PROPERTIES OF FUNCTIONS WITH MONOTONE GRAPHS

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Abstract. A metric space (X,d) is *monotone* if there is a linear order $<$ on X and a constant $c>0$ such that $d(x,y)\leqslant cd(x,z)$ for all $x<y<z\in X$. Properties of continuous functions with monotone graph (considered as a planar set) are investigated. It is shown, e.g., that such a function can be almost nowhere differentiable, but must be differentiable at a dense set, and that Hausdorff dimension of the graph of such a function is 1.

1. Introduction

A metric space (X,d) is called monotone if there is a linear order $<$ on X and a constant $c>0$ such that $d(x,y)\leqslant cd(x,z)$ for all $x<y<z\in X$.

Suppose f is a continuous real-valued function defined on an interval. The graph $\mathfrak{f}$ of f is a subset of the plane. The goal of this paper is to investigate differentiability of f assuming that the graph $\mathfrak{f}$ is a monotone space.

Monotone metric spaces. Monotone metric spaces were introduced in [16, 9, 8]. Some applications are given in [16, 15, 6].

Definition 1.1. Let (X,d) be a metric space.

(X,d) is called *monotone* if there is a linear order $<$ on X and a constant $c>0$ such that for all $x,y,z\in X$

$$d(x,y)\leqslant cd(x,z)\qquad\text{whenever } x<y<z. \tag{1}$$

The order $<$ is called a *witnessing order* and c is called a *witnessing constant.*

(X,d) termed *σ-monotone* if it is a countable union of monotone subspaces.

It is easy to check that if (X,d), c and $<$ satisfy (1), then $d(y,z)\leqslant(c+1)d(x,z)$ for all $x<y<z$. It follows that replacing condition (1) by

$$\max\bigl(d(x,y),d(y,z)\bigr)\leqslant cd(x,z)\qquad\text{whenever } x<y<z \tag{2}$$

gives an equivalent definition of a monotone space. Since we will be occasionally interested in the value of c, we introduce the following notions.

2000 *Mathematics Subject Classification.* 26A24, 26A27, 26A46.

Key words and phrases. Monotone metric space, continuous function, graph, derivative, approximate derivative, absolutely continuous function, σ-porous set.

Some of the work on this project was conducted during O. Zindulka's sabbatical stay at the Instituto de matemáticas, Unidad Morelia, Universidad Nacional Autonóma de México supported by CONACyT grant no. 125108. M. Hrušák gratefully acknowledges support from PAPIIT grant IN101608 and CONACYT grant 80355. A. Nekvinda was supported by MSM 6840770010 and the grant 201/08/0383 of the Grant Agency of the Czech Republic. V. Vlasák was supported by the grant 22308/B-MAT/MFF of the Grant Agency of the Charles University in Prague and by grant 201/09/0067 of the Grant Agency of the Czech Republic.

Definition 1.2. Let $c > 0$. A metric space (X, d) is called

(i) *c-monotone* if there is a linear order $<$ such that (1) holds,
(ii) *symmetrically c-monotone* if there is a linear order $<$ such that (2) holds.

It is clear that (X, d) is monotone iff it is c-monotone for some c iff it is symmetrically c-monotone for some c. It is also clear that if a space is c-monotone, then it is symmetrically $(c + 1)$-monotone and that a symmetrically c-monotone space is c-monotone.

Topological properties of monotone and σ-monotone spaces are investigated in [9]. We recall the relevant facts proved therein. A monotone metric space is suborderable, i.e. embeds in a linearly orderable metric space. In particular, if $<$ is a witnessing order, then every open interval (a, b) is open in the metric topology, i.e. the metric topology is finer than the order topology.

If a metric space contains a dense monotone subspace, then the space itself is monotone. It follows that every σ-monotone subset of a metric space is contained in a σ-monotone F_σ-subset. This fact will be utilized at several occasions.

Though the topological dimension of a monotone metric space is at most one, in a general context of a separable metric space there is nothing one can say about the Hausdorff dimension of a monotone space. Indeed, there are 1-monotone compact spaces of arbitrary Hausdorff dimension, including ∞. However, when one considers only monotone subspaces of Euclidean spaces, there is, as proved in the oncoming paper [14], an upper estimate of Hausdorff dimension by means of the witnessing constant. On the other hand, by a result from [6], every Borel set in $\mathbb{R}^n$ contains a σ-monotone subset of the same Hausdorff dimension. Thus a monotone set can have Hausdorff dimension greater than 1. The same holds for curves: by an unpublished result of Pieter Allaart and Ondřej Zindulka, the von Koch curve is monotone.

The interplay between porosity and monotonicity in the plane are investigated in [5, 14]. In particular, by [5, Theorem 4.2], every monotone set in $\mathbb{R}^n$ is strongly porous (see Section 7 for the definition). This fact is utilized in Section 7.

Monotone graphs. We will focus on properties of continuous functions that have monotone graph. Our hope was that such a function must be differentiable at a substantial portion of its domain. It, however, turned out that the interplay between monotonicity of graph and differentiability is more delicate and definitely not straightforward. Our goal is to study this interplay.

It turns out that such a graph has σ-finite 1-dimensional Hausdorff measure and in particular, in contrast with the just mentioned von Koch curve property, has Hausdorff dimension 1. Can one go further and prove for instance that a continuous function with a monotone graph is differentiable at a large set, say, almost everywhere? Or, in the other direction, that a differentiable function has a monotone or σ-monotone graph? We provide answers to these questions.

We outline a few results. In sections 3 and 4 we show, e.g., that a differentiable function has a σ-monotone graph and that a continuous function with monotone graph has knot points (i.e. both upper/lower Dini derivatives are $\infty/-\infty$) almost everywhere where it does not posses a derivative.

In section 6 continuous functions with a 1-monotone graph are investigated. The strongest result says that such a function on a compact interval is of finite variation and in particular is differentiable almost everywhere.

However, in Section 5 we construct a continuous function that exhibits that, perhaps surprisingly, this theorem completely fails for monotone graph: an almost nowhere differentiable function with a monotone graph. Consequently, almost all points of the domain are knot points.

So a continuous function with monotone graph can be rather wild. But not completely: in Section 4 we show that such a function is differentiable at an uncountable dense set and its graph is of σ-finite length and in particular of Hausdorff dimension 1.

As proved at the beginning of Section 7, a graph of an absolutely continuous function is σ-monotone except a set of linear measure zero. The following result is thus perhaps surprising: there is an absolutely continuous function whose graph is not σ-monotone. Moreover, such a function can be constructed so that the graph is a porous set.

The concluding Section 8 lists some open problems.

2. Monotone graphs

A topological closure and interior of a set A in a metric space are denoted by $\overline{A}$ and $\operatorname{int} A$, respectively.

For $A \subseteq \mathbb{R}^2$ denote $\dim_{\mathsf{H}} A$ the Hausdorff dimension of A. Lebesgue measure on the line is denoted by $\mathscr{L}$. Given $A \subseteq \mathbb{R}^2$, its linear measure, i.e. 1-dimensional Hausdorff measure, is denoted $\mathscr{H}^1(A)$ and referred to as *Hausdorff length*.

We will be concerned with monotonicity of graphs of continuous functions. The symbol I is used to denote a non-degenerate interval of real numbers. Let $f : I \to \mathbb{R}$ be a continuous function. Formally there is no difference between f and its graph, but confusion may arise for instance from "f is monotone". Therefore we use $\mathfrak{f}$ when referring to the graph of f as a pointset in the plane (and likewise $\mathfrak{g}$ for the graph of g etc.). Given a set $E \subseteq I$, denote $\mathfrak{f}|E$ the graph of f restricted to E.

We write $\psi_f(x) = (x, f(x))$ (or just $\psi(x)$ if there is no danger of confusion) to denote the natural parametrization of $\mathfrak{f}$. The graph $\mathfrak{f}$ is obviously a connected linearly ordered space. By [3, Theorem II], if a space is linearly orderable and connected, then the order is unique up to reversing. Therefore there are only two orders on $\mathfrak{f}$ that can witness monotonicity of C: the order given by $\psi(t) < \psi(s)$ if $t < s$ and its reverse. Since being symmetrically c-monotone is invariant with respect to reversing the witnessing order, it does not matter which of the two orders we choose. Overall, given the conditions

$$\text{for all } x < y < z \in I \quad |\psi(x) - \psi(y)| \leqslant c|\psi(x) - \psi(z)|, \tag{3}$$

$$\text{for all } x < y < z \in I \quad |\psi(z) - \psi(y)| \leqslant c|\psi(x) - \psi(z)|, \tag{4}$$

we have

Lemma 2.1. *If $f : I \to \mathbb{R}$ is continuous, then*

(i) *$\mathfrak{f}$ is c-monotone if and only if at least one of* (3), (4) *holds,*
(ii) *$\mathfrak{f}$ is symmetrically c-monotone if and only if both* (3) *and* (4) *hold.*

The following simple condition equivalent to monotonicity of $\mathfrak{f}$ will turn useful.

Definition 2.2. Given $c > 0$, say that f satisfies condition P_c if

$$\max_{x \leqslant t \leqslant y} |f(x) - f(t)| \leqslant c|x - y| \text{ whenever } x < y \text{ and } f(x) = f(y). \tag{P_c}$$

Lemma 2.3. *Let $f : I \to \mathbb{R}$ be a continuous function and $c \geqslant 1$.*

(i) *If $\mathfrak{f}$ is c-monotone, then f satisfies P_c,*
(ii) *if f satisfies P_{c-1}, then $\mathfrak{f}$ is symmetrically c-monotone.*

Proof. (i) Let $x < y$ satisfy $f(x) = f(y)$. c-monotonicity of $\mathfrak{f}$ yields for all $t \in [x, y]$

$$|f(x) - f(t)| \leqslant |\psi(x) - \psi(t)| \leqslant c|\psi(x) - \psi(y)| = c|x - y|.$$

(ii) We prove only condition (3), as condition (4) is proved in the same manner. Let $x < y < z \in I$. Suppose that $f(x) \leqslant f(z) \leqslant f(y)$, all other cases are trivial or similar. Find $w \in [x, y]$ such that $f(w) = f(z)$. Condition P_{c-1} yields $|f(z) - f(y)| \leqslant (c-1)|z - w|$. Therefore

$$\begin{aligned}|\psi(x) - \psi(y)| &\leqslant |(x - z, f(x) - f(y))| \\ &\leqslant |\psi(x) - \psi(z)| + |(z - z, f(z) - f(y))| \\ &\leqslant |\psi(x) - \psi(z)| + (c-1)|z - w| \leqslant c|\psi(x) - \psi(z)|. \qquad \square\end{aligned}$$

Monotonicity and σ-monotonicity are clearly global properties. It turns out that the following pointwise counterpart of monotonicity is worth investigation.

Definition 2.4. Let $f : I \to \mathbb{R}$ be a continuous function and $c \geqslant 1$.

• Say that $\mathfrak{f}$ is *c-monotone at $y \in \mathfrak{f}$* if there is a neighborhood $U \subseteq \mathfrak{f}$ of y such that if $x < y < z$ and $x, z \in U$, then $|x - y| \leqslant c|x - z|$, and *monotone at y* if it is c-monotone at y for some $c \geqslant 1$. The set of all points where $\mathfrak{f}$ is c-monotone is denoted $\mathsf{Mon}_c(\mathfrak{f})$. The set of all points where $\mathfrak{f}$ is monotone is denoted $\mathsf{Mon}(\mathfrak{f})$.

