &= -\int_{\mathbb{R}^1}\int_\Omega \partial_t[\eta_h*(\varrho\zeta_\delta\boldsymbol{v})]\cdot[\eta_h*(\zeta_\delta\boldsymbol{v})]\, dxdt\\
&= -\int_{\mathbb{R}^1}\int_\Omega \frac{(\varrho\zeta_\delta\boldsymbol{v})(t+h) - (\varrho\zeta_\delta\boldsymbol{v})(t)}{h}\cdot[\eta_h*(\zeta_\delta\boldsymbol{v})]\, dxdt\\
&= -\int_{\mathbb{R}^1}\int_\Omega \frac{(\varrho\zeta_\delta\boldsymbol{v})(t+h) - (\varrho\zeta_\delta\boldsymbol{v})(t)}{h}\cdot[\eta_h*(\zeta_\delta\boldsymbol{v})]\, dxdt\\
&\quad + \int_{\mathbb{R}^1}\int_\Omega \varrho\frac{(\zeta_\delta\boldsymbol{v})(t+h) - (\zeta_\delta\boldsymbol{v})(t)}{h}\cdot[\eta_h*(\zeta_\delta\boldsymbol{v})]\, dxdt\\
&\quad - \int_{\mathbb{R}^1}\int_\Omega \frac{1}{2}\varrho\,\partial_t[\eta_h*(\zeta_\delta\boldsymbol{v})]^2\, dxdt\\
&= -\int_{\mathbb{R}^1}\int_\Omega \frac{\varrho(t+h) - \varrho(t)}{h}\,(\zeta_\delta\boldsymbol{v})(t+h)\cdot[\eta_h*(\zeta_\delta\boldsymbol{v})]\, dxdt\\
&\quad - \frac{1}{2}\int_{\mathbb{R}^1}\int_\Omega \varrho\,\partial_t[\eta_h*(\zeta_\delta\boldsymbol{v})]^2\, dxdt.
\end{aligned}$$

We observe that

$$\partial_t(\eta_h * \varrho) + \mathrm{div}_x[\eta_h * (\varrho \boldsymbol{v})] = 0 \text{ for all } t \in \mathbb{R},$$

where we set

$$(\varrho, \boldsymbol{v})(t) = (\varrho_0, \mathbf{0}) \text{ for } t < 0, \quad (\varrho, \boldsymbol{v})(t) = (\varrho(T), \mathbf{0}) \text{ for } t > T.$$

Accordingly, using also the continuity equation (1.1), the second member on the right-hand side of (4.17) reads

$$\int_{\mathbb{R}^1} \int_\Omega [\eta_h * (\varrho \zeta_\delta \boldsymbol{v})] \cdot \partial_t [\eta_h * (\zeta_\delta \boldsymbol{v})] \, dxdt = \int_{\mathbb{R}^1} \int_\Omega -(\eta_h * \varrho \boldsymbol{v}) \cdot \nabla_x \Big((\zeta_\delta \boldsymbol{v})(t+h) \cdot [\eta_h * (\zeta_\delta \boldsymbol{v})] \Big) \, dxdt$$
$$+ \frac{1}{2} \int_{\mathbb{R}^1} \int_\Omega \varrho \boldsymbol{v} \cdot \nabla_x [\eta_h * (\zeta_\delta \boldsymbol{v})]^2 \, dxdt =: J_{\delta,h} \, .$$

Finally, it is not difficult to observe that

$$\lim_{\delta \to 0} \lim_{h \to 0} \left(\int_0^\tau \int_\Omega \varrho \boldsymbol{v} \otimes \boldsymbol{v} : \nabla \boldsymbol{\varphi}_{h,\delta} \, dxdt + J_{\delta,h} \right) = 0 \, .$$

To conclude, choosing $\boldsymbol{\varphi} = \boldsymbol{\varphi}_{h,\delta}$ in (4.15) and then letting $h, \delta \to 0$, we may infer that

$$\int_0^\tau \int_\Omega \overline{p(\varrho)} \, \mathrm{div} \, \boldsymbol{v} - \overline{p(\varrho) \, \mathrm{div} \, \boldsymbol{v}} \, dxdt \le 0 \quad \text{for } a.a. \tau \in [0,T]. \tag{4.18}$$