• If $\mathfrak{f}$ is c-monotone (monotone) at $\psi_f(y)$, we call y an *$\mathcal{M}_c$-point* (*$\mathcal{M}$-point*) of f. The set of all $\mathcal{M}_c$-points of f is denoted $\mathcal{M}_c(f)$ or just $\mathcal{M}_c$. The set of all $\mathcal{M}$-points of f is denoted $\mathcal{M}(f)$ or just $\mathcal{M}$.

It is clear that $\mathsf{Mon}(\mathfrak{f}) = \mathfrak{f}|\,\mathcal{M}(f)$. Since the natural parametrization ψ_f is a homeomorphism, it thus makes no difference whether we investigate topological properties of $\mathsf{Mon}(\mathfrak{f})$ or $\mathcal{M}(f)$.

Obviously, $y \in I$ is an $\mathcal{M}_c$-point if and only if there is $\varepsilon > 0$ such that

$$\text{for all } x \in (y - \varepsilon),\ z \in (y + \varepsilon) \quad |\psi(x) - \psi(y)| \leqslant c|\psi(x) - \psi(z)|. \tag{5}$$

By the reasoning preceding (2), the inequality $|\psi(x) - \psi(y)| \leqslant c|\psi(x) - \psi(z)|$ in condition (5) can be replaced with $|\psi(y) - \psi(z)| \leqslant c'|\psi(x) - \psi(z)|$, possibly with another constant c'. Thus the definition of $\mathcal{M}$-point is "symmetric", in that it is invariant under reversing the orientation of x- or y-axis.

Another equivalent definition: y is an $\mathcal{M}$-point if and only if there is c and $\varepsilon > 0$ such that

$$\text{for all } x \in (y - \varepsilon),\ z \in (y + \varepsilon) \quad |f(x) - f(y)| \leqslant c\big(|f(x) - f(z)| + |z - x|\big). \tag{6}$$

Let us clarify the relation of monotonicity, σ-monotonicity, pointwise monotonicity and $\mathcal{M}$-points. The proof of the following is straightforward.

Proposition 2.5. *If $f : I \to \mathbb{R}$ is continuous, then*

(i) *$\mathcal{M}$ and $\mathcal{M}_c$ are F_σ-sets, and so are $\mathsf{Mon}(\mathfrak{f})$ and $\mathsf{Mon}_c(\mathfrak{f})$,*
(ii) *$\mathsf{Mon}(\mathfrak{f}) = \mathfrak{f}|\,\mathcal{M}$ is σ-monotone,*
(iii) *$\mathsf{Mon}_c(\mathfrak{f}) = \mathfrak{f}|\,\mathcal{M}_c$ is a countable union of closed c-monotone sets.*

Needles to say that if a continuous function has a monotone graph, then all points are $\mathcal{M}$-points. However, there is a continuous function f on $[0,1]$ with $\mathcal{M}_1(f) = [0,1]$, i.e. $\mathfrak{f}$ is 1-monotone at each point, but $\mathfrak{f}$ is not monotone: let $f(x) = |x|^{3/2} \sin \frac{1}{x}$ for $x \neq 0$, $f(0) = 0$. It is easy to check that condition (P_c) fails for each c and thus $\mathfrak{f}$ is not monotone. On the other hand, f is differentiable everywhere and hence, by Theorem 6.3 below, all points are $\mathcal{M}_1$-points.

By [9, Corollary 2.6], every monotone set has a monotone closure. Using this fact, the above proposition and Baire category theorem one can easily prove the following facts on relation between monotonicity, σ-monotonicity and pointwise monotonicity.

Lemma 2.6. *If $f : I \to \mathbb{R}$ is continuous, with a σ-monotone graph, then for any interval $J \subseteq I$ there is a subinterval $J' \subseteq I$ such that $\mathfrak{f}|J'$ is monotone.*

Corollary 2.7. *Let $f : I \to \mathbb{R}$ be continuous.*

- (i) *If f has a σ-monotone graph, then $\operatorname{int} \mathcal{M}$ is dense in I, i.e. $\textsf{Mon}(\mathfrak{f})$ contains an open dense subset of $\mathfrak{f}$.*
- (ii) *If all points of I are $\mathcal{M}$-points, i.e. if $\mathfrak{f}$ is monotone at each point, then $\mathfrak{f}$ is σ-monotone.*

Part (i) of this corollary cannot be strengthened: As shown in 8.1, there is a continuous function f on $[0,1]$ with a σ-monotone graph, and a perfect set of non-$\mathcal{M}$-points.

There is a profound connection between the set of $\mathcal{M}$-points and monotone subspaces of $\mathfrak{f}$. Its proof is straightforward.

Proposition 2.8. *If $f : I \to \mathbb{R}$ is continuous, then the following are equivalent.*

- (i) *Every monotone set $M \subseteq \mathfrak{f}$ is nowhere dense in $\mathfrak{f}$,*
- (ii) *every monotone set $M \subseteq \mathfrak{f}$ is meager in $\mathfrak{f}$,*
- (iii) *$\mathcal{M}(f)$ is meager in I,*
- (iv) $\operatorname{int} \mathcal{M}(f) = \emptyset$.

3. Differentiability vs. pointwise monotonicity

We now investigate if pointwise monotonicity is related to differentiability. Recall definitions of derivatives and related notation. The *upper right Dini derivative* of a function $f : I \to \mathbb{R}$ at point x is denoted and defined by $\overline{f}^+(x) = \limsup_{y \to x+} \frac{f(y)-f(x)}{y-x}$. The other three Dini derivatives $\underline{f}^+(x)$, $\overline{f}^-(x)$ and $\underline{f}^-(x)$ are defined likewise. If the four Dini derivatives at x equal, the common value is of course the derivative $f'(x)$. If the two right Dini derivatives at x are equal, the common value is called the *right derivative* and denoted $f^+(x)$; and likewise for the left side. The set of points where the derivative of f exists (infinite values are allowed) is denoted $\mathcal{D}(f)$ or just $\mathcal{D}$.

A point $x \in I$ is called a *knot point of f* if $\overline{f}^-(x) = \overline{f}^+(x) = \infty$ and $\underline{f}^-(x) = \underline{f}^+(x) = -\infty$. The set of knot points of f is denoted $\mathcal{K}(f)$ or just $\mathcal{K}$.

The *approximate upper right Dini derivative* of f at point x is denoted and defined by

$$\overline{f}^+_{\mathsf{app}}(x) = \inf\Big\{t : \lim_{\delta \to 0+} \frac{1}{\delta} \mathscr{L}\big(\big\{y \in (x, x+\delta) : \tfrac{f(y)-f(x)}{y-x} \leqslant t\big\}\big) = 1\Big\}.$$

The other three approximate Dini derivatives $\underline{f}^+_{\mathsf{app}}(x)$, $\overline{f}^-_{\mathsf{app}}(x)$ and $\underline{f}^-_{\mathsf{app}}(x)$ are defined likewise, as well as the *approximate derivative* $f'_{\mathsf{app}}(x)$ and *right* and *left approximate derivatives.* The set of points where the approximate derivative of f exists is denoted $\mathcal{D}_{\mathsf{app}}(f)$ or just $\mathcal{D}_{\mathsf{app}}$. Approximate knot points are defined in the obvious way. The set of approximate knot points of f is denoted $\mathcal{K}_{\mathsf{app}}(f)$ or just $\mathcal{K}_{\mathsf{app}}$.

Lemma 3.1. *Let $f : I \to \mathbb{R}$ be continuous and $y \in I$.*

(i) *If y is not an $\mathcal{M}$-point, then $\overline{f}^+(y) = -\underline{f}^-(y) = \infty$, or $\overline{f}^-(y) = -\underline{f}^+(y) = \infty$.*

(ii) *If $\underline{f}^-(y) = -\infty$ and $\underline{f}^+_{\mathsf{app}}(y) = \infty$, then y is not an $\mathcal{M}$-point.*

Proof. (i) Let $y = f(y) = 0$. Since 0 is not an $\mathcal{M}$-point, it follows from (6) that there are sequences $x_n \nearrow 0$ and $z_n \searrow 0$ such that

$$|f(x_n)| \geqslant n\big(|f(x_n) - f(z_n)| + |x_n - z_n|\big). \tag{7}$$

In particular, $|f(x_n)| \geqslant n|x_n - z_n| \geqslant n|x_n|$. *Mutatis mutandis* we may assume that all $f(x_n)$'s have the same sign. Suppose that $f(x_n) > 0$ for all n; the other case is treated likewise. Hence $f(x_n) \geqslant n|x_n|$. Clearly $f(z_n) \geqslant f(x_n) - |f(z_n) - f(x_n)|$, and (7) yields $|f(z_n)-f(x_n)| \leqslant \frac{1}{n}f(x_n)$. Therefore $f(z_n) \geqslant f(x_n)(1-\frac{1}{n})$. Apply (7) again to get $f(z_n) \geqslant n(1 - \frac{1}{n})z_n = (n-1)z_n$. In summary, $f(x_n) \geqslant n|x_n|$ and $f(z_n) \geqslant (n-1)z_n$, which is enough for $\overline{f}^+(y) = -\underline{f}^-(y) = \infty$.

(ii) Suppose for contrary that y is an $\mathcal{M}$-point and assume without loss of generality that $y = f(y) = 0$. Let ε and c be such that (6) holds. Let $a = 4c$ and $\beta = 1/a$.

Since $\underline{f}^+_{\mathsf{app}}(0) > a$, there is $\delta < \varepsilon$ such that for all $s < \delta$

$$\mathscr{L}\big\{t \in (0,s) : \tfrac{f(t)}{t} > a\big\} > s(1-\beta). \tag{8}$$

Since $\underline{f}^-(0) < -a$, there is $s \in (0,\delta)$ such that $\frac{f(-s)}{s} \geqslant a$. Therefore there is $x \in [-s, 0)$ such that $f(x) = as$. Now use (8) to conclude that there is $t \in (s(1-\beta), s)$ such that $\frac{f(t)}{t} > a$, i.e. $f(t) > at$, and choose $z \in (0,t)$ such that $f(z) = at$. Clearly $x \in (-\varepsilon, 0)$ and $z \in (0, \varepsilon)$. However, $|f(x) - f(y)| = f(x) = as$, $|f(x) - f(z)| = as - at \leqslant as - as(1-\beta) = as\beta = s$ and $|z - x| \leqslant 2s$. Thus (6) yields $as \leqslant c(s + 2s)$, which is contradicted by $a = 4c$. □

Theorem 3.2. *If $f : I \to \mathbb{R}$ is continuous, then*

(i) *$\mathcal{D}(f) \subseteq \mathcal{M}(f)$,*

(ii) *there is a set $E \subseteq I$ such that $\mathscr{H}^1(\mathfrak{f}|E) = 0$ and $\mathcal{M}(f) \subseteq \mathcal{D}_{\mathsf{app}}(f) \cup \mathcal{K}_{\mathsf{app}}(f) \cup E$. In particular, almost every $\mathcal{M}$-point $x \notin \mathcal{D}_{\mathsf{app}}(f)$ is a knot point.*

Proof. (i) It follows from Lemma 3.1(i) that if x is not an $\mathcal{M}$-point, then there are two Dini derivatives at x that differ. Therefore $x \notin \mathcal{D}(f)$.

(ii) We employ the approximate derivative version of the famous Denjoy–Young–Saks Theorem due to Alberti, Csornyei, Laczkovich and Preiss [1] that strengthens the Denjoy–Khintchine Theorem:

If f is measurable, then there is a set $E \subseteq I$ such that $\mathscr{H}^1(\mathfrak{f}|E) = 0$ and for every point $x \notin E$ either $f'_{\mathsf{app}}(x)$ exists and is finite, or else all approximate Dini derivatives are infinite.