In order to finish the proof, we have to establish point-wise convergence of the densities ϱ_n. To this end, we use the renormalized equation (3.10) in the form

$$\partial_t [P(\varrho)] + \mathrm{div}_x [P(\varrho) \boldsymbol{v}] + p(\varrho) \, \mathrm{div}_x \, \boldsymbol{v} = 0,$$

with $P(\varrho) = \varrho \psi(\varrho)$ - a strictly convex function (as follows from (3.4)). Noting that we also have

$$\partial_t [\overline{P(\varrho)}] + \mathrm{div}_x [\overline{P(\varrho)} \boldsymbol{v}] + \overline{p(\varrho) \, \mathrm{div}_x \, \boldsymbol{v}} = 0,$$

and taking (4.18) into account, we conclude that

$$\left[\int_\Omega [\overline{P(\varrho)} - P(\varrho)] \, dx \right] \Big|_0^\tau = - \int_0^\tau \int_\Omega \overline{p(\varrho) \, \mathrm{div} \, \boldsymbol{v}} - p(\varrho) \, \mathrm{div} \, \boldsymbol{v} \, dxdt \le - \int_0^\tau \int_\Omega (\overline{p(\varrho)} - p(\varrho)) \, \mathrm{div} \, \boldsymbol{v} \, dxdt, \tag{4.19}$$

where

$$[\overline{P(\varrho)} - P(\varrho)](0, \cdot) = 0.$$

Now, we have, by virtue of convexity of P,

$$\int_\Omega \overline{P(\varrho)} - P(\varrho) \, dx \ge d \limsup_{n \to \infty} \int_\Omega |\varrho_n - \varrho|^2 \, dx \text{ for a certain } d > 0,$$

while

$$- \int_0^\tau \int_\Omega (\overline{p(\varrho)} - p(\varrho)) \, \mathrm{div} \, \boldsymbol{v} \, dxdt = - \lim_{n \to \infty} \int_0^\tau \int_\Omega (p(\varrho_n) - p(\varrho)) \, \mathrm{div} \, \boldsymbol{v} \, dxdt$$

$$\le \lim_{n \to \infty} \int_0^\tau \int_\Omega p'(\varrho)(\varrho - \varrho_n) \, \mathrm{div} \, \boldsymbol{v} \, dxdt + C \limsup_{n \to \infty} \int_0^\tau \int_\Omega |\varrho_n - \varrho|^2 \, dxdt.$$

Thus we may use (4.19), together with the standard Gronwall argument, in order to conclude that $\overline{P(\varrho)} = P(\varrho)$. In particular,

$$\varrho_n \to \varrho \text{ in } L^2((0,T)\times\Omega), \quad \overline{p(\varrho)} = p(\varrho),$$
$$\int_0^\tau \int_\Omega \overline{p(\varrho)\,\mathrm{div}\,\boldsymbol{v}} - p(\varrho)\,\mathrm{div}\,\boldsymbol{v} = 0.$$

Finally, as above, by choosing $\boldsymbol{\varphi} = \boldsymbol{v}$ in (4.15) one arrives at

$$\int_0^\tau \int_\Omega \overline{\mathbb{S}(\boldsymbol{v}) : \mathbb{D}^d(\boldsymbol{v})} - \overline{\mathbb{S}(\boldsymbol{v})} : \mathbb{D}^d(\boldsymbol{v}) \le 0.$$

This together with the inequality $\int_0^T \|\mathbb{D}^d \boldsymbol{v}_n - \mathbb{D}^d \boldsymbol{v}\|_r^r \le C \int_0^T \int_\Omega (\mathbb{S}(\boldsymbol{v}_n) - \mathbb{S}(\boldsymbol{v})) : (\mathbb{D}(\boldsymbol{v}_n) - \mathbb{D}(\boldsymbol{v}))\, dxdt$ implies that

$$\mathbb{D}^d \boldsymbol{v}_n \to \mathbb{D}^d \boldsymbol{v} \text{ in } L^r((0,T)\times\Omega;\mathbb{R}^{3\times 3}).$$

We have proved Theorem 3.1.

References

[1] M. Bulíček, P. Gwiazda, J. Málek, K. R. Rajagopal, and A. Świerczewska Gwiazda. On flows of fluids described by an implicit constitutive equation characterized by a maximal monotone graph. In *Partial differential equations and fluid mechanics*, volume 402 of *London Math. Soc. Lecture Note Ser.*, pages 23–51. Cambridge Univ. Press, Cambridge, 2012.

[2] M. Bulíček, P. Gwiazda, J. Málek, and A. Świerczewska. On unsteady flows of implicitly constituted incompressible fluids. *SIAM J. Math. Anal.*, 44(4):2756–2801, 2012.