It follows that if $x \notin \mathcal{D}_{\mathsf{app}} \cup \mathcal{K}_{\mathsf{app}} \cup E$, then all possible configurations of the Dini derivatives obtain by reversing the x- or y-axis from the following two cases:

- $\underline{f}^-_{\mathsf{app}}(x) = \overline{f}^-_{\mathsf{app}}(x) = -\infty$, $\underline{f}^+_{\mathsf{app}}(x) = \overline{f}^+_{\mathsf{app}}(x) = +\infty$,

- $\underline{f}^-_{\mathsf{app}}(x) = -\infty$, $\overline{f}^-_{\mathsf{app}}(x) = \underline{f}^+_{\mathsf{app}}(x) = \overline{f}^+_{\mathsf{app}}(x) = +\infty$.

Both satisfy the hypotheses of Lemma 3.1(ii). Hence x is not an $\mathcal{M}$-point.

The second statement of (ii) follows from the obvious inclusion $\mathcal{K}_{\mathsf{app}} \subseteq \mathcal{K}$. □

The last goal of this section is to derive from Theorem 3.2(ii) that the set of points where the graph is monotone has σ-finite length and in particular Hausdorff dimension 1. We need the following folklore covering lemma. Instead of reference we provide a brief proof.

Lemma 3.3. *Let X be a metric space and $E \subseteq X$. Let $\{r_x : x \in E\}$ be a set of positive reals such that $\sup_{x\in E} r_x < \infty$. Then for each $\delta > 2$ there is a set $D \subseteq E$ such that the family $\{B(x, r_x) : x \in D\}$ is disjoint and the family $\{B(x, \delta r_x) : x \in D\}$ covers E.*

Proof. We may assume that $r_x < 1$ for all $x \in E$. Define recursively

$$A_n = \{x \in E : (\delta-1)^{-n+1} > r_x \geqslant (\delta-1)^{-n}\},$$
$$B_n = \{x \in A_n : B(x, r_x) \cap \bigcup_{i<n} \bigcup \mathcal{A}_i = \emptyset\}$$

and let $\mathcal{A}_n \subseteq \{B(x, r_x) : x \in B_n\}$ be a maximal disjoint family. It is easy to check that $D = \{x \in E : B(x, r_x) \in \bigcup_{n=0}^{\infty} \mathcal{A}_n\}$ is the required set. □

Lemma 3.4. *Suppose that $f : I \to \mathbb{R}$ is continuous.*

$$E^+ = \{x \in [0,1] : \exists x_n \downarrow x \textit{ such that } f(x_n) = f(x)\}.$$

If $A \subseteq E^+ \cap \mathcal{M}_c(f)$, then $\mathscr{H}^1(\mathfrak{f}|A) \leqslant 4c\mathscr{L}(A)$.

Proof. Let $A \subseteq E^+$. For $\varepsilon > 0$ let A_ε be the set of points $y \in A$ satisfying (5). Fix $\eta \in (0, \varepsilon)$. Let $\{U_i : i \in \mathbb{N}\}$ be a cover of A_ε by open intervals of length $< \eta$ such that $\sum_i \operatorname{diam}(U_i) < \mathscr{L}(A_\varepsilon) + \eta$.

Now fix $i \in \mathbb{N}$. For each $x \in A_\varepsilon \cap U_i$ choose $z_x > x$, $z_x \in U_i$ such that $f(z_x) = f(x)$. If $y \in [x, z_x] \cap A_\varepsilon$, then, since $|z_x - x| \leqslant \eta < \varepsilon$, condition (5) with $z - z_x$ is met. Hence

$$|\psi(y) - \psi(x)| \leqslant c|\psi(z_x) - \psi(x)| = c|z_x - x|.$$

It follows that letting $r_x = c(z_x - x)$ we have

$$\mathfrak{f}|([x, z_x] \cap A_\varepsilon) \subseteq B(\psi(x), r_x).$$

The family $\mathcal{B} = \{B(\psi(x), r_x) : x \in A_\varepsilon \cap U_i\}$ thus covers $A_\varepsilon \cap U_i$. Apply Lemma 3.3: for any $\delta > 2$ there is a set $A' \subseteq A_\varepsilon \cap U_i$ such that the family $\{B(\psi(x), r_x) : x \in A'\}$ is pairwise disjoint and $\mathfrak{f}|(A_\varepsilon \cap U_i) \subseteq \bigcup_{x\in A'} B(\psi(x), \delta r_x)$. We claim that the family of intervals $\{[x, z_x] : x \in A'\}$ is pairwise disjoint. Indeed, if $x, y \in A'$ were such that $[x, z_x] \cap [y, z_y] \neq \emptyset$, then either $x \in [y, z_y] \cap A_\varepsilon$ or $y \in [x, z_x] \cap A_\varepsilon$. Suppose the former. Then $\psi(x) \in \psi([y, z_y] \cap A_\varepsilon) \subseteq B(\psi(y), r_y)$. Therefore the balls $B(\psi(x), r_x)$ and $B(\psi(y), r_y)$ would not be disjoint.

It follows that $\sum_{x\in A'} |x - z_x| \leqslant \operatorname{diam}(U_i)$, which yields

$$\sum_{x\in A'} \operatorname{diam}(B(\psi(x), \delta r_x)) \leqslant 2\delta \sum_{x\in A'} r_x \leqslant 2\delta c \sum_{x\in A'} |x - z_x| \leqslant 2\delta c \operatorname{diam}(U_i).$$

Moreover, the diameters of $B(\psi(x), \delta r_x)$ do not exceed $2\delta c\eta$. Consequently

$$\mathscr{H}^1_{2\delta c\eta}(\mathfrak{f}|(A_\varepsilon \cap U_i) \leqslant 2\delta c \operatorname{diam}(U_i).$$

Summing over i yields

$$\mathscr{H}^1_{2\delta c\eta}(\mathfrak{f}|A_\varepsilon) \leqslant 2\delta c \sum_{i\in\mathbb{N}} \operatorname{diam}(U_i) \leqslant 2\delta c(\mathscr{L}(A_\varepsilon)+\eta).$$

$\mathscr{H}^1(\mathfrak{f}|A_\varepsilon) \leqslant 4c\mathscr{L}(A_\varepsilon)$ now follows on letting $\eta \to 0$ and $\delta \to 2$, and $\mathscr{H}^1(\mathfrak{f}|A) \leqslant 4c\mathscr{L}(A)$ on letting $\varepsilon \to 0$. □

Theorem 3.5. *If $f : I \to \mathbb{R}$ is continuous, then $\mathscr{H}^1(\mathsf{Mon}(\mathfrak{f}))$ is σ-finite. In particular,* $\dim_{\mathsf{H}} \mathsf{Mon}(\mathfrak{f}) = 1$.

Proof. Let $A = \{\psi(x) : x \notin \mathcal{K}\} \subseteq \mathfrak{f}$. It is clear that for any point $a \in A$ there is a (one-sided) cone V with vertex at a and a ball B centered at a such that the only point of $\mathfrak{f}$ within $V \cap B$ is a. Such a set is by [7, Lemma 15.13] *rectifiable*, i.e. covered by countably many Lipschitz curves. In particular, A has σ-finite length. In view of Theorem 3.2(ii) it thus remains to show that $\mathscr{H}^1(\mathfrak{f}|\,\mathcal{M}\cap\mathcal{K})$ is σ-finite. But that follows at once from Lemma 3.4, since any knot point belongs to the set E^+. □

Corollary 3.6. *If $f : I \to \mathbb{R}$ is continuous with a monotone graph, then $\mathscr{H}^1(\mathfrak{f})$ is σ-finite. In particular,* $\dim_{\mathsf{H}} \mathfrak{f} = 1$.

4. Functions with a monotone or σ-monotone graph

In this section we investigate differentiability of continuous functions with monotone or σ-monotone graph.

Our first theorem claims that if $\mathfrak{f}$ is monotone, then the approximate derivatives coincide with derivatives.

Proposition 4.1. *If $f : I \to \mathbb{R}$ is continuous with a monotone graph, then $\overline{f}^+_{\mathsf{app}}(x) = \overline{f}^+(x)$ for all $x \in I$. A similar statement holds for all Dini derivatives.*

Proof. Assume for contrary that there is x such that $\overline{f}^+_{\mathsf{app}}(x) < \overline{f}^+(x)$. *Mutatis mutandis* we may suppose that $x = f(x) = 0$. Choosing suitable constants α, β the function $g(y) = \alpha f(y) - \beta y$ satisfies $\overline{g}^+_{\mathsf{app}}(0) < 0$ and $\overline{g}^+(0) > 1$. Since the graph of g is an affine transform of the graph of f and an affine transform is bi-Lipschitz, the graph of g is by [9, Proposition 2.2] a monotone set. Therefore there is $c \geqslant 1$ such that g satisfies condition P_c.

Since $\overline{g}^+_{\mathsf{app}}(0) < 0$, the set $M = \{y \in I : g(y) < 0\}$ satisfies

$$\forall\varepsilon > 0\ \exists\delta_0\ \forall\delta \in (0,\delta_0) \quad \mathscr{L}\big([0,\delta]\setminus M\big) < \varepsilon\delta \tag{9}$$

Let $\varepsilon = \frac{1}{2c}$ and let δ_0 satisfy (9). Since $\overline{g}^+(0) > 1$, there is $t \in (0, \delta_0/2)$ such that $g(t) > t$. Put $\delta = 2t$. Since $\varepsilon \leqslant \frac{1}{2}$ and $\delta < \delta_0$, (9) yields $M \cap (0,t) \neq \emptyset$ and $M \cap (t, 2t) \neq \emptyset$. Therefore the numbers $a = \sup(0,t)\cap M$, $b = \inf(t,2t)\cap M$ satisfy $0 < a < t < b < \delta$. Also $g(a) = g(b) = 0$ by the continuity of g. Obviously $[a,b] \cap M = \emptyset$. Hence (9) yields $|b-a| < \varepsilon\delta = t/c$. Therefore $c|b-a| < t < g(t) = |g(t) - g(a)|$ and thus condition P_c fails: the desired contradiction. □

This theorem together with Theorem 3.2 yield

Corollary 4.2. *If $f : I \to \mathbb{R}$ is continuous function with a monotone graph, then there is a set $E \subseteq I$ such that $\mathscr{H}^1(\mathfrak{f}|E) = 0$ and $I = \mathcal{D}(f) \cup \mathcal{K}(f) \cup E$. In particular, almost all points $x \notin \mathcal{D}(f)$ are knot points.*

Corollary 4.3. *If $f : I \to \mathbb{R}$ is a continuous function with a c-monotone graph, then $\mathscr{H}^1(\mathfrak{f}|A) \leqslant 4c\mathscr{L}(A)$ for every $A \subseteq I \setminus \mathcal{D}(f)$. In particular, $\mathscr{H}^1(\mathfrak{f}|I \setminus \mathcal{D}(f)) < \infty$.*

In the next section we present an example of function with a monotone graph that has derivative almost nowhere, hence the set of knot points is rather large. However, a σ-monotone graph yields a dense set of differentiability.

Lemma 4.4. *Let $f : I \to \mathbb{R}$ be a continuous function with a σ-monotone graph. Then $\mathscr{L}(f[\mathcal{D}(f) \cap J]) > 0$ for each interval $J \subseteq I$ where f is not constant.*

Proof. Using Lemma 2.6 it is clearly enough to prove that if $f : [0,1] \to \mathbb{R}$, $f(0) \neq f(1)$, and $\mathfrak{f}$ is monotone, then $\mathscr{L}(f[\mathcal{D}]) > 0$. Suppose the contrary: $\mathscr{L}(f[\mathcal{D}]) = 0$. Let $0 \leqslant x < y \leqslant 1$. Use the assumption and Corollary 4.3 to estimate $|f(x) - f(y)|$:

$$\begin{aligned}|f(x) - f(y)| &= \mathscr{L}([f(x), f(y)]) \leqslant \mathscr{L}(f[x,y]) \\ &\leqslant \mathscr{L}(f[[x,y] \setminus \mathcal{D}]) + \mathscr{L}(f[[x,y] \cap \mathcal{D}]) \\ &\leqslant 4c\,\mathscr{L}([x,y] \setminus \mathcal{D}) + \mathscr{L}(f[\mathcal{D}]) \leqslant 4c\,\mathscr{L}([x,y]) = 4c|x-y|.\end{aligned}$$

It follows that f is a Lipschitz function. Therefore it is differentiable almost everywhere. Use Corollary 4.3 again to get $\mathscr{L}(f[[0,1] \setminus \mathcal{D}]) \leqslant \mathscr{H}^1(\mathfrak{f}|[0,1] \setminus \mathcal{D}) = 0$. Thus

$$\mathscr{L}(f[\mathcal{D}]) \geqslant \mathscr{L}(f[0,1]) - \mathscr{L}(f[I \setminus \mathcal{D}]) = \mathscr{L}(f[0,1]) \geqslant |f(0) - f(1)| > 0,$$

which contradicts the assumption. □

Let us call a set *perfectly dense* if its intersection with any nonempty open set contains a perfect set.

Theorem 4.5. *If $f : I \to \mathbb{R}$ is a continuous function with a σ-monotone graph, then f is differentiable at a perfectly dense set.*

Proof. If f is constant on I, there is nothing to prove. Otherwise Lemma 4.4 yields $\mathscr{L}(f[\mathcal{D}(f) \cap I]) > 0$. Therefore $\mathcal{D}(f) \cap I$ is an uncountable Borel set. Thus it contains, by the Perfect Set Theorem, a perfect set. □

Corollary 4.6. *If $f : I \to \mathbb{R}$ is a continuous function, then* $\operatorname{int} \mathcal{M}(f) \subseteq \overline{\mathcal{D}(f)}$.

We now present several examples illustrating that one cannot prove much more than Theorem 3.2 and Corollary 4.6 about differentiability properties of $\mathcal{M}$-points. The first two examples are nowhere differentiable functions. Note that by the above corollary and Proposition 2.8 such a function must have a small set of $\mathcal{M}$-points:

Corollary 4.7. *If $f : I \to \mathbb{R}$ is a continuous, nowhere differentiable function, then $\mathcal{M}(f)$ is meager, i.e. every monotone set $M \subseteq \mathfrak{f}$ is nowhere dense.*

For $y \in \mathbb{R}$ let $\|y\| = \operatorname{dist}(y, \mathbb{Z})$.

Proposition 4.8. *The function $f(y) = \sum_{k=0}^{\infty} 2^{-k} \|2^{k^2} y\|$ is continuous and has no $\mathcal{M}$-points. Therefore every monotone subset of $\mathfrak{f}$ is meager and f is nowhere differentiable.*

Proof. Continuity of f is easy. Fix $y \in \mathbb{R}$, $\varepsilon > 0$ and $c > 0$. We want to disprove condition (6). Let $n \in \mathbb{N}$ be large enough (this will be specified later). It is easy to check that there is $i \in \{1,3\}$ such that

$$\big|\|2^{n^2} y\| - \|2^{n^2} y - \tfrac{i}{4}\|\big| \geqslant \tfrac{1}{4}. \tag{10}$$

Set $x = y - \frac{i}{4}2^{-n^2}$, $z = x + 2^{-n^2}$. Clearly if n is large enough, then $x \in (y - \varepsilon)$ and $z \in (y, y + \varepsilon)$. We show that x, z witness failure of (6).

(a) If $k > n$, then $-\frac{i}{4}2^{k^2-n^2}$ is an integer. Therefore $\|2^{k^2}x\| = \|2^{k^2}y\| = \|2^{k^2}z\|$.

(b) Since $2^{n^2}z - 2^{n^2}x = 1$, we have $\|2^{n^2}x\| = \|2^{n^2}z\|$.

(c) If $k < n$ and $t = y$ or $t = z$, then $\big|\|2^{k^2}t\| - \|2^{k^2}x\|\big| \leqslant 2^{k^2}|t-x| \leqslant 2^{k^2-n^2}$. Thus

$$\sum_{k<n} 2^{-k}\big|\|2^{k^2}t\| - \|2^{k^2}x\|\big| \leqslant \sum_{k<n} 2^{-k}2^{k^2-n^2} \leqslant n2^{2-3n}.$$

It follows that

$$|f(z) - f(x)| \overset{(a,b)}{\leqslant} \sum_{k<n} 2^{-k}\big|\|2^{k^2}z\| - \|2^{k^2}x\|\big| \overset{(c)}{\leqslant} n2^{2-3n},$$

$$|f(y) - f(x)| \overset{(a)}{\geqslant} 2^{-n}\big|\|2^{n^2}y\| - \|2^{n^2}x\|\big| - \sum_{k<n} 2^{-k}\big|\|2^{k^2}y\| - \|2^{k^2}x\|\big|$$

$$\overset{(10,c)}{\geqslant} 2^{-n-2} - n2^{2-3n}.$$

Combine these estimates to get

$$\frac{|f(y) - f(x)|}{|f(z) - f(x)| + |z - x|} \geqslant \frac{2^{-n-2} - n2^{2-3n}}{n2^{2-3n} + 2^{-n^2}}.$$

With a proper choice of n the rightmost expression is as large as needed, in particular greater than c. Therefore (6) fails. □

Example 4.9. Let f be the above function. Define $g(x) = (x - \frac{1}{2})\sin\frac{1}{2x-1}f(x)$. It is easy to derive from the above that g has no $\mathcal{M}$-points except $x = \frac{1}{2}$. Straightforward calculation of Dini derivatives at $x = \frac{1}{2}$ gives $\overline{g}^+(\frac{1}{2}) = \overline{g}^-(\frac{1}{2}) = \frac{1}{2}$ and $\underline{g}^+(\frac{1}{2}) = \underline{g}^-(\frac{1}{2}) = -\frac{1}{2}$. Therefore $\frac{1}{2}$ is an $\mathcal{M}$-point (actually an $\mathcal{M}_1$-point). It also follows from Theorem 3.2 that g is differentiable at no point. In particular $\mathcal{D}(g)$ is not dense in $\mathcal{M}(g)$.

Example 4.10. Let $T(x) = \sum_{n=0}^{\infty} 2^{-n}\|2^n x\|$ be the *Takagi function*. The following facts can be found in [2]. T does not possess a finite one-sided derivative at any point. However, if x is a dyadic rational, then $T^+(x) = +\infty$ and $T^-(x) = -\infty$. Also $T'(x) = +\infty$ at a dense set.

It follows that the sets $\mathcal{D}(T)$, $\mathcal{M}_1(T)$, $\mathcal{M}(T)$ as well as their complements are dense.

5. A non-differentiable function with a monotone graph

In this section we provide an example of a continuous, almost nowhere differentiable function on $[0, 1]$ with a monotone graph. Note that it follows from the above results that such a function necessarily have the following properties:

- Every point of $[0, 1]$ is an $\mathcal{M}$-point,
- the function is almost nowhere approximately differentiable,
- almost all points are knot points (actually approximate knot points),
- the function has a derivative at a perfectly dense set.

Theorem 5.1. *For any $c > 1$ there is a continuous, almost nowhere differentiable function $f : [0, 1] \to \mathbb{R}$ with a symmetrically c-monotone graph.*

The proof is a bit involved. The function f we construct satisfies condition P_1. That is enough, because given any $c > 1$, the function $x \mapsto (c-1)f(x)$ satisfies obviously condition P_{c-1} and is thus by Lemma 2.3 c-monotone. We first construct the function and then prove its properties in a sequence of lemmas.

Construction of the function. The function f is defined as a limit of a sequence of piecewise linear continuous functions $f_n : [0,1] \to [0,1]$ that we now define.

We recursively specify finite sets $\mathcal{A}_n = \{a_n^k : k = 0, \dots, r_n\} \subseteq [0,1]$ such that

$$0 = a_n^0 < a_n^1 < \dots < a_n^{r_n} = 1$$

and values of f_n at each point of $\mathcal{A}_n$. The function f_n is then defined as the unique function that is linear between consecutive points of $\mathcal{A}_n$.

For $n = 0$ put $\mathcal{A}_0 = \{0, 1\}$ and $f_0(0) = f_0(1) = 0$.

The induction step: Suppose f_n and $\mathcal{A}_n = \{a_n^k : k = 0, \dots, r_n\}$ are constructed. Let $k < r_n$ be arbitrary. For $l = 0, \dots, 5$ set $x_l = \frac{l}{5} a_n^{k+1} + (1 - \frac{l}{5}) a_n^k$.
If $f_n(a_n^k) = f_n(a_n^{k+1})$, set $A_n^k = \{x_l : l = 1, \dots, 5\}$ and

$$\begin{aligned} f_{n+1}(x_0) = f_{n+1}(x_1) = f_{n+1}(x_4) = f_{n+1}(x_5) = f_n(a_n^k), \\ f_{n+1}(x_2) = f_{n+1}(x_3) = f_n(a_n^k) + \tfrac{1}{6}|a_n^{k+1} - a_n^k|. \end{aligned}$$

If $f_n(a_n^k) \neq f_n(a_n^{k+1})$, set $A_n^k = \{x_0, x_1, x_4, x_5\}$ and

$$\begin{aligned} f_{n+1}(x_0) &= f_n(a_n^k), \\ f_{n+1}(x_5) &= f_n(a_n^{k+1}), \\ f_{n+1}(x_1) &= f_{n+1}(x_4) = \tfrac{1}{2}\big(f_n(a_n^k) + f_n(a_n^{k+1})\big) \end{aligned}$$

and let $\mathcal{A}_{n+1} = \bigcup_{k=0}^{r_n - 1} A_n^k$.