[3] M. Bulíček and J. Málek. On unsteady internal flows of Bingham fluids subject to threshold slip on the impermeable boundary. Preprint MORE/2013/06 (accepted for publication in "Recent Developments of Mathematical Fluid Mechanics", series: Advances in Mathematical Fluid Mechanics, Birkhauser-Verlag, 2014.

[4] R. J. DiPerna and P.-L. Lions. Ordinary differential equations, transport theory and Sobolev spaces. *Invent. Math.*, 98(3):511–547, 1989.

[5] E. Feireisl. *Dynamics of viscous compressible fluids*, volume 26 of *Oxford Lecture Series in Mathematics and its Applications*. Oxford University Press, Oxford, 2004.

[6] J. Frehse and M. Růžička. Non-homogeneous generalized Newtonian fluids. *Math. Z.*, 260(2):355–375, 2008.

[7] J. Frehse and J. Málek and M. Růžička. Large data existence result for unsteady flows of inhomogeneous shear-thickening heat-conducting incompressible fluids. *Comm. Partial Differential Equations*, 35(10):1891–1919, 2010.

[8] F. Guillén-González, M. Poblete-Cantellano, and M. A. Rojas-Medar. On the variational inequalities related to viscous density-dependent incompressible fluids. *Ann. Univ. Ferrara Sez. VII Sci. Mat.*, 56(1):163–180, 2010.

[9] D. Hoff and J. Smoller. Non-formation of vacuum states for compressible Navier-Stokes equations. *Comm. Math. Phys.*, **216**(2):255–276, 2001.

[10] O. A. Ladyzhenskaya. New equations for description of viscous incompressible fluids and the global solvability of the boundary-value problems for them. *Trudy Matem. Inst. Akad. Nauk SSSR*, 102:85–104, 1967.

[11] J. L. Lions. *Quelques méthodes de résolution des problèmes aux limites non linéaires*. Dunod, Paris, 1969.

[12] P. L. Lions. *Mathematical topics in fluid mechanics. Vol. 2*, volume 3 of *Oxford Lecture Series in Mathematics and its Applications*. The Clarendon Press Oxford University Press, New York, 1998. Incompressible models, Oxford Science Publications.

[13] J. Málek and K. R. Rajagopal. Compressible generalized Newtonian fluids. *Z. Angew. Math. Phys.*, 61:1097–1110, 2010.

[14] A. E. Mamontov. Existence of global solutions of multidimensional equations for a compressible Bingham fluid. *Mat. Zametki*, 82(4):560–577, 2007.

[15] A. E. Mamontov. Global solvability of the multidimensional equations of a compressible non-Newtonian fluid, the transport equation, and Orlicz spaces. *Sib. Èlektron. Mat. Izv.*, 6:120–165, 2009.

[16] K. R. Rajagopal. On implicit constitutive theories. *Appl. Math.*, 48:279–319, 2003.

[17] K. R. Rajagopal. On implicit constitutive theories for fluids. *Journal of Fluid Mechanics*, 550:243–249, 2006.

[18] K. R Rajagopal. Non-linear elastic bodies exhibiting limiting small strain. *Math. Mech. Solids*, 16:122–139, 2011.

[19] K. R Rajagopal. On a new class of models in elasticity. *J. Math. Comp. Appl.*, 15:506–528, 2011.

[20] K. R. Rajagopal and A. R. Srinivasa. On the thermodynamics of fluids defined by implicit constitutive relations. *Z. Angew. Math. Phys.*, 59:715–729, 2008.

[21] S. A. Sazhenkov. The problem of the motion of rigid bodies in a non-Newtonian incompressible fluid. *Sibirsk. Mat. Zh.*, 39(1):146–160, iii, 1998.

[22] V. V. Zhikov and S. E. Pastukhova. On the solvability of the Navier-Stokes system for a compressible non-Newtonian fluid. *Dokl. Akad. Nauk*, 427(3):303–307, 2009.

E. Feireisl
Institute of Mathematics
Academy of Sciences of the Czech Republic
Žitná 25
116 21 Prague 1, Czech Republic
e-mail: `feireisl@math.cas.cz`

X. Liao and J. Málek
Charles University, Faculty of Mathematics and Physics
Mathematical Institute
Sokolovská 83
186 75 Prague 8, Czech Republic
e-mail: `liaoxian925@gmail.com, malek@karlin.mff.cuni.cz`