Lemma 5.2. *Let $n \in \mathbb{N}$ and $k < r_n$. Then the following holds:*

(i) *If $k > 0$, then $|a_n^{k-1} - a_n^k| \leqslant 3|a_n^{k+1} - a_n^k| \leqslant 9|a_n^{k-1} - a_n^k|$,*
(ii) $|a_{2n}^{k+1} - a_{2n}^k| \leqslant \left(\frac{3}{25}\right)^n$,
(iii) $|a_{2n+1}^{k+1} - a_{2n+1}^k| \leqslant \frac{1}{5}\left(\frac{3}{25}\right)^n$,
(iv) $|a_n^{k+1} - a_n^k| \geqslant \left(\frac{1}{5}\right)^n$,
(v) $f_i(a_n^k) = f_n(a_n^k)$ *if $i \geqslant n$,*
(vi) $|f_n(a_n^{k+1}) - f_n(a_n^k)| \leqslant \frac{1}{6}\left(\frac{1}{2}\right)^n$,
(vii) $\frac{|f_n(a_n^k) - f_n(a_n^{k+1})|}{|a_n^k - a_n^{k+1}|} = 0$ *or* $\frac{|f_n(a_n^k) - f_n(a_n^{k+1})|}{|a_n^k - a_n^{k+1}|} \geqslant \frac{5}{6}$,
(viii) *if $i > 0$ and $x \in [a_n^k, a_n^{k+1}]$, then*

$$\begin{aligned} \min\big(f_n(a_n^k), f_n(a_n^{k+1})\big) &\leqslant f_{n+i}(x) \\ &\leqslant \max\big(f_n(a_n^k), f_n(a_n^{k+1})\big) + |a_n^{k+1} - a_n^k| \sum_{j=1}^{i} 6^{-j}, \end{aligned}$$

(ix) *if $i > 0$, $x \in (a_n^k, a_n^{k+1})$ and $f_n(a_n^k) \neq f_n(a_n^{k+1})$, then*

$$f_{n+i}(x) < \max\big(f_n(a_n^k), f_n(a_n^{k+1})\big),$$

(x) *f_n is continuous and $f_n(x) \in [0,1]$ for all $x \in [0,1]$.*

Proof. (i)–(v) follows right away from the construction of functions f_n. (vi) can be easily proved from the construction using (ii) and (iii).

(vii): Case $n = 0$ is trivial. Assume (vii) holds for some $n \geqslant 0$ and we prove it for $n+1$. Let $i < r_{n+1}$ be arbitrary. There exists $k < r_n$ such that $a^i_{n+1}, a^{i+1}_{n+1} \in [a^k_n, a^{k+1}_n]$.

- If $\dfrac{|f_n(a^k_n) - f_n(a^{k+1}_n)|}{|a^k_n - a^{k+1}_n|} = 0$, then

$$\frac{|f_{n+1}(a^i_{n+1}) - f_{n+1}(a^{i+1}_{n+1})|}{|a^i_{n+1} - a^{i+1}_{n+1}|} = 0 \quad \vee \quad \frac{|f_{n+1}(a^i_{n+1}) - f_{n+1}(a^{i+1}_{n+1})|}{|a^i_{n+1} - a^{i+1}_{n+1}|} = \frac{5}{6}.$$

- If $\dfrac{|f_n(a^k_n) - f_n(a^{k+1}_n)|}{|a^k_n - a^{k+1}_n|} \neq 0$, then

$$\frac{|f_{n+1}(a^i_{n+1}) - f_{n+1}(a^{i+1}_{n+1})|}{|a^i_{n+1} - a^{i+1}_{n+1}|} = 0$$

or

$$\frac{|f_{n+1}(a^i_{n+1}) - f_{n+1}(a^{i+1}_{n+1})|}{|a^i_{n+1} - a^{i+1}_{n+1}|} = \frac{5}{2} \frac{|f_n(a^k_n) - f_n(a^{k+1}_n)|}{|a^k_n - a^{k+1}_n|} \geqslant \frac{5}{6}.$$

(viii): The first inequality is obvious. The second inequality is proved by induction over i. Case $i = 1$ easily follows from the construction. Suppose that this statement is true for $i = p$. We show that it is also true for $i = p+1$. Find $l < r_{n+1}$ such that $x \in [a^l_{n+1}, a^{l+1}_{n+1}]$ and use the induction hypothesis to compare $f_n(a^k_n), f_n(a^{k+1}_n)$ with $f_{n+1}(a^l_{n+1}), f_{n+1}(a^{l+1}_{n+1})$ (which is the case $i = 1$) and $f_{n+1}(a^l_{n+1}), f_{n+1}(a^{l+1}_{n+1})$ with $f_{n+p+1}(x)$ (which is the case $i = p$).

(ix): This is similar to (viii). Case $i = 1$ easily follows from the construction. Proceed by induction: Assume that the statement is true for $i = p$. We show that it is also true for $i = p+1$. Find $l < r_{n+1}$ such that $x \in [a^l_{n+1}, a^{l+1}_{n+1}]$.

If $f(a^l_{n+1}) \neq f(a^{l+1}_{n+1})$ then use the statement to compare $f_n(a^k_n), f_n(a^{k+1}_n)$ with $f_{n+1}(a^l_{n+1}), f_{n+1}(a^{l+1}_{n+1})$ (which is the case $i = 1$) and $f_{n+1}(a^l_{n+1}), f_{n+1}(a^{l+1}_{n+1})$ with $f_{n+p+1}(x)$ (which is the case $i = p$).

If $f(a^l_{n+1}) = f(a^{l+1}_{n+1})$ then by the construction and (vii) we have

$$\begin{aligned}\frac{25}{36}|a^l_{n+1} - a^{l+1}_{n+1}| &= \frac{5}{12}|a^{k+1}_n - a^k_n| \\ &\leqslant \frac{1}{2}\max\big(f_n(a^k_n), f_n(a^{k+1}_n)\big) - \min\big(f_n(a^k_n), f_n(a^{k+1}_n)\big) \\ &= \max\big(f_n(a^k_n), f_n(a^{k+1}_n)\big) - f_{n+1}(a^l_{n+1}).\end{aligned}$$

By (viii) we have

$$f_{n+p+1}(x)) \leqslant f_{n+1}(a^l_{n+1}) + \frac{1}{5}|a^l_{n+1} - a^{l+1}_{n+1}|.$$

Thus $f_{n+p+1}(x) < \max\big(f_n(a^k_n), f_n(a^{k+1}_n\big)$.

(x) can be easily proved from the construction using (viii). $\square$

Lemma 5.3. *The functions f_i satisfy condition P_1 for every i.*

Proof. Let $x < y \in [0,1]$ and i be arbitrary such that $f_i(x) = f_i(y)$. We show that

$$\max_{x \leqslant t \leqslant y} |f_i(x) - f_i(t)| \leqslant |x - y|. \tag{11}$$

Since f_i is piecewise linear, level sets are finite. We may thus assume that there is no $w \in (x,y)$ such that $f_i(w) = f_i(x)$. Let $z \in (x,y)$ be such that $\max_{x\leqslant t\leqslant y}|f_i(x) - f_i(t)| = |f_i(x) - f_i(z)|$. The case $f_i(x) = f_i(z)$ is trivial. We may thus assume that either $f_i(x) < f_i(z)$ or $f_i(x) > f_i(z)$.

Assume first $f_i(x) < f_i(z)$. By the construction of f_i we can find minimal $n \leqslant i$ and $k < r_n - 1$ such that $z \in (a_n^k, a_n^{k+1}) \subset (x,y)$ and $f_i(a_n^k) = f_i(a_n^{k+1}) \in (f_i(x), f_i(z)]$. By Lemma 5.2(v) we have $f_n(a_n^k) = f_n(a_n^{k+1}) = f_i(a_n^k)$. We show that $f_n(a_n^{k-1}) < f_n(a_n^k)$.

Suppose the contrary: $f_n(a_n^{k-1}) > f_n(a_n^k)$. By Lemma 5.2(viii) we have $f_i(t) \geqslant f_n(a_n^k)$ for all $t \in (a_n^{k-1}, a_n^k)$. So, $x \notin [a_n^{k-1}, a_n^k]$. Thus $a_n^{k-1} \in (x,y)$. By Lemma 5.2(vii) and (i) we have

$$f_n(a_n^{k-1}) \geqslant f_n(a_n^k) + \frac{5}{6}|a_n^{k-1} - a_n^k| \geqslant f_n(a_n^k) + \frac{5}{18}|a_n^k - a_n^{k+1}|.$$

By Lemma 5.2(viii) we have $f(z) \leqslant f_n(a_n^k) + \frac{1}{5}|a_n^k - a_n^{k+1}|$. Thus $f_i(z) < f_i(a_n^{k-1})$, which contradicts that $f_i(t) \leqslant f_i(z)$ for all $t \in (x,y)$.

Similarly, we have $f_n(a_n^{k+1}) > f_n(a_n^{k+2})$.

By the construction we have that there exists $l < r_{n-1}$ such that $a_{n-1}^l = a_n^{k-2}$, $a_{n-1}^{l+1} = a_n^{k+3}$) and $f_n(a_{n-1}^l) = f_n(a_{n-1}^{l+1})$. By the minimality of n we have $(x,y) \not\supset (a_{n-1}^l, a_{n-1}^{l+1})$. Thus $x \in [a_{n-1}^l, a_{n-1}^{l+1}]$ or $y \in [a_{n-1}^l, a_{n-1}^{l+1}]$. We can assume $x \in [a_{n-1}^l, a_{n-1}^{l+1}]$. By Lemma 5.2(viii) and $x, z \in [a_{n-1}^l, a_{n-1}^{l+1}]$ we have

$$\max_{x\leqslant t\leqslant y}|f_i(x) - f_i(t)| = |f_i(x) - f_i(z)| \leqslant \frac{1}{5}|a_{n-1}^l - a_{n-1}^{l+1}| = |a_n^k - a_n^{k+1}| \leqslant |x-y|.$$

Now assume $f_i(x) > f_i(z)$. By the construction of f_i and Lemma 5.2(viii) we can find minimal $n \leqslant i$ and $k < r_n - 1$ such that $a_n^k, a_n^{k+1} \in (x,y)$ and $f_i(a_n^k) = f_i(a_n^{k+1}) = f_i(z)$. By Lemma 5.2(v) we have $f_n(a_n^k) = f_n(a_n^{k+1}) = f_i(z)$. Since $f_i(t) \geqslant f_n(a_n^k)$ for all $t \in (x,y)$ and Lemma 5.2(ix) we have $f_n(a_n^{k-1}), f_n(a_n^{k+2}) > f_n(a_n^k)$. By the construction there is no $l < r_{n-1}$ such that $(a_{n-1}^l, a_{n-1}^{l+1}) \supset (a_n^{k-1}, a_n^{k+2})$. Thus there are two possible cases:

(i) There exists $l < r_{n-1}$ such that $a_{n-1}^l = a_n^{k+1}$ and $f(a_{n-1}^{l-1}) = f(a_{n-1}^l)$.
(ii) There exists $l < r_{n-1}$ such that $a_{n-1}^l = a_n^k$ and $f(a_{n-1}^{l+1}) = f(a_{n-1}^l)$.

We prove only (i), as the case (ii) is similar. By minimality of n we have $x \in [a_{n-1}^{l-1}, a_{n-1}^l]$. Lemma 5.2(viii) yields

$$\begin{aligned}\max_{x\leqslant t\leqslant y}|f_i(x) - f_i(t)| &= |f_i(x) - f_n(a_{n-1}^l)| \\ &\leqslant \frac{1}{5}|a_{n-1}^{l-1} - a_{n-1}^l| = |a_n^k - a_n^{k+1}| < |x-y|. \qquad \square\end{aligned}$$

Lemma 5.4. *The sequence $\{f_n\}$ is uniformly Cauchy.*

Proof. Fix $n \in \mathbb{N}$ and let $k < r_n$. If $a_n^k \leqslant x \leqslant a_n^{k+1}$, then by construction of f_{n+1}

$$|f_{n+1}(x) - f_n(x)| \leqslant \frac{1}{6}|a_n^{k+1} - a_n^k| + \frac{3}{10}|f_n(a_n^{k+1}) - f_n(a_n^k)|.$$

Estimate $|a_n^{k+1} - a_n^k|$ using Lemma 5.2(ii) and (iii) and $|f_n(a_n^{k+1}) - f_n(a_n^k)|$ using Lemma 5.2(vi) and combine the estimates to get

$$\frac{1}{6}|a_n^{k+1} - a_n^k| + \frac{3}{10}|f_n(a_n^{k+1}) - f_n(a_n^k)| \leqslant 2^{-n}.$$

Thus $|f_{n+1}(x)-f_n(x)|\leqslant 2^{-n}$, irrespective of the particular k. Since the intervals $[a_n^k,a_n^{k+1}]$, $k<r_n$, cover $[0,1]$, we have $|f_{n+1}(x)-f_n(x)|\leqslant 2^{-n}$ for all x, which is clearly enough. $\square$

This lemma lets us define $f=\lim_{n\to\infty} f_n$. We claim that thus defined f is the required function. It is of course continuous. By Lemma 5.3 the functions f_n satisfy condition P_1. It is easy to check that since f is a limit of f_n's, it satisfies P_1 as well. We thus have

Proposition 5.5. *f is a continuous function satisfying P_1.*

It remains to show that f fails to have a derivative at almost all points. For $n\in\mathbb{N}$ define

$$\begin{aligned}
A_n &= \overline{\{x\in[0,1]: f_n'(x)=0\}},\\
B_n &= \overline{[0,1]\setminus A_n},\\
B &= \bigcup_{i\in\mathbb{N}}\bigcap_{n\geqslant i} B_n,\\
D &= \left\{x\in[0,1];\ \forall n\in\mathbb{N}: x\cdot 5^n \pmod 1 \notin \left(\tfrac15,\tfrac45\right)\right\}.
\end{aligned}$$

Lemma 5.6. $\mathscr{L}(B)=0$.

Proof. For every n set $\mathcal{M}_n=\left\{i<r_n: f_n'\left(\frac{a_n^i+a_n^{i+1}}{2}\right)\neq 0\right\}$. It is easy to see that

$$B=\bigcup_{n\in\mathbb{N}}\bigcup_{i\in\mathcal{M}_n}\{x\cdot|a_n^{i+1}-a_n^i|+a_n^i : x\in D\}$$

and since obviously $\mathscr{L}(D)=0$, we are done. $\square$

Proposition 5.7. (i) *If $x\notin B$, then $f^+(x)$ and $f^-(x)$ do not exist.*
(ii) *If $x\in B$, then at least one of the Dini derivatives of f at x is infinite.*

Proof. (i): Let $x\notin B$. We show that $f^+(x)$ does not exists, the proof for $f^-(x)$ is similar. Let $\delta>0$ be arbitrary. Since $x\notin B$ there exist $n\in\mathbb{N}$ and $k_i<r_{n+i}$ such that $x\in(a_n^{k_0},a_n^{k_0+1})$, $f_n(a_n^{k_0})=f_n(a_n^{k_0+1})$, $|a_n^{k_0+1}-a_n^{k_0}|<\delta$ and $a_{n+i}^{k_i}=a_n^{k_0}$ for all $i\in\mathbb{N}$. By the construction of functions f_n we have $f_{n+i}(a_{n+i}^{k_i})=f_{n+i}(a_{n+i}^{k_i+1})$. Since $x\neq a_n^{k_0}$ there exists $i\in\mathbb{N}$ such that $x\notin[a_{n+i}^{k_i},a_{n+i}^{k_i+1}]$. We may assume that $x\notin(a_{n+1}^{k_1},a_{n+1}^{k_1+1})$. By Lemma 5.2(v) and (viii) we have

$$\begin{aligned}
\left|\tfrac{f(a_{n+2}^{k_2+3})-f(x)}{a_{n+2}^{k_2+3}-x}-\tfrac{f(a_{n+2}^{k_2+4})-f(x)}{a_{n+2}^{k_2+4}-x}\right| &= \left|\tfrac{f_{n+2}(a_{n+2}^{k_2+3})-f(x)}{a_{n+2}^{k_2+3}-x}-\tfrac{f_{n+2}(a_{n+2}^{k_2+4})-f(x)}{a_{n+2}^{k_2+3}-x}\right|\\
&\geqslant \left|\tfrac{f_{n+2}(a_{n+2}^{k_2+3})-f(x)}{a_{n+2}^{k_2+3}-x}-\tfrac{f_{n+2}(a_{n+2}^{k_2+4})-f(x)}{a_{n+2}^{k_2+3}-x}\right|\\
&\geqslant \left|\tfrac{f_{n+2}(a_{n+2}^{k_2+3})-f_{n+2}(a_{n+2}^{k_2+4})}{|a_n^{k_0+1}-a_n^{k_0}|}\right| \geqslant \frac{1}{30}.
\end{aligned}$$

Thus, $f^+(x)$ does not exists.

(ii): Since $x\in B$ there exist $n\in\mathbb{N}$ and $k_i<r_{n+i},\ i\in\mathbb{N}$, such that

- $x\in[a_{n+i}^{k_i},a_{n+i}^{k_i+1}]$ for all $i\in\mathbb{N}$,
- $f_n(a_n^{k_0})=f_n(a_n^{k_0+1})$,
- $f_{n+i}(a_{n+i}^{k_i})\neq f_{n+i}(a_{n+i}^{k_i+1})$ for all $i>0$.

By the construction of functions f_n we have, for all $i > 0$,

$$\left|\frac{f_{n+i}(a_{n+i}^{k_i+1}) - f_{n+i}(a_{n+i}^{k_i})}{a_{n+i}^{k_i+1} - a_{n+i}^{k_i}}\right| = \frac{5}{6}\left(\frac{5}{2}\right)^{i-1}.$$

By Lemma 5.2(v) we have, for all $i > 0$,

$$\left|\frac{f(a_{n+i}^{k_i+1}) - f(x)}{a_{n+i}^{k_i+1} - x}\right| \geqslant \frac{5}{6}\left(\frac{5}{2}\right)^{i-1}$$

or

$$\left|\frac{f(x) - f(a_{n+i}^{k_i})}{x - a_{n+i}^{k_i}}\right| \geqslant \frac{5}{6}\left(\frac{5}{2}\right)^{i-1},$$

which is clearly enough. □

Theorem 5.1 now follows from Proposition 5.5, Lemma 5.6 and Proposition 5.7.

6. $\mathcal{M}_1$-points

It turns out that being an $\mathcal{M}_1$-point is a particularly simple and strong property: it is, modulo negligible set, equivalent to differentiability. We begin with an elementary lemma.

Lemma 6.1. *Let $f : I \to \mathbb{R}$ be continuous and $y \in I$. Suppose $\varepsilon > 0$ is such that condition (5) holds with $c = 1$. If there is $x \in (y-\varepsilon, y)$ such that $f(x) > f(y)$, then $\overline{f}^+(y) \leqslant \frac{y-x}{f(x)-f(y)}$.*

Proof. Let C be the open disc centered at $\psi(x)$ whose boundary circle passes through $\psi(x)$. If $z \in (y, y+\varepsilon)$, then $\psi(z) \notin C$. Therefore $\overline{f}^+(y)$ is less than or equal to the slope of the line tangent to C at $\psi(y)$. This slope is clearly equal to $\frac{y-x}{f(x)-f(y)}$, as required. □

Corollary 6.2. *Let $f : I \to \mathbb{R}$ be continuous and $y \in I$ an $\mathcal{M}_1$-point. If $\underline{f}^-(y) < 0$, then $\overline{f}^+(x) \leqslant \frac{1}{|\underline{f}^-(y)|}$.*

Theorem 6.3. *If $f : I \to \mathbb{R}$ is continuous, then there is a set $E \subseteq I$ such that $\mathscr{H}^1(\mathfrak{f}|E) = 0$ and $\mathcal{D}(f) \subseteq \mathcal{M}_1(f) \subseteq \mathcal{D}(f) \cup E$. In particular, f is differentiable at almost every $\mathcal{M}_1$-point.*

Proof. If f has a derivative, finite or infinite, at y, then there is obviously $\varepsilon > 0$ such that if $y - \varepsilon \leqslant x < y < z \leqslant y + \varepsilon$, then the angle spanned by the vectors $\psi(x) - \psi(y)$ and $\psi(z) - \psi(y)$ is obtuse and consequently

$$|\psi(y) - \psi(x)| \leqslant |\psi(z) - \psi(x)|, \tag{12}$$

which is nothing but condition (5) with $c = 1$. Hence y is an $\mathcal{M}_1$-point.

To prove the latter inclusion we employ the famous Denjoy–Young–Saks Theorem, cf. [10, IX(4.2)]: *There is a set $E \subseteq I$ such that $\mathscr{H}^1(\mathfrak{f}|E) = 0$ and for every point $x \notin E$ one of the following cases occurs:* (a) *$f'(x)$ exists,* (b) *x is a knot point,* (c) *$\overline{f}^+(x) = -\underline{f}^-(x) = \infty$ and $\overline{f}^-(x) = \underline{f}^+(x)$ are finite,* (d) *$\overline{f}^-(x) = -\underline{f}^+(x) = \infty$ and $\overline{f}^+(x) = \underline{f}^-(x)$ are finite.*

Suppose for contrary that there is $y \in \mathcal{M}_1(f) \setminus (\mathcal{D} \cup E)$. Then one of cases (b), (c), (d) occurs. Since (d) obtains from (c) by reversing the y-axis, we only have to consider (b) and (c). In either case, $\underline{f}^-(y) = -\infty$ and $\overline{f}^+(y) = \infty$. The above corollary yields $\overline{f}^+(x) \leqslant 0$: a contradiction. $\square$

The set $\mathfrak{f}|\,\mathcal{D}(f)$ is, by this theorem and Proposition 2.5(iii), a countable union of closed 1-monotone sets. An easy symmetry argument gives a bit more:

Corollary 6.4. *If $f : I \to \mathbb{R}$ is continuous, then $\mathfrak{f}|\,\mathcal{D}(f)$ admits a countable cover by symmetrically 1-monotone sets.*

1-monotone graphs behave particularly nice:

Theorem 6.5. *If I is compact and $f : I \to \mathbb{R}$ is continuous with a 1-monotone graph , then f is of bounded variation.*

Proof. Set $A = \{x \in I : \forall y \in [0, x)\ f(y) - f(x) < x - y\}$ and let $g(x) = f(x) + x$. Obviously $A = \{x \in I : \forall y \in [0, x)\ g(y) < g(x)\}$, hence g is increasing on A. By [10, VII(4.1)] there is a non-decreasing extension $g^* : I \to \mathbb{R}$ of g. If follows that $f^*(x) = g^*(x) - x$ is of bounded variation on $[0, 1]$ and clearly $f^*(x) = f(x)$ for all $x \in A$. Therefore $\mathscr{H}^1(\mathfrak{f}|A) \leqslant \mathscr{H}^1(\mathfrak{f}^*) < \infty$. The same argument shows that letting $B = \{x \in I : \forall y \in [0, x)\ f(x) - f(y) < x - y\}$ we have $\mathscr{H}^1(\mathfrak{f}|B) \leqslant \mathscr{H}^1(\mathfrak{f}^*) < \infty$.

Now suppose $x \notin A$, i.e. there is $y < x$ such that $f(y) - f(x) \geqslant x - y$. Lemma 6.1 yields $\overline{f}^+(x) \leqslant 1$. The same argument shows that if $x \notin B$, then $\underline{f}^+(x) \geqslant -1$. In summary, if $x \notin A \cup B$, then $|\overline{f}^+(x)| \leqslant 1$. By the remark following [10, IX(4.6)]

$$\mathscr{H}^1(\mathfrak{f}|I \setminus (A \cup B)) \leqslant \int_{I \setminus (A \cup B)} \sqrt{1 + (\overline{f}^+(x))^2} \leqslant \sqrt{2}\mathscr{L}(I) < \infty.$$

Altogether $\mathscr{H}^1(\mathfrak{f}) \leqslant \mathscr{H}^1(\mathfrak{f}|A) + \mathscr{H}^1(\mathfrak{f}|B) + \mathscr{H}^1(\mathfrak{f}|I \setminus (A \cup B)) < \infty$. In particular, f is of bounded variation. $\square$

Since every nondecreasing function has a 1-monotone graph, we have the following characterization of bounded variation.

Corollary 6.6. *A continuous function $f : [0, 1] \to \mathbb{R}$ is of bounded variation if and only if it is a sum of two continuous functions with 1-monotone graphs.*

7. An absolutely continuous function with a non-σ-monotone graph

If $f : I \to \mathbb{R}$ is absolutely continuous, then it is differentiable almost everywhere and, moreover, the set $\mathfrak{f}|I \setminus \mathcal{D}(f)$ is of length zero. Thus Corollary 6.4 yields:

Corollary 7.1. *If $f : I \to \mathbb{R}$ is absolutely continuous, then there is a countable family $\{M_n\}$ of symmetrically 1-monotone sets such that*

$$\mathscr{H}^1\Big(\mathfrak{f} \setminus \bigcup\nolimits_{n \in \mathbb{N}} M_n\Big) = 0.$$

We want to show that this fact cannot be sharpened by providing an example of an absolutely continuous function whose graph is not σ-monotone.

Recall the notion of strong porosity, as defined in [5]. A set $X \subseteq \mathbb{R}^2$ is termed *strongly porous* if there is $p > 0$ such that for any $x \in \mathbb{R}^2$ and any $r \in (0, \operatorname{diam} X)$ there is $y \in \mathbb{R}^2$ such that $B(y, pr) \subseteq B(x, r) \setminus X$. The constant p is termed a *porosity constant* of X. As proved in [5, Theorem 4.2], every monotone set in $\mathbb{R}^2$ is strongly porous. More information on porosity properties of monotone sets in $\mathbb{R}^n$ can be found in [14].

M. Zelený [13] found an example of an absolutely continuous function whose graph is not σ-porous[1], and since a countable union of strongly porous sets is σ-porous, we have, in view of [5, Theorem 4.2] mentioned above, the following theorem.

Theorem 7.2. *There is an absolutely continuous function on* $[0,1]$ *whose graph is not* σ*-monotone.*

Zelený's example is rather involved. We provide another example that is much simpler and moreover it exhibits that the implication monotone $\Rightarrow$ strongly porous cannot be reversed even for graphs.

Theorem 7.3. *There is an absolutely continuous function* $f : [0,1] \to \mathbb{R}$ *whose graph is strongly porous, but every monotone subset of* $\mathfrak{f}$ *is nowhere dense. In particular,* $\mathfrak{f}$ *is not* σ*-monotone.*

The function is built of single peak functions. Let $\|x\| = \operatorname{dist}(x, \mathbb{R} \setminus [-1,1])$. Fix two sequences of positive reals $\langle a_n \rangle$ and $\langle b_n \rangle$. Suppose that $\sum_n a_n < \infty$ and let the sequence $\langle q_n \rangle$ enumerate all rationals within $[0,1]$. The following formula defines a real-valued function $f : [0,1] \to \mathbb{R}$.

$$f(x) = \sum_{n \in \mathbb{N}} a_n \Big\| \frac{x - q_n}{b_n} \Big\|$$

We will show that with a proper choice of the two sequences the function f possesses the required properties.

For simplicity stake write $f_n(x) = a_n \big\| \frac{x-q_n}{b_n} \big\|$ and $s_n = \frac{a_n}{b_n}$.

Lemma 7.4. *If* $\sum_n a_n < \infty$*, then* f *is absolutely continuous.*

Proof. Fix $\varepsilon > 0$. Choose $m \in \mathbb{N}$ such that $\sum_{n>m} a_n \leqslant \varepsilon$ and let $\delta = \frac{\varepsilon}{\sum_{n \leqslant m} s_n}$. Suppose $x_0 < y_0 < x_1 < y_1 < \cdots < x_k < y_k$ satisfy $\sum_{i=0}^{k} y_i - x_i < \delta$. Since $|f_n(x_i) - f_n(y_i)| \leqslant s_n(y_i - x_i)$ for all i and n, we have, for all n,

$$\sum_{i=0}^{k} |f_n(x_i) - f_n(y_i)| \leqslant \sum_{i=0}^{k} s_n(y_i - x_i) < \delta s_n \tag{13}$$

and since the function f_n is unimodal and ranges between 0 and a_n, also

$$\sum_{i=0}^{k} |f_n(x_i) - f_n(y_i)| \leqslant 2a_n. \tag{14}$$

Use (13) for $n \leqslant m$ and (14) for $n > m$ to get

$$\begin{aligned}\sum_{i=0}^{k} |f(x_i) - f(y_i)| &\leqslant \sum_{n \leqslant m} \sum_{i=0}^{k} |f_n(x_i) - f_n(y_i)| + \sum_{n>m} \sum_{i=0}^{k} |f_n(x_i) - f_n(y_i)| \\ &\overset{(13,14)}{<} \sum_{n \leqslant m} \delta s_n + \sum_{n>m} 2a_n \leqslant \varepsilon + 2\varepsilon = 3\varepsilon,\end{aligned}$$

the last inequality by the choice of ε and δ. $\square$

[1] See [13] or [11, 12] for the definition of σ-porous.

Lemma 7.5. *If* $\lim_{m\to\infty} \frac{\sum_{n>m} a_n}{a_m} = 0$ *and* $\lim_{m\to\infty} \frac{\sum_{n<m} s_n}{s_m} = 0$, *then* $\mathcal{M}(f)$ *is meager.*

Proof. It is clear that if $s_m > 2c$, then the points $q_m - b_m < q_m < q_m + b_m$ witness that the graph f_m is not c-monotone. We want to show that the same argument works for the entire sum $f = \sum_n f_n$. The former condition ensures that the terms f_n, $n > m$, contribute to the sum negligible quantities because of their small magnitudes. The latter condition ensures that also the terms f_n, $n < m$, are negligible because of their small slopes.

Write

$$\varepsilon_m = \frac{\sum_{n>m} a_n}{a_m} + \frac{\sum_{n<m} s_n}{s_m} = \frac{\sum_{n>m}^{\infty} a_n + b_m \sum_{n<m} s_n}{a_m}.$$

According to Propositions 2.7, 2.5 and Lemma 2.6 it is enough to show that $\mathfrak{f}|I$ is monotone for no interval I.

Fix $c > 0$ and an interval I. The hypotheses ensure that $\varepsilon_m \to 0$ and $s_m \to \infty$. Therefore if m is large enough m, then

$$\frac{1-\varepsilon_m}{2(\frac{1}{s_m} + \varepsilon_m)} > c. \tag{15}$$

Choose such na m subject to $[q_m - b_m, q_m + b_m] \subseteq I$. Write $x = q_m - b_m$, $z = q_m + b_m$. If we succeed to prove that

$$|\psi(z) - \psi(q_m)| > c|\psi(z) - \psi(x)|, \tag{16}$$

we will be done, because the points $x < q_m < z$ will witness that $\mathfrak{f}|I$ is not c-monotone. Estimate the term on the left

$$\begin{aligned}
|\psi(z) - \psi(q_m)| &\geqslant |f(z) - f(q_m)| \geqslant |f_m(z) - f_m(q_m)| - \sum_{n\neq m} |f_n(z) - f_n(q_m)| \\
&\geqslant a_m - \Big(\sum_{n<m} |f_n(z) - f_n(q_m)| + \sum_{n>m} |f_n(z) - f_n(q_m)|\Big) \\
&\geqslant a_m - \Big(\sum_{n<m} s_n b_m + \sum_{n>m} a_n\Big) = a_m - \varepsilon_m a_m = a_m(1-\varepsilon_m),
\end{aligned}$$

and the term on the right

$$\begin{aligned}
|\psi(z) - \psi(x)| &\leqslant 2b_m + |f(z) - f(y)| \\
&\leqslant 2b_m + \sum_{n<m} |f_n(z) - f_n(x)| + \sum_{n>m} |f_n(z) - f_n(x)| \\
&\leqslant 2b_m + 2b_m \sum_{n<m} s_n + \sum_{n>m} a_n \leqslant 2b_m + 2\Big(b_m \sum_{n<m} s_n + \sum_{n>m} a_n\Big) \\
&\leqslant 2(b_m + \varepsilon_m a_m) = 2a_m\Big(\frac{1}{s_m} + \varepsilon_m\Big).
\end{aligned}$$

Thus (15) yields

$$\frac{|\psi(z) - \psi(q_m)|}{|\psi(z) - \psi(x)|} \geqslant \frac{a_m(1-\varepsilon_m)}{2a_m\big(\frac{1}{s_m} + \varepsilon_m\big)} = \frac{1-\varepsilon_m}{2\big(\frac{1}{s_m} + \varepsilon_m\big)} > c$$

and (16) follows. ∎

The next goal is to show that with a proper choice of $\langle a_n\rangle$ and $\langle b_n\rangle$ the graph of f is porous. To that end we introduce the following system of rectangles. Let $\mathcal{R}$ denote the family of all planar rectangles $I \times J$, where I, J are compact intervals, with aspect ratio $5:3$, i.e. $\frac{\mathscr{L}(I)}{\mathscr{L}(J)} = \frac{5}{3}$. Each $R \in \mathcal{R}$ is covered in a natural way by 15 non-overlapping closed squares with side one fifth of the length of the base of R. The family of these squares will be denoted $\mathcal{S}(R)$. These squares determine in a natural way five closed columns and three closed rows.

Given $R \in \mathcal{R}$, the length of the base of R is denoted $\ell(R)$.

Lemma 7.6. *There are sequences $\langle a_n\rangle$ and $\langle b_n\rangle$ satisfying hypotheses of Lemma 7.5 such that for each $R \in \mathcal{R}$ there is a square $S \in \mathcal{S}(R)$ such that* $\operatorname{int} S \cap \mathfrak{f} = \emptyset$.

Proof. We build the sequences recursively. Let $g_n = \sum_{i \leqslant n} f_i$, $n \in \mathbb{N}$, be the partial sums of f; graphs of g_n are denoted $\mathfrak{g}_n$. Our goal is to find a_n's and b_n's so that for each n the following holds:

(C_n) For each $R \in \mathcal{R}$ there is a square $S \in \mathcal{S}(R)$ disjoint with $\mathfrak{g}_n$.

Choose a_0 and b_0 so that $s_0 > 3$. The graph of f_0 is obviously covered by three lines: two skewed and one horizontal. Let $R \in \mathcal{R}$. Each of the two skewed lines, because of their big slopes, can meet at most two out of the five columns. Therefore one column remains left. The horizontal line meets at worst two of the three squares forming this column. Thus one square remains disjoint with each of the three lines and thus with the graph $\mathfrak{g}_0$ of $g_0 = f_0$. Thus condition C_0 is met.

Now suppose that a_i and b_i are set up for all $i < n$ so that condition C_{n-1} is met. Let $\varepsilon_n = \min\{|q_i - q_j| : 0 \leqslant i < j \leqslant n\}$.

Claim. *There is $\delta_n > 0$ such that if $\ell(R) \geqslant \varepsilon_n$, then there is $S \in \mathcal{S}(R)$ that is at least δ_n far apart from $\mathfrak{g}_{n-1}$.*

Proof. Suppose the contrary: For each m there is $R_m \in \mathcal{R}$ such that $\ell(R_m) \geqslant \varepsilon_n$ and the distance of S from $\mathfrak{g}_{n-1}$ is less than $\frac{1}{m}$ for each square $S \in \mathcal{S}(R_m)$. In particular, $\ell(R_m) \leqslant 5$ for all $m \geqslant 1$ and there is a bounded set that contains all rectangles R_m. Thus passing to a subsequence we may suppose that $\langle R_m\rangle$ is convergent in the Hausdorff metric. The limit R of this sequence is clearly a rectangle with aspect ratio $5:3$ or a point. But the latter cannot happen, because $\ell(R_m) \geqslant \varepsilon_n$ for each m. Thus $R \in \mathcal{R}$. The distance of $\mathfrak{g}_{n-1}$ from each of the squares $S \in \mathcal{S}(R)$ is obviously zero. Since the squares are compact, $\mathfrak{g}_{n-1}$ meets all of them: the desired contradiction. □

Choose $a_n < \delta_n$ and b_n subject to

$$a_n \leqslant \frac{2^{-n}}{n}, \qquad s_n > 2^n \sum_{i<n} s_i. \tag{17}$$

We need to show that thus chosen values ensure condition C_n.

Suppose first that $\ell(R) \geqslant \varepsilon_n$. There is $S \in \mathcal{S}(R)$ such that $\operatorname{dist}(S, \mathfrak{g}_{n-1}) \geqslant \delta_n$. Consequently

$$\begin{aligned}\operatorname{dist}(S, \mathfrak{g}_n) &\geqslant \operatorname{dist}(S, \mathfrak{g}_{n-1}) - \operatorname{dist}(\mathfrak{g}_{n-1}, \mathfrak{g}_n)\\ &\geqslant \delta_n - \max|g_{n-1} - g_n| = \delta_n - \max|f_n| = \delta_n - a_n > 0.\end{aligned}$$

Thus S is disjoint with $\mathfrak{g}_n$.

To treat the case $\ell(R) < \varepsilon_n$ we first prove

Claim. *If $g_n(x) > 0$ and a local maximum of g_n occurs at x, then $x = q_j$ for some $j \leqslant n$.*

Proof. Suppose $g_n(x) > 0$ and there is a local maximum of g_n at x. We examine the left-sided derivative $g_n^-(x)$. Clearly $g_n^-(x) = \sum_{i \leqslant n} f_i^-(x)$ and each $f_i^-(x)$ is either 0, or s_i, or $-s_i$. If all of them were 0, the value $g_n(x)$ would be 0, so there is $i \leqslant n$ such that $f_i^-(x) \neq 0$. Let $j = \max\{i \leqslant n : f_i^-(x) \neq 0\}$. First of the conditions (17) yields $|\sum_{i<j} f_i^-(x)| < s_j$. Since $g_n^-(x) \geqslant 0$, it follows that $f_j^-(x) = s_j$.

By the same analysis of the right-sided derivative, letting $k = \max\{i \leqslant n : f_i^+(x) \neq 0\}$ we have $f_k^+(x) = -s_k$.

Suppose that $j < k$. Then, by the definition of j, $f_k^-(x) = 0$ and $f_k^+(x) = -s_k$. But there is no such point. Thus $j < k$ fails. The same argument proves that $j > k$ fails as well. Therefore $j = k$. Overall, $f_j^-(x) = s_j$ and $f_j^+(x) = -s_j$. The only point with this property is q_j. $\square$

Now suppose $R = I \times J \in \mathcal{R}$ and that $\ell(R) < \varepsilon_n$. It is clear that if the graph $\mathfrak{g}_n$ passes through all squares $S \in \mathcal{S}(R)$, then g_n has at least two positive local maxima in I. Therefore, by the above Claim, there are $i < j \leqslant n$ such that both q_i and q_j belong to I. Consequently $|q_i - q_j| \leqslant \mathscr{L}(I) = \ell(R) < \varepsilon_n$, which contradicts the definition of ε_n. Thus $\mathfrak{g}_n$ misses at least one of the squares $S \in \mathcal{S}(R)$. The proof of condition C_n is finished.

It remains to draw the statement of the lemma from conditions C_n. Fix $R \in \mathcal{R}$. Since there are only finitely many squares in $\mathcal{S}(R)$, there is $S \in \mathcal{S}(R)$ such that the set $F = \{n : \mathfrak{g}_n \cap S = \emptyset\}$ is infinite. Since $f = \lim_{n \in F} g_n$, we have $\mathfrak{f} \subseteq \overline{\bigcup_{n \in F} \mathfrak{g}_n}$. Therefore $\mathfrak{f}$ does not meet $\operatorname{int} S$.

Conditions (17) ensure that f satisfies hypotheses of Lemma 7.5. $\square$

Proof of Theorem 7.3. The required function f is of course the one constructed in the above lemma. Let $B(x, r)$ be any closed ball in $\mathbb{R}^2$. Inscribe in $B(x, r)$ a rectangle $R \in \mathcal{R}$, as big as possible. By the above lemma there is a square $S \in \mathcal{S}(r)$ such that $\operatorname{int} S$ misses $\mathfrak{f}$. Inscribe into S an open ball B. This ball is disjoint with $\mathfrak{f}$. The radius of this ball is by trivial calculation $r/\sqrt{34}$. The closed ball concentric with B and of radius $\frac{r}{6}$ is thus disjoint with $\mathfrak{f}$. We proved that $\mathfrak{f}$ is strongly porous.

The function f is absolutely continuous by Lemma 7.4 and $\mathfrak{f}$ is not σ-monotone by Lemma 7.5. $\square$

Since any monotone function has trivially a 1-monotone graph, and since every absolutely continuous function is a difference of two increasing functions, we have

Corollary 7.7. *A sum of two functions with* 1*-monotone graphs need not have a σ-monotone graph.*

8. Remarks and questions

We conclude with several remarks and questions that we consider interesting.

Hausdorff dimension. If a continuous function $f : I \to \mathbb{R}$ has a monotone graph, then $\dim_{\mathsf{H}} \mathfrak{f} = 1$ by Corollary 3.6. The analogy for σ-monotone graph fails:

Proposition 8.1. *There is a continuous function $f : [0, 1] \to \mathbb{R}$ with a σ-monotone graph such that $\dim_{\mathsf{H}} \mathfrak{f} > 1$. Any such function admits a perfect set of non-$\mathcal{M}$-points.*

Proof. There is a continuous function $g : [0,1] \to \mathbb{R}$ such that $\dim_{\mathsf{H}} \mathfrak{g} > 1$, cf. e.g. [4, Chapter 11]. By [6], there is a monotone compact set $K \subseteq \mathfrak{g}$ such that $\dim_{\mathsf{H}} K > 1$. Let $C = \{x \in [0,1] : \psi(x) \in K\}$. Define f to coincide with g on C and on each component of the complement of C let f be linear and so that it is continuous on $[0,1]$. Since there are only countably many components, the resulting function has a σ-monotone graph.

To prove the second statement notice that Theorem 3.5 yields $\dim_{\mathsf{H}} \mathfrak{f}|\,\mathcal{M}(f) = 1$ and thus if $\dim_{\mathsf{H}} \mathfrak{f} > 1$, then the set of non-$\mathcal{M}$-points certainly contains a perfect set. □

Nowhere differentiable functions. The nowhere differentiable function of Proposition 4.8 has no $\mathcal{M}$-points. Though there is a plethora of other nowhere differentiable functions with the same property and the argument for nonexistence of $\mathcal{M}$-points seems similar to that for nonexistence of derivatives, in general we know about nowhere differentiable functions only Corollary 4.7: the set of $\mathcal{M}$-points is meager.

Question 8.2. Is there a continuous nowhere differentiable function with a dense or even perfectly dense set of $\mathcal{M}$-points? What about $\mathcal{M}_1$-points?

The Baire category arguments used cannot be adapted to subsets of a graph that are of positive measure, since such sets may be totally disconnected and thus have way too many candidates for witnessing order to check.

Question 8.3. Let f be the function of Proposition 4.8. Is there a set $A \subseteq [0,1]$ of positive measure such that $\mathfrak{f}|A$ is monotone?

Bounded variation. By Theorem 6.5, a continuous function with a 1-monotone graph is of bounded variation. It also follows from Proposition 4.1 that a continuous function with a monotone, rectifiable graph is differentiable almost everywhere.

Question 8.4. Is there a continuous function on $[0,1]$ with a monotone, rectifiable graph that is not of bounded variation?

Luzin property. Recall that f satisfies *Luzin condition* if $\mathscr{L}(f(A)) = 0$ whenever $\mathscr{L}(A) = 0$. Note that if f has a monotone graph, then it satisfies Luzin condition "almost everywhere": Letting $\mathcal{D}_\infty = \{x \in \mathcal{D}(f) : |f'(x)| = \infty\}$, we have $\mathscr{L}(\mathcal{D}_\infty) = 0$ and if $A \cap \mathcal{D}_\infty = \emptyset$, then $\mathscr{L}(A) = 0$ implies $\mathscr{L}(f(A)) = 0$. Hence f satisfies Luzin condition if and only if $\mathscr{L}(f(\mathcal{D}_\infty)) = 0$.

The following easily follows from Theorem 4.4.

Proposition 8.5. *A continuous function satisfying Luzin condition with a σ-monotone graph is differentiable at a set that has positive measure within each interval.*

Question 8.6. Is a continuous function satisfying Luzin condition with a monotone graph differentiable almost everywhere?

Porosity constant. We know from [5, Theorem 4.2] that any monotone set in $\mathbb{R}^2$ is strongly porous, and from Theorem 7.3 that the converse fails. In our proof we showed that the porosity constant of $\mathfrak{f}$ can be pushed to $1/\sqrt{34}$. Perhaps a set must be σ-monotone if it is strongly porous and its porosity constant is large enough? For compact sets in the plane it is not so: Let $C \subseteq \mathbb{R}$ be a strongly porous perfect set such that every $p < \frac{1}{2}$ is a porosity constant of C. The set $C \times [0,1]$ clearly has the same property. On the other hand, by [5, Lemma 2.1] it is not σ-monotone.

Hence there is a strongly porous compact set $X \subseteq \mathbb{R}^2$ such that every $p < \frac{1}{2}$ is its porosity constant and yet X is not σ-monotone. But what about curves and graphs?

Question 8.7. Is there $p < \frac{1}{2}$ such that every strongly porous curve in $\mathbb{R}^2$ with porosity constant p is monotone or σ-monotone? What about graphs of continuous functions?

Monotone graph vs. continuity. Say that a function $f : I \to \mathbb{R}$ is *σ-continuous* if there is a partition $\{D_n : n \in \mathbb{N}\}$ of I such that $f \restriction D_n$ is continuous for each n. We claim that monotone graph does not imply σ-continuity. To see that, let $C \subseteq [0,1]$ be the usual Cantor ternary set and $g : C \to C$ a non-σ-continuous function. By [5, Proposition 4.6] $C \times C$ is monotone. Therefore so is the graph of g. Now extend g to $f : [0,1] \to \mathbb{R}$ by $f(x) = -1$ for $x \notin C$. Easy to check that $\mathfrak{f}$ is monotone and yet not σ-continuous.

How about 1-monotone graphs? Consider the function $f : [0,1] \to \mathbb{R}$ defined by $f(x) = -1$ if x is rational and $f(x) = x$ otherwise. The graph of f is 1-monotone, but f is continuous at no point. However, f is continuous on both rationals and irrationals.

Question 8.8. Is a function with a 1-monotone graph σ-continuous?

